LECTURE 2: HYPERKÄHLER METRICS AND THE KS WCF.

1. HYPERKÄHLER METRICS & TWISTORS
2. USING THE TORUS FIBRATION STRUCTURE OF $\mathcal{M} \longrightarrow \mathcal{B} \Rightarrow$ DEFINITION OF THE $\mathcal{X}_\gamma$.
3. ONE-LOOP COMPUTATION OF SINGLE-PARTICLE CORRECTIONS.
4. HOW TO INCLUDE QUANTUM EFFECTS OF MUTUALLY NONLOCAL PARTICLES.
5. KSWCF & CONTINUITY OF THE METRIC
6. PHYSICAL PROOF OF KSWCF FROM WARD IDENTITIES FOR LINE OPERATORS
7. SUMMARY OF LECT'S I & II; SUMMARY OF LECT'S III & IV.
8. OPEN PROBLEMS

1. HK METRICS & TWISTORS

DEF: A HYPERKÄHLER MANIFOLD IS A RIEMANNIAN MANIFOLD WITH THREE ORTHOGONAL TMNS. OF THE TANGENT BUNDLE

$$J_a \in End(TM) \qquad a = 1,2,3$$

SUCH THAT

1.) J_a SATISFY THE ALGEBRA OF THE QUATERNIONS:

$$J_a^2 = -1 \qquad J_a J_b = \epsilon_{abc} J_c$$

2.) $\nabla J_a = 0$

REMARKS

1.) M IS KÄHLER WRT EACH COMPLEX STRUCTURE $\Longrightarrow$ 3 KÄHLER FORMS ω_a $a = 1, 2, 3$

2.) IN COMPLEX STRUCTURE J_3 AT ANY POINT $p \in M$ WE CAN CHOOSE AN ORTHONORMAL BASIS FOR THE QUATERNIONIC VECTORSPACE

$T_p^* M$: $\quad (dz^I, dw_I) \quad I = 1, \dots, r$

$$J_3 : \ (dz^I, dw_I) \longrightarrow (i\, dz^I, i\, dw_I)$$

$$J_1 : \ (dz^I, dw_I) \rightarrow (-\overline{dw_I}, \overline{dz^I})$$

$$J_2 : \ (dz^I, dw_I) \rightarrow (i\,\overline{dw_I}, -i\,\overline{dz^I})$$

AT THE POINT p

$$\omega_3 = \frac{i}{2} dz^I \wedge \overline{dz^I} + \frac{i}{2} dw_I \wedge \overline{dw_I}$$

$$\omega_1 + i\omega_2 = dz^I \wedge dw_I$$

IS TYPE (2,0).

MOREOVER IT IS HOLOMORPHIC AND SYMPLECTIC

3.) IN FACT M HAS A WHOLE S^2 WORTH OF COMPLEX STRUCTURES

$$(n^a J_a)^2 = -1 \qquad \text{FOR} \quad n^a n^a = 1$$

4.) CHOOSE A NORTH POLE FOR STEREOGRAPHIC PROJECTION:

$$S^2 \cong \mathbb{C}P^1 \longrightarrow \mathbb{C} \ni \zeta$$

SO THAT $\zeta = 0$ CORRESPONDS TO J_3

THEN EXPLICIT ROTATION –

EXPRESSED IN TERMS OF ζ

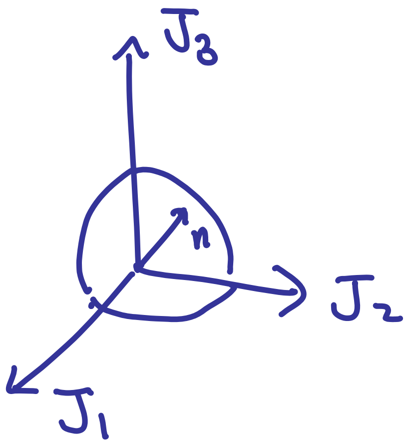
$$(n^1, n^2, n^3) = \left(\frac{\zeta+\bar{\zeta}}{1+|\zeta|^2}, \frac{i(\zeta-\bar{\zeta})}{1+|\zeta|^2}, \frac{1-|\zeta|^2}{1+|\zeta|^2} \right)$$

SHOWS THAT:

FOR GENERAL COMPLEX STRUCTURE ζ

THE HOLOMORPHIC SYMPLECTIC FORM IS:

$$\boxed{\omega_\zeta := -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-}$$



NOTE: I AM USING THE SAME SYMBOL ζ AS IN THE DISCUSSION OF THE KSWCF

HITCHIN'S TWISTOR THEOREM

THE TWISTOR SPACE OF $\mathcal{M}$ IS $Z = \mathcal{M} \times S^2$

IT CAN BE GIVEN THE STRUCTURE OF A COMPLEX MANIFOLD BY TAKING $S^2 \cong \mathbb{CP}^1$ BUT THE COMPLEX STRUCTURE IN THE $\mathcal{M}$ DIRECTIONS DEPENDS ON $\zeta \in \mathbb{CP}^1$.

HITCHIN'S THEOREM IS AN EQUIV. OF HOLOMORPHIC DATA FOR Z WITH THE HK METRIC ON $\mathcal{M}$:

THEOREM: IF $(\mathcal{M}, g)$ IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p : Z \longrightarrow \mathbb{CP}^1$$

$$M^\zeta = p^{-1}(\zeta) = M \text{ IN COMPLEX STRUCTURE } \zeta$$

2. $\exists$ HOLOMORPHIC SECTION

ϖ OF $\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$

$\varpi_\zeta := \varpi\big|_{M^\zeta}$ = HOLOMORPHIC SYMPLECTIC FORM ON M^ζ

3. $\exists$ ANTI-HOLOMORPHIC $\sigma : Z \to Z$

COVERING $\zeta \to -1/\bar{\zeta}$

4. $\forall x \in M$, $\exists$ HOLOMORPHIC SECTION

$s_x : \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL

BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

CONVERSELY,

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

FOR $\zeta \in \mathbb{C}^*$:

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

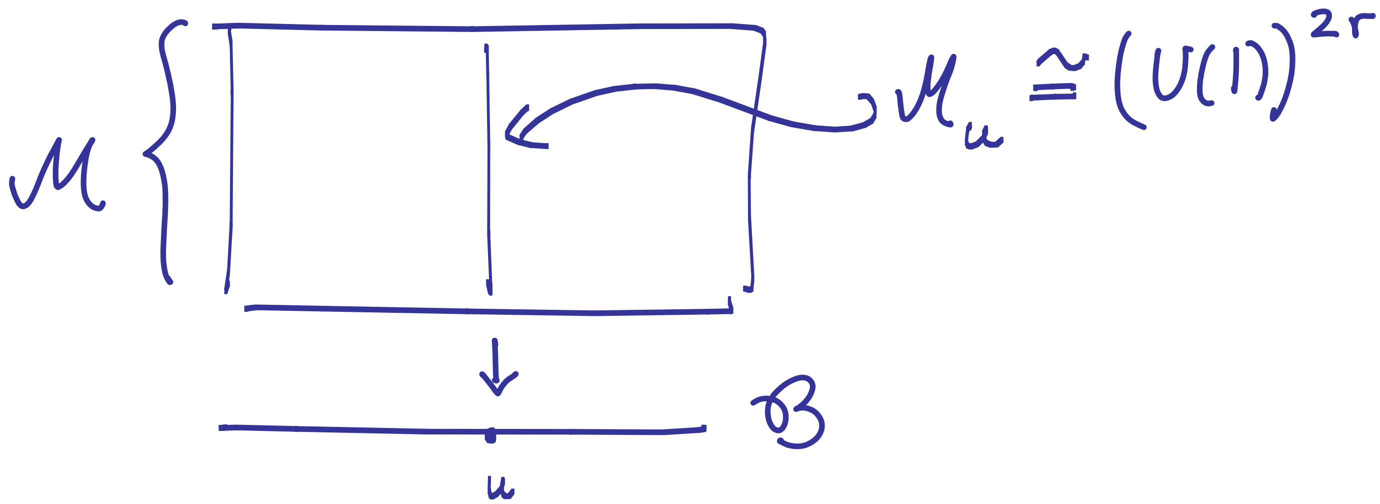
KÄHLER FORM

$$\omega_+ = \omega_1 + i\omega_2$$

OUR STRATEGY IS TO CONSTRUCT ϖ_ζ EXPLICITLY FOR $\mathcal{M}^5$ USING A "NICE" SET OF HOLOMORPHIC FUNCTIONS ON TWISTOR SPACE:

$$\mathcal{X}_\gamma \quad , \quad \gamma \in \Gamma$$

2. EXTRA STRUCTURE FROM THE TORUS FIBRATION OF $\mathcal{M}$:



HEURISTICALLY:

AT LARGE R $\mathcal{X}_\gamma$, WHEN RESTRICTED TO $\mathcal{M}_u$ WILL BE APPROXIMATELY THE FOURIER MODES

$$\mathcal{X}_\gamma \sim e^{i\theta_\gamma} \cdot \text{const.}$$

$$\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

IS CANONICALLY DEFINED

FOR $\zeta \neq 0,\infty$ $\mathcal{M}_u$ IS NOT HOLOMORPHIC

SO CAUCHY RIEMANN FIXES u-DEPENDCE

THE COMPLEX TORUS FIBRATION $\mathcal{T}$

WHAT WE REALLY DO IS COMPARE THE REAL TORUS $\Gamma_u^* \otimes \mathbb{R}/2\pi\mathbb{Z}$ WITH THE COMPLEX TORUS $\Gamma_u^* \otimes \mathbb{C}^*$

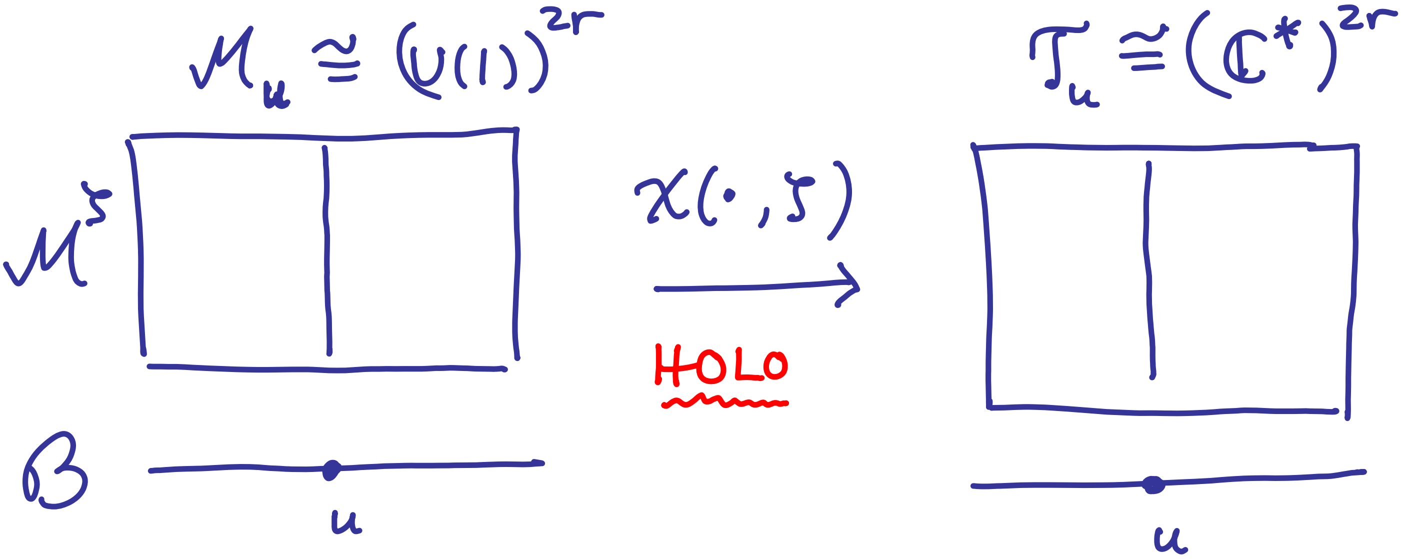
SO CONSIDER $\mathcal{T} := \Gamma^* \otimes \mathbb{C}^*$

- $\mathcal{T}$ HAS A FIXED COMPLEX STRUCTURE WITH HOLOMORPHIC FIBERS $\cong (\mathbb{C}^*)^{2r}$
- $\mathcal{T}$ HAS HOLOMORPHIC FUNCTIONS X_γ
- $\mathcal{T}$ HAS A FIBERWISE HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^{\mathcal{T}} := \frac{1}{2}\epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}}$$

WE SEARCH FOR A HOLOMORPHIC MAP

$$\mathcal{X} : \mathcal{Z} \longrightarrow \mathcal{T} = \Gamma^* \otimes \mathbb{C}^*$$



SO THAT: $\varpi_\zeta = \mathcal{X}(\cdot, \zeta)^* (\varpi^T)$

IF WE DEFINE $\mathcal{X}_\gamma := \mathcal{X}^* (X_\gamma)$

$$\Rightarrow \quad \varpi_\zeta = \frac{1}{2} \epsilon^{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

WE CAN VIEW

$$\varpi_\zeta = \frac{1}{2}\epsilon^{ij}\,\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

IN TWO WAYS:

- KNOW THE METRIC $\Rightarrow$ CONSTRUCT $\mathcal{X}_\gamma$

– DO THIS FOR THE SEMIFLAT METRIC AND FIRST QUANTUM CORRECTION

- ULTIMATELY, WE DEFINE THE $\mathcal{X}_\gamma$ AND USE THEM TO DEFINE ϖ_ζ (AND HENCE THE HK METRIC)

EXAMPLE: SEMI-FLAT LIMIT

WE KNOW g^{sf} $\Longrightarrow$ COMPUTE

$$\varpi_\zeta = -\frac{i}{2\zeta}\,\omega_+ + \omega_3 - \frac{i}{2}\,\zeta\,\omega_-$$

FROM THE EXPLICIT METRIC IN LECTURE 1 IN COMPLEX STRUCTURE $\zeta = 0$ WHERE

$$da^I \quad \& \quad dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

ARE TYPE (1,0) WE FIND

$$\omega_+ \sim da^I \wedge dz_I$$

$$\omega_3 \sim R\, \mathrm{Im}\tau_{IJ}\, da^I \wedge d\bar{a}^J + \frac{1}{R}(\mathrm{Im}\tau)^{-1,IJ}\, dz_I \wedge d\bar{z}_J$$

SO NOW WE

FIND $\mathcal{X}_{\gamma_i}$ SO THAT:

$$\varpi_\zeta = \frac{1}{8\pi^2 R} \epsilon^{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

SOLUTION: DEFINE $\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$

$$\boxed{\mathcal{X}_\gamma^{sf} = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

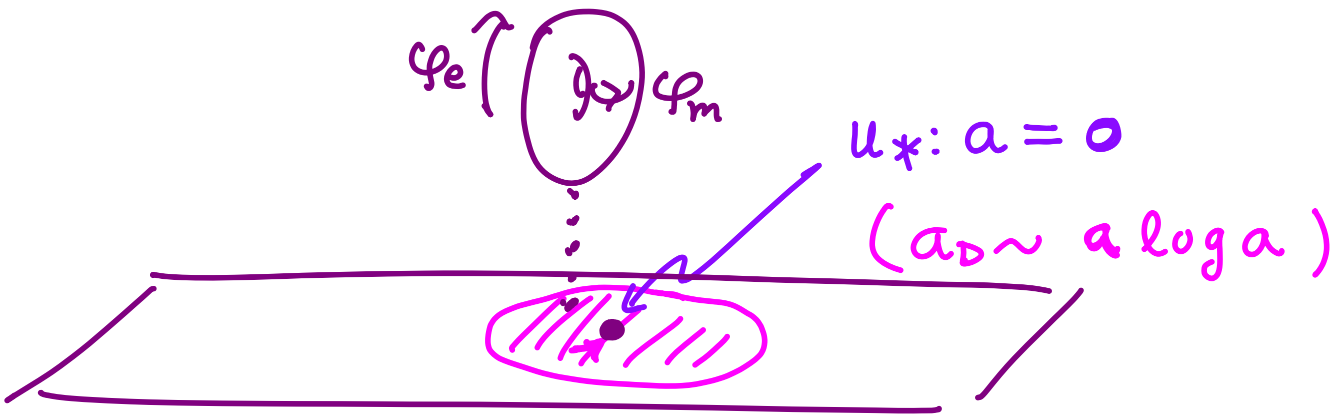
- LEADING APPXT. TO $\mathcal{X}_\gamma$ FOR $R \to \infty$
- NO Q.C.'s FROM BPS STATES.
- NOTE THIS IS <u>HOLOMORPHIC</u> IN COMPLEX STRUCTURE ζ !

3. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.
- CONSIDER A POINT $u_* \in \mathcal{B}$ WHERE A SINGLE HM HAS $M \to 0$

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$ & CENTRAL CHARGE $Z = qa$



SCALING LIMIT: $aR = \varepsilon \ll 1$, $R \longrightarrow \infty \Longrightarrow$

DOMINANT CONTRIBUTION FROM A SINGLE 4D BPS H.M.

COMPUTATION OF THE METRIC

- INCLUDE H.M. Φ IN 4D EFF. ACTION
- THEN DO KK REDUCTION KEEPING ALL FOURIER MODES
- THEN INTEGRATE OUT THE MASSIVE MODES

$$\int dx^{0123}\, R \cdot \Big\{ g^{MN} \left(\partial_M \Phi - iq A_M \Phi\right)\left(\partial_N \bar{\Phi} + iq A_N \bar{\Phi}\right) + 2q^2 |a|^2 |\Phi|^2 + \dots \Big\}$$

$$\text{KK:} \quad \Phi = \sum_{n\in\mathbb{Z}} e^{inx^3}\, \Phi^{(n)}(x^\mu)$$

$$= \int dx^{012}\, 2\pi R \sum_{n\in\mathbb{Z}} \Big\{ \left|\partial_\mu \Phi^{(n)} - iq A_\mu \Phi^{(n)}\right|^2 + $$

$$+ 2q^2 \cdot |a|^2 |\Phi^{(n)}|^2 + \frac{1}{R^2}\left(n + q\varphi_e\right)^2 |\Phi^{(n)}|^2 \Big\}$$

COMPUTING THE 1-LOOP CORRECTION TO THE METRIC:

- THE SEMI-FLAT METRIC HAS $u(1) \oplus u(1)$ TRANSLATION SYMMETRY IN φ_e & φ_m.

- THE CORRECTIONS FROM Φ COUPLE TO φ_e BUT NOT TO φ_m $\Longrightarrow$ $U(1)$ ISOMETRY OF φ_m TRANSLATION IS PRESERVED $\Longrightarrow$

QUANTUM CORRECTED METRIC SHOULD BE OF GIBBONS HAWKING FORM:

(SEE SEIBERG & WITTEN, OOGURI & VAFA, SEIBERG & SHENKER FOR FURTHER DISCUSSION. THE LAST PAPER CLAIMS THE RESULT IS ONE-LOOP EXACT IN A $U(1)$ THEORY WITH N_f HYPERMULTIPLETS.)

GIBBONS-HAWKING ANSATZ:

HK METRIC ON LINE BUNDLE OVER A REGION IN $\mathbb{R}^3$:

$$ds^2 = V^{-1}(\vec{x})\left(d\frac{\varphi_m}{2\pi} + A\right)^2 + V(\vec{x})\,(d\vec{x})^2$$

Hyperkähler $\Longleftrightarrow$ $F = dA = *dV$

$V(\vec{x}) =$ HARMONIC. (POSITIVE)

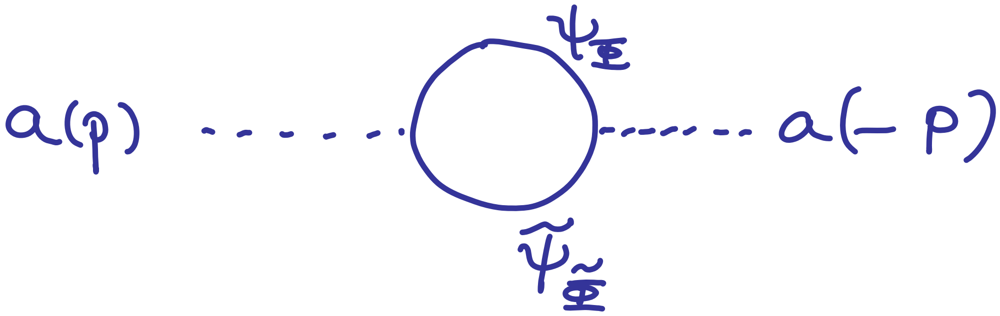
HK STRUCTURE: $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A\right) + \frac{1}{2} V \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

(WARNING: CHANGE OF NORMALIZATION OF φ_e, φ_m TO PERIODICITY 2π.)

THEREFORE, WE NEED ONLY FIND $V(\vec{x})$. THIS CAN BE MOST EASILY DETERMINED BY COMPUTING THE K.E. OF $a(x^\mu)$:

$$W = \sqrt{2}\, a \Phi \tilde{\Phi}$$



$$\Longrightarrow$$

$$V(\vec{x}) = \frac{g^2 R}{4\pi}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{g^2R^2|a|^2+\left(g\frac{\varphi_e}{2\pi}+n\right)^2}} - k_n\right)$$

UV cutoff.

$a = x^1 + i x^2$

$\varphi_e = 2\pi R\, x^3$ PERIODIC

POISSON RESUMMATION $\Rightarrow$ NICE PHYSICAL INTERPRETATION:

$$V(\vec{x}) = V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi}\sum_{n\neq 0} e^{inq\varphi_e} K_0\left(2\pi R|nqa|\right)$$

- $K_0(x) \sim e^{-x} \quad x \to +\infty$

- BPS WORLDLINE ACTION:

$$e^{-2\pi R|Z_\gamma(u)| + i\theta_\gamma}$$

$$\boxed{\Rightarrow \text{COMPUTE } \varpi_3 = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-}$$

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\mathcal{X}_\gamma\}$ ON TWISTOR SPACE IS GENERATED

$$\mathcal{X}_e := \mathcal{X}_{(1,0)} = \exp\{i\varphi_e + \cdots\}$$

$$\mathcal{X}_m := \mathcal{X}_{(0,1)} = \exp\{i\varphi_m + \cdots\}$$

$$\mathcal{X}_{(a,b)} = \mathcal{X}_e^a \mathcal{X}_m^b$$

DETERMINE $\mathcal{X}_e$ AND $\mathcal{X}_m$

FROM A DIFFERENTIAL EQUATION

$$\boxed{\varpi_\zeta = -\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m}{\mathcal{X}_m}}$$

WE FIND:

$$\chi_e = \chi_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i\varphi_e + \pi R \zeta \overline{a}\right]$$

BUT

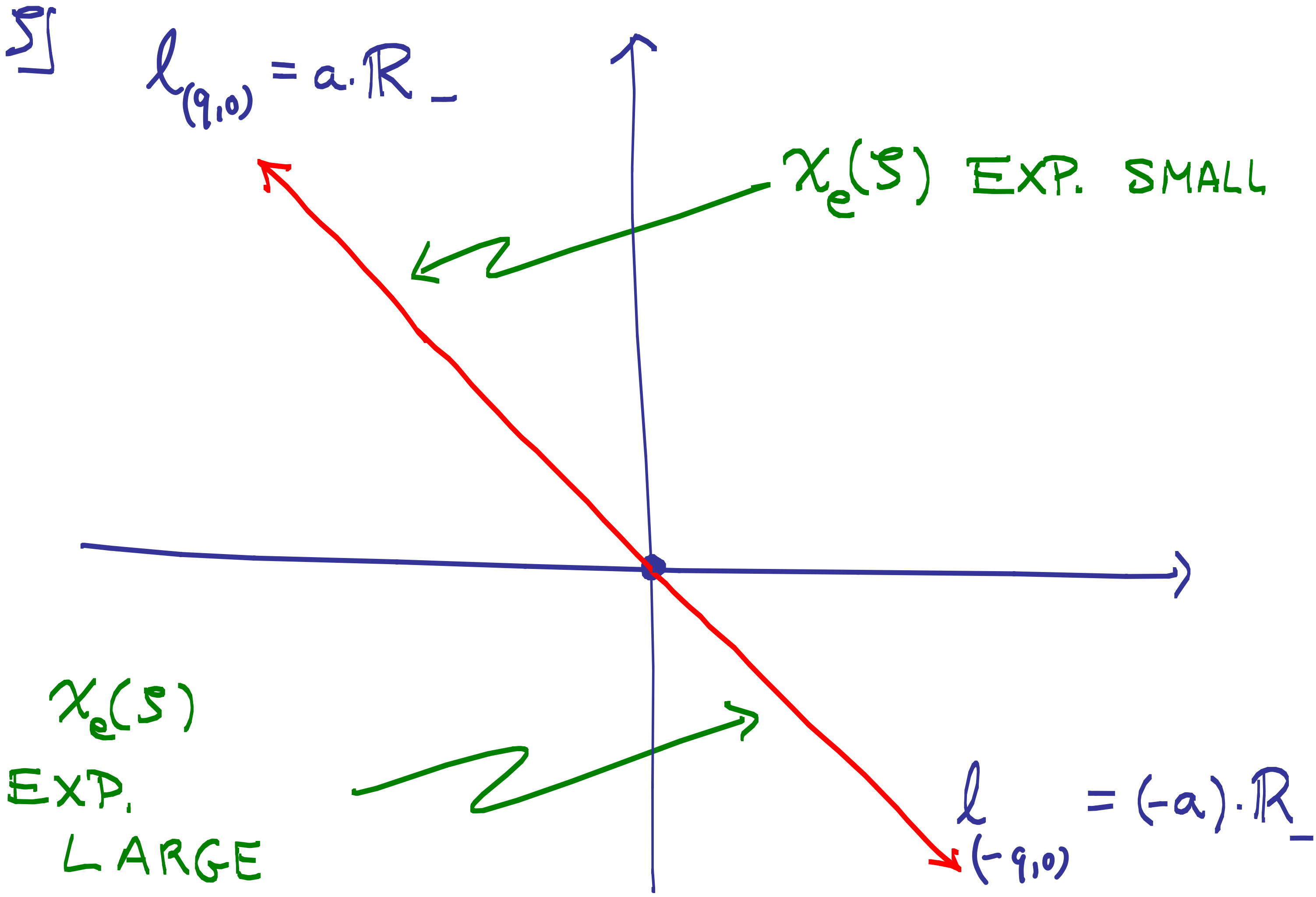
$$\chi_m = \chi_m^{s.f.} \cdot \chi_m^{inst.}$$

$$\chi_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i\varphi_m + \pi R \zeta \overline{a_D}\right]$$

$$a_D = \frac{g^2}{2\pi i}\left(a \log \frac{a}{e\Lambda}\right)$$

χ_m^{inst} = "INSTANTON CONTRIBUTION"

$$\boxed{\mathcal{X}_m^{inst}(\zeta) = \exp\left\{ \frac{iq}{4\pi}\int_{\ell_{(q,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\,\log\left(1-\mathcal{X}_e(\zeta')^{q}\right) - \frac{iq}{4\pi}\int_{\ell_{(-q,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\,\log\left(1-\mathcal{X}_e(\zeta')^{-q}\right)\right\}}$$



WHY IS IT AN INSTANTON CONTRIBUTION?

CONSIDER

$$\mathcal{X}_\gamma^{sf} = \exp\left(\pi R \frac{Z_\gamma}{\zeta} + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right)$$

ON THE BPS RAY:

$$\ell_{\gamma,u} = \{\zeta \mid Z_\gamma/\zeta \in \mathbb{R}_-\}$$

$$Z_\gamma = e^{i\alpha_\gamma}|Z_\gamma|, \quad \zeta = -e^{\eta + i\alpha_\gamma}$$

BPS RAY PARAMETRIZED BY $\eta \in \mathbb{R}$

$$\boxed{\mathcal{X}_\gamma^{sf} = \exp\left(-2\pi R|Z_\gamma|\cosh\eta + i\theta_\gamma\right)}$$

So integrals of order

$$e^{i\theta_\gamma}\, e^{-2\pi R|Z_\gamma|}$$

= action of BPS world line.

EMERGENCE OF THE KS TRANSFORMATION

AS A FUNCTION OF ζ, $\mathcal{X}_m$ IS DISCONTINUOUS ACROSS THE BPS RAYS OF THE HYPERMULTIPLET OF CHARGE $(\pm q, 0)$

$$\ell_{\gamma,u} := \left\{ \zeta \;\middle|\; \frac{Z_\gamma(u)}{\zeta} \in \mathbb{R}_- \right\}$$

ACROSS THESE RAYS:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = \left(\mathcal{X}_e, \mathcal{X}_m (1 - \mathcal{X}_e^{\pm q})^{\mp q}\right)^{ccw}$$

$$= K_{(\pm q, 0)} (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= K_{(\pm q, 0)}^{\Omega(q,0)} (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

6 KEY PROPERTIES OF $\mathcal{X}_\gamma$

1. $\mathcal{X}_\gamma$ ARE HOLOMORPHIC ON $\mathcal{Z}$

2. $\mathcal{X}_\gamma \cdot \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$

3. $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$

4. $\mathcal{X}_\gamma \sim \mathcal{X}_\gamma^{s.f.}$ FOR $R \to \infty$

5. $$\left.\begin{array}{l} \lim\limits_{\zeta\to 0} \mathcal{X}_\gamma \exp\left(-\frac{\pi R}{\zeta} Z_\gamma(u)\right) \\ \lim\limits_{\zeta\to\infty} \mathcal{X}_\gamma \exp\left(-\pi R \zeta \overline{Z_\gamma(u)}\right) \end{array}\right\} \text{FINITE}$$

6. $\mathcal{X}_{\gamma'}(\zeta)$ TRANSFORMS BY

$S_\gamma = \prod_{\gamma'' \| \gamma} K_{\gamma''}^{\Omega(\gamma'';u)}$ AS ζ CROSSES THE BPS RAY $\ell_{\gamma,u}$.

4. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: PROPERTIES 1–6

HOLD FOR THE <u>EXACT</u> FUNCTIONS $\mathcal{X}_{\gamma}$, USING ALL THE BPS RAYS ℓ_{γ} WITH DISCONTINUITY $K_{\gamma}^{\Omega(\gamma;u)}$

THIS WILL DETERMINE THEM UNIQUELY

NOW: FINDING $\mathcal{X}_\gamma$ SATISFYING PROPERTIES $1 \to 6$ IS <u>EQUIVALENT</u> TO SOLVING A RIEMANN-HILBERT PROBLEM:

<u>RH</u>: FIND A PIECEWISE HOLOMOR. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.

SUMMARIZING THE $\mathcal{X}_\gamma$ BY A SINGLE MAP $\mathcal{X}$ (RECALL $\mathcal{X}_\gamma = \mathcal{X}^*(X_\gamma)$)

$\Rightarrow$ A RIEMANN-HILBERT PROBLEM IN THE ζ-PLANE FOR THE MAP

$$\mathcal{X}(\cdot, \zeta) : \mathcal{M}^\zeta \longrightarrow \mathcal{T} = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ :

$$\mathcal{X}^{cw} = S_\gamma\left(\mathcal{X}^{ccw}\right)$$

[RECALL: $S_\gamma = \prod_{\ell_{\gamma'}=\ell_\gamma} K_{\gamma'}^{\Omega(\gamma';u)}$]

2.) $\mathcal{X}(\zeta)$ HAS ASYMPTOTICS FOR $\zeta \longrightarrow 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $\mathcal{O}(1)$ CORRECTIONS

$$Y := (\mathcal{X}^{sf})^{-1}\mathcal{X} \; : \mathcal{M} \longrightarrow \mathcal{M}$$

i.e.

$$Y_0 = \lim_{\zeta\to 0} Y(\zeta) \;\; \& \;\; Y_\infty = \lim_{\zeta\to\infty} Y(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{sf}_\gamma(\zeta) \cdot$$

$$\exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_{\gamma'}(\zeta')\right] \right\}$$

NOW, RECALL:

$$\boxed{\mathcal{X}^{sf}_\gamma = \exp\left(-2\pi R |Z_\gamma| \cosh\eta + i\theta_\gamma\right)}$$

ALONG BPS RAYS $\ell_{\gamma,u}$

$\Longrightarrow$ INTEGRALS ARE BEUTIFULLY CONVERGENT

AND $|\chi_\gamma^{sf}| \ll 1$ AT LARGE R IN THE INTEGRAND...

$\Longrightarrow$ THEREFORE WE CAN ITERATE THIS EQUATION

WHILE IT LOOKS COMPLICATED ONE CAN ORGANIZE THE EXPANSION AS A SUM OVER TREES...

GIVES THE FULL INSTANTON EXPANSION!

$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S, AT LEAST AT LARGE R

(THE EXPANSION MIGHT BREAK DOWN AT SMALL R .... ?)

5. KS WCF & CONTINUITY OF THE METRIC

NOW WE CONSTRUCT THE METRIC FROM:

$$\varpi_\zeta = \frac{1}{4\pi^2 R} \mathcal{X}^*(\cdot,\zeta)(\varpi^T)$$

EXPLICIT LEADING ORDER CORRECTIONS EXPRESSED AS AN ∞ SUM OF BESSEL FUNCTIONS.

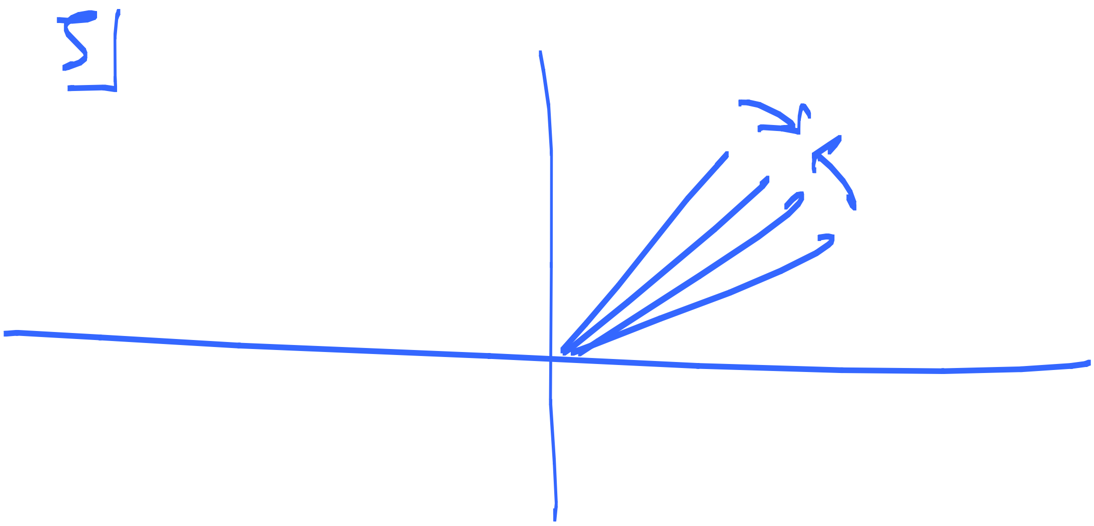
- CONTINUITY IN ζ ?

O.K. BECAUSE DISCONTINUITIES OF $\mathcal{X}(\cdot,\zeta)$ ARE SYMPLECTOMORPHISMS

- BUT WHAT ABOUT CONTINUITY AS A FUNCTION OF u ??

ROLE OF THE KS WCF:

- AS u CROSSES A WALL OF MS BPS RAYS PILE UP



$u \to u^+$: DISCONTINUITY IN RH PROBLEM ALONG $\ell_{\gamma_1} = \ell_{\gamma_2}$

$$= \prod_{\substack{n \geq 0 \\ m \geq 0}}^{\curvearrowright} K_{n\gamma_1 + m\gamma_2}^{\Omega(n\gamma_1 + m\gamma_2; u^+)}$$

$u \to u^-$: DISCONTINUITY IS:

$$= \prod^{\curvearrowleft} K_{n\gamma_1 + m\gamma_2}^{\Omega(n\gamma_1 + m\gamma_2; u^-)}$$

THUS, THE RH PROBLEM REMAINS <u>UNCHANGED</u> AS u CROSSES THE WALL IF THE $\Omega(\gamma;u)$ OBEY THE KSWCF.

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE HK. METRIC ACROSS WALLS OF M.S. !

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM?

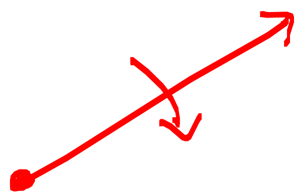
6. DIFFL EQUATIONS AND A PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$A_\zeta = X^{-1} \zeta \partial_\zeta X$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$X^{-1} \zeta \partial_\zeta X \to (SX)^{-1} \zeta \partial_\zeta (SX)$$

$$= X^{-1} \zeta \partial_\zeta X$$

$\Rightarrow A_\zeta$ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Longrightarrow \quad \zeta \partial_\zeta \mathcal{X} = \mathcal{X} \overleftarrow{\mathcal{A}}_\zeta$$

STRUCTURE GROUP: SYMPL$(\mathbb{T})$

ASYMPTOTICS $\Rightarrow$

$$\mathcal{A}_\zeta = \zeta^{-1} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(+1)}$$

(NOTE: $\mathcal{A}_\zeta^{(-1)}$ CONJUGATE TO $i\pi Z_{\gamma^i} \overleftarrow{\frac{\partial}{\partial \theta^i}}$)

SINCE S_γ IS INDPT. OF $R, u, \Lambda \dots$

SAME ARGUMENT $\Rightarrow$ $\mathcal{X}$ SATISFIES A

SET OF DIFFERENTIAL EQATIONS:

$$\frac{\partial}{\partial u} \chi = \chi \mathcal{A}_u$$

$$\frac{\partial}{\partial \bar{u}} \chi = \chi \bar{\mathcal{A}}_u$$

$$\Lambda \frac{\partial}{\partial \Lambda} \chi = \chi \mathcal{A}_\Lambda$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}} \chi = \chi \mathcal{A}_{\bar{\Lambda}}$$

$$R \frac{\partial}{\partial R} \chi = \chi \mathcal{A}_R$$

$$\zeta \frac{\partial}{\partial \zeta} \chi = \chi \mathcal{A}_\zeta$$

$$\mathcal{A}_i = \zeta^{-1} \mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta \mathcal{A}_i^{(+1)}$$

SUMMARY OF THE LOGIC:

PROPERTIES 1–6 $\iff$ R-H PROBLEM

R-H PROBLEM WITH PRESCRIBED DISCONTINUITIES

$\Updownarrow$

SYSTEM OF DIFF. EQS. WITH PRESCRIBED "MONODROMY" (STOKES DATA).

(KNOWING $\mathcal{A}_i$ $\iff$ KNOWING THE "MONODROMY")

KEY POINT: <u>THE DIFF. EQS. ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!</u>

THE χ_γ HAVE NICE PHYSICAL INTERPRETATIONS:

$$\begin{tikzcd}
 & \boxed{D=6 \ \mathcal{N}=1 \ \text{SYM}} \arrow[dl, "T^2"'] \arrow[ddr, "T^3"] & \\
\boxed{D=4 \ \mathcal{N}=2 \ \text{SYM}} \arrow[drr, "S^1"'] & & \\
 & & \boxed{D=3,\ \mathcal{N}=4 \ \sigma\text{-MODEL}}
\end{tikzcd}$$

- 6D LIGHTLIKE LINE OPERATORS
- 4D 't HOOFT-WILSON-MALDACENA LOOP OPERATORS ANNIHILATED BY Q_ζ
- 3D CHIRAL RING OPERATORS ANN. BY Q_ζ.

- $$v^M = R\left(0,0,0,1,\frac{\zeta+\zeta^{-1}}{\sqrt{2}},\frac{i(\zeta-\zeta^{-1})}{\sqrt{2}}\right)$$

$$\oint dx^3\, v^M A_M = \oint dx^3\left(A_3 + \zeta R \Phi + \zeta^{-1} R \bar{\Phi}\right)$$

- FOR $SPIN(1,5)$ $(4 \otimes 4)_a \cong 6_v$

$V^M V_M = 0 \Rightarrow V^{rs}$ HAS KERNEL η_s

$$\Rightarrow \left[\eta_s Q_s, \; V^{rr'} A_{rr'}\right]$$

$$= \qquad V^{rr'}\left(\eta_r \lambda_{r'} - \eta_{r'} \lambda_r\right) = 0.$$

- CALL THE SUSY Q_ζ FOR ABOVE NULL VECTOR

* $Q_\zeta = \overline{Q} + \zeta K_3$ IN 4D DONALDSON - WITTEN TWIST

* $Q_\zeta = Q_1 + \zeta Q_2$ IN 3D TFT FOR ROZANSKY - WITTEN TWIST

$$\chi_e^I = \exp \oint \left(A_3^I + R\zeta a^I + R\bar{\zeta}^{-1}\bar{a}^I\right) dx^3$$

$$\chi_{m,I} = \exp \oint \left((A_3^D)_I + R\zeta\, a_{D,I} + R\bar{\zeta}^{-1}\, \overline{a_{D,I}}\right) dx^3$$

ARE Q_ζ - CLOSED AND GENERATE THE CHIRAL RING OF THE 3D TFT.

- ANOMALOUS $U(1)_R$ WARD IDENTITIES

+ ANOMALOUS SCALING WARD IDENT'S

$\Longrightarrow$ R, ζ DIFFERENTIAL EQUATION.

$$\left.\begin{aligned} \frac{\partial}{\partial u}\chi &= \chi\mathcal{A}_u \\ \frac{\partial}{\partial \bar{u}}\chi &= \chi\mathcal{A}_{\bar{u}} \end{aligned}\right\}$$

HOLOMORPHY ON $\mathcal{M}^5$

ALSO HOLOMORPHY...

$$\left.\begin{aligned} \Lambda\frac{\partial}{\partial \Lambda}\chi &= \chi\mathcal{A}_\Lambda \\ \bar{\Lambda}\frac{\partial}{\partial \bar{\Lambda}}\chi &= \chi\mathcal{A}_{\bar{\Lambda}} \end{aligned}\right\}$$

VIEW Λ AS BACKGROUND VEV OF A VM.

$$\left.\begin{aligned} R\frac{\partial}{\partial R}\chi &= \chi\mathcal{A}_R \\ \zeta\frac{\partial}{\partial \zeta}\chi &= \chi\mathcal{A}_\zeta \end{aligned}\right\}$$

ANOMALOUS SCALE AND R-SYMMETRY

(ASIDE: WHY THE 3-TERM LAURENT EXPANSION ?

CAUCHY-RIEMANN EQS $\Rightarrow$ STRUCTURE

$$\mathcal{A}_u = \zeta^{-1} \mathcal{A}_u^{(-)} + \mathcal{A}_u^{(0)} + \zeta \mathcal{A}_u^{(+)}$$

BY DIRECT COMPUTATION IN HK GEOMETRY.

$\Rightarrow$ SAME FOR $\mathcal{A}_\Lambda$ (WEAKLY GAUGED SYMM.)

SCALE SYMMETRY:

$$\left(R\frac{\partial}{\partial R} - u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}} - \Lambda\frac{\partial}{\partial \Lambda} - \bar{\Lambda}\frac{\partial}{\partial \bar{\Lambda}} \right)\mathcal{X} = 0$$

$U(1)_R$ SYMMETRY:

$$\left(\zeta\frac{\partial}{\partial \zeta} - u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}} - \Lambda\frac{\partial}{\partial \Lambda} + \bar{\Lambda}\frac{\partial}{\partial \bar{\Lambda}} \right)\mathcal{X} = 0$$

$\Rightarrow$ $\mathcal{A}_R, \mathcal{A}_\zeta$ HAVE 3-TERM LAURENT EXPANSIONS IN ζ.)

TO COMPLETE THE STORY WE MUST DERIVE THE CONNECTION $\mathcal{A}_i$, OR EQUIVALENTLY – SINCE IT IS A FLAT CONNECTION ON $\mathcal{B} \times \mathbb{CP}^1 \times \mathbb{R}_+$ – ITS "MONODROMY"

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.*

$\mathcal{A}_\zeta^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS ℓ_γ

DENOTE STOKES FACTORS BY S_γ

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\mathcal{S}_\gamma$ ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1-INSTANTON APPROXIMATION:

$$\mathcal{S}_\gamma = S_\gamma^{K.S.}$$

THIS COMPLETES THE PROOF OF THE KS WCF FOR FIELD THEORY $\blacksquare$

* CRASH COURSE ON STOKES PHENOM.

CONSIDER A FIRST ORDER MATRIX ODE IN THE COMPLEX t-PLANE:

$$\frac{d}{dt}\psi = A(t)\psi$$

A SOLUTION ψ GAUGE-TRANSFORMS THE FLAT CONNECTION $A(t)\,dt$ TO ZERO.

IF $A(t)$ IS REGULAR NEAR t_0 THEN SO IS THE SOLUTION ψ.

SUPPOSE $A(t)$ HAS A SINGULAR POINT, SAY, AT $t = 0$:

- A REGULAR SINGULAR POINT HAS

$$A(t) = \frac{A_{-1}}{t} + \cdots$$

THEN:

1. $\exists$ CONVERGENT SERIES SOLUTIONS IN A DISK AROUND $t=0$. IF A_{-1} IS DIAGONAL:

$$A_{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

THEN

$$\Psi = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix} \left(1 + \psi_1 t + \psi_2 t^2 + \dots \right)$$

2. $\Psi(t)$ WILL HAVE MONODROMY AROUND $t=0$.

- AN IRREGULAR SINGULAR POINT HAS A POLE OF ORDER >1.

CONSIDER THE SIMPLEST CASE:

$$A(t) = \frac{Z}{t^2} + \frac{A_{-1}}{t} + \cdots$$

ASSUME: $Z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$

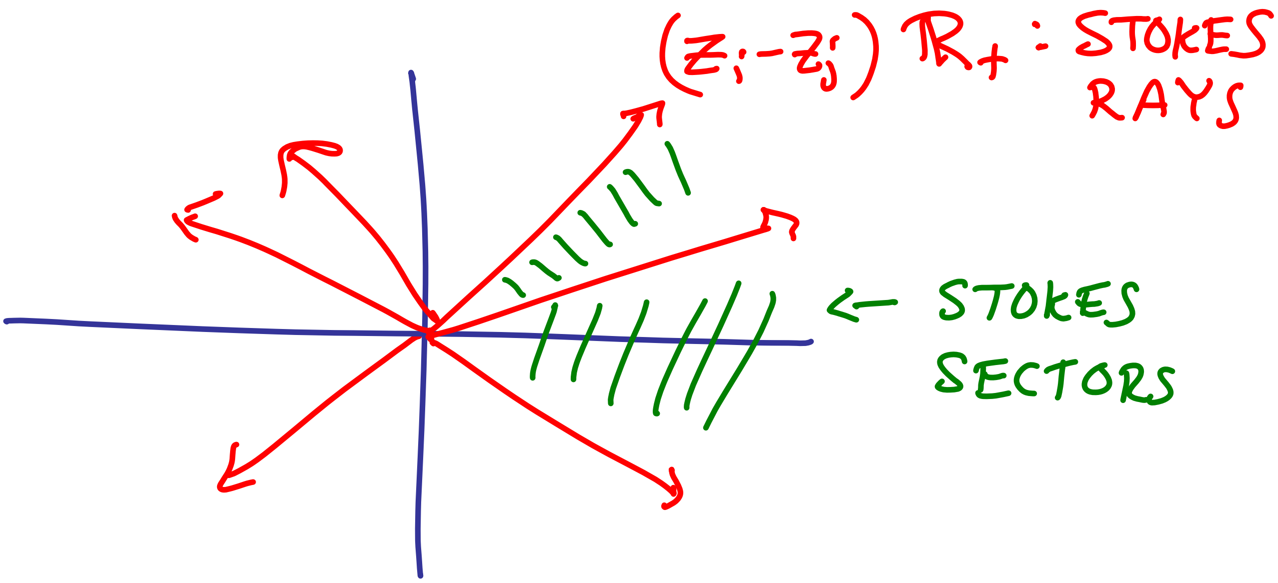
THEN THE SERIES METHOD LEADS TO A <u>FORMAL</u> <u>SOLUTION</u>:

$$\underline{\Psi}_f = e^{-Z/t}\left(1 + \psi_1 t + \psi_2 t^2 + \cdots\right)$$

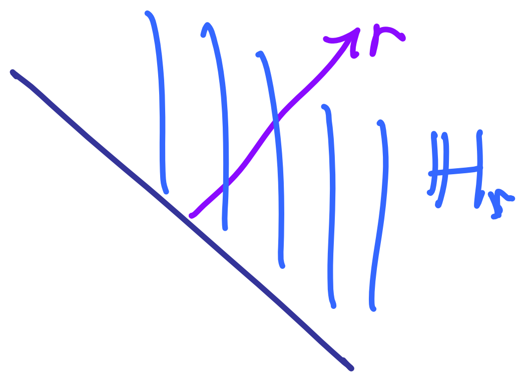
BUT NOW THE SERIES IS JUST <u>ASYMPTOTIC</u> FOR $t \to 0$; IT DOES <u>NOT</u> CONVERGE.

WHAT ABOUT TRUE SOLUTIONS?

DEF: THE



THM: LET r BE A RAY $\neq$ STOKES RAY AND $\mathbb{H}_r$ THE HALF-PLANE CONTAINING r:

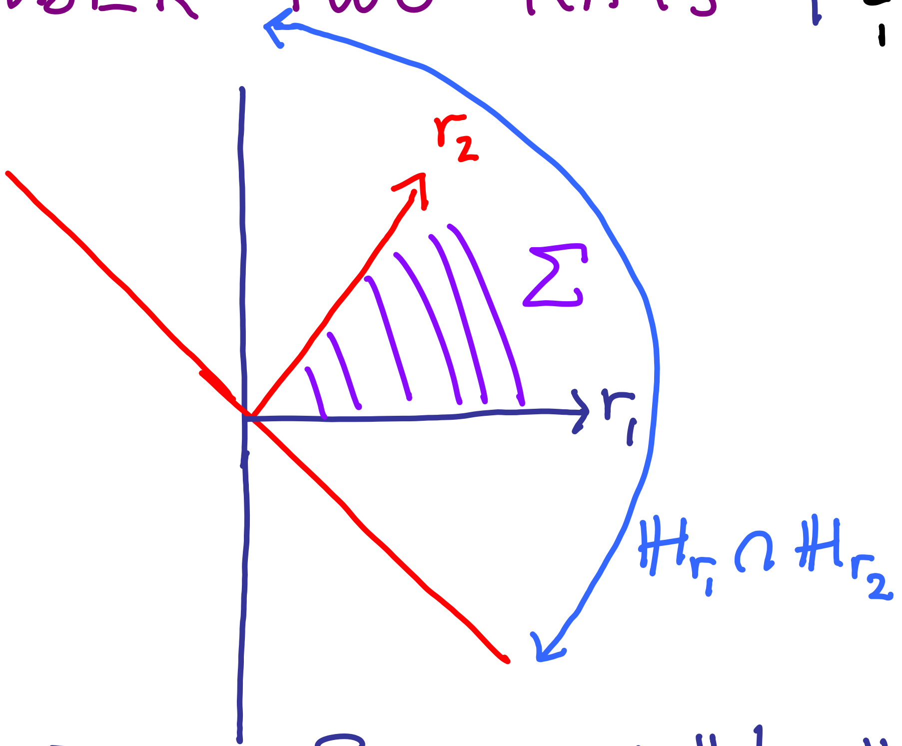


THEN, $\exists !$ TRUE SOLUTION Φ_r ASYMPTOTIC TO THE FORMAL SOLUTION FOR ALL $t \to 0$ IN $\mathbb{H}_r$

$$\Phi_r \, e^{Z/t} \longrightarrow 1 \qquad t \to 0$$

STOKES PHENOMENON:

CONSIDER TWO RAYS r_1 & r_2



$$\Phi_{r_1} = \Phi_{r_2} \cdot S_\Sigma \quad \text{ON } \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$$

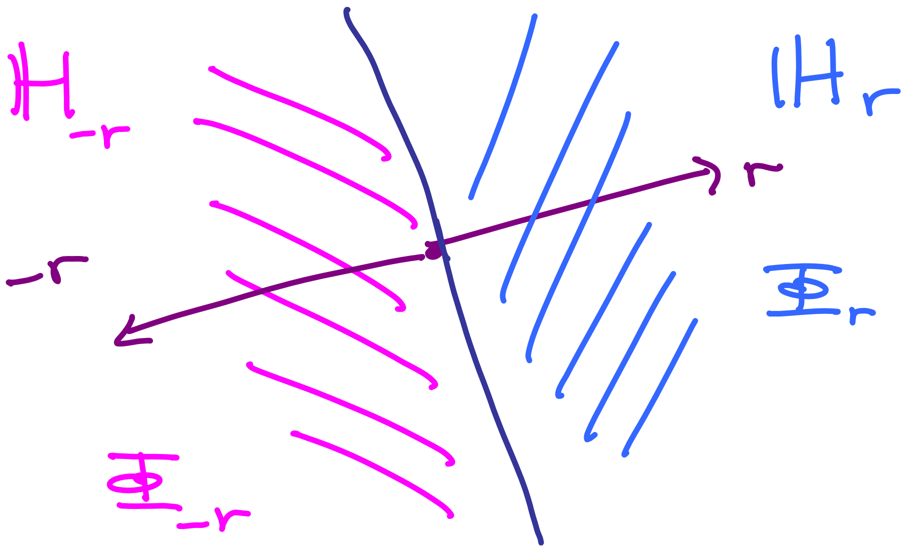
STOKES PHENOMENON:

$S_\Sigma = 1$ IF THERE IS NO STOKES RAY IN Σ, BUT $S_\Sigma \neq 1$ IF $\exists$ STOKES RAYS IN Σ.

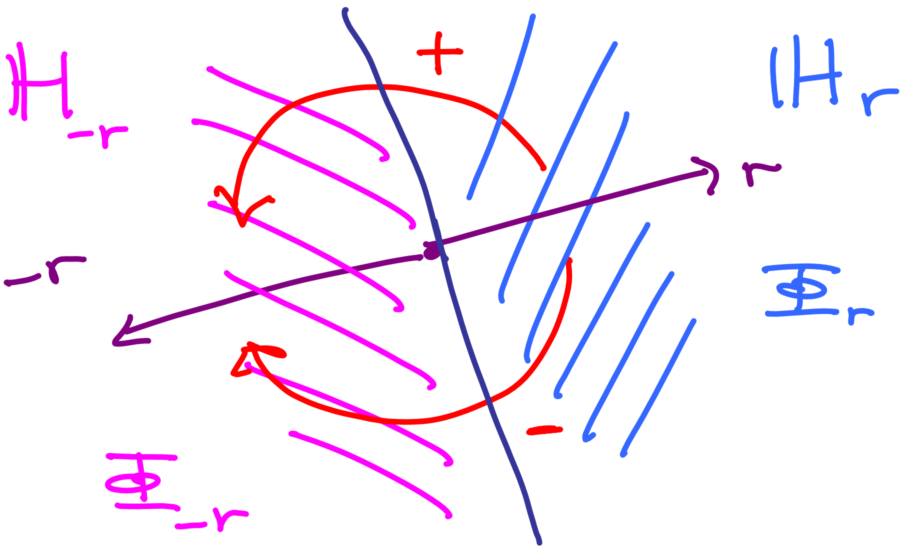
IF ONE RAY ℓ THEN $S_\Sigma = S_\ell$ IS THE <u>STOKES</u> <u>FACTOR</u> FOR ℓ.

ANALOG OF MONODROMY

CHOOSE $\pm r \neq$ STOKES RAY



THERE ARE TWO ANALYTIC CONT'S OF Φ_r TO $\mathbb{H}_{-r}$



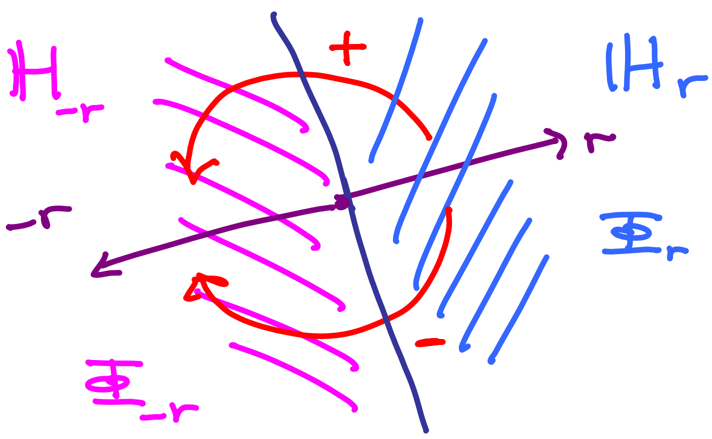
THM 2:

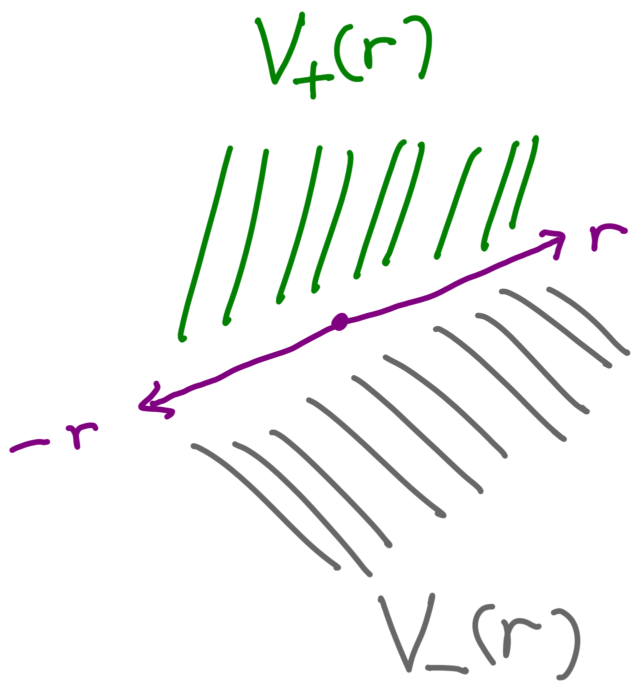
$$\Phi_r^+ = \Phi_{-r} \cdot S_+ \quad \text{ON} \quad \mathbb{H}_{-r}$$

$$\Phi_r^- = \Phi_{-r} \cdot S_- \quad \text{ON} \quad \mathbb{H}_{-r}$$

WITH

$$S_+ = \overrightarrow{\prod_{\ell \in V_+(r)}} S_\ell \qquad S_- = \overrightarrow{\prod_{\ell \in V_-(r)}} S_\ell^-$$





7. SUMMARY OF THE LECTURES

7A: TAKE-HOME SUMMARY OF I & II

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE HYPERKÄHLER METRIC FOLLOWS FROM THE KS WCF.

4. KS TMNS APPEAR AS DISCONTINUITIES OF HOLOMORPHIC FUNCTIONS $\mathcal{X}_\gamma(\zeta)$ THAT HAVE THE INTERPRETATION OF LINE OPER'S.

7B SUMMARY OF LECTURES 3 & 4

MOTIVATION: WE WANTED TO FIND ANOTHER CONSTRUCTION OF $\mathcal{X}_\gamma$ NOT MAKING USE OF THE LARGE R EXPANSION, AND PERHAPS MORE COMPUTABLE.

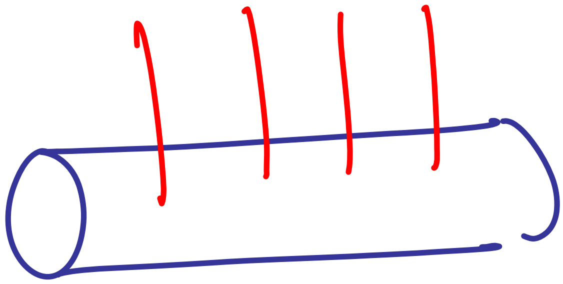
WE FOUND INDEED A NEW CONSTRUCTION FOR A CLASS OF $D=4, N=2$ THEORIES.

WE CONSIDER K M5 BRANES ON

$\mathbb{R}^{1,3} \times C$, C = RIEMANN SURFACE

WITH DEFECTS

C_α — SINGLE M5



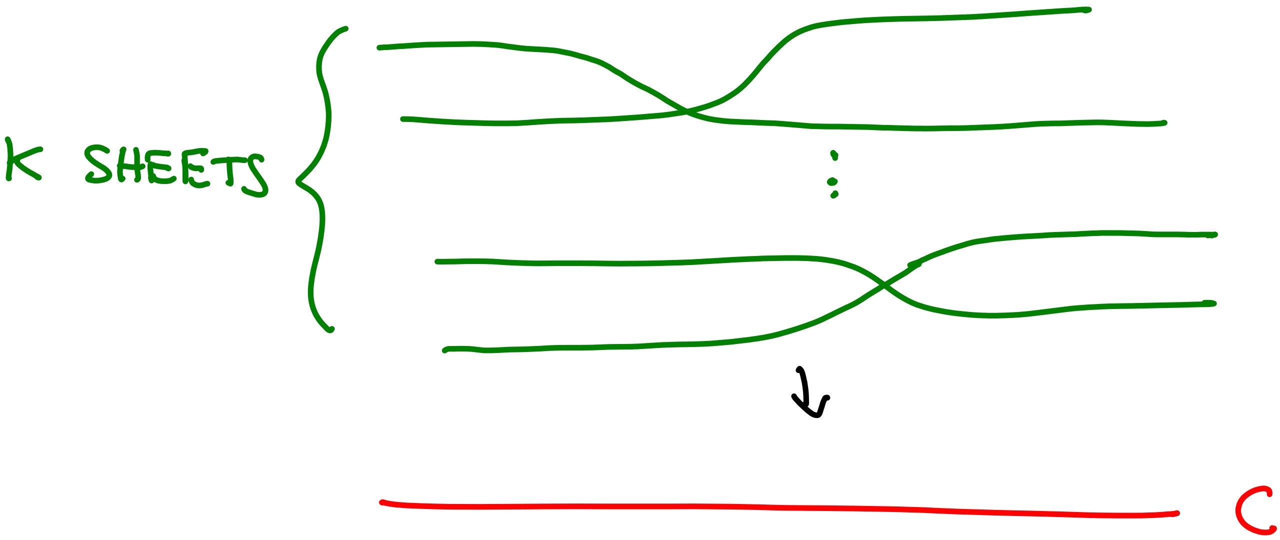
k M5 $\Rightarrow$ SU(k) (2,0) THEORY

LOW ENERGY LIMIT: $D=4$, $N=2$ FIELD THEORY.

SEIBERG WITTEN CURVE IS A RIEMANN SURFACE

$$\Sigma \subset T^*C$$

Σ IS A K-FOLD BRANCHED COVER



λ = CANONICAL 1-FORM ON T^*C

LET $\lambda_i = \lambda$ ON SHEET i, THEN:

BPS STATES FROM OPEN M2 BRANES $\Rightarrow$ STRING WEBS ON C FOLLOWING CURVES

$$\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{i\vartheta_*}$$

WHICH ARE CLOSED, OR END ON BRANCH POINTS.

FOR $K=2 \Rightarrow \langle \lambda, \partial_t \rangle = e^{i\vartheta_*}$

NOW RECALL WE COMPACTIFY ON S^1_R TO GET OUR HK σ-MODEL.

DO IT IN THE OTHER ORDER:

$$
\begin{tikzcd}[column sep=small]
 & \text{6D } (2,0)\ A_{K-1} \,/\, \mathbb{R}^{1,2}\times S^1_R \times C \ \text{+ DEFECTS} \arrow[dl, "\ell_C \ll \ell_{S^1},\, \ell_{\mathbb{R}^{1,2}}"'] \arrow[dr, "\ell_{S^1} \ll \ell_C,\, \ell_{\mathbb{R}^{1,2}}"] & \\
\text{4D } \mathcal{N}=2 \text{ GAUGE THEORY} \,/\, \mathbb{R}^{1,2}\times S^1_R \arrow[dr, "\ell_{S^1} \ll \ell_{\mathbb{R}^{1,2}}"'] & & \dfrac{\text{5D } U(K) \text{ SYM}}{\mathbb{R}^{1,2}\times C} \arrow[dl, "\ell_C \ll \ell_{\mathbb{R}^{1,2}}"] \\
 & \sigma\text{-MODEL: } \mathbb{R}^{1,2} \longrightarrow \mathcal{M} &
\end{tikzcd}
$$

THIS IMPLIES WE CAN ALSO TAKE $\mathcal{M}$ TO BE HITCHIN MODULI SPACE:

$$\begin{cases} F + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}\varphi + [\bar{A}, \varphi] = 0 \end{cases}$$

A = $SU(k)$ GAUGE FIELD ON C

φ = $k \times k$ $(1,0)$-FORM ON C

$$\varphi \sim \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_k \end{pmatrix}$$

$v_i \sim$ POSITIONS OF "IR M5-BRANE" IN $\Sigma \subset T^*C$

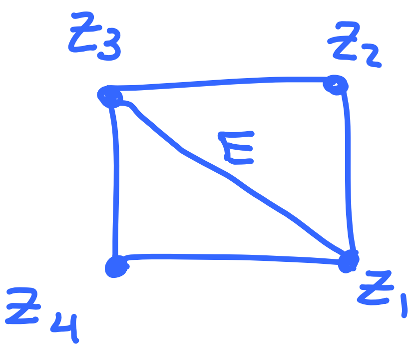
NOW IF (A, φ) SOLVE HITCHIN THEN

$$\mathcal{A} = \zeta^{-1} R \varphi + A + \zeta R \bar{\varphi}$$

IS FLAT : $\quad d\mathcal{A} + \mathcal{A}^2 = 0.$

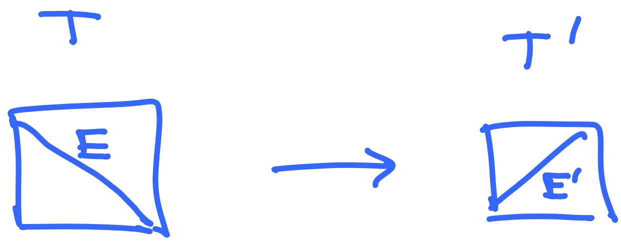
$\mathcal{M}$ = MODULI SPACE OF FLAT $SL(K, \mathbb{C})$ CONNECTIONS ON C WITH PRESCRIBED MONODROMIES.

FOCK & GONCHAROV CONSTRUCT A BEAUTIFUL SET OF COORDINATES ON $\mathcal{M}$ GIVEN A "DECORATED TRIANGULATION" ON C. FOR $K = 2$:



$$\Rightarrow \quad \mathcal{X}_E^T = - \frac{S_1 \wedge S_2 \; S_3 \wedge S_4}{S_4 \wedge S_1 \; S_2 \wedge S_3}$$

FLIP:



$$\chi^{T'}_{E'} = 1/\chi^{T}_{E}$$

$$\chi^{T'}_{E_{12}} = \chi^{T}_{E_{12}}\left(1+\chi^{T}_{E}\right)$$

$$\vdots$$

FOR GOOD $\zeta \to 0, \infty$ ASYMPTOTICS WE NEED "WKB TRIANGULATIONS" $T(\vartheta, u)$:

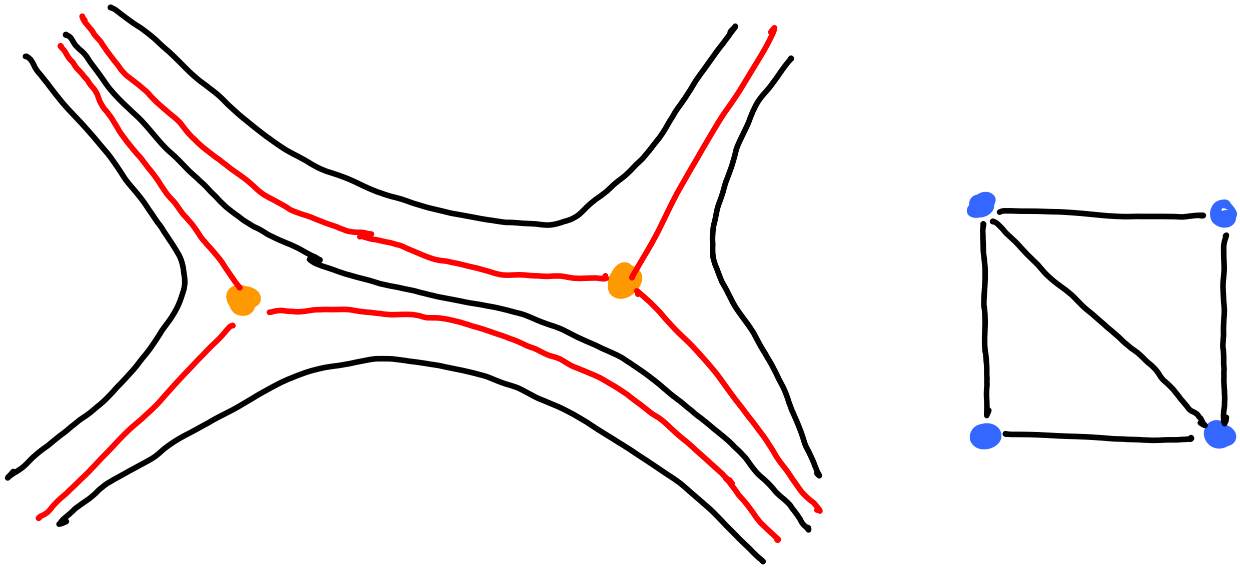
$$\langle \lambda, \partial_t \rangle = e^{i\vartheta}$$

THEN WE CAN COMPUTE $\zeta \to 0, \infty$ FOR $\chi_E^{T(\vartheta,u)}$ USING WKB.

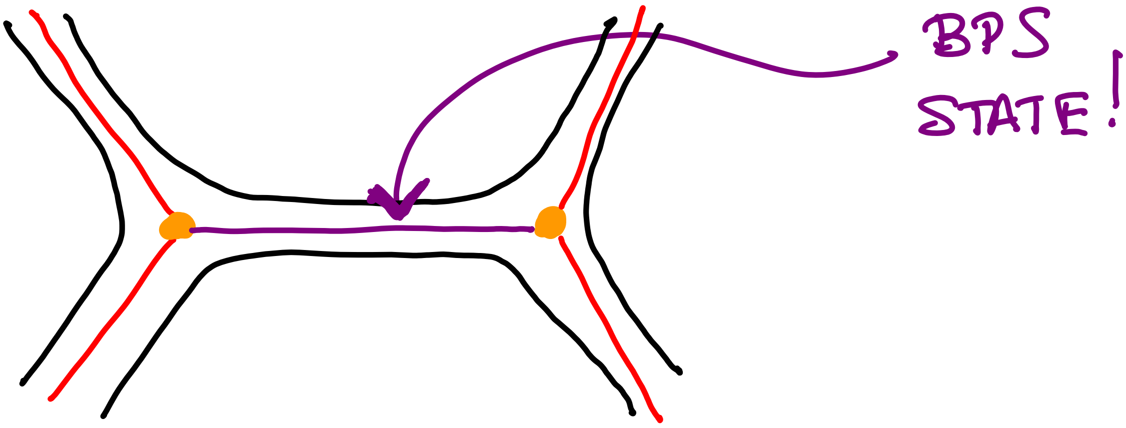
NOW ϑ- DEPENDENCE SHOWS JUMPS AT CRITICAL VALUES $e^{i\vartheta_c}$ = PHASES OF Z FOR BPS STATES!

HYPERMULTIPLET JUMP:

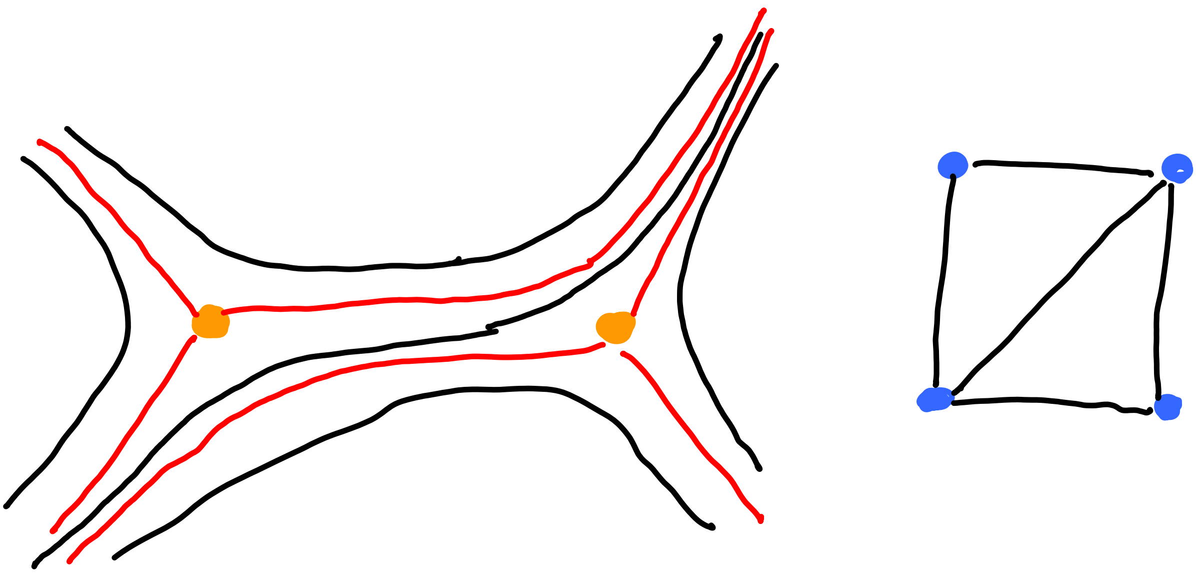
$\vartheta < \vartheta_c$



$\vartheta = \vartheta_c$



$\vartheta > \vartheta_c$



THIS IS JUST A FLIP

AND SOMETHING ANALOGOUS — BUT FANCIER — HAPPENS FOR VECTORMULTIPLETS

WE CONSTRUCT $\mathcal{X}_\gamma$ FROM $\mathcal{X}_E^{T(\vartheta,u)}$.

THE JUMPS IN $T(\vartheta,u)$ INDUCE KS TRANSFORMATIONS!

COMPARING ϑ & $\vartheta+\pi$ GIVES A <u>SPECTRUM-GENERATING STOKES MATRIX</u>

$$\mathbb{S}: \quad \mathcal{X}_\gamma \longrightarrow \tilde{\mathcal{X}}_\gamma$$

1.) EXPLICITLY COMPUTABLE

2.) BUT ALSO:

$$\mathbb{S} = \prod_{\vartheta < -\arg Z < \vartheta+\pi} K_\gamma^{\Omega(\gamma;u)}$$

$\Longrightarrow$ Ω IS DETERMINED!

8. NEW DIRECTIONS & OPEN PROBLEMS

- EXTENSION OF $\mathbf{S}$ TO HIGHER RANK HITCHIN SYSTEMS (PARTIALLY DONE.)

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAFA.

$\mathcal{B}$ = FAMILY OF MASSIVE $d=2$ $\mathcal{N}=(2,2)$ Q.F.T.

$V \longrightarrow \mathcal{B}$ BUNDLE OF (c,c) OPERATORS ($\Longleftrightarrow$ R GROUNDSTATES)

ANALOGY $V = \mathcal{C}^\infty(\mathcal{M}_u)$

CECOTTI & VAFA DEFINE tt^* CONNECTION

$$\nabla_i \Psi = \left(\frac{\partial}{\partial t^i} + \zeta^{-1} C_i + g^{-1} \partial_i g \right) \Psi = 0$$

$$\bar{\nabla}_i \Psi = \left(\frac{\partial}{\partial \bar{t}^i} + \zeta \bar{C}_i \right) \Psi = 0$$

FLATNESS $\Rightarrow$ tt^* EQUATIONS

ANALOGY: HOLOMORPHY OF $\mathcal{X}$ ON $\mathcal{M}^\zeta$

MOREOVER, ANOMALOUS SCALE + $U(1)_R$ SYMMETRY $\Longrightarrow$

$$\zeta \frac{\partial}{\partial \zeta} \Psi = \left(R\zeta C + Q - R\zeta^{-1} \bar{C} \right) \Psi$$

$$R \frac{\partial}{\partial R} \Psi = \left(R\zeta C + Q + R\zeta^{-1} \bar{C} \right) \Psi$$

BPS WALL-CROSSING EXPRESSED THROUGH STOKES FACTORS

STOKES FACTORS $S_{ij} = \mathbb{1} + e_{ij} \quad i \neq j$

BPS CHARGES LABELED BY PAIRS $i \neq j$

EXAMPLE $\mathrm{RK}(V) = 3$. CROSSING A WALL FOR BPS STATE (13):

$$S_{12}^{\Omega_{12}} S_{13}^{\Omega_{13}^{+}} S_{23}^{\Omega_{23}} = S_{23}^{\Omega_{23}} S_{13}^{\Omega_{13}^{-}} S_{12}^{\Omega_{12}}$$

$$\Longleftrightarrow \qquad \Omega_{13}^{+} = \Omega_{13}^{-} - \Omega_{12}\Omega_{23}$$

NOTE $\quad Q_{ij} = \mathrm{Tr}_{\mathcal{H}_{ij}}\left(F(-1)^F e^{-R \cdot H}\right)$

PROBLEM: WHAT IS THE 4D ANALOG OF THIS FORMULA? CAN WE WRITE $\mathcal{A}_j^{(0)}$ AS A TRACE?

- RELATION TO INTEGRABLE SYSTEMS.

THE INTEGRAL EQUATION FOR THE $\mathcal{X}_\gamma$ IS ACTUALLY A FORM OF THE THERMODYNAMIC BETHE ANSATZ!

$$Z_\gamma = e^{i\alpha_\gamma}|Z_\gamma| \qquad \zeta = -e^{i\alpha_\gamma + \eta}$$

$$\log \mathcal{X}_\gamma^{sf} = -2\pi R |Z_\gamma| \cosh\eta + i\varphi_\gamma$$

$$\beta\mu_\gamma = i\varphi_\gamma + \log(-\sigma(\gamma))$$

$$\mathcal{X}_\gamma = -\sigma(\gamma)\, e^{\beta\mu_\gamma - \epsilon_\gamma(\eta)}$$

$$S_{\gamma\gamma'}(\theta) = \left(\sinh \tfrac{1}{2}(\eta + i\alpha_\gamma - i\alpha_{\gamma'})\right)^{\langle\gamma,\gamma'\rangle}$$

"S-MATRIX"

OPEN PROBLEMS HERE:

1. RELATION TO TBA OF Y-SYSTEMS IN $D=4, \mathcal{N}=4$? OR INTEGRABLE SYSTEMS OF NEKRASOV+ SHATASHVILI?

2. CAN WE EXPLOIT TECHNIQUES IN INTEGRABLE SYSTEMS TO FIND $\mathcal{X}_\gamma$ FOR SMALL R?

- ANALOG FOR SUPERGRAVITY:

$\mathcal{M} \longrightarrow$ Q.K. MANIFOLD FIBERED BY HEISENBERG GROUPS

$\mathcal{T} \longrightarrow$ CANONICAL H.G. EXTENSION OF $\Gamma^* \otimes \mathbb{C}^*$ GIVEN BY Q.R.

HITCHIN THM. $\longrightarrow$ LE BRUN THM.

$\varpi_\zeta \longrightarrow$ CONTACT STRUCTURE

BUT.... CONVERGENCE ???

<u>BUT!</u>

- $\Omega(\lambda\gamma) \overset{\lambda\to\infty}{\sim} e^{\text{const.}\,\lambda^2}$

- $\int_{\ell_{\lambda\gamma}} [d\zeta'] \log\left(1 - \mathcal{X}^{sf}_{\lambda\gamma}\right) \sim e^{-\text{const.}\,\lambda}$

- $\Longrightarrow$ SUM ON γ' DOES NOT CONVERGE!!

RELATION TO MATHEMATICAL WORK

- RELATION TO WORK OF BRIDGELAND + TOLEDANO LAREDO. (THEY HAVE ALL THE ELEMENTS OF W.C., STOKES PHENOMENA, ISOMONODROMIC DIFFL EQS. ...)

- RELATION TO GENERATING FUNCTIONS OF D. JOYCE (SIMILAR PRINCIPLE OF WCF FROM CONTINUITY)

- RELATION TO q-DEFORMED "MOTIVIC" FORMULAE OF K & S

  (SEE DIMOFTE+GUKOV FOR A RECENT PROPOSAL)