# Algebra of the Infrared

SCGP, October 15, 2013

Gregory Moore, Rutgers University

*…<u>work in progress</u> ….*

collaboration with
Davide Gaiotto & Edward Witten

# Three Motivations

1. Two-dimensional N=2 Landau-Ginzburg models.

2. Knot homology.

3. Categorification of 2d/4d wall-crossing formula.

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

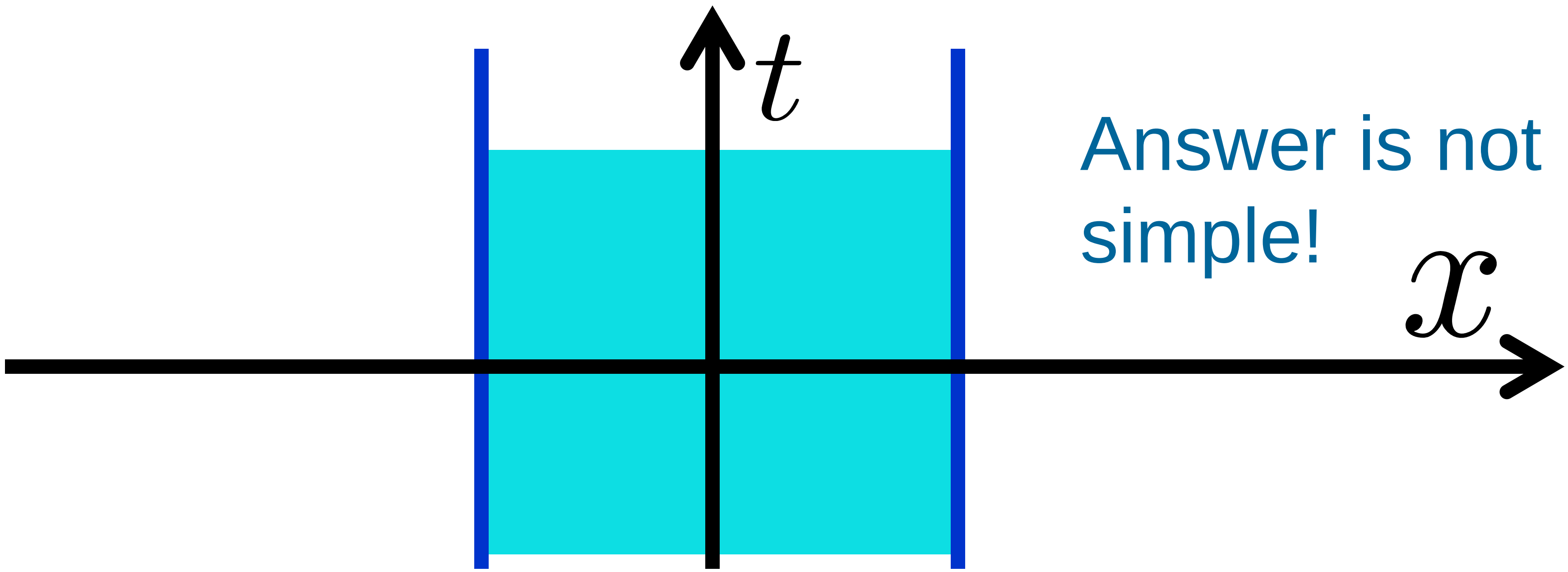
# D=2, $\mathcal{N}$=2 Landau-Ginzburg Theory

X: Kähler manifold

W: X $\longrightarrow \mathbb{C}$ Superpotential (A holomorphic Morse function)

Simple question:

What is the space of BPS states on an interval ?




Answer is not simple!

Witten (2010) reformulated knot homology in terms of Morse complexes.

This formulation can be further refined to a problem in the categorification of Witten indices in certain LG models (Haydys 2010, Gaiotto-Witten 2011)

Gaiotto-Moore-Neitzke studied wall-crossing of BPS degeneracies in 4d gauge theories. This leads naturally to a study of Hitchin systems and Higgs bundles.

When adding surface defects one is naturally led to a "nonabelianization map" inverse to the usual abelianization map of Higgs bundle theory. A "categorification" of that map should lead to a categorification of the 2d/4d wall-crossing formula.

# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

# Definition of a Plane Web

We begin with a purely mathematical construction.

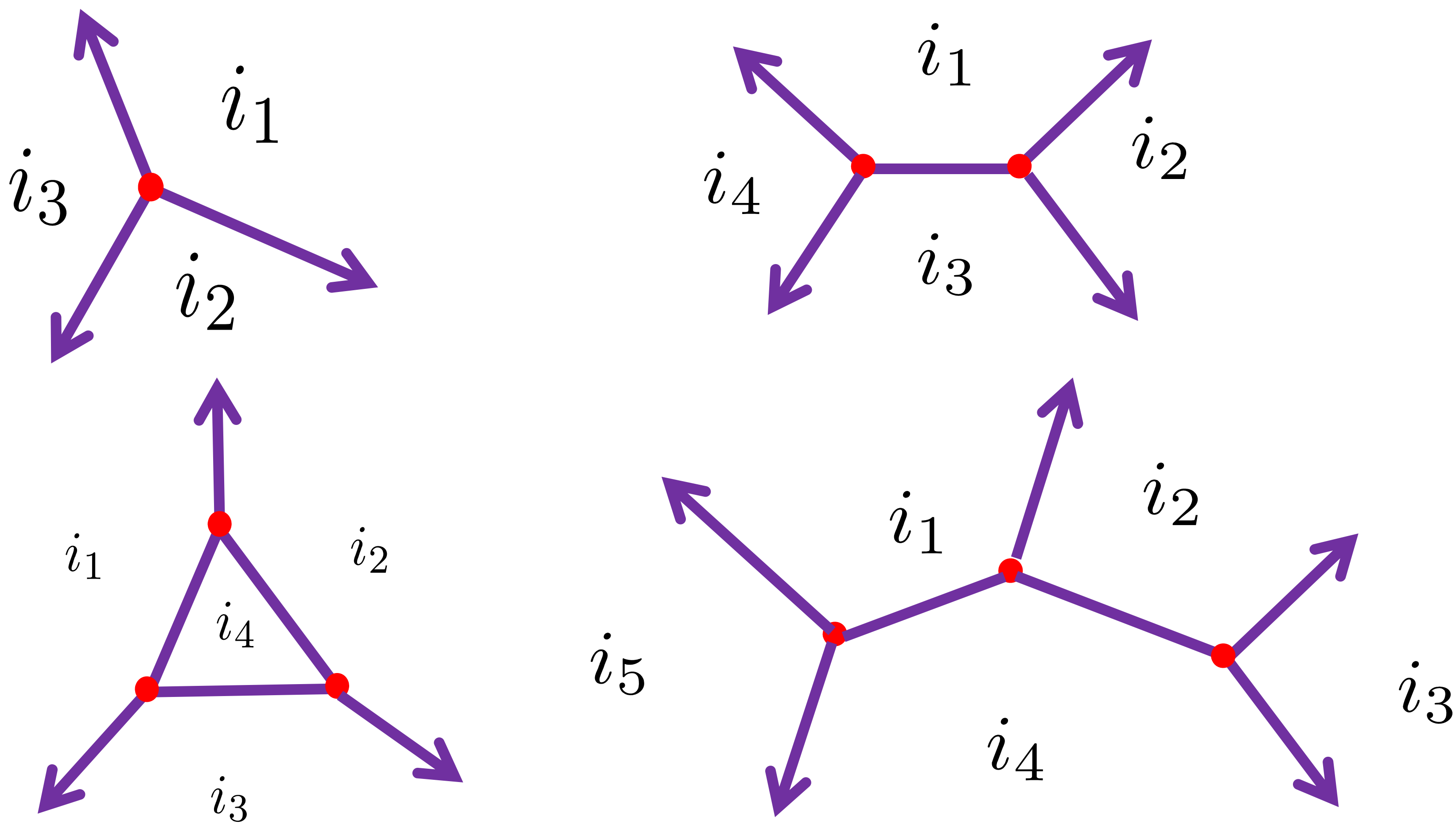
We show later how it emerges from LG field theory.

Vacuum data:

1. A finite set of ``vacua'': $i, j, k, \cdots \in \mathbb{V}$

2. A set of weights $z : \mathbb{V} \to \mathbb{C}$

**Definition:** A *plane web* is a graph in $\mathbb{R}^2$, together with a labeling of faces by vacua so that across edges labels differ and if an edge is oriented so that *i* is on the left and *j* on the right then the edge is parallel to $z_{ij} = z_i - z_j$ .

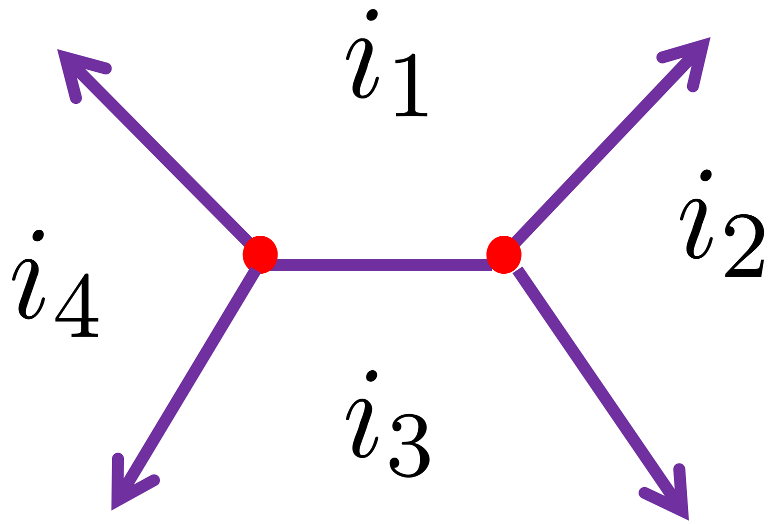


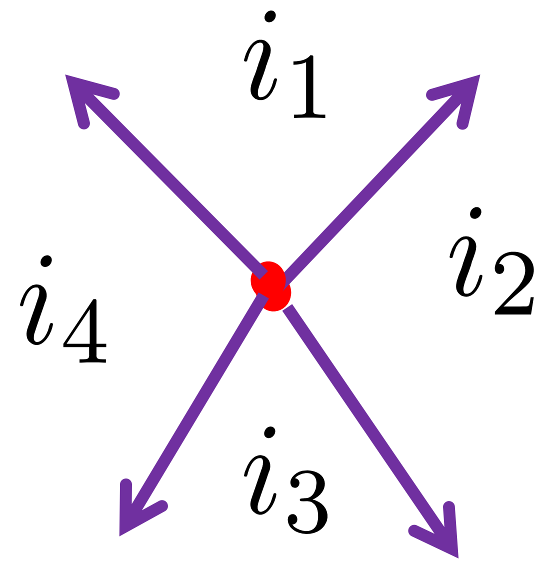

Useful intuition: We are joining together straight strings under a tension $z_{ij}$. At each vertex there is a no-force condition:

$$z_{i_1,i_2} + z_{i_2,i_3} + \cdots z_{i_n,i_1} = 0$$

# Deformation Type

Equivalence under translation and stretching (but not rotating) of strings subject to no-force constraint defines ***deformation type***.



# Moduli of webs with fixed deformation type

$$\dim \mathcal{D}(\mathfrak{w}) = 2V(\mathfrak{w}) - E(\mathfrak{w})$$

$V(\mathfrak{w}), E(\mathfrak{w})$ Number of vertices, internal edges.

($z_i$ in generic position)

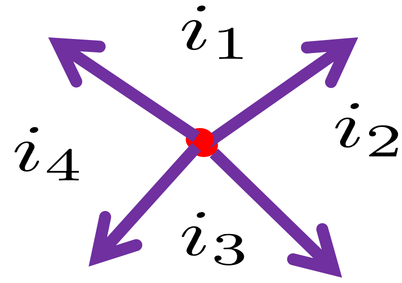
$$\mathcal{D}^{\mathrm{red}}(\mathfrak{w}) = \mathcal{D}(\mathfrak{w})/\mathbb{R}^2_{\mathrm{transl}}$$

$$\dim \mathcal{D}^{\mathrm{red}}(\mathfrak{w}) = d(\mathfrak{w})$$

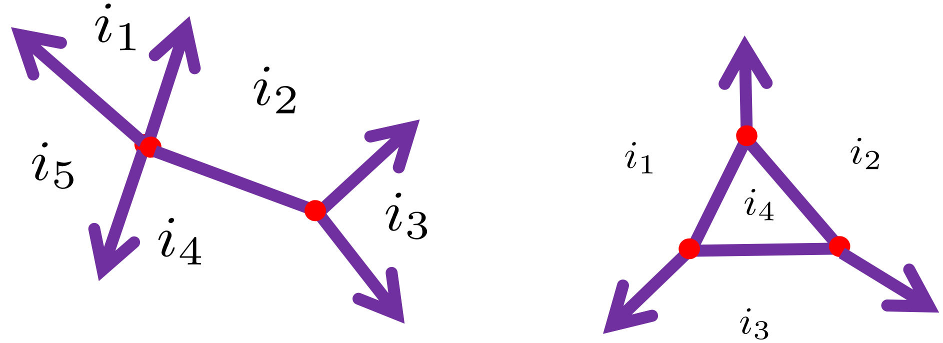
$$d(\mathfrak{w}) := 2V(\mathfrak{w}) - E(\mathfrak{w}) - 2$$

# Rigid, Taut, and Sliding

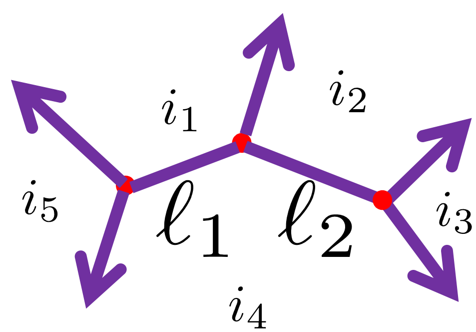
A ***rigid web*** has d($\mathfrak{w}$) = 0.
It has one vertex:




A ***taut web*** has
d($\mathfrak{w}$) = 1:




A ***sliding web*** has
d($\mathfrak{w}$) = 2



# Cyclic Fans of Vacua

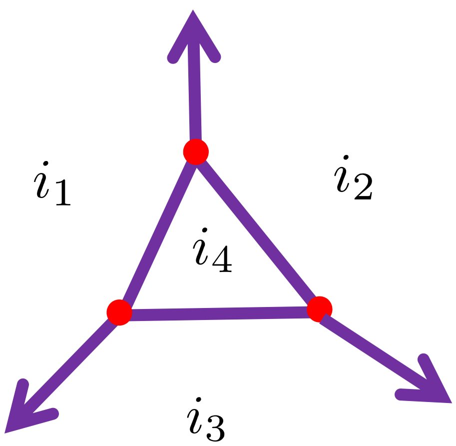
***Definition***: A cyclic fan of vacua is a cyclically-ordered set

$$I = \{i_1, \dots, i_n\}$$

so that the rays $z_{i_k, i_{k+1}} \mathbb{R}_+$ are ordered clockwise

Local fan of vacua at a vertex *v*:

$$I_v(\mathfrak{w})$$




and at $\infty$

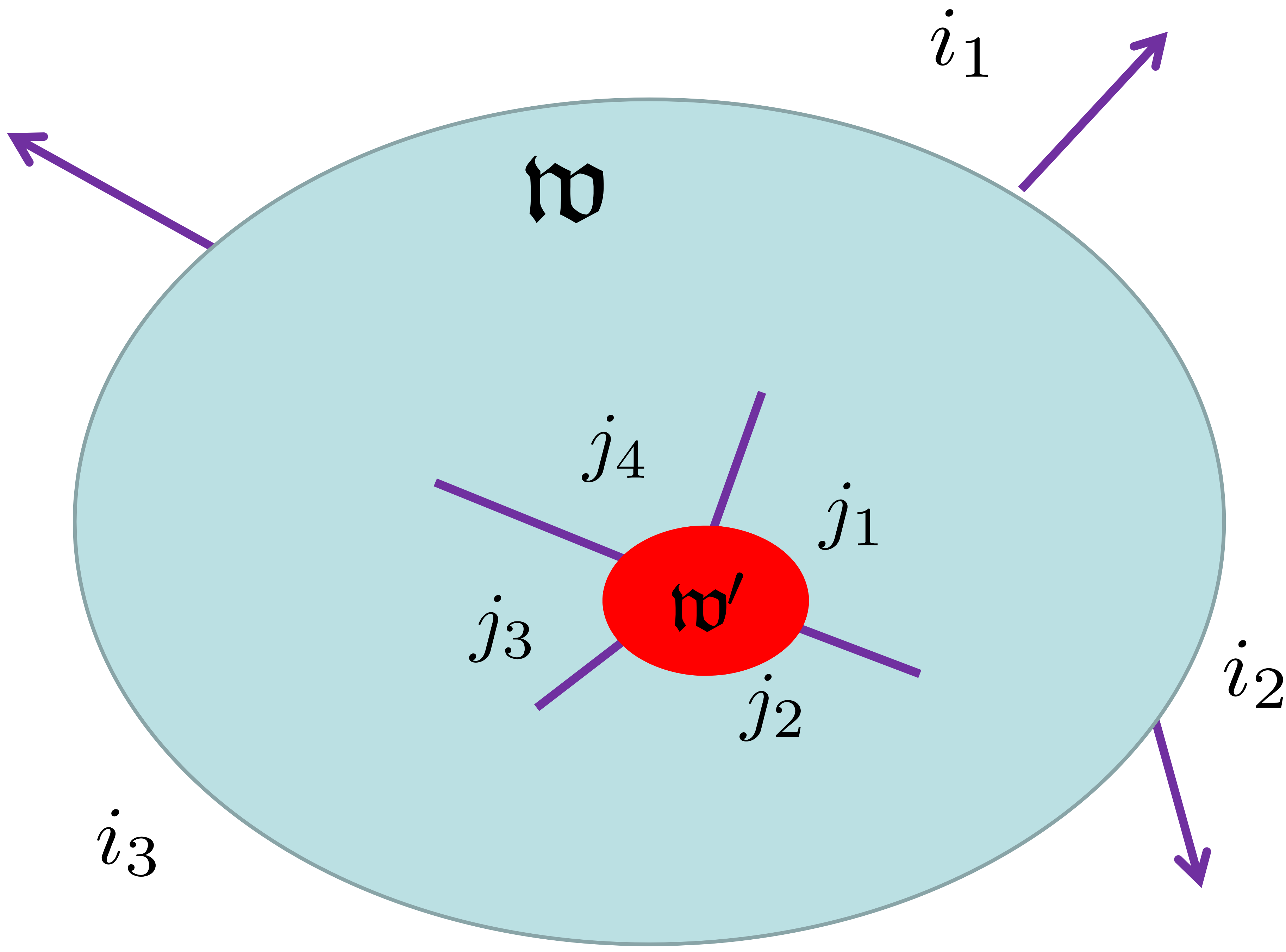
$$I_\infty(\mathfrak{w})$$

# Convolution of Webs

***Definition:*** Suppose $\mathfrak{w}$ and $\mathfrak{w}'$ are two plane webs and $v \in \mathcal{V}(\mathfrak{w})$ such that

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}')$$

The *convolution of $\mathfrak{w}$ and $\mathfrak{w}'$*, denoted $\mathfrak{w} *_v \mathfrak{w}'$ is the deformation type where we glue in a copy of $\mathfrak{w}'$ into a small disk cut out around *v*.



$i_1$
$\mathfrak{w}$
$j_4$
$j_1$
$\mathfrak{w}'$
$j_3$
$i_2$
$j_2$
$i_3$

# The Web Ring

$\mathcal{W}$ Free abelian group generated by oriented deformation types of plane webs.

``oriented'': Choose an orientation o($\mathfrak{w}$) of $\mathcal{D}^{\mathrm{red}}(\mathfrak{w})$

$$* : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$$

$$I_v(\mathfrak{w}_1) \neq I_\infty(\mathfrak{w}_2) \Rightarrow \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

$$\mathfrak{w}_1 * \mathfrak{w}_2 := \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2$$

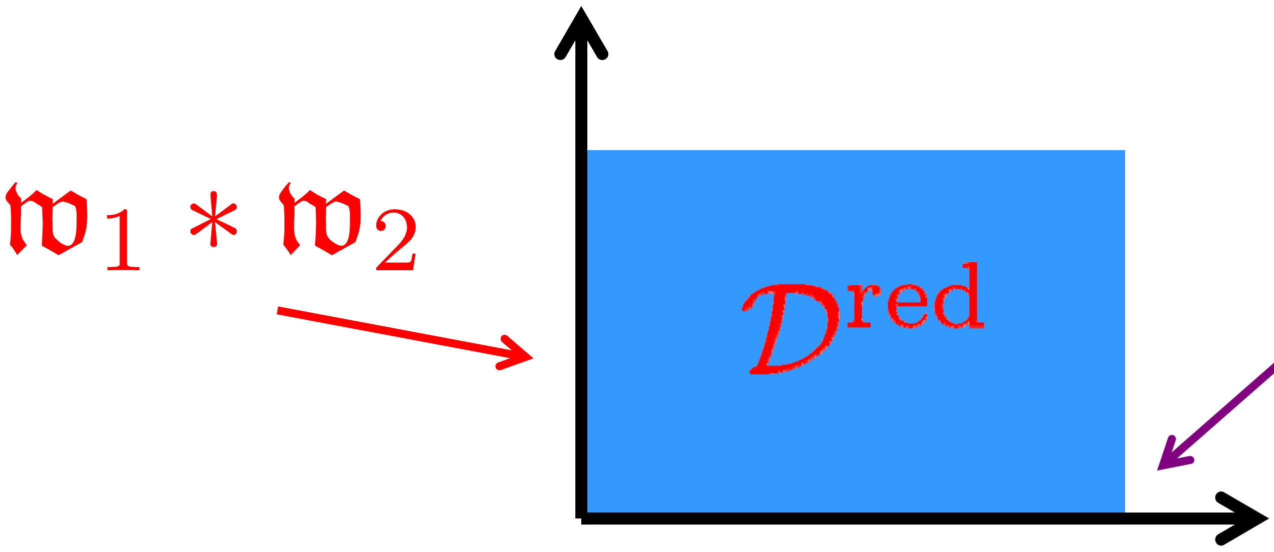
$$o(\mathfrak{w} *_v \mathfrak{w}') = o(\mathfrak{w}) \wedge o(\mathfrak{w}')$$

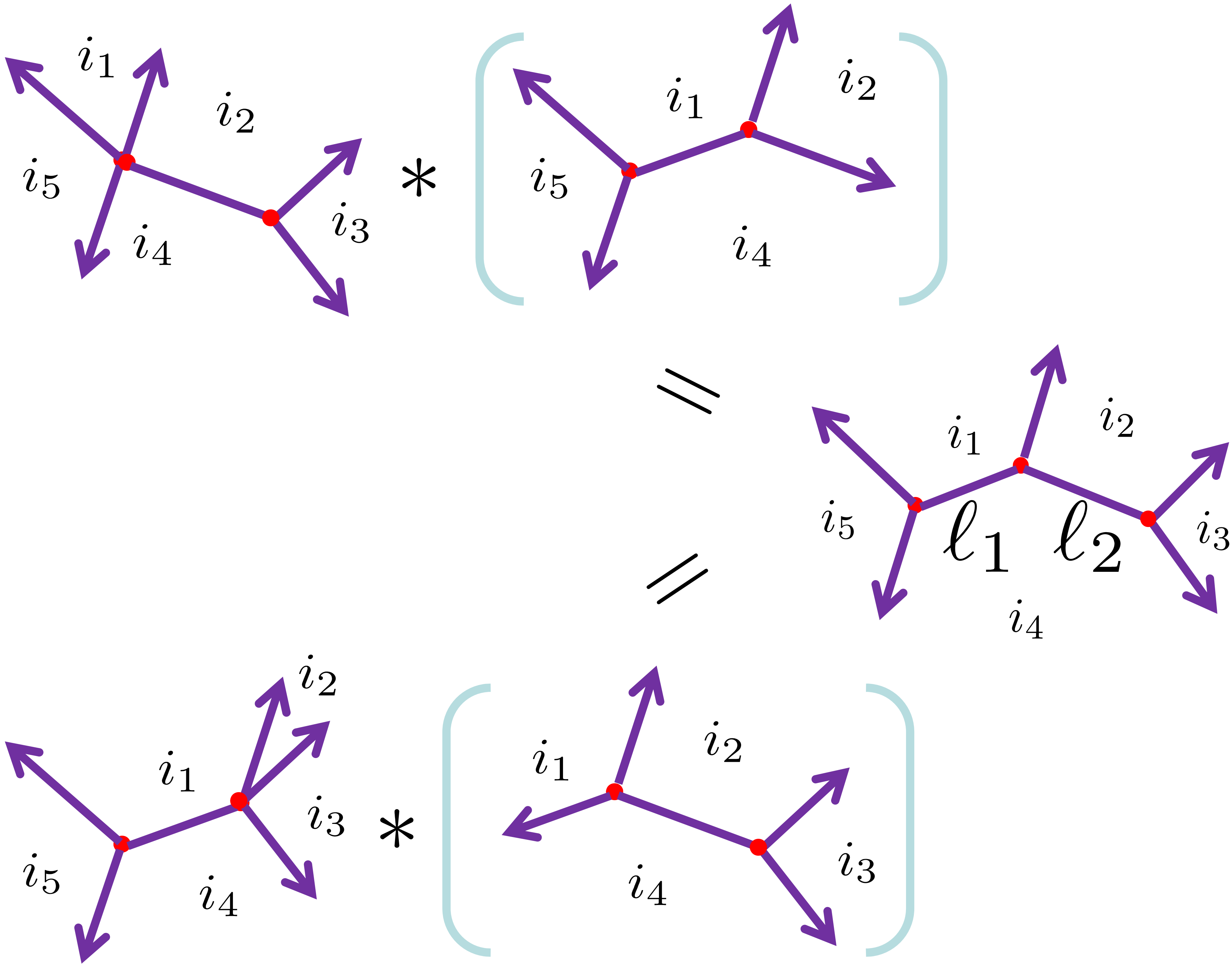
# The taut element

***Definition:*** The taut element $\mathfrak{t}$ is the sum of all taut webs with standard orientation

$$\mathfrak{t} := \sum_{d(\mathfrak{w})=1} \mathfrak{w}$$

Theorem: $\mathfrak{t} * \mathfrak{t} = 0$





i1
i2
i5
i4
i3
*
i1
i2
i5
i4
=
i1
i2
i5
ℓ1
ℓ2
i3
i4
=
i2
i1
i3
i5
i4
*
i1
i2
i4
i3

# Extension to the tensor algebra

Define an operation by taking an unordered set $\{v_1, \ldots, v_m\}$ and an ordered set $\{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$ and saying

$$\mathfrak{w} *_{\{v_1, \ldots, v_m\}} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$$

- vanishes unless there is some ordering of the $v_i$ so that the fans match up.
- when the fans match up we take the appropriate convolution.

$$T\mathcal{W} := \mathcal{W} \oplus \mathcal{W}^{\otimes 2} \oplus \mathcal{W}^{\otimes 3} \oplus \cdots$$

$$T(\mathfrak{w}) : T\mathcal{W} \to \mathcal{W}$$

$$T(\mathfrak{w})[\mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_n] := \mathfrak{w} *_{\mathcal{V}(\mathfrak{w})} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

# Convolution Identity on Tensor Algebra

$$\mathfrak{t} * \mathfrak{t} = 0 \quad \Longrightarrow \quad T(\mathfrak{t})$$ satisfies $L_\infty$ relations

$$\sum_{\mathrm{Sh}_2(S)} \epsilon\, T(\mathfrak{t})[T(\mathfrak{t})[S_1], S_2] = 0.$$

$$S = \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

Two-shuffles: $\mathrm{Sh}_2(S)$ $\quad S = S_1 \amalg S_2$

This makes $\mathcal{W}$ into an $L_\infty$ algebra

# Half-Plane Webs

Same as plane webs, but they sit in a half-plane $\mathcal{H}$.

Some vertices (but no edges) are allowed on the boundary.

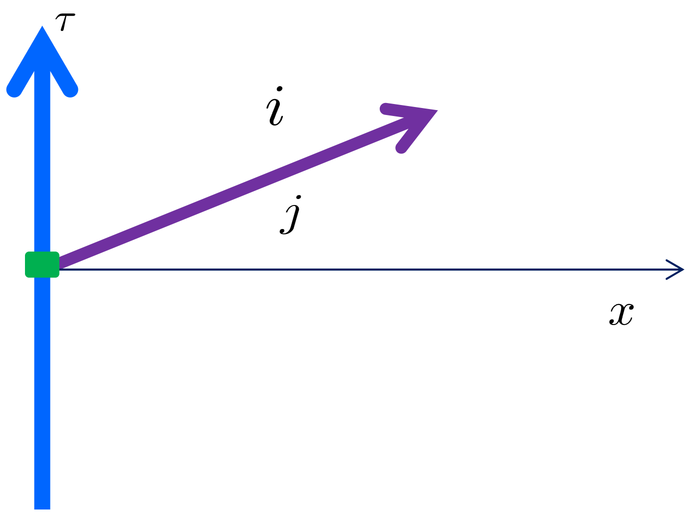
$$\mathcal{V}_i(\mathfrak{u})$$ Interior vertices

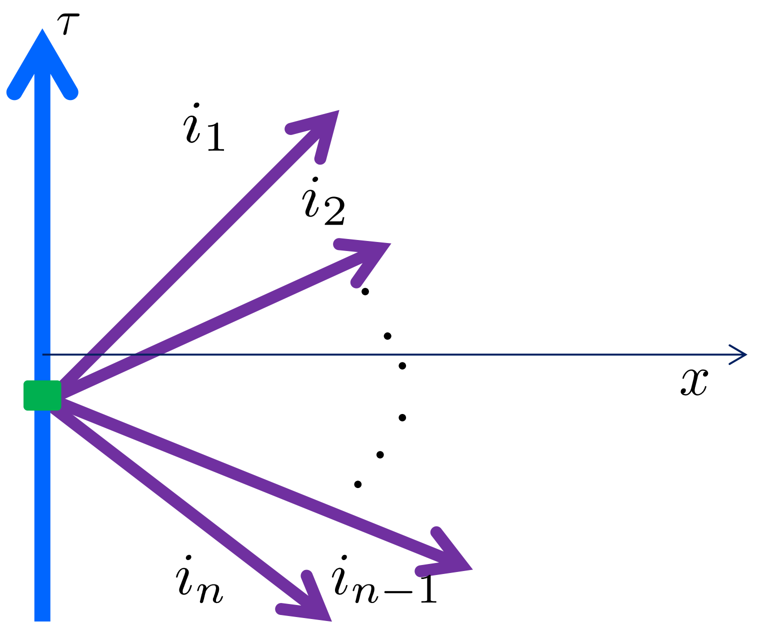
$$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1, \dots, v_n\}$$ ***time-ordered*** boundary vertices.

deformation type, reduced moduli space, etc. ….

$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}) - 1$$

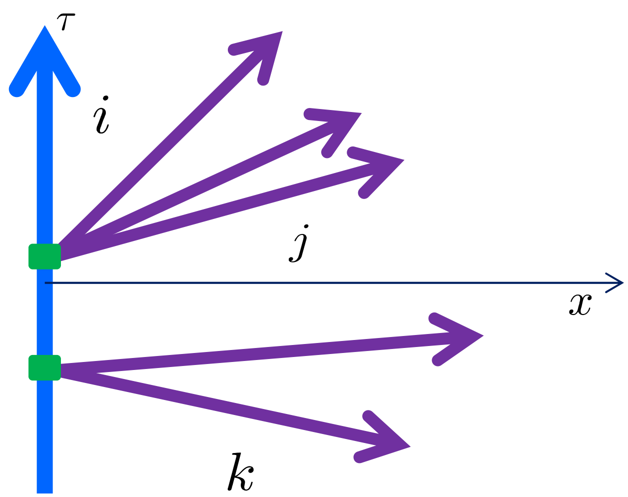
# Rigid Half-Plane Webs

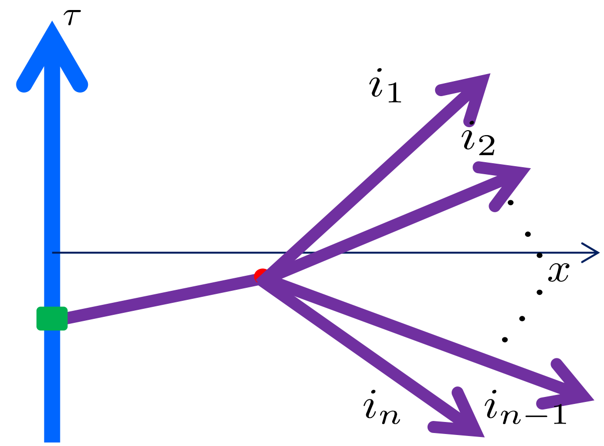


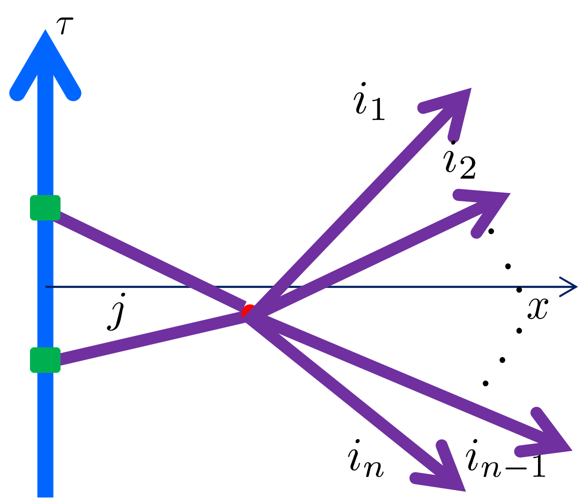

$$d(\mathfrak{u}) = 0$$

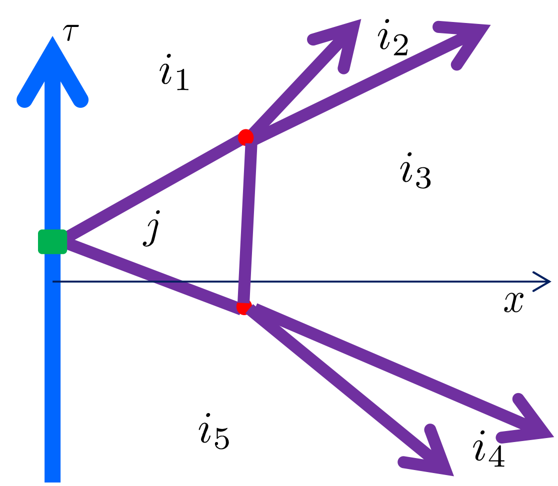
# Taut Half-Plane Webs



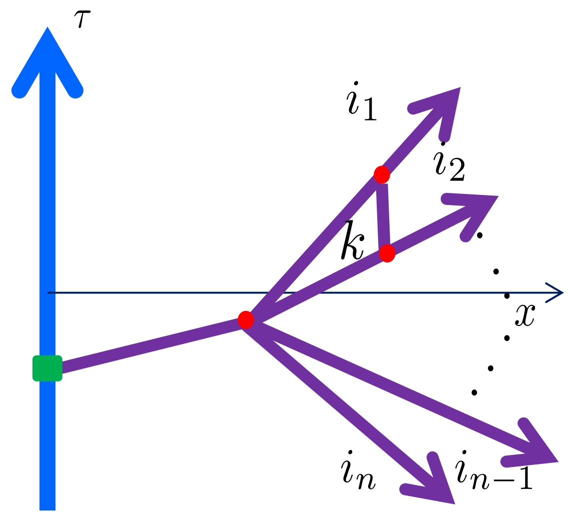




$$d(\mathfrak{u}) = 1$$

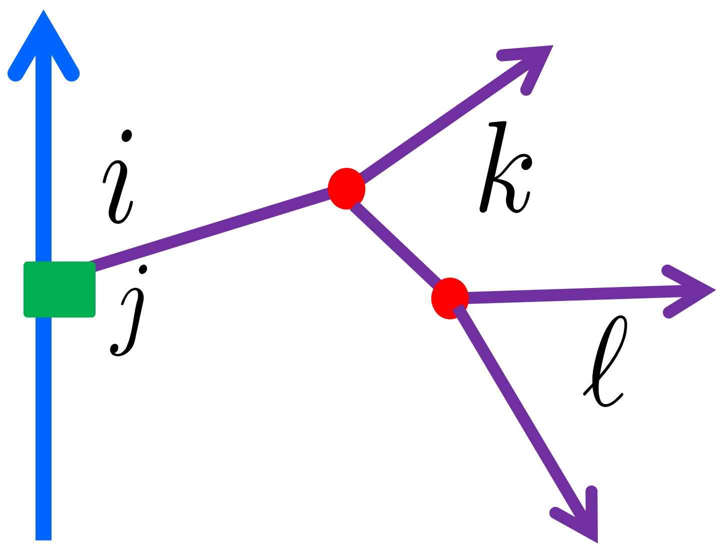


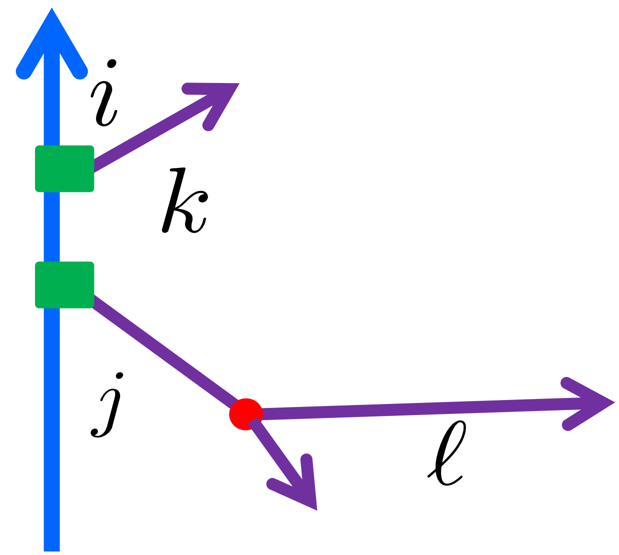



# Sliding Half-Plane webs




$$d(\mathbf{u}) = 2$$



# Half-Plane fans

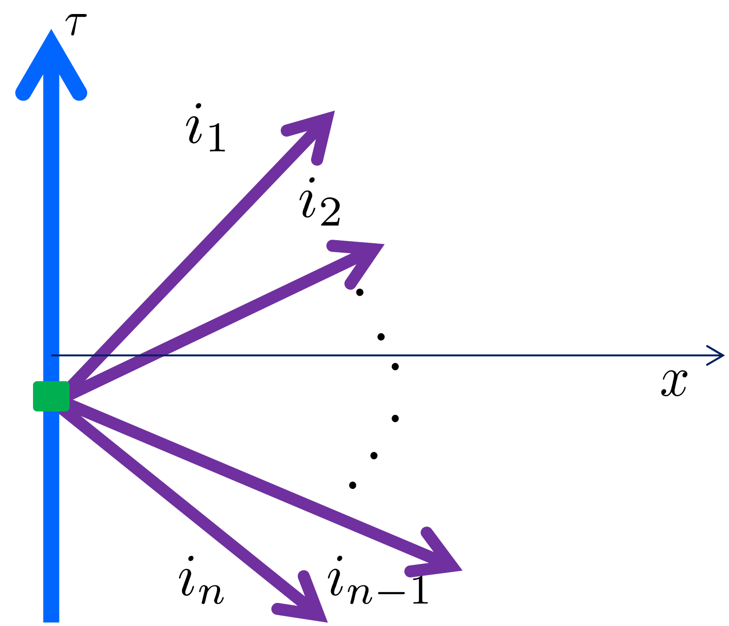
A half-plane fan is an ordered set of vacua,

$$J = \{i_1, \dots, i_n\}$$

such that successive vacuum weights:

$$z_{i_s, i_{s+1}}$$

are ordered clockwise:



# Convolutions for Half-Plane Webs

We can now introduce a convolution at boundary vertices:

Local half-plane fan at a boundary vertex *v*: $J_v(\mathfrak{u})$

Half-plane fan at infinity: $J_\infty(\mathfrak{u})$

$\mathcal{W}_{\mathcal{H}}$ Free abelian group generated by oriented def. types of half-plane webs

There are now two convolutions:

$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \to \mathcal{W}_{\mathcal{H}}$$

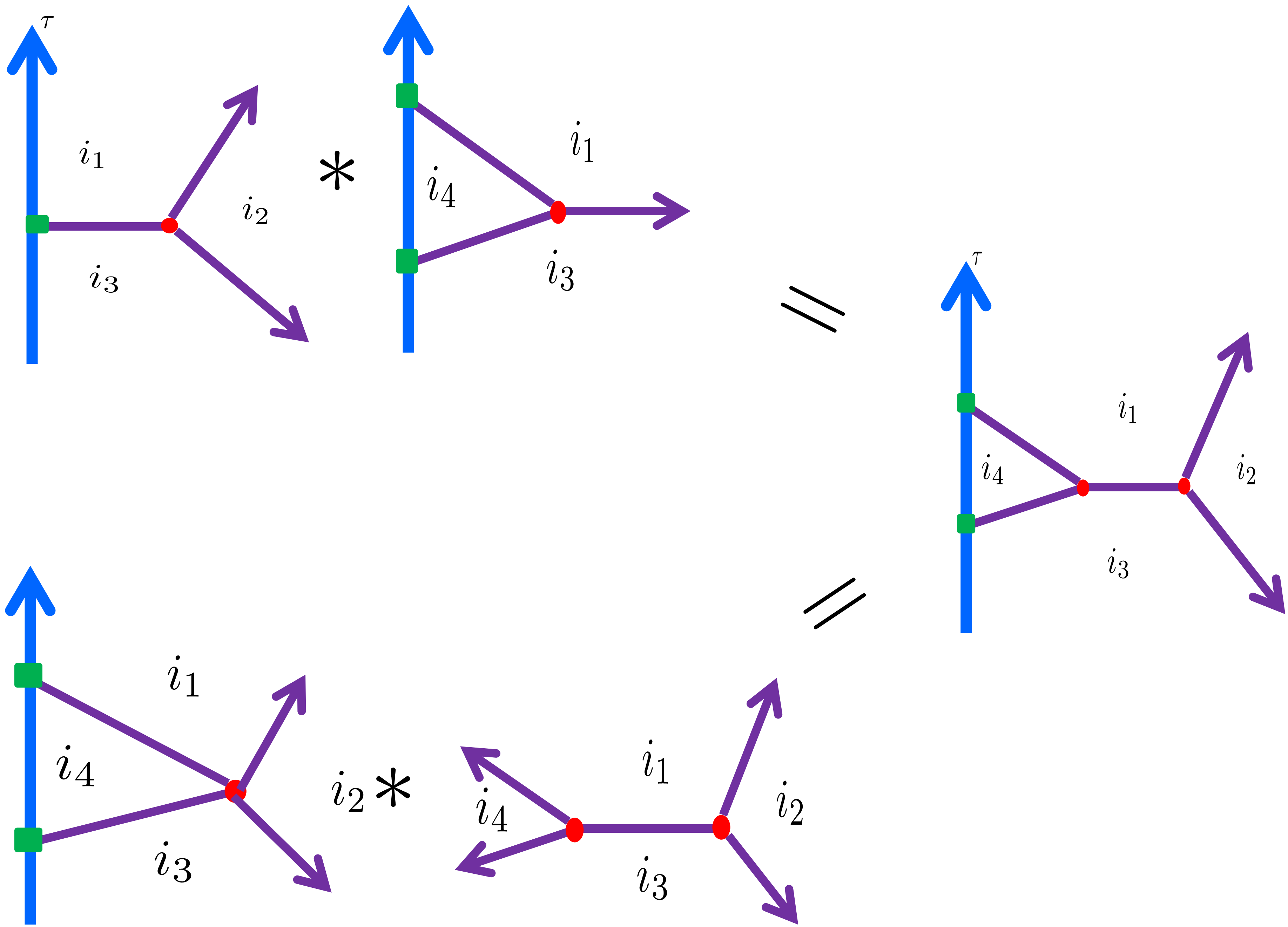
$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

# Convolution Theorem

Define the half-plane taut element: $\mathfrak{t}_{\mathcal{H}} := \sum_{d(\mathfrak{u})=1} \mathfrak{u}$

Theorem: $\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{p} = 0$

Proof: A sliding half-plane web can degenerate (in real codimension one) in two ways: Interior edges can collapse onto an interior vertex, or boundary edges can collapse onto a boundary vertex.



$\tau$
$i_1$
$i_2$
$i_3$
$*$
$i_1$
$i_4$
$i_3$
$=$
$\tau$
$i_1$
$i_4$
$i_2$
$i_3$
$=$
$i_1$
$i_4$
$i_3$
$i_2$
$*$
$i_1$
$i_4$
$i_2$
$i_3$

# Tensor Algebra Relations

Extend $\mathfrak{t}_{\mathcal{H}}{}^*$ to tensor algebra operator

$$T(\mathfrak{t}_{\mathcal{H}}) : T\mathcal{W}_{\mathcal{H}} \otimes T\mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

$$\sum \epsilon\, T(\mathfrak{t}_{\mathcal{H}})[P_1, T(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon\, T(\mathfrak{t}_{\mathcal{H}})[P; T(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\} \quad P = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_m\}$$

Sum over ordered partitions:

$$P = P_1 \amalg P_2 \amalg P_3$$

# Conceptual Meaning

$\mathcal{W}_{\mathcal{H}}$ is an L∞ module for the $L_\infty$ algebra $\mathcal{W}$

$\mathcal{W}_{\mathcal{H}}$ is an A∞ algebra

There is an $L_\infty$ morphism from the $L_\infty$ algebra $\mathcal{W}$ to the $L_\infty$ algebra of the Hochschild cochain complex of $\mathcal{W}_{\mathcal{H}}$

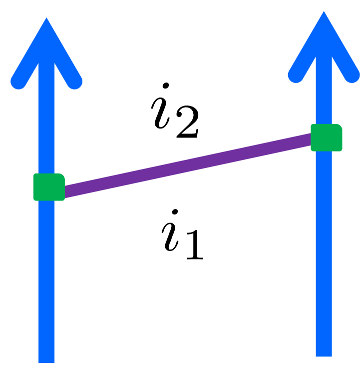
# Strip-Webs

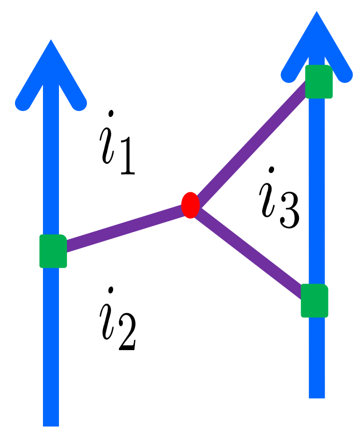
Now consider webs in the strip $\mathbb{R} \times [x_\ell, x_r]$

$$d(\mathfrak{s}) := 2V_i(\mathfrak{s}) + V_\partial(\mathfrak{s}) - E(\mathfrak{s}) - 1$$

Now ***taut*** and ***rigid strip-webs*** are the same, and have d(s)=0.

***sliding strip-webs*** have d(s)=1.



# Convolution Identity for Strip $\mathfrak{t}$'s

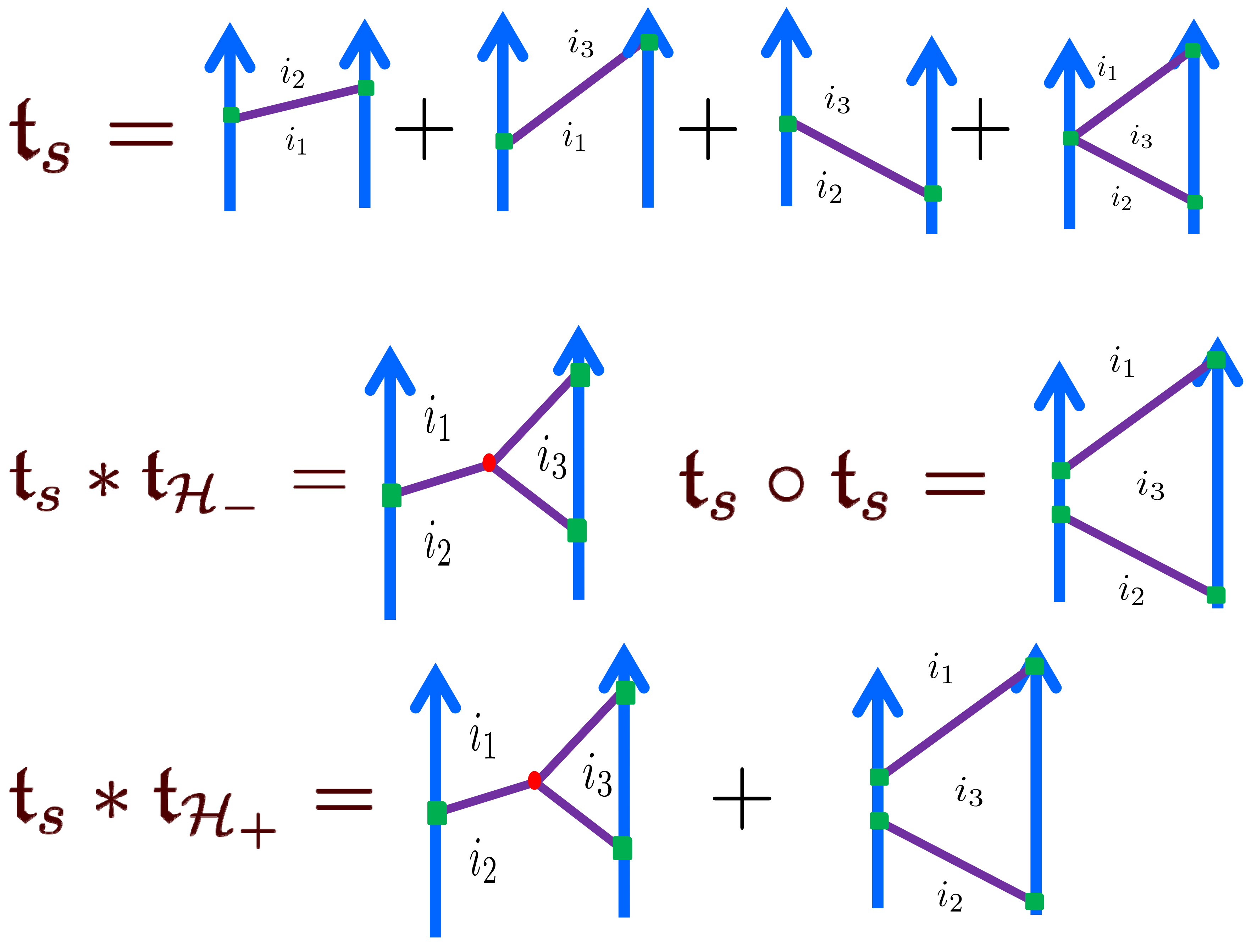
$$\mathfrak{t}_s := \sum_{d(\mathfrak{s})=0} \mathfrak{s}$$

Convolution theorem:

$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

where for strip webs we denote time-concatenation by

$$\mathfrak{s}_1 \circ \mathfrak{s}_2$$



$\mathfrak{t}_s =$
$i_2$
$i_1$
$+$
$i_3$
$i_1$
$+$
$i_3$
$i_2$
$+$
$i_1$
$i_3$
$i_2$
$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} =$
$i_1$
$i_3$
$i_2$
$\mathfrak{t}_s \circ \mathfrak{t}_s =$
$i_1$
$i_3$
$i_2$
$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} =$
$i_1$
$i_3$
$i_2$
$+$
$i_1$
$i_3$
$i_2$

# Conceptual Meaning

$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

$\mathcal{W}_S$ : Free abelian group generated by oriented def. types of strip webs.

There is a corresponding elaborate identity on tensor algebras …

$\mathcal{W}_S$ is an $A_\infty$ bimodule

+ … much more

# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

# Web Representations

**Definition:** A *representation of webs* is

a.) A choice of $\mathbb{Z}$-graded $\mathbb{Z}$-module $R_{ij}$ for every ordered pair ij of distinct vacua.

b.) A degree = -1 pairing $$K : R_{ij} \otimes R_{ji} \to \mathbb{Z}$$

For every cyclic fan of vacua introduce a *fan representation*:

$$I = \{i_1, \dots, i_n\} \Rightarrow$$

$$R_I := R_{i_1, i_2} \otimes \cdots \otimes R_{i_n, i_1}$$

# Web Rep & Contraction

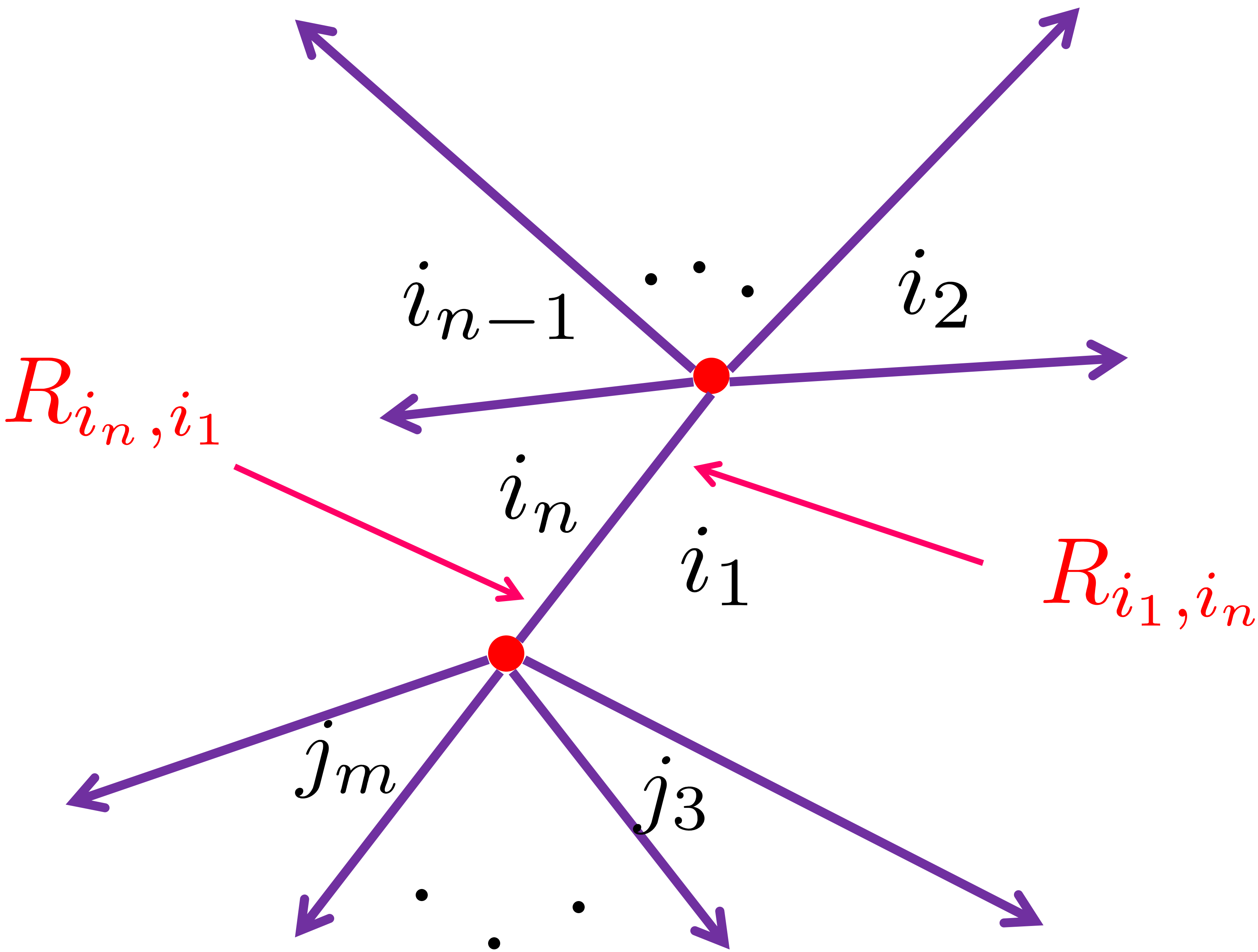
Given a rep of webs and a deformation type $\mathfrak{w}$ we define the *representation of $\mathfrak{w}$*:

$$R(\mathfrak{w}) := \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})}$$

There is a natural contraction operator:

$$\rho(\mathfrak{w}) : R(\mathfrak{w}) \to R_{I_\infty(\mathfrak{w})}$$

by applying the contraction K to the pairs $R_{ij}$ and $R_{ji}$ on each edge:


$i_{n-1}$
$i_2$
$R_{i_n,i_1}$
$i_n$
$i_1$
$R_{i_1,i_n}$
$j_m$
$j_3$

# $L_\infty$ -algebras, again

$$R^{\mathrm{int}} := \oplus_I R_I$$ Rep of the rigid webs.

$$\rho(\mathfrak{t}_p) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

$$\mathfrak{t}_p * \mathfrak{t}_p = 0 \quad \Longrightarrow$$

$$\sum_{\mathrm{Sh}_2(S)} \epsilon\, \rho(\mathfrak{t}_p)[\rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

Now, $S = \{r_1, \dots, r_n\}$ $\quad r_i \in R^{\mathrm{int}}$

# Half-Plane Contractions

A rep of a half-plane fan: $J = \{j_1, \ldots, j_n\}$

$$R_J := R_{j_1, j_2} \otimes \cdots \otimes R_{j_{n-1}, j_n}$$

$\rho(\mathfrak{u})$ now contracts

$$\underbrace{\bigotimes_{v \in \mathcal{V}_\partial(\mathfrak{u})} R_{J_v(\mathfrak{u})}}_{\text{time ordered!}} \bigotimes_{v \in \mathcal{V}_i(\mathfrak{u})} R_{I_v(\mathfrak{u})}$$

$$\to R_{J_\infty(\mathfrak{u})}$$

# The Vacuum $A_\infty$ Category

(For the positive half-plane $\mathcal{H}_+$)

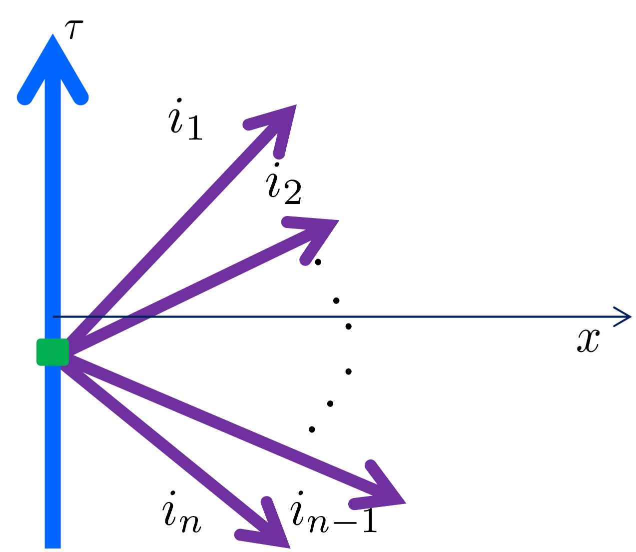
Objects: i $\in \mathbb{V}$.

Morphisms: $$\mathrm{Hom}(j,i) = \begin{cases} \widehat{R}_{ij} & \mathrm{Re}(z_{ij}) > 0 \\ \mathbb{Z} & i = j \\ 0 & \mathrm{Re}(z_{ij}) < 0 \end{cases}$$

$$\widehat{R}_{i_1,i_n} := \oplus_J R_J$$

$$J = \{i_1, \dots, i_n\}$$

$$\widehat{R}_{i_1,i_n} = R_{i_1,i_n} \oplus \cdots$$



# Hint of a Relation to Wall-Crossing

The morphism spaces can be defined by a Cecotti-Vafa/Kontsevich-Soibelman-like product as follows:

Suppose $\mathbb{V}$ = { 1, …, K}.
Introduce the elementary K x K matrices $e_{ij}$

$$1 + \oplus_{\mathrm{Re}(z_{ij})>0} \widehat{R}_{ij} e_{ij} = \underbrace{\otimes_{\mathrm{Re}(z_{ij})>0} (1 + R_{ij} e_{ij})}_{\text{phase ordered!}}$$

# Defining $A_\infty$ Multiplications

Sum over cyclic fans: $R^{\rm int} := \oplus_I R_I$

$$\rho(\mathfrak{t}_p) : TR^{\rm int} \to R^{\rm int}$$

***Interior amplitude:*** $\beta \in R^{\rm int}$ Satisfies the $L_\infty$ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t}_p)(e^\beta) = 0$$

$$m_n^\beta[r_1, \dots, r_n] := \rho(\mathfrak{t}_{\mathcal{H}})[r_1, \dots, r_n; e^\beta]$$

$$r_s \in \mathrm{Hom}(i_{s-1}, i_s)$$

# Proof of $A_\infty$ Relations

$$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_p = 0 \quad \Longrightarrow$$

$$\sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{r_1, \dots, r_m\} \qquad S = S_1 \amalg S_2$$

$$P = \{r_1^{\partial}, \dots, r_n^{\partial}\} \qquad P = P_1 \amalg P_2 \amalg P_3$$

$$r_a \in R^{\mathrm{int}} \qquad r_s^{\partial} \in \widehat{R}_{i_{s-1}, i_s}$$

$$\sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$\cancel{+ \sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2]} = 0.$$

$$S = \{\beta, \ldots, \beta\}$$

and the second line vanishes.

Hence we obtain the A∞ relations for :

$$m^{\beta}[P] := \rho(\mathfrak{t}_{\mathcal{H}})[P; e^{\beta}]$$

Defining an A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta)$

# Enhancing with CP-Factors

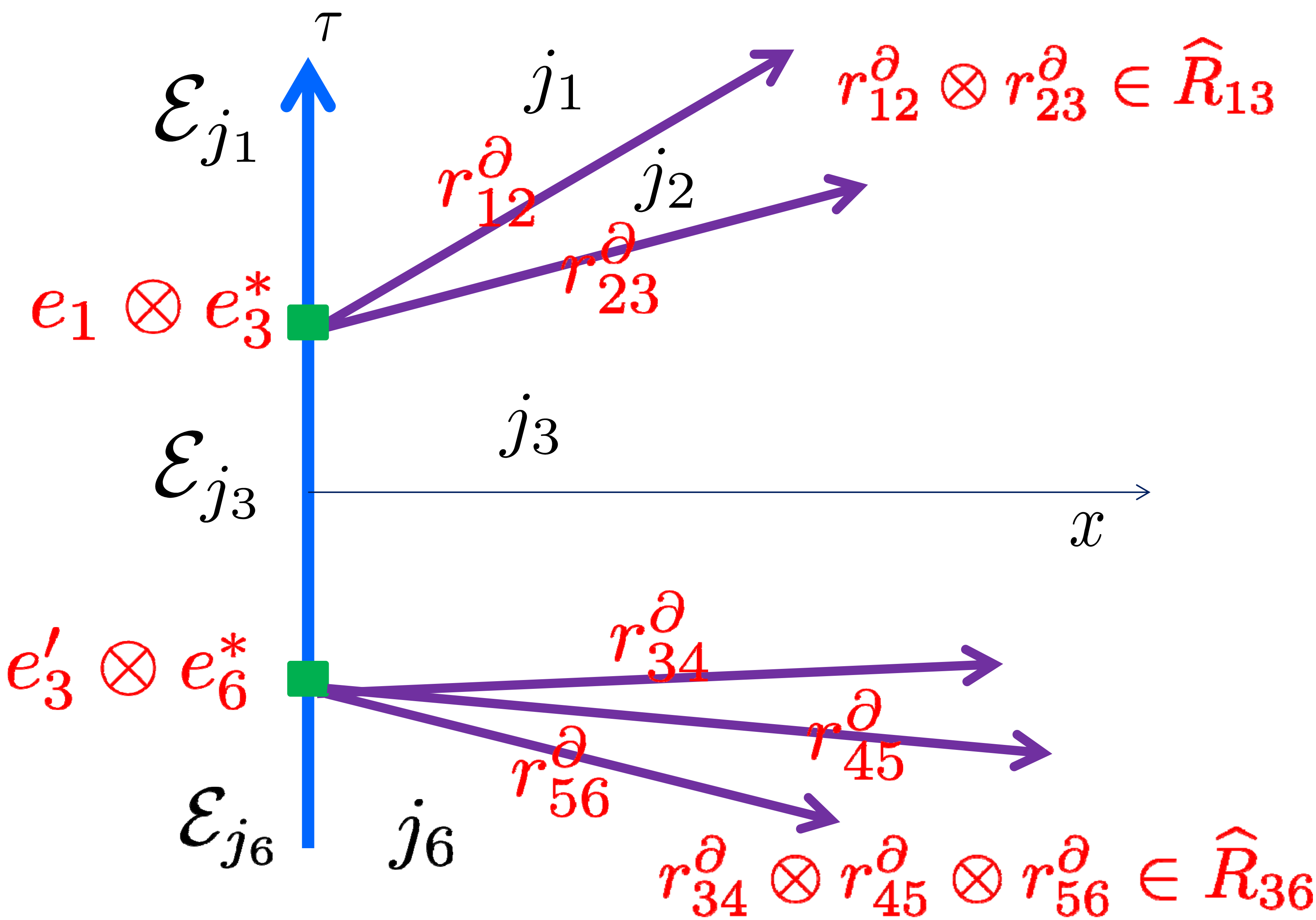
CP-Factors: $i \in \mathbb{V}$ $\longrightarrow$ $\mathcal{E}_i$ $\mathbb{Z}$-graded module

$$\mathrm{Hop}(i,j) \Longrightarrow \mathcal{E}_i \otimes \mathrm{Hop}(i,j) \otimes \mathcal{E}_j^*$$

$$m_n^\beta \Longrightarrow m_n^\beta \otimes m_2^{\mathrm{CP}}$$

Enhanced A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta; \mathcal{E}_*)$

# Example: Composition of two morphisms



# Boundary Amplitudes

A Boundary Amplitude $\mathcal{B}$ (defining a Brane) is a solution of the $A_\infty$ MC:

$$\mathcal{B} \in \oplus_{i,j} \mathrm{Hop}(i,j)$$

$$\mathcal{B} \in \oplus_{\mathrm{Re}(z_{ij})>0} \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*$$

$$\sum_{n=1}^{\infty} m_n^{\beta}[\mathcal{B}^{\otimes n}] = 0$$

$$\rho(\mathfrak{t}_{\mathcal{H}})[\frac{1}{1-\mathcal{B}}; e^{\beta}] = 0$$

# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

# Constructions with Branes

Strip webs with Brane boundary conditions help answer the physics question at the beginning.

The Branes themselves are objects in an $A_\infty$ category

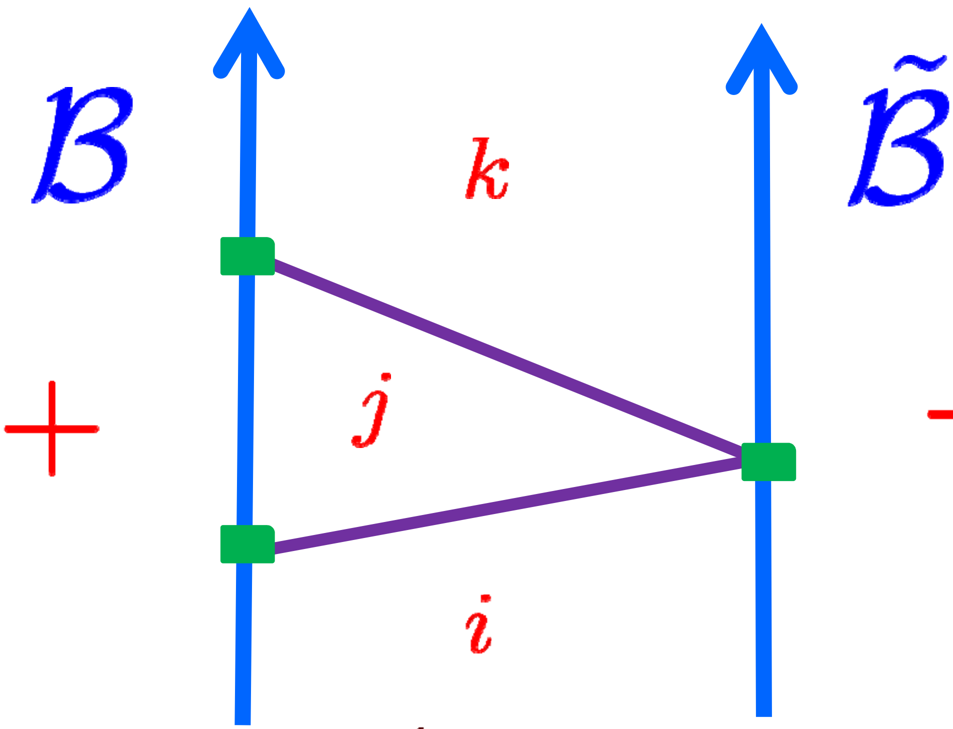
$$\mathfrak{Br}(\mathbb{V}, z, R, K, \beta)$$

("Twisted complexes": Analog of the derived category.)

Given a (suitable) continuous path of data $(\mathbb{V}, z, R, K, \beta)(x)$ we construct an invertible functor between Brane categories, only depending on the homotopy class of the path.

(Parallel transport of Brane categories.)

$$\mathfrak{t}_s = \cdots +$$




$$+ \cdots$$

$$d(v) := \rho(\mathfrak{t}_s)\left(\frac{1}{1-\mathcal{B}}; e^{\beta}; v; \frac{1}{1-\tilde{\mathcal{B}}}\right)$$

$$d : \oplus_{i\in\mathbb{V}}\mathcal{E}_i \otimes \widetilde{\mathcal{E}}_i \to \oplus_{i\in\mathbb{V}}\mathcal{E}_i \otimes \widetilde{\mathcal{E}}_i$$

$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

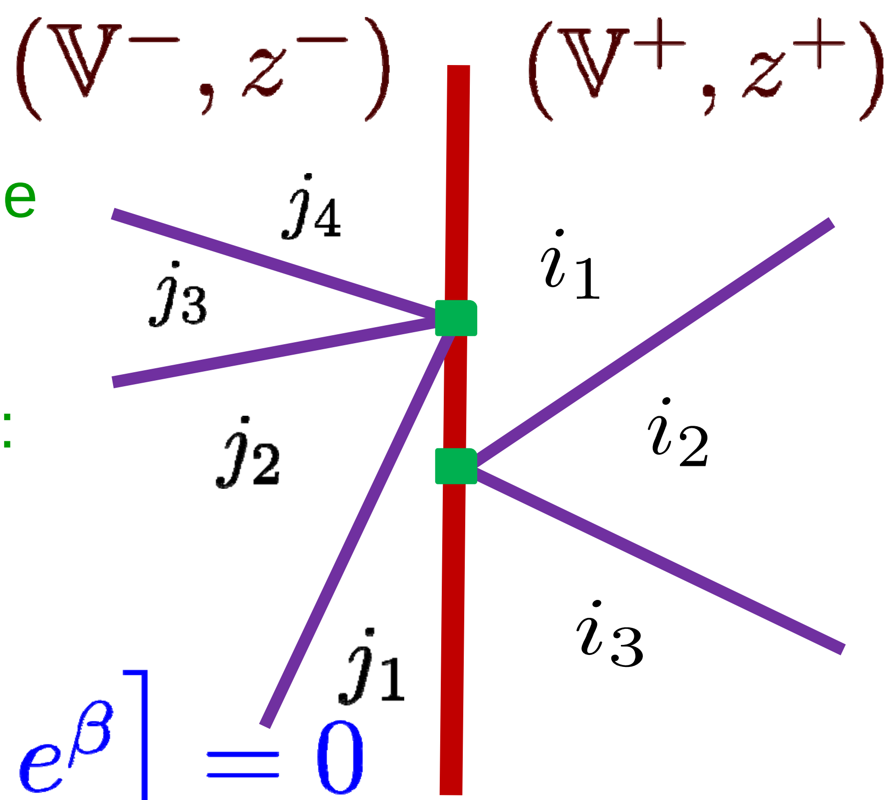
Convolution identity implies: $d^2 = 0$

# Interfaces webs & Interfaces

Given data $(\mathbb{V}^{\pm}, z^{\pm}, R^{\pm}, K^{\pm}, \beta^{\pm})$

Introduce a notion of ``interface webs''

These behave like half-plane webs and we can define an <u>*Interface*</u> <u>*Amplitude*</u> to be a solution of the MC equation:

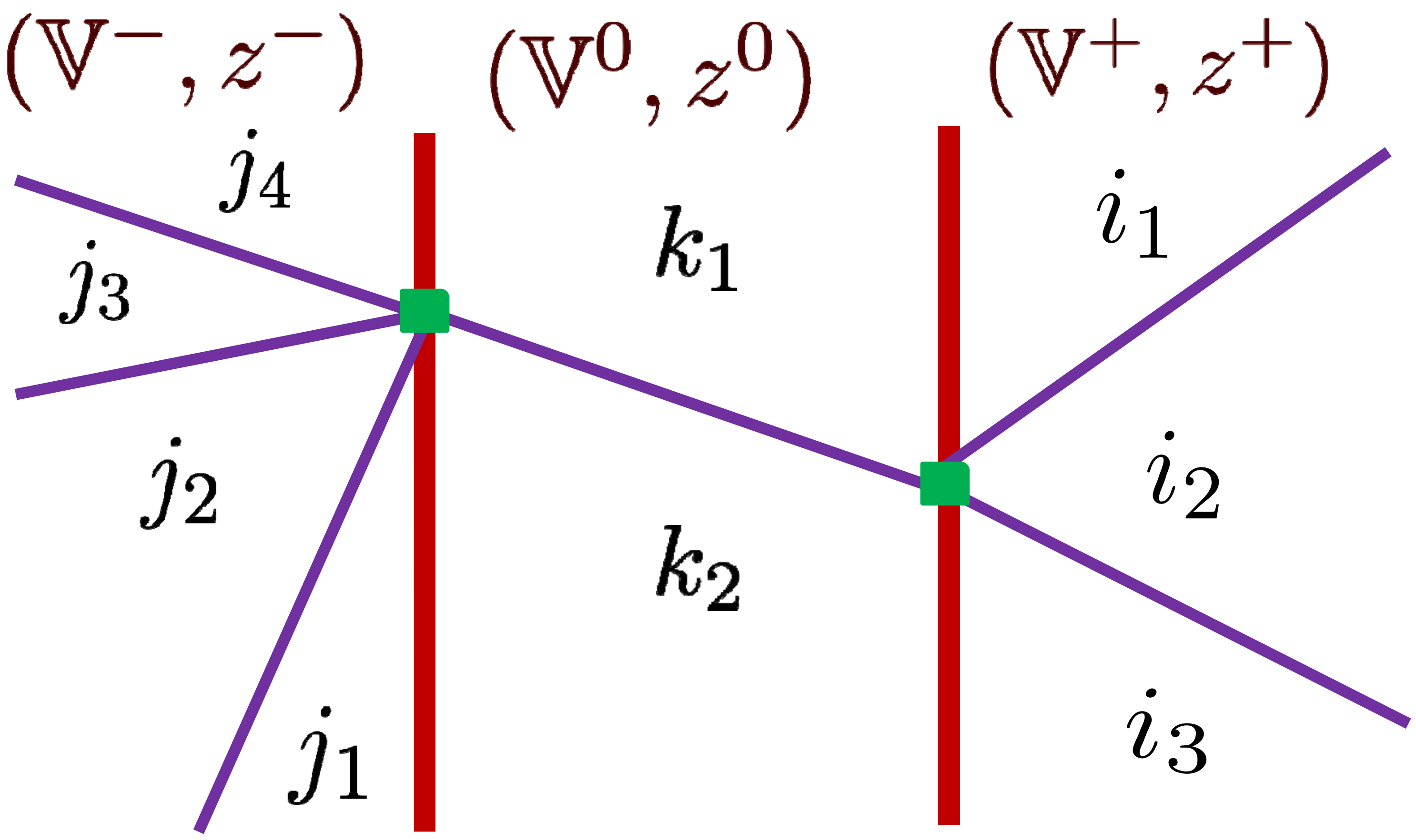



$$\rho(\mathfrak{t}^{-,+})\left[\frac{1}{1-\mathcal{B}^{-,+}}; e^{\beta}\right] = 0$$

# Composite webs

Given data $(\mathbb{V}, z, R, K, \beta)^{-,0,+}$

Introduce a notion of ``composite webs''



# Composition of Interfaces

A convolution identity implies:

$$\rho(\mathfrak{t}^{-,0,+})\left[\frac{1}{1-\mathcal{B}^{-,0}},\frac{1}{1-\mathcal{B}^{0,+}};e^{\beta}\right]\in\mathfrak{Br}^{-,+}$$

Defines a family of $A_\infty$ bifunctors:

$$\mathfrak{Br}^{-,0}\times\mathfrak{Br}^{0,+}\to\mathfrak{Br}^{-,+}$$

$$\mathfrak{Br}^{-,0}\times\mathfrak{Br}^{0,1}\times\mathfrak{Br}^{1,+}\to\mathfrak{Br}^{-,+}$$

Product is associative up to homotopy

Composition of such bifunctors leads to categorified parallel transport

# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

# Physical ``Theorem''

## Data

$(X,\omega)$: Kähler manifold (exact)

W: $X \longrightarrow \mathbb{C}$ Holomorphic Morse function

Finitely many critical points with critical values in general position.

## We construct an explicit realization of above:

- Vacuum data.
- Interior amplitudes.
- Chan-Paton spaces and boundary amplitudes.
- "Parallel transport" of Brane categories.

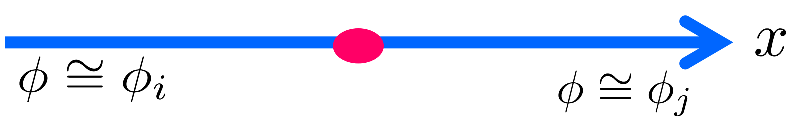
# Vacuum data:

$\mathbb{V}$ Morse critical points $\phi_i$ $dW(\phi_i) = 0$

$$z_i \sim W_i := W(\phi_i) \quad \left[ \text{Actually,} \quad z_i = \mathrm{i}\zeta \overline{W}_i \right]$$

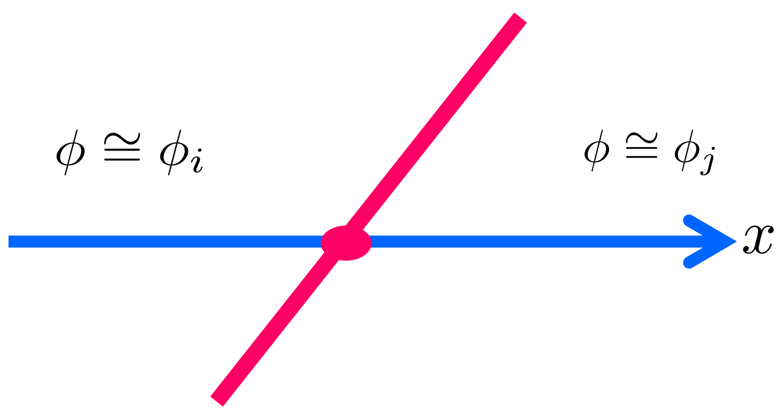
## Connection to webs uses BPS states:

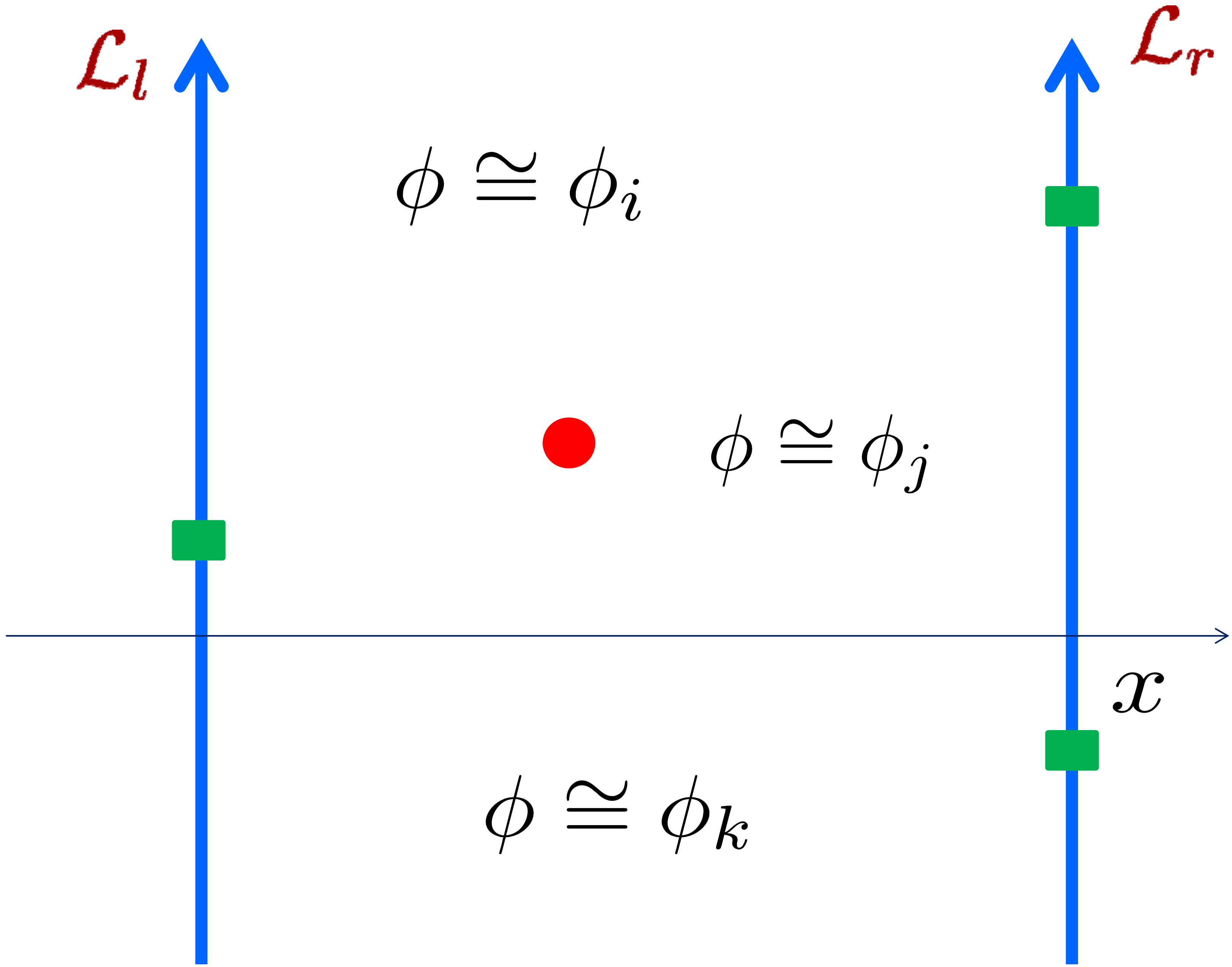
Semiclassically, they are solitonic particles.




Worldlines preserving "ζ-supersymmetry"
are solutions of the "ζ-instanton equation"

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$




$\mathcal{L}_l$
$\mathcal{L}_r$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$x$
$\phi \cong \phi_k$

Now, we explain this more systematically …

# SQM & Morse Theory
# (Witten: 1982)

*M*: Riemannian;  h: *M* $\longrightarrow \mathbb{R}$,  Morse function

SQM:  $q : \mathbb{R}_{\rm time} \to M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

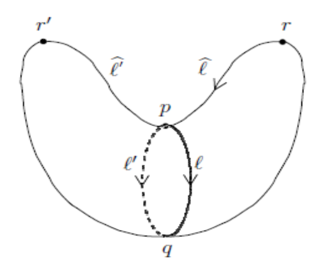
$$L = g_{IJ}\dot{q}^I \dot{q}^J - g^{IJ}\partial_I h \partial_J h$$

$$+g_{IJ}\bar{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \bar{\chi}^I \chi^J \; - R_{IJKL}\bar{\chi}^I \chi^J \bar{\chi}^K \chi^L$$

MSW complex:

$$\mathbb{M}^\bullet := \oplus_{p:dh(p)=0} \mathbb{Z} \cdot \Psi(p)$$

$$F(\Psi(p)) = \frac{1}{2}(d_\uparrow(p) - d_\downarrow(p))$$




$$d(\Psi(p)) = \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

# 1+1 LG Model as SQM

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega$$

Recover the standard 1+1 LG model with superpotential:
Two –dimensional $\zeta$-susy algebra is manifest.

# Boundary conditions for $\phi$

Boundaries at infinity:

$$\phi \to \phi_i \qquad \phi \to \phi_j$$

$$x \to -\infty \qquad x \to +\infty$$

$$\phi|_{\partial D} \in \mathcal{L} \subset X$$

Boundaries at finite distance: Preserve $\zeta$-susy:

$$\iota_{\mathcal{L}}^{*}(\lambda) = dk$$

$$\pm \mathrm{Im}(\zeta^{-1} W) \geq \Lambda$$

# Lefshetz Thimbles

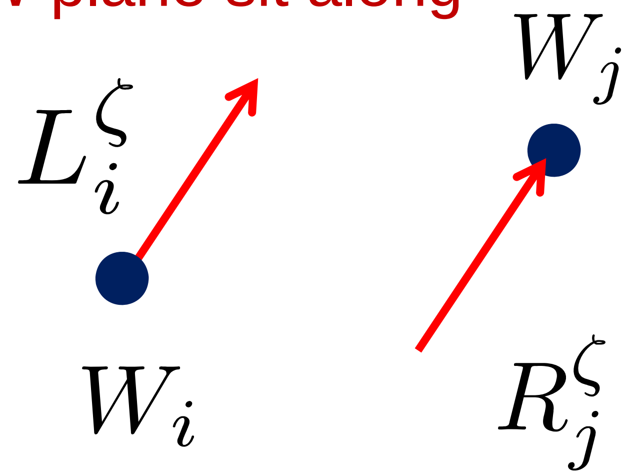
Stationary points of h are solutions to the differential equation

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

The projection of solutions to the complex W plane sit along straight lines of slope $\zeta$

If D contains x $\longrightarrow$ -∞ $\quad \phi \rightarrow \phi_i$

If D contains x $\longrightarrow$ +∞ $\quad \phi \rightarrow \phi_j$



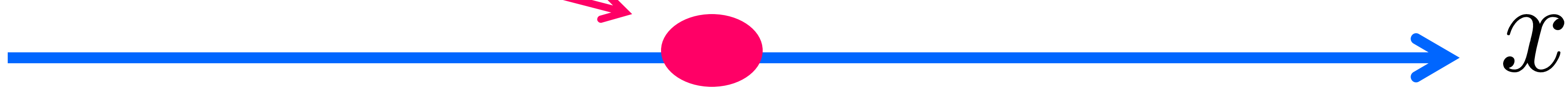

Inverse image in X defines left and right Lefshetz thimbles

They are Lagrangian subvarieties of X

# Solitons For D=$\mathbb{R}$

Scale set by W

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$



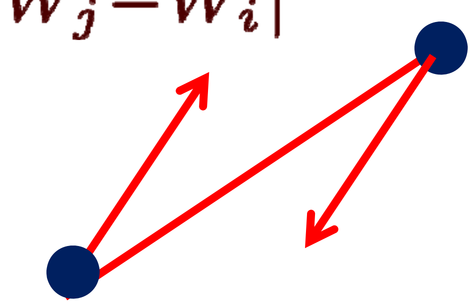

$\phi \cong \phi_i$ $\qquad\qquad$ $\phi \cong \phi_j$

For general $\zeta$ there is no solution.

But for a suitable phase there is a solution

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

$W_j$



$W_i$

This is the classical soliton. There is one for each intersection (Cecotti & Vafa)

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

# MSW Complex

$$R_{ij} = \oplus_{\mathbf{solitons}} \mathbb{Z} \cdot \phi_{ij}$$

(Taking some shortcuts here….)

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

$$F = -\frac{1}{2}\eta(D - \epsilon)$$

# Instantons

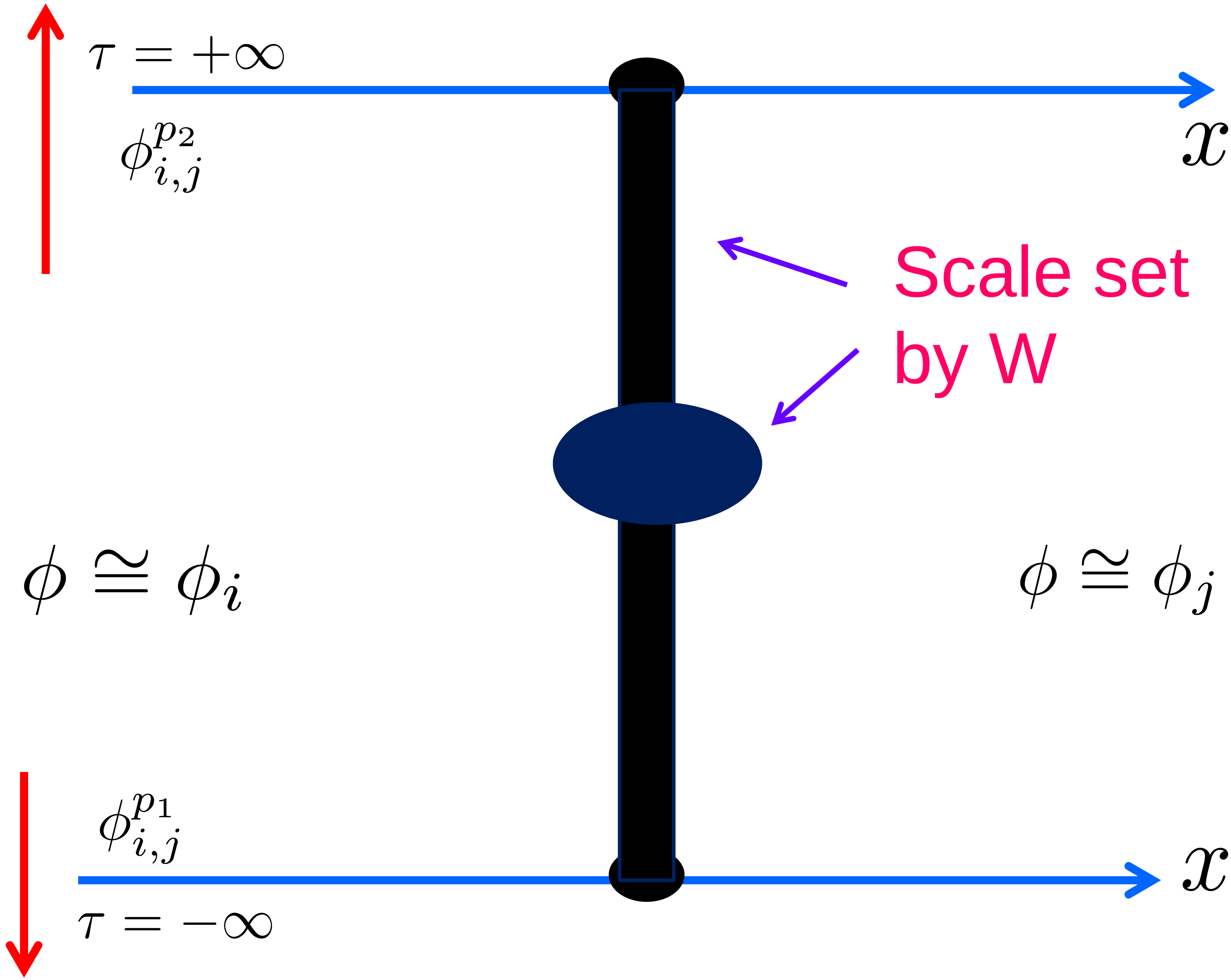
Instanton equation $$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

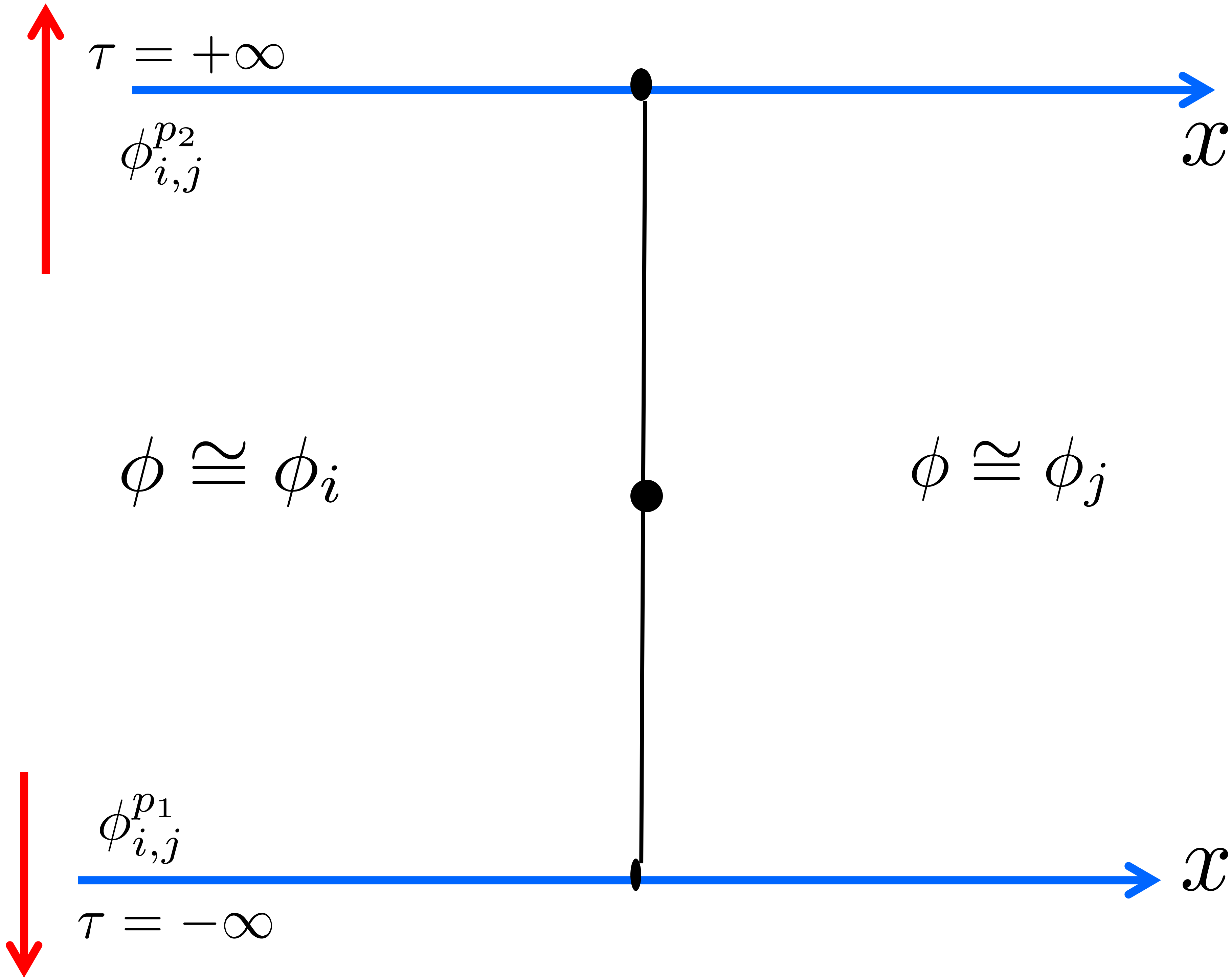
$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\bar{\partial}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

At short distance scales W is irrelevant and we have the usual holomorphic map equation.

At long distances the theory is almost trivial since it has a mass scale, and it is dominated by the vacua of W.


$\tau = +\infty$
$\phi^{p_2}_{i,j}$
$x$
Scale set
by W
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi^{p_1}_{i,j}$
$x$
$\tau = -\infty$



$\tau = +\infty$
$\phi_{i,j}^{p_2}$
$x$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
$x$
$\tau = -\infty$

# The Boosted Soliton - 1

We are interested in the $\zeta$-instanton equation for a fixed generic $\zeta$

We can still use the soliton to produce a solution for phase $\zeta$

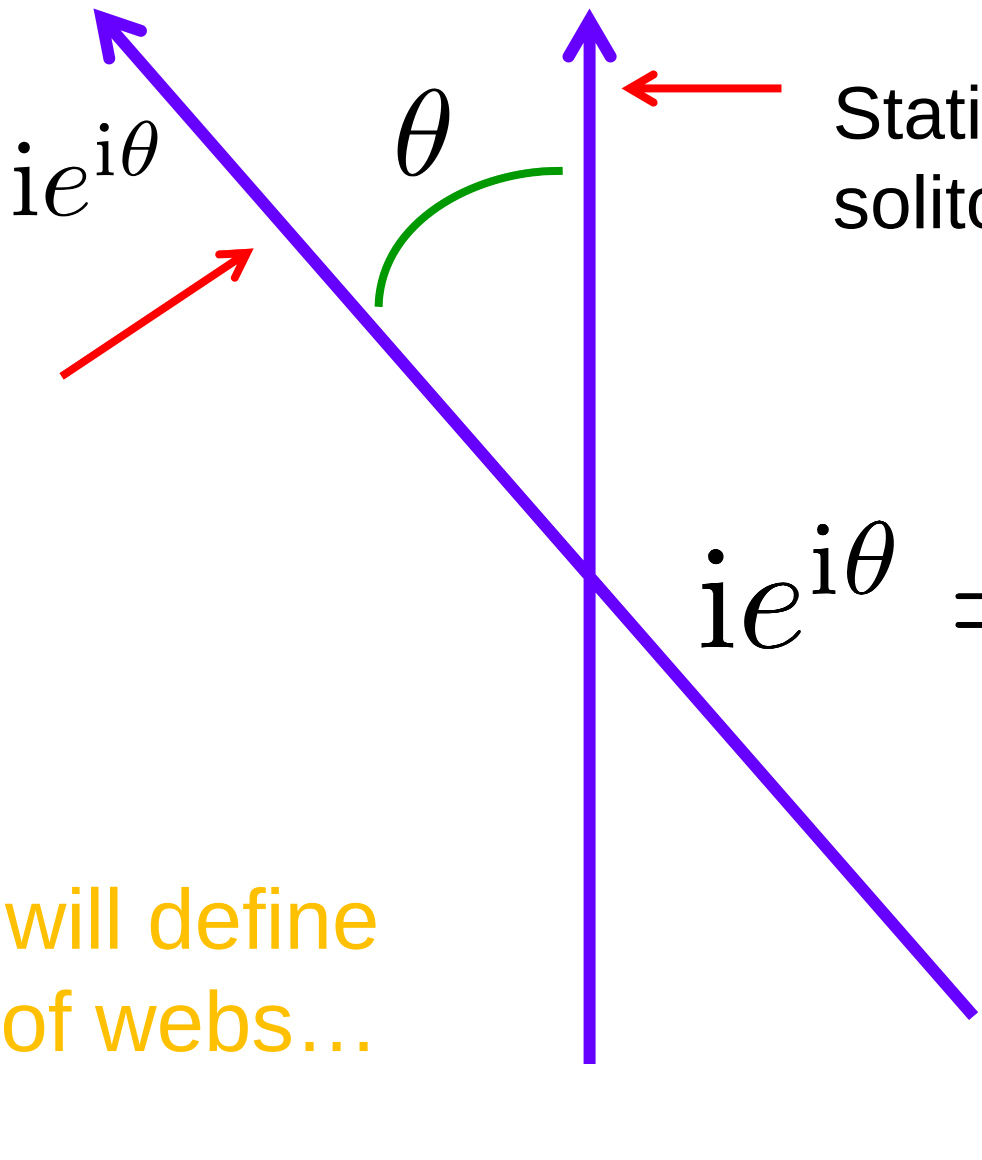
$$\phi_{ij}^{\rm inst}(x,\tau) := \phi_{ij}^{\rm sol}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial\tau}\right)\phi_{ij}^{\rm inst} = e^{\mathrm{i}\theta}\zeta_{ji}\frac{\partial\bar{W}}{\partial\bar{\phi}}$$

Therefore we produce a solution of the instanton equation with phase $\zeta$ if

$$\zeta = e^{\mathrm{i}\theta}\zeta_{ji} \qquad\qquad \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

# The Boosted Soliton -2




Stationary
soliton

$$\phi \cong \phi_j$$

Boosted
soliton

$$\phi \cong \phi_i$$

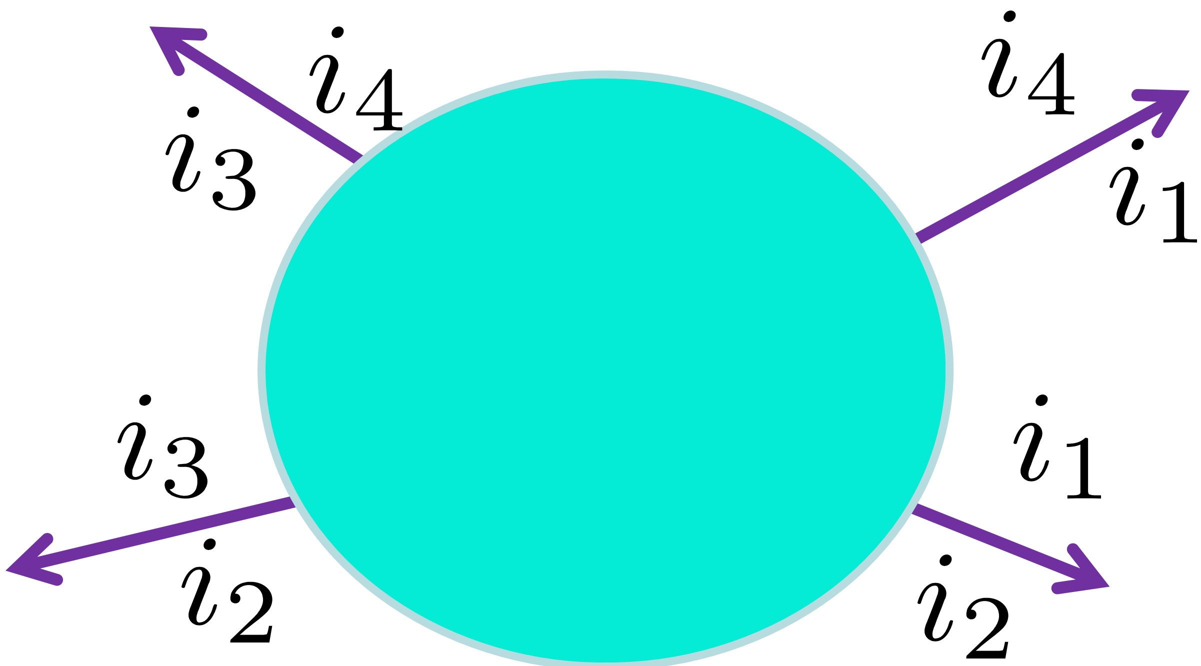
$$\mathrm{i}e^{\mathrm{i}\theta} = \frac{z_i - z_j}{|z_i - z_j|}$$

$$z_i = \mathrm{i}\zeta \overline{W}_i$$

These will define
edges of webs…

# Path integral on a large disk

Consider the path integral on a large disk:




Choose boundary conditions preserving ζ-supersymmetry:

Consider a cyclic fan of vacua $I = \{i_1, \dots, i_n\}$.

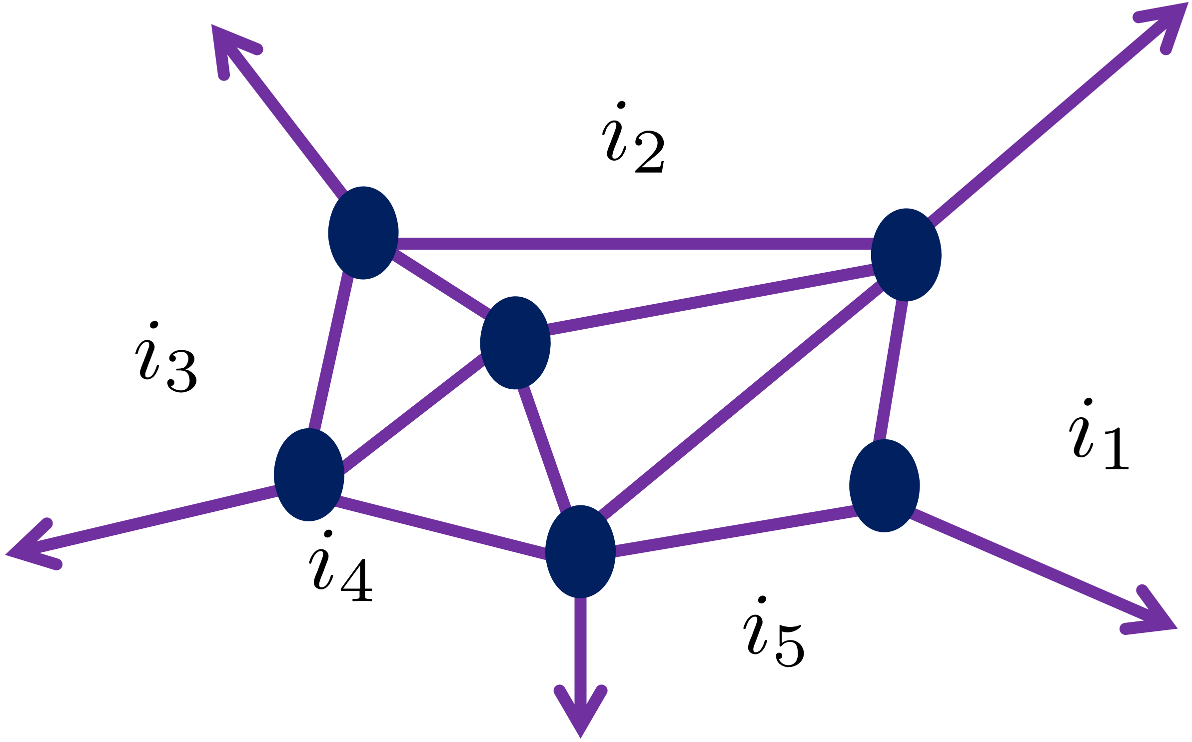
$$\phi^{\text{inst}}_{i_1 i_2} \otimes \cdots \otimes \phi^{\text{inst}}_{i_n i_1} \in R_I$$

# Ends of moduli space

Path integral localizes on moduli space of $\zeta$-instantons with these boundary conditions:

$$\mathcal{M}(\mathcal{F})$$

This moduli space has several "ends" where solutions of the $\zeta$-instanton equation look like



# Interior Amplitude From Path Integral

Label the ends by webs $\mathfrak{w}$. Each end produces a wavefunction $\Psi(\mathfrak{w})$ associated to a web $\mathfrak{w}$.

The total wavefunction is Q-invariant

$$Q \sum_{\mathfrak{w}} \Psi(\mathfrak{w}) = 0$$

The wavefunctions $\Psi(\mathfrak{w})$ are themselves constructed by gluing together wavefunctions $\Psi(\mathfrak{r})$ associated with rigid webs $\mathfrak{r}$

→ $L_\infty$ identities on the interior amplitude

$$\beta \sim \sum_{\mathrm{rigid}\,\mathfrak{r}} \Psi(\mathfrak{r})$$

→ $\mathfrak{Vac}(X, W)$

# Half-Line Solitons

Classical solitons on the right half-line are labeled by: $p \in \mathcal{L} \cap R_j^{\zeta}$

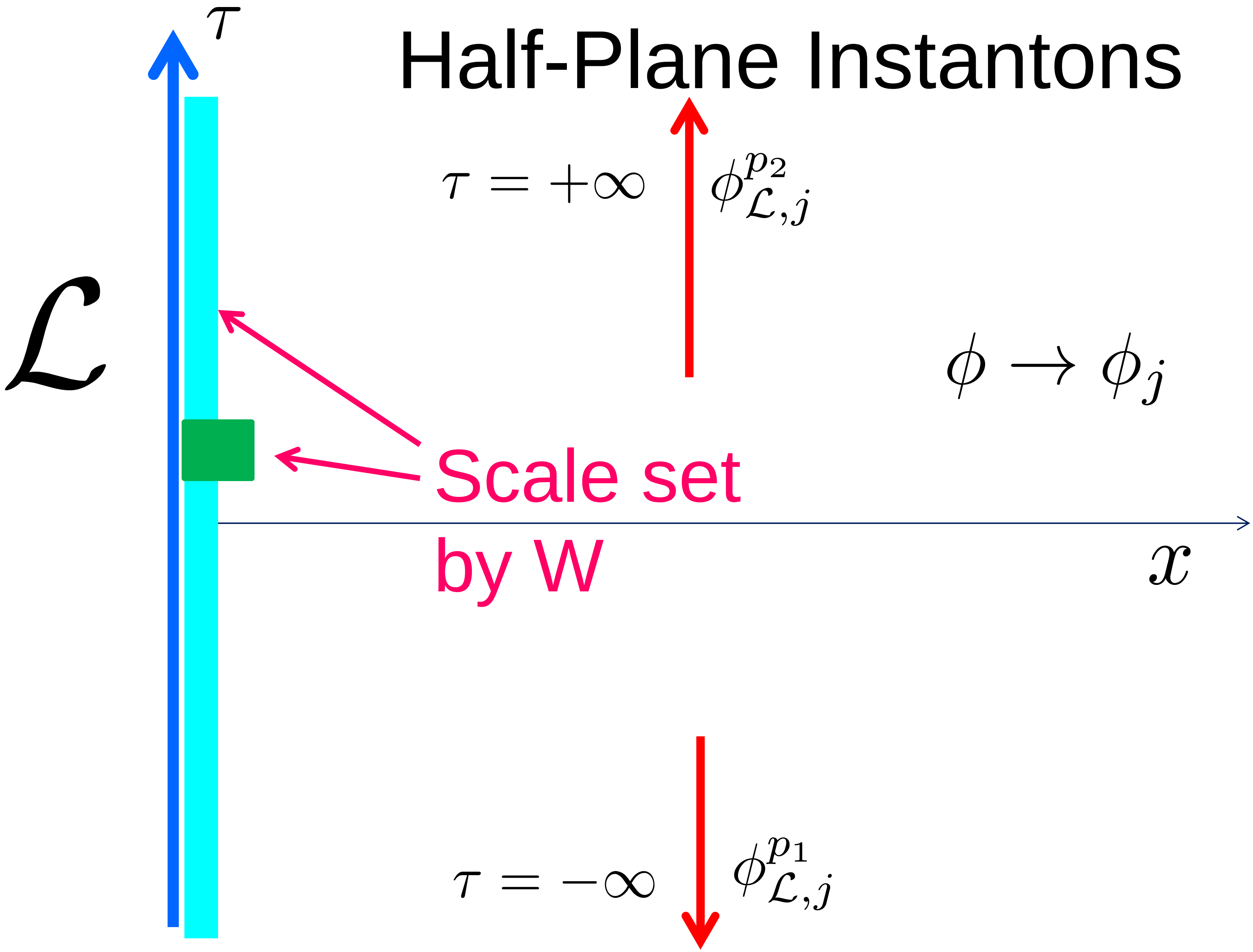
MSW complex: $\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$

Grading the complex: Assume X is CY and that we can find a logarithm:

$$w = \mathrm{Im} \log \frac{\iota^*(\Omega^{d,0})}{\mathrm{vol}(\mathcal{L})}$$

Then the grading is by $f = \eta(D) - w$

# Half-Plane Instantons



# The Morse Complex on $\mathbb{R}_+$ Gives Chan-Paton Factors

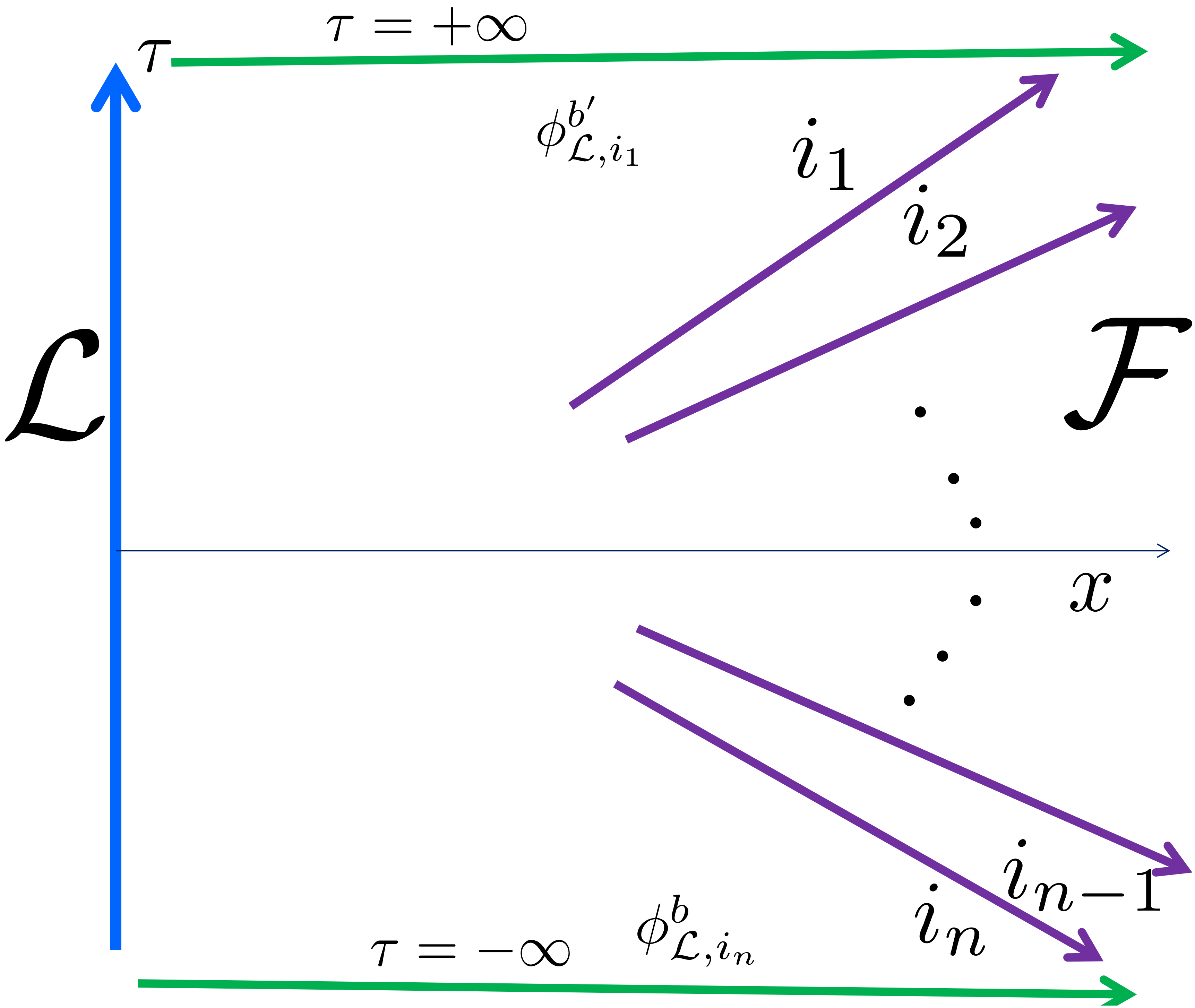
Now introduce Lagrangian boundary conditions $\mathcal{L}$ :

$$\mathcal{E}_j := \mathbb{M}^{\bullet}_{\mathcal{L},j}$$

Half-plane fan of solitons:

$$\phi^{\mathrm{inst}}_{i_1 i_2} \otimes \cdots \otimes \phi^{\mathrm{inst}}_{i_{n-1} i_n}$$

define boundary conditions for the $\zeta$-instanton equation:



$\tau = +\infty$
$\tau$
$\phi^{b'}_{\mathcal{L},i_1}$
$i_1$
$i_2$
$\mathcal{L}$
$\mathcal{F}$
$x$
$i_{n-1}$
$i_n$
$\tau = -\infty$
$\phi^{b}_{\mathcal{L},i_n}$

# Boundary Amplitude from Path Integral

Again Q$\Psi$=0 implies that counting solutions to the instanton equation constructs a boundary amplitude with CP spaces

$$\mathcal{E}_j := \mathbb{M}^{\bullet}_{\mathcal{L},j}$$

$$\rho(\mathrm{t}_{\mathcal{H}})[\frac{1}{1-\mathcal{B}}; e^{\beta}] = 0$$

- Construct differential on the complex on the strip.
- Construct objects in the category of Branes

# A Natural Conjecture

Following constructions used in the Fukaya category, Paul Seidel constructed an $A_\infty$ category FS[X,W] associated to a holomorphic Morse function W: X to $\mathbb{C}$.

Tw[FS[X,W]] is meant to be the category of A-branes of the LG model.

But, we also think that Br[Vac[X,W]] is the category of A-branes of the LG model!

So it is natural to conjecture an equivalence of $A_\infty$ categories:

$$\mathrm{Tw}[\mathrm{FS}[X,W]] \cong \mathrm{Br}[\mathrm{Vac}[X,W]]$$

"ultraviolet"

"infrared"

# Solitons On The Interval

Now consider the finite interval $[x_l, x_r]$ with boundary conditions $\mathcal{L}_l$, $\mathcal{L}_r$

When the interval is much longer than the scale set by W the MSW complex is

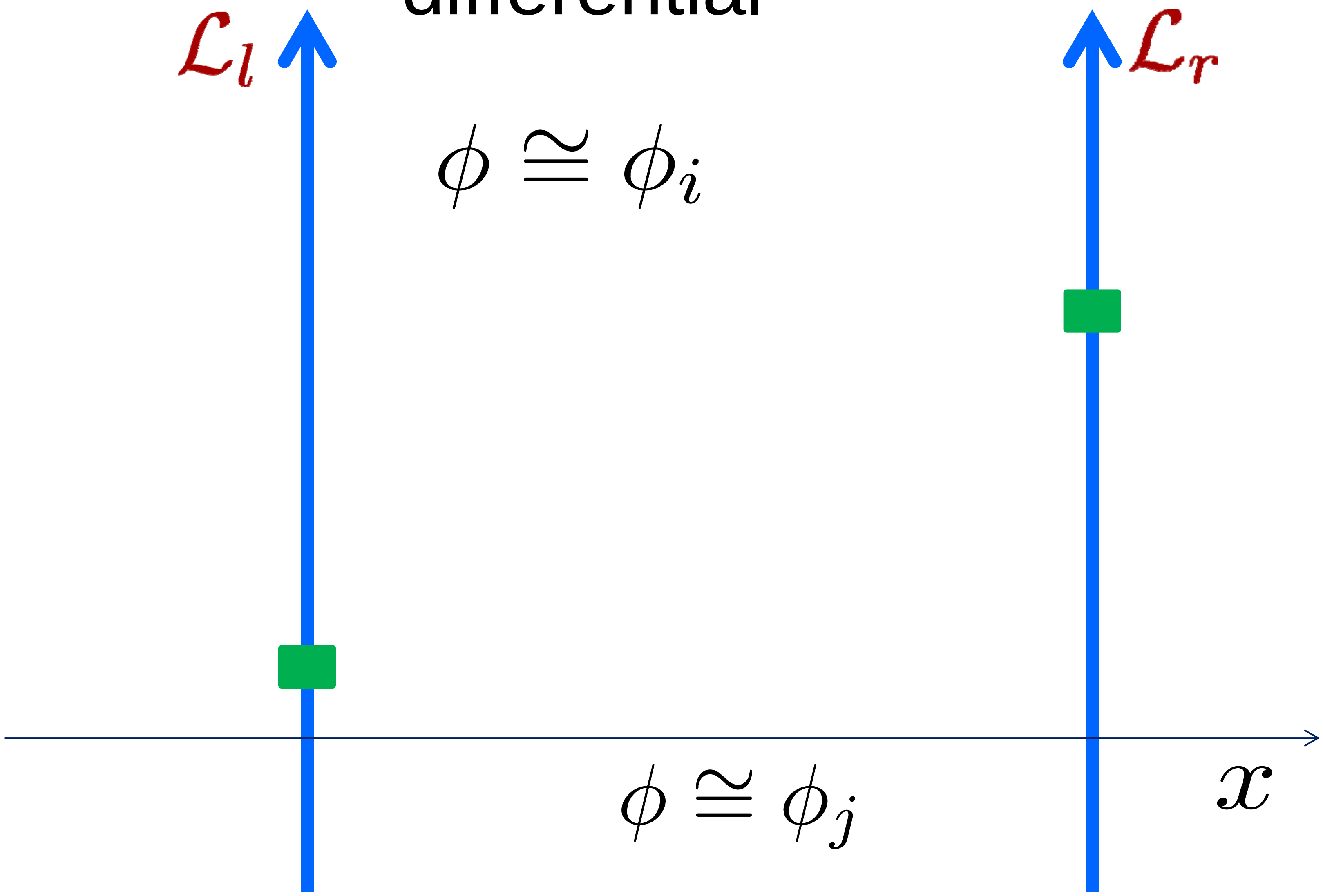
$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i}\otimes\mathbb{M}_{i,\mathcal{L}_r}$$

The Witten index factorizes nicely: $\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$

But the differential $d_{\mathcal{L}_\ell,i}\otimes 1 + 1\otimes d_{i,\mathcal{L}_r}$

is too naïve !

# Instanton corrections to the naïve differential



# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

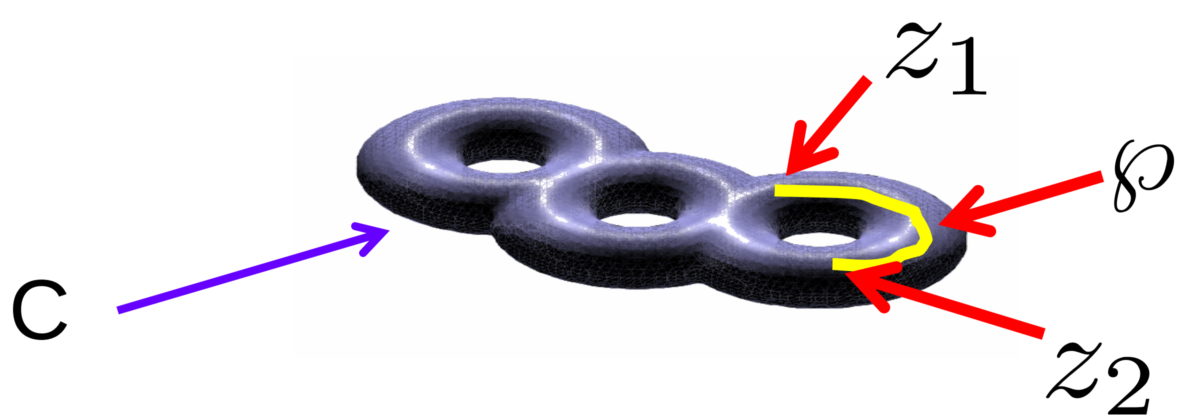
# Families of Theories

Now consider a *family* of Morse functions

$$W(\phi; z) \qquad\qquad z \in C$$

Let $\wp$ be a path in C connecting $z_1$ to $z_2$.

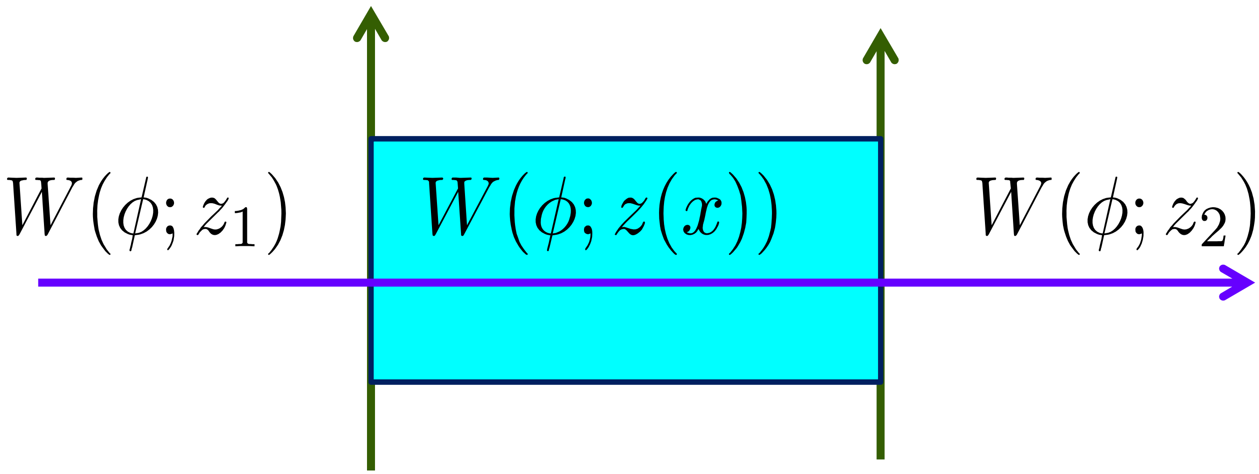
View it as a map z: $[x_l, x_r] \longrightarrow C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



# Domain Wall/Interface

Using z(x) we can still formulate our SQM!

$$h = \int_D \left( pdq + \mathrm{Re}(\zeta^{-1} W(\phi; z(x))dx \right)$$




From this construction it manifestly preserves two supersymmetries.

# Parallel Transport of Categories

To $\wp$ we associate an $A_\infty$ functor

$$\mathbb{F}(\wp) : Br[Vac[W_1]] \to Br[Vac[W_2]]$$

(Relation to GMN: "Categorification of S-wall crossing")

To a composition of paths we associate a composition of $A_\infty$ functors:

$$\mathbb{F}(\wp_1 \circ \wp_2) = \mathbb{F}(\wp_1) \circ \mathbb{F}(\wp_2)$$

To a homotopy of $\wp_1$ to $\wp_2$ we associate an equivalence of $A_\infty$ functors. ( Categorifies CVWCF.)

# Outline

- Introduction & Motivations
- Webs, Convolutions, and Homotopical Algebra
- Web Representations
- Web Constructions with Branes
- Landau-Ginzburg Models & Morse Theory
- Supersymmetric Interfaces
- Summary & Outlook

# Summary

1. We gave a viewpoint on instanton corrections in 1+1 dimensional LG models based on IR considerations.

2. This naturally leads to $L_\infty$ and $A_\infty$ structures.

3. As an application, one can construct the (nontrivial) differential which computes BPS states on the interval.

4. When there are families of LG superpotentials there is a notion of parallel transport of the $A_\infty$ categories.

# Outlook

1. Finish proofs of parallel transport statements.

2. Relation to S-matrix singularities?

3. Are these examples of universal identities for massive 1+1 N=(2,2) QFT?

4. Generalization to 2d4d systems: Categorification of the 2d4d WCF.

5. Computability of Witten's approach to knot homology? Relation to other approaches to knot homology?