Preprint typeset in JHEP style - HYPER VERSION

Quantum Symmetries and Compatible Hamiltonians

Gregory W. Moore

Abstract: Adapted from Notes for Physics 695, Rutgers University. Fall 2013. Revised version used for ESI Lectures, at the ESI, Vienna, August 2014. Version: August 14, 2014

-TOC-

Contents

---

1. Introduction

These are lecture notes for a course I gave at Rutgers University during the Fall of 2013. The main goal of the notes is to give mathematical background necessary for an understanding of a specific point of view on the recent developments in the theory of topological insulators and superconductors. This viewpoint, which builds on the work of C. Kane et. al., A. Kitaev, A. Ludwig et. al., and A. Altland and M. Zirnbauer, was developed in [22]. The main theme is how symmetries are implemented in quantum mechanics and how the presence of symmetries constrains the possible Hamiltonians that a quantum system with a specified symmetry can have. I have tried to explain how the results follow simply from the basic principles of quantum mechanics.

I have aimed the notes at graduate students in both physics and mathematics, with the idea that a solid grounding in some of the topics chosen will serve them well in their future research careers, even if their interests are far removed from topological states of matter. If one's purpose is simply to understand the recent developments in topological insulators then, for example, the extensive discussions of Chapters 3-8 and Chapters 12, 13, 17 and 18 are clearly overkill. But the mathematics developed here is very useful in a wide variety of areas in Physical Mathematics. In some places I have used the approach of first-rate mathematicians writing about physics. For example, the treatment of Clifford algebras in Chapters 13 and 17 is slightly nonstandard for physicists since it emphasizes the role of $\mathbb{Z}_2$-graded or super-linear algebra. Some sections rely heavily on the masterful treatment by P. Deligne [16]. As I learned in my (unpublished) work with J. Distler and D. Freed on the K-theory approach to orientifolds of string theory, this is an excellent way to approach the subject of twisted equivariant K-theory. (And, in turn, as explained in [22], the classification of topological insulators properly relies on twisted equivariant K-theory.) In some parts of the chapter on fermions and the spin representation I have borrowed liberally from the beautiful book of Pressley and Segal [35].

One thing I have stressed which, in my opinion, is not very well appreciated in the literature, is that there are many conceptually distinct "10-fold ways." There is a straightforward generalization of Dyson's 3-fold way which applies to all quantum systems, interacting or not, bosonic, fermionic - whatever. This is rather nicely based on the fact that there are 10 superdivision algebras over the real numbers, in close analogy to Dyson's ensembles. (As explained in Chapters 8-10, Dyson's classification follows from simple group theory and the Frobenius theorem, which identifies the 3 (associative) division algebras over the real numbers as $\mathbb{R}, \mathbb{C}, \mathbb{H}$.) This 10-fold way is described in Chapters 14- 16. I do not believe it has been properly explained in the literature before, although [20] and [22] came close.

Another "10-fold way" is associated with the work of Altland and Zirnbauer and is discussed in Chapter 19. The AZ classification involves classical Cartan symmetric spaces. The large N limit of these spaces are classifying spaces of K-theory and this is briefly discussed in Chapter 20.

I do make some effort to connect the various "10-fold ways." For example, Dyson's original paper [18] entitled "The Threefold Way:..." in fact contains a 10-fold classification of what he called "corepresentations." [1] Dyson's 10-fold classification of irreducible ϕ-reps can be related again to the 10 real superdivision algebras, although that precise relation relies on a conjecture, not fully proven in [22]. It seems to me to be conceptually distinct from the 10-fold way of Chapter 15, although it is clear that both trace their existence back to the 10 real superdivision algebras. Similarly, the AZ classification described in Chapter 19 asks a physically different question from that answered by Dyson's classification or the above-mentioned 10-fold classifications. Since the symmetric spaces can be related to the classifying spaces of K-theory (Chapter 20) and the latter are related to Clifford algebras there is once again a connection to the 10 real super-division algebras. From the viewpoint of K-theory, $10 = 2+8$. From the viewpoint of Chapter 15 on the other hand, $10 = 3+7$. In the problems discussed in these notes these decompositions are unnatural. The underlying unifying concept is that of a real super-division algebra.

In one of those delicious ironies, with which the history of mathematics and physics is so pregnant, the relation of the Clifford algebras to K-theory was developed by Atiyah, Bott, and Shapiro [7] and Wall [40] almost simultaneously with Dyson's work.

The original plan for the lecture series was to expand a little on two lectures given at a school in St. Ottilien (July 2012) [32] and on a lecture given at a conference on topological insulators at the SCGP in May 2013 [33]. Time flies, all too often I stopped to smell the flowers, and so the final Chapters 22-25 have not yet been written (although they correspond to definite slides in the talk [33]). As the course was ending I was just beginning to write Chapter 21 which extends the AZ classification to free bosonic systems. (This possibility was also noted in [44].) This chapter is even more incomplete than the previous ones. I do think it is very likely these ideas could be very profitably applied to

[1] Since the term "corepresentations" has many misleading connotations I have deprecated this usage in favor of "ϕ-twisted representations" or "ϕ-representations". See Chapter 8. I'm still looking for a better name.

systems of ultracold atoms and Bose-Einstein condensates which are the subject of many exciting current experimental discoveries. But I leave that for the future.

I hope to finish these notes at some point in the future. In the meantime, I hope they will be useful to students, even in this manifestly unfinished state. So they will remain available on my homepage.

2. Quantum Automorphisms

sec:QuantAut

2.1 States, operators and probabilities

We begin with first principles. The Dirac-von Neumann axioms of quantum mechanics posit that to a physical system we associate a complex Hilbert space $\mathcal{H}$ such that

1. Physical states are identified with traceclass positive operators ρ of trace one. They are usually called *density matrices*. We denote the space of physical states by $\mathcal{S}$.

2. Physical observables are identified with self-adjoint operators. We denote the set of (bounded) self-adjoint operators by $\mathcal{O}$.

Recall that *pure states* are the extremal points of $\mathcal{S}$. They are the dimension one projection operators. They are often identified with rays in Hilbert space for the following reason:

If $\psi \in \mathcal{H}$ is a nonzero vector then it determines a line

$$\ell_\psi := \{z\psi | z \in \mathbb{C}\} := \psi\mathbb{C} \tag{2.1}$$

Note that the line does not depend on the normalization or phase of ψ, that is, $\ell_\psi = \ell_{z\psi}$ for any nonzero complex number z. Put differently, the space of such lines is projective Hilbert space

$$\mathbb{P}\mathcal{H} := (\mathcal{H} - \{0\})/\mathbb{C}^* \tag{2.2}$$

Equivalently, this can be identified with the space of rank one projection operators. Indeed, given any line $\ell \subset \mathcal{H}$ we can write, in Dirac's bra-ket notation: [2]

$$P_\ell = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \tag{2.3}$$

where ψ is any nonzero vector in the line ℓ.

The "Born rule" states that when measuring the observable O in a state ρ the probability of measuring value $e \in E \subset \mathbb{R}$, where E is a Borel-measurable subset of $\mathbb{R}$, is

$$P_{\rho,O}(E) = \mathrm{Tr} P_O(E)\rho. \tag{2.4}$$

Here P_O is the projection-valued-measure associated to the self-adjoint operator O by the spectral theorem.

[2] We generally denote inner products in Hilbert space by $(x_1, x_2) \in \mathbb{C}$ where $x_1, x_2 \in \mathcal{H}$. Our convention is that it is complex-linear in the second argument. However, we sometimes write equations in Dirac's bra-ket notation because it is very popular. In this case, identify x with $|x\rangle$. Using the Hermitian structure there is a unique anti-linear isomorphism of $\mathcal{H}$ with $\mathcal{H}^*$ which we denote $x \mapsto \langle x|$. Sometimes we denote vectors by Greek letters $\psi, \chi, \dots$, and scalars by Latin letters $z, w, \dots$. But sometimes we denote vectors by Latin letters, $x, w, \dots$ and scalars by Greek letters, $\alpha, \beta, \dots$.

2.2 Automorphisms of a quantum system

Now we state the formal notion of a general "symmetry" in quantum mechanics:

Definition An *automorphism* of a quantum system is a pair of bijective maps $s_1 : \mathcal{O} \to \mathcal{O}$ and $s_2 : \mathcal{S} \to \mathcal{S}$ where s_1 is real linear on $\mathcal{O}$ such that (s_1, s_2) preserves probability measures:

$$P_{s_1(O),s_2(\rho)} = P_{O,\rho} \tag{2.5}$$

This set of mappings forms a group which we will call the group of *quantum automorphisms*.

♣Need to state some appropriate continuity properties. ♣

The meaning of s_1 being linear on $\mathcal{O}$ is that if $T_1, T_2 \in \mathcal{O}$ and $D(T_1) \cap D(T_2)$ is a dense domain such that $\alpha_1 T_1 + \alpha_2 T_2$, with α_1, α_2 real has a unique self-adjoint extension then $s_1(\alpha_1 T_1 + \alpha_2 T_2) = \alpha_1 s_1(T_1) + \alpha_2 s_1(T_2)$. A consequence of the symmetry axiom is that s_2 is affine linear on states:

$$s_2(t\rho_1 + (1-t)\rho_2) = ts_2(\rho_1) + (1-t)s_2(\rho_2) \tag{2.6}$$

eq:Aff-Lin

The argument for this is that (s_1, s_2) must preserve expectation values $\langle T \rangle_\rho = \mathrm{Tr}(T\rho)$. However, positive self-adjoint operators of trace one are themselves observables and we have $\langle \rho_1 \rangle_{\rho_2} = \langle \rho_2 \rangle_{\rho_1}$, so the restriction of s_1 to $\mathcal{S}$ must agree with s_2. Now apply linearity of s_1 on the self-adjoint operators. From (2.6) it follows [3] that s must take extreme states to extreme states, and hence s_2 induces a single map

$$s : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}. \tag{2.7}$$

eq:QuantAut

Moreover, the preservation of probabilities, restricted to the case of self-adjoint operators given by rank one projectors and pure states (also given by rank one projectors) means that the function

$$\mathfrak{o} : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0,1] \tag{2.8}$$

eq:overlap-1

defined by

$$\mathfrak{o}(\ell_1, \ell_2) := \mathrm{Tr} P_{\ell_1} P_{\ell_2} \tag{2.9}$$

eq:overlap-2

must be invariant under s:

$$\mathfrak{o}(s(\ell_1), s(\ell_2)) = \mathfrak{o}(\ell_1, \ell_2) \tag{2.10}$$

eq:overlap-3

Definition The function defined by (2.8) and (2.9) is known as the *overlap function*.

Remarks

1. The upshot of our arguments above is that the quantum automorphism group of a system with Hilbert space $\mathcal{H}$ can be identified with the group of (suitably continuous) maps (2.7) such that (2.10) holds for all lines ℓ_1, ℓ_2. We denote the group of such maps by $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

[3] For some interesting discussion of related considerations see [37].

2. The reason for the name "overlap function" or "transition probability" which is also used, is that if we choose representative vectors $\psi_1 \in \ell_1$ and $\psi_2 \in \ell_2$ we obtain the - perhaps more familiar - expression:

$$\mathrm{Tr} P_{\ell_1} P_{\ell_2} = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle} \tag{2.11}$$

2.3 Overlap function and the Fubini-Study distance

ubiniStudyDist

If $\mathcal{H}$ is finite dimensional then we can identify it as $\mathcal{H} \cong \mathbb{C}^N$ with the standard hermitian metric. Then $\mathbb{P}\mathcal{H} = \mathbb{C}P^{N-1}$ and there is a well-known metric on $\mathbb{CP}^{N-1}$ known as the "Fubini-Study metric" from which one can define a minimal geodesic distance $d(\ell_1, \ell_2)$ between two lines (or projection operators). When the FS metric is suitably normalized the overlap function $\mathfrak{o}$ is nicely related to the Fubini-Study distance d by

$$\mathfrak{o}(\ell_1, \ell_2) = \left(\cos \frac{d(\ell_1, \ell_2)}{2}\right)^2 \tag{2.12}$$ eq:OL-FS

Let us first check this for the case $N = 2$. Then we claim that for the case

$$\mathbb{P}\mathcal{H}^2 = \mathbb{C}P^1 \cong S^2 \tag{2.13}$$

d is just the usual round metric on the sphere and the proper normalization will be unit radius. Let us first check this:

First we write the most general general density matrix in two dimensions. Any 2×2 Hermitian matrix is of the form $a + \vec{b} \cdot \vec{\sigma}$ where $\vec{\sigma}$ is the vector of "Pauli matrices":

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{2.14}$$

$a \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^3$. Now a density matrix ρ must have trace one, and therefore $a = \frac{1}{2}$. Then the eigenvalues are $\frac{1}{2} \pm |\vec{b}|$ so positivity means it must have the form

$$\rho = \frac{1}{2}(1 + \vec{x} \cdot \vec{\sigma}) \tag{2.15}$$

where $\vec{x} \in \mathbb{R}^3$ with $\vec{x}^2 \leq 1$.

The extremal states, corresponding to the rank one projection operators are therefore of the form

$$P_{\vec{n}} = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \tag{2.16}$$

where $\vec{n}$ is a unit vector. This gives the explicit identification of the pure states with elements of S^2. Moreover, we can easily compute:

$$\mathrm{Tr} P_{\vec{n}_1} P_{\vec{n}_2} = \frac{1}{2}(1 + \vec{n}_1 \cdot \vec{n}_2) \tag{2.17}$$

and $\vec{n}_1 \cdot \vec{n}_2 = \cos(\theta_1 - \theta_2)$ where $|\theta_1 - \theta_2|$ (with θ's chosen so this is between 0 and π) is the geodesic distance between the two points on the unit sphere. Thus we obtain (2.12). eq:OL-FS

There is another viewpoint which is useful. Nonzero vectors in $\mathbb{C}^2$ can be normalized to be in the unit sphere S^3. Then the association of projector to state given by

$$|\psi\rangle \to |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{n}\cdot\vec{\sigma}) \tag{2.18}$$

defines a map $\pi : S^3 \to S^2$ known as the *Hopf fibration.*

The unit sphere is a principal homogeneous space for $SU(2)$ and we may coordinatize $SU(2)$ by the Euler angles:

$$u = e^{-i\frac{\phi}{2}\sigma^3} e^{-i\frac{\theta}{2}\sigma^2} e^{-i\frac{\psi}{2}\sigma^3} \tag{2.19}$$

with range $0 \le \theta \le \pi$ and identifications:

$$(\phi, \psi) \sim (\phi + 4\pi, \psi) \sim (\phi, \psi + 4\pi) \sim (\phi + 2\pi, \psi + 2\pi) \tag{2.20}$$

We can make an identification with the unit sphere in $\mathbb{C}^2$ by viewing it as a homogeneous space:

$$\psi = \begin{pmatrix} e^{-i\frac{\psi+\phi}{2}}\cos\theta/2 \\ e^{-i\frac{\psi-\phi}{2}}\sin\theta/2 \end{pmatrix} = u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.21}$$

The projector onto the line through this space is

$$P_{\ell_\psi} = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{n}\cdot\vec{\sigma}) \tag{2.22}$$

with $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ as usual. Alternatively, we could map $\pi : S^3 \to S^2$ by $\pi(\psi) = [\psi_1 : \psi_2] \cong \mathbb{C}P^1$, and this will correspond to the point in S^2 by the usual stereographic projection. ♣from which pole? ♣

In any case, for the case $N = 2$ we see that $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ is just the group of isometries of S^2 with its round metric. This group is well known to be the orthogonal group $O(3)$.

Moving on to higher N we can define the FS metric in a number of ways:

1. Identify $\mathbb{C}P^N$ as a homogeneous space

$$\mathbb{C}P^N \cong U(N+1)/U(N) \times U(1) \cong SU(N+1)/SU(N) \times U(1) \tag{2.23}$$

This follows from the stabilizer-orbit theorem: There is a transitive action of $U(N+1)$ on the set of lines in $\mathbb{C}^{N+1}$ and the stabilizer of a line ℓ is the product of the unitary group of ℓ (which is $U(1)$) and the unitary group of $\ell^\perp$ (which is $U(N)$). If we give an orthogonal decomposition of the Lie algebras using a Cartan-Killing metric on $SU(N+1)$: [4]

$$su(N+1) \cong su(N) \oplus u(1) \oplus \mathfrak{p} \tag{2.24}$$

then we can identify $\mathfrak{p}$ with the tangent space at the origin. The restriction of the Cartan-Killing form to $\mathfrak{p}$, then made left-invariant by group translation defines the FS metric.

[4] Since $SU(N+1)$ is simple the CK metric is unique up to scale.

2. We can identify the holomorphic tangent space to $\ell \in \mathbb{PC}^{N+1}$ as

$$T_\ell \mathbb{PC}^{N+1} \cong \mathrm{Hom}(\ell, \ell^\perp) \tag{2.25}$$

Put this way, a tangent vector is a linear map $t : \ell \to \ell^\perp$, and we can define an Hermitian metric by the formula

$$h(t_1, t_2) := \mathrm{Tr}(t_1^\dagger t_2) \tag{2.26}$$

This viewpoint has the advantage that it works in infinite dimensions if t_1, t_2 are traceclass operators. ♣Check the proper class of operators. ♣

3. Indeed, the Hermitian metric just defined is a Kähler metric and one choice of Kähler potential is $K = \log \sum_i X_i \bar{X}_i$ where X_i are homogeneous coordinates.

It is known that the FS metric on $\mathbb{C}P^N$ has the property that the submanifolds $\mathbb{C}P^k \to \mathbb{C}P^N$ embedded by $[z_1 : \cdots : z_{k+1}] \to [z_1 : \cdots : z_{N+1}]$ are *totally geodesic submanifolds.*

Definition If (M, g) is a Riemannian manifold a submanifold $M_1 \subset M$ is said to be *totally geodesic* if the geodesics between any two points in M_1 with respect to the induced metric (the pullback of g) are the same as the geodesics between those two points considered as points of M.

Example: If (M, g) is the two-dimensional Euclidean plane then the totally geodesic one-dimensional manifolds are straight lines. Any one-dimensional submanifold which bends affords a short-cut in the ambient space.

If M_1 is the fixed point set of an isometry of (M, g) then it is totally geodesic. Now note that the submanifolds $\mathbb{C}P^k$ are fixed points of the isometry ♣simple proof or ref? ♣

$$[z_1 : \cdots : z_{N+1}] \to [z_1 : \cdots : z_{k+1} : -z_{k+2} : \cdots : -z_{N+1}] \tag{2.27}$$

Another way to see this from the viewpoint of homogeneous spaces is that if we exponentiate a Lie algebra element in $\mathfrak{p}$ to give a geodesic in $U(N+1)$ and project to the homogeneous space we get all geodesics on the homogeneous space. But for any $t \in \mathfrak{p}$ we can put it into a $U(2)$ subalgebra.

Now, any two lines ℓ_1, ℓ_2 span a two-dimensional sub-Hilbert space of $\mathcal{H}$, so, thanks to the totally geodesic property of the FS metric, our discussion for $\mathcal{H} \cong \mathbb{C}^2$ suffices to check eq:0L-FS (2.12) in general.

2.4 From (anti-) linear maps to quantum automorphisms

Now, there is one fairly obvious way to make elements of $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. Suppose $u \in U(\mathcal{H})$ is a unitary operator. Then it certainly takes lines to lines and hence can be used to define a map (which we also denote by u) $u : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$. For example if we identify ℓ as ℓ_ψ for some nonzero vector ψ then we can define

$$u(\ell_\psi) := \ell_{u(\psi)} \tag{2.28}$$

One checks that which vector ψ we use does not matter and hence the map is well-defined. In terms of projection operators:

$$u : P \mapsto uPu^\dagger \tag{2.29}$$

and, since u is unitary, the overlaps $\mathrm{Tr}(P_1 P_2)$ are preserved.

Now - very importantly - this is not the *only* way to make elements of $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

We call a map $a : \mathcal{H} \to \mathcal{H}$ *anti-linear* if

$$a(\psi_1 + \psi_2) = a(\psi_1) + a(\psi_2) \tag{2.30}$$

but

$$a(z\psi) = z^* a(\psi) \tag{2.31}$$

where z is a complex scalar. It is in addition called *anti-unitary* if it is norm-preserving:

$$\| a(\psi) \|^2 = \| \psi \|^2 \tag{2.32}$$

Exercise
Show that

$$(a(\psi_1), a(\psi_2)) = (\psi_2, \psi_1) \tag{2.33}$$

Now, anti-unitary maps also can be used to define quantum automorphisms. If we try to define $a(\ell)$, $\ell \in \mathbb{C}\mathcal{H}$ by

$$a(\ell_\psi) = \ell_{a(\psi)} \tag{2.34}$$

then the map is indeed well-defined because if $\ell_{\psi'} = \ell_\psi$ then $\psi' = z\psi$ for some $z \neq 0$ and then

$$a(\ell_{\psi'}) = \ell_{a(\psi')} = \ell_{a(z\psi)} = \ell_{z^* a(\psi)} = \ell_{a(\psi)} \tag{2.35}$$

Moreover,

$$\frac{|(a(\psi_1), a(\psi_2))|^2}{(a(\psi_1), a(\psi_1))(a(\psi_2), a(\psi_2))} = \frac{|(\psi_1, \psi_2)|^2}{(\psi_1, \psi_1)(\psi_2, \psi_2)} \tag{2.36}$$

and hence the induced map on $\mathbb{P}\mathcal{H}$ does indeed preserve overlaps.

Remark: One may ask why we don't simply say that a induces a map on projection operators $P \mapsto aPa^\dagger$. Indeed we can, if we define the adjoint by $(\psi_1, a\psi_2) = (\psi_2, a^\dagger \psi_1)$.

2.5 Wigner's theorem

In the previous subsection we showed how unitary and antiunitary operators on Hilbert space induce quantum automorphisms. Are there other ways of making quantum automorphisms? Wigner's theorem says no:

Theorem: Every quantum automorphism $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ is induced by a unitary or antiunitary operator on Hilbert space, as above.

I don't know of a simple intuitive proof of Wigner's theorem. In addition to Wigner's own argument the paper [39] cites 26 references with alternative proofs! (And there are others, for examples [41][21].)

We will indicate two proofs.

Let us first consider the case of a two-dimensional Hilbert space. In this case we identified $\mathbb{P}\mathcal{H} \cong S^2$ and the isometry group is just $O(3)$. Now,

$$O(3) = \mathbb{Z}_2 \times SO(3) \tag{2.37}$$

Let us first consider the connected component of the identity.

There is a standard homomorphism

$$\pi : SU(2) \to SO(3) \tag{2.38}$$

defined by $\pi(u) = R$ where

$$u\vec{x} \cdot \vec{\sigma} u^{-1} = (R\vec{x}) \cdot \vec{\sigma} \tag{2.39}$$ eq:SU2-to-SO3

Therefore, under the Hopf fibration

$$|\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \tag{2.40}$$

we see - using the Euler angle parametrization - that any proper rotation on $\vec{n}$ is induced by some $SU(2)$ action on $|\psi\rangle$. Elements in the connected component of $O(3)$ not containing the identity can be written as PR where $R \in SO(3)$ and P is any reflection in a plane. It will be convenient to choose P to be reflection in the plane $y = 0$ so that it transforms $(\phi, \theta) \to (-\phi, \theta)$. But this just corresponds to complex conjugation of $\psi(\vec{n})$, which establishes the theorem for two-dimensional Hilbert space. [5]

Having established Wigner's theorem for $N = 2$ one can now proceed by induction on dimension. See [39] for details.

A second proof, due to V. Bargmann [11], (and which also works for separable infinite dimensional $\mathcal{H}$) proceeds as follows

Let S_ρ denote the sphere of radius ρ inside Hilbert space:

$$S_\rho = \{\psi \in \mathcal{H} | \parallel \psi \parallel^2 = \rho^2\} \tag{2.41}$$

Now $S_\rho/U(1) \cong \mathbb{P}\mathcal{H}$ for $\rho \neq 0$, as we henceforth assume. We will denote equivalence classes in $S_\rho/U(1)$, by $[\psi]$ where $\parallel \psi \parallel^2 = \rho^2$. These equivalence classes are often called "rays" in physics, although in fact such an equivalence class is a circle of vectors in the Hilbert space.

Given a quantum automorphism $s : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ we can unambiguously define a corresponding map

$$s : S_\rho/U(1) \to S_\rho/U(1) \tag{2.42}$$ eq:sOnSrho

We will also denote it by s to avoid cluttering the notation. The meaning should be clear from context. To define s in (2.42) consider $[\psi] \in S_\rho/U(1)$. Then ℓ_ψ, the line through ψ, is well-defined, so we can consider $\ell' = s(\ell_\psi)$. Choose any nonzero vector $\psi' \in \ell'$. We can always choose ψ' to be of norm ρ. For any such choice define $s[\psi] := [\psi']$. This map does

[5]We stress that there is no basis-independent notion of "complex conjugation." But in the above description of the unit sphere as a homogeneous space for $SU(2)$ we made an explicit choice of basis, so then complex conjugation is well-defined.

not depend on the choice of ψ' and is therefore well-defined. Note that $\parallel \psi \parallel^2 = \parallel \psi' \parallel^2$. If we define the overlap function $\mathfrak{o} : S_{\rho_1}/U(1) \times S_{\rho_2}/U(1) \to \mathbb{R}_+$ by

$$\mathfrak{o}([\psi_1],[\psi_2]) := |(\psi_1,\psi_2)|^2 \tag{2.43}$$

then $\mathfrak{o}$ is well-defined and preserved by s.

Now note a key

Lemma: If ℓ_n, $n = 1,2,\dots$ is a set of orthogonal lines, so, $\mathfrak{o}(\ell_n,\ell_m) = \delta_{n,m}$, then $s(\ell_n) = \ell_n'$ is another set of orthogonal lines. Therefore if we choose nonzero vectors $f_n \in \ell_n$ then we claim that for *any* set of vectors $f_n' \in \ell_n'$ such that

$$s([f_n]) = [f_n'] \tag{2.44}$$

we have $|(f_n',f_m')| = |(f_n,f_m)| = \delta_{n,m} \parallel f_n \parallel^2$ and moreover if

$$v = \sum \alpha_n f_n \tag{2.45}$$

then for any v' such that $s([v]) = [v']$ we have

$$v' = \sum \alpha_n' f_n' \tag{2.46}$$

with $|\alpha_n| = |\alpha_n'|$.

Proof of Lemma: Note that

$$|\alpha_n|^2 = \mathfrak{o}([f_n],[v]) = \mathfrak{o}(s[f_n],s[v]) = |\alpha_n'|^2. \tag{2.47}$$

Now, choose any unit vector $e \in \mathcal{H}$. Then *choose* another unit vector $e' \in \mathcal{H}$ so that $s([e]) = [e']$. We will construct a unitary or anti-unitary operator T on $\mathcal{H}$ which induces s. To begin, we set $T(e) = e'$, so T will depend on the choice of e'.

Let $\mathcal{P} := \ell_e^\perp \subset \mathcal{H}$ and $\mathcal{P}' := \ell_{e'}^\perp \subset \mathcal{H}$. Our first aim is to construct a map $T : \mathcal{P} \to \mathcal{P}'$. To do this consider a nonzero vector $p \in \mathcal{P}$. Since ℓ_e and ℓ_p are orthogonal lines we know that $s(\ell_e)$ and $s(\ell_p)$ are orthogonal lines. Since $s(\ell_e) = \ell_{e'}$ there must exist a vector $p' \in \mathcal{P}'$ with

$$s([p]) = [p'] \tag{2.48}$$

and moreover $\parallel p' \parallel = \parallel p \parallel$. We choose such a vector p'. Two different choices p' and $\tilde{p}'$ are related by a phase $\tilde{p}' = e^{i\theta_1}p'$.

Similarly, consider the vector $v = e + p \in \mathcal{H}$, and choose a v' so that

$$s([v]) = [v'] \tag{2.49}$$

Any two choices of v' and $\tilde{v}'$ are related by a phase $\tilde{v}' = e^{i\theta_2}v'$. By our Lemma with $f_1 = e, f_2 = p$ we know that we must have

$$v' = \alpha' e' + \beta' p' \tag{2.50}$$

with $|\alpha'| = 1$ and $|\beta'| = 1$. The only ambiguity in choosing v' was an overall phase so if we divide by α' we get a canonical vector:

$$v'' = e' + \frac{\beta'}{\alpha'} p' \tag{2.51}$$

In particular *the vector* $p'' := \frac{\beta'}{\alpha'} p'$ *is independent of the choices of phase in* p' *and* v'. That is, having made a choice of e, e' and p there is a *canonically defined* vector $p'' \in \mathcal{P}'$. We now define $T(p)$ by

$$T(p) := p'' \tag{2.52}$$

Note that

$$\| T(p) \| = \| p'' \| = \| p' \| = \| p \| . \tag{2.53}$$

so we can extend to $p = 0$ by $T(0) = 0$. We have now defined a map $T : \mathcal{P} \to \mathcal{P}'$. Moreover, we also define

$$T(e + p) := e' + p'' = e' + T(p) \tag{2.54}$$

To summarize, for any nonzero $p \in \mathcal{P}$ we have defined $T(p) \in \mathcal{P}'$ and $T(e+p)$ so that

$$\begin{aligned} s([p]) &= [T(p)] \\ s([e+p]) &= [T(e+p)] = [e' + T(p)] \end{aligned} \tag{2.55}$$

Now, the invariance of overlaps under s means that if $p_1, p_2 \in \mathcal{P}$ then

$$\begin{aligned} |(p_1, p_2)|^2 &= |(T(p_1), T(p_2))|^2 \\ |(e + p_1, e + p_2)|^2 &= |(e' + T(p_1), e' + T(p_2))|^2 \end{aligned} \tag{2.56}$$

and therefore:

1. For all $p_1, p_2 \in \mathcal{P}$ we have

$$\mathrm{Re}((T(p_1), T(p_2)) = \mathrm{Re}(p_1, p_2) \tag{2.57}$$

eq:cond-1

2. Moreover, if $(p_1, p_2) \in \mathbb{R}$ then

$$(T(p_1), T(p_2)) = (p_1, p_2). \tag{2.58}$$

eq:cond-2

Now assume that $\dim\mathcal{P} > 1$. Otherwise, we are in the two-dimensional case which we have already covered.

Given any vector $w \in \mathcal{P}$ define a function $\chi_w : \mathbb{C} \to \mathbb{C}$ by

$$T(\alpha w) := \chi_w(\alpha) T(w) \tag{2.59}$$

and since T is norm-preserving on $\mathcal{P}$ we have $|\chi_w(\alpha)| = |\alpha|$. We are going to show that in fact this function is independent of w. To this end choose any ON set of vectors $\{f_i\}$ in $\mathcal{P}$. Then we know that $f_i' := T(f_i)$ are ON. For brevity write $T(\alpha f_i) = \chi_i(\alpha) f_i'$. Apply (eq:cond-1 2.57) to $p_1 = \alpha f_i$ and $p_2 = \beta f_i$ (same i) to get:

$$\mathrm{Re}(\chi_i(\alpha)^* \chi_i(\beta)) = \mathrm{Re}(\alpha^* \beta) \tag{2.60}$$

eq:cond-1p

Since $\chi_i(1)=1$, we can take $\alpha=1$ in (2.60), and hence

$$\mathrm{Re}(\chi_i(\beta)) = \mathrm{Re}(\beta) \tag{2.61}$$

eq:rlprt

Now, we saw before in our lemma that if $p=\sum_i \alpha_i f_i$ then $T(p)=\sum \alpha_i' f_i'$ with $|\alpha_i'|=|\alpha_i|$. We claim that in fact $\alpha_i'=\chi_i(\alpha_i)$. This is trivial if α_i is zero. If it is not zero then let $\gamma_i = 1/\alpha_i^*$ so that $(\gamma_i f_i, \alpha_i f_i)=1$. But then by (2.58) $\chi_i(\gamma_i)^*\chi_i(\alpha_i)=1$. On the other hand, it is also true that $(\gamma_i f_i, p)=1$ so again by (2.58) we have

$$1 = (\chi_i(\gamma_i) f_i', \sum_j \alpha_j' f_j') = \chi_i(\gamma_i)^*\alpha_i' \tag{2.62}$$

and hence $\alpha_i'=\chi_i(\alpha_i)$. Next, we also claim that $\chi_i(\alpha)$ is independent of i. To see this let $w=f_i+f_j$ with $i\neq j$. Then $T(w)=\chi_i(1)f_i'+\chi_j(1)f_j' = f_i'+f_j'$. Then

$$T(\alpha w) = \chi_i(\alpha)f_i' + \chi_j(\alpha) f_j' = T(w)\chi_w(\alpha) = (f_i'+f_j')\chi_w(\alpha) \tag{2.63}$$

Now, another simple little lemma: Suppose that v_1, v_2 are two linearly independent vectors and $\alpha_1,\alpha_2,\alpha_3$ are complex numbers such that

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_3 v_3 \tag{2.64}$$

eq:splefact

Then $\alpha_1=\alpha_2=\alpha_3$. Proof: Let P_i be the orthogonal projection onto the plane perpendicular to v_i, $i=1,2$. Then $P_1 v_2$ and $P_2 v_1$ are nonzero vectors. Applying P_1 and then P_2 to (2.64) gives the statement.

So, invoking (2.64) we have $\chi_1(\alpha)=\chi_2(\alpha)=\chi_w(\alpha)$. Denote this common function as $\chi(\alpha)$. Using the properties we proved above we know that $|\chi(i)|=1$ and $\mathrm{Re}(\chi(i))=0$. Therefore $\chi(i)=\eta i$ with $\eta=\pm 1$. Therefore

♣At this point in the argument does it still depend on f_1, f_2? ♣

$$\mathrm{Im}(\chi(\beta)) = \mathrm{Re}(i^*\chi(\beta)) = \eta \mathrm{Re}(\chi(i)^*\chi(\beta) = \eta\mathrm{Re}(i^*\beta) = \eta \mathrm{Im}(\beta) \tag{2.65}$$

and combining this with (2.61) we learn that for any $\beta\in\mathbb{C}$

$$\chi(\beta) = \begin{cases} \beta & \eta = 1 \\ \beta^* & \eta=-1 \end{cases} \tag{2.66}$$

In particular, it follows that χ is real linear: $\chi(\alpha_1+\alpha_2)=\chi(\alpha_1)+\chi(\alpha_2)$ and $\chi(r\alpha)=r\chi(\alpha)$ for $r\in\mathbb{R}$ and $\alpha\in\mathbb{C}$. Therefore $T:\mathcal{P}\to\mathcal{P}'$ is also real-linear. Now we can extend T to the entire Hilbert space: If $v\in\mathcal{H}$ then it has a unique decomposition

$$v = \alpha e + p \tag{2.67}$$

with $\alpha\in\mathbb{C}$ and $p\in\mathcal{P}$. We then define

$$T(v) := \chi(\alpha)e' + T(p) \tag{2.68}$$

One can check that $T(v)$ is either $\mathbb{C}$ linear or anti-linear. Moreover:

$$\parallel T(v)\parallel^2 = |\alpha|^2 + \parallel T(p)\parallel^2 = |\alpha|^2 + \parallel p \parallel^2 = \parallel v\parallel^2 \tag{2.69}$$

Finally:

$$
\begin{aligned}
s([v]) &= s([\alpha e + p]) \\
&= s[|\alpha|(\frac{\alpha}{|\alpha|}e + \frac{1}{|\alpha|}p)] \\
&= s[|\alpha|(e + \frac{1}{\alpha}p)] \\
&= [|\alpha|(e' + T(\frac{1}{\alpha}p))] \\
&= [|\alpha|(e' + \frac{1}{\chi(\alpha)}T(p))] \\
&= [\chi(\alpha)e' + T(p))] \\
&= [T(v)]
\end{aligned}
\tag{2.70}
$$

so T really does induce the original map s. This concludes the proof of Wigner's theorem.

Theorem: Any two lifts $T, \tilde{T}$ of s differ by a phase.

This is clear from the construction above: The only essential choice was the choice of e'. Any two choices of e' differ by a phase. The dependence on e is not so obvious, so let us simply consider two anti-unitary operators T_1, T_2 which induce the same s. Then $[T_1(v)] = [T_2(v)]$ for every v and hence $T_1(v) = \alpha(v)T_2(v)$, where $|\alpha(v)| = 1$. One might worry that this phase could depend on v, however, invoking the simple fact (eq:splefact 2.64) above we see that - at least when $\dim\mathcal{H} > 1$, the phase is independent of v.

Exercise
Simplify the above proof of Wigner's theorem!

---

3. A little bit about group extensions

roupExtensions

We assume a basic familiarity with abstract group theory. However, let us recall that a *group homomorphism* is a map $\varphi : G_1 \to G_2$ between two groups such that

$$\varphi(g_1 g_1') = \varphi(g_1)\varphi(g_1') \qquad \forall g_1, g_1' \in G_1 \tag{3.1}$$

We define the *kernel* of φ to be $\ker\varphi := \{g \in G_1 | \varphi(g) = 1\}$ and the image to be $\mathrm{Im}\ \varphi := \{g_2 \in G_2 | \exists g_1 \in G_1, \varphi(g_1) = g_2\}$. These are natural subgroups of G_1 and G_2 respectively. Given three groups G_1, G_2, G_3 and a pair of homomorphisms φ_1 and φ_2 we say the sequence

$$G_1 \stackrel{\varphi_1}{\to} G_2 \stackrel{\varphi_2}{\to} G_3 \tag{3.2}$$

is *exact at* G_2 if $\ker\varphi_2 = \mathrm{Im}\ \varphi_1$.

If N, G, and Q are three groups and ι and π are homomorphisms such that

$$1 \to N \quad \stackrel{\iota}{\to} \quad G \quad \stackrel{\pi}{\to} \quad Q \to 1 \tag{3.3}$$ eq:central

is exact at N, G and Q then the sequence is called a *short exact sequence* and we say that G is an *extension of Q by N*. This is equivalent to the three conditions:

1. ι is an *injective* homomorphism.

2. π is a *surjective* homomorphism.

3. $\ker(\pi) = \mathrm{Im}\ (\iota)$.

Note that since ι is injective we can identify N with its image in G. Then, N is a kernel of a homomorphism (namely π) and is hence a *normal* or *invariant* subgroup (hence the notation). Then it is well-known that G/N is a group and is in fact isomorphic to the image of π. That group Q is thus a quotient of G (hence the notation).

There is a notion of *homomorphism of two group extensions*

$$1 \to N \overset{\iota_1}{\to} G_1 \overset{\pi_1}{\to} Q \to 1 \tag{3.4}$$ eq:ext1

$$1 \to N \overset{\iota_2}{\to} G_2 \overset{\pi_2}{\to} Q \to 1 \tag{3.5}$$ eq:ext1

This means that there is a group homomorphism $\varphi : G_1 \to G_2$ so that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\iota_1"] \arrow[d, equal] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"] & Q \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & N \arrow[r, "\iota_2"] & G_2 \arrow[r, "\pi_2"] & Q \arrow[r] & 1 \end{tikzcd} \tag{3.6}$$ eq:ExtensionMorph

When there is a homomorphism of group extensions based on $\psi : G_2 \to G_1$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity then the group extensions are said to be *isomorphic extensions*.

Given group N and Q it can certainly happen that there is more than one nonisomorphic extension of Q by N. Classifying all extensions of Q by N is a difficult problem.

We would encourage the reader to think geometrically about this problem, even in the case when Q and N are finite groups, as in Figure fig:GroupExtension 1. In particular we will use the important notion of a *section*, that is, a right-inverse to π: It is a map $s : Q \to G$ such that $\pi(s(q)) = q$ for all $q \in Q$. Such sections always exist.[6] Note that in general $s(\pi(g)) \neq g$. This is obvious from Figure fig:GroupExtension 1: The map π projects the entire "fiber over q" to q. The section s chooses just one point above q in that fiber.

Now, given an extension and a choice of section s we define a map

$$\omega : Q \to \mathrm{Aut}(N) \tag{3.7}$$ eq:OmegaQN

$$q \mapsto \omega_q \tag{3.8}$$

The definition is given by

$$\iota(\omega_q(n)) = s(q)\iota(n)s(q)^{-1} \tag{3.9}$$

Because $\iota(N)$ is normal the RHS is again in $\iota(N)$. Because ι is injective $\omega_q(n)$ is well-defined. Moreover, for each q the reader should check that indeed $\omega_q(n_1 n_2) = \omega_q(n_1)\omega_q(n_2)$,

[6] By the axiom of choice. For continuous groups such as Lie groups there might or might not be continuous sections.

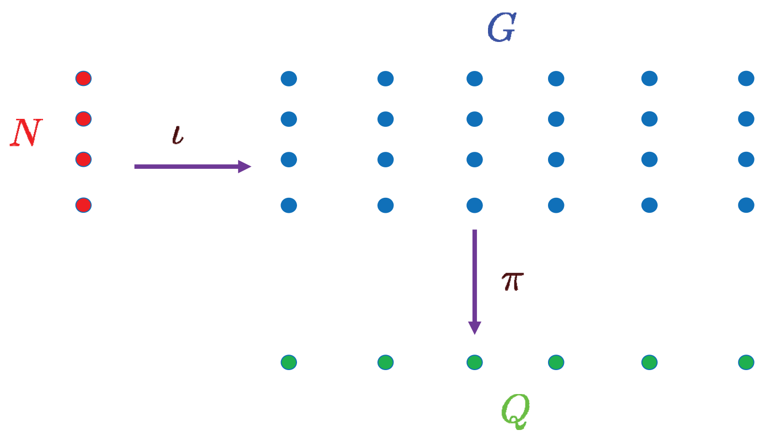

Figure 1: Illustration of a group extension $1 \to N \to G \to Q \to 1$ as an N-bundle over Q. The fiber over $q \in Q$ is just the preimage under π. fig:GroupExtensio

therefore we really have homomorphism $N \to N$. Moreover ω_q is invertible (show this!) and hence it is an automorphism.

Remark: Clearly the ι is a bit of a nuisance and leads to clutter and it can be safely dropped if we consider N simply to be a subgroup of G. The confident reader is encouraged to do this. The formulae will be a little cleaner. However, we will be pedantic and retain the ι in most of our formulae.

Let us stress that the map $\omega : Q \to \mathrm{Aut(N)}$ *in general is not a homomorphism* and *in general depends on the choice of section s*. Let us see how close ω comes to being a group homomorphism:

$$\begin{aligned} \iota\left(\omega_{q_1} \circ \omega_{q_2}(n)\right) &= s(q_1)\iota(\omega_{q_2}(n))s(q_1)^{-1} \\ &= s(q_1)s(q_2)\iota(n)(s(q_1)s(q_2))^{-1} \end{aligned} \tag{3.10}$$ eq:comp-omeg

In general the section is not a homomorphism, but clearly something nice happens when it is:

Definition: We say an extension *splits* if there is a section $s : Q \to G$ which is *also* a *group homomorphism.*

Theorem: An extension is isomorphic to a semidirect product iff there is a splitting.

Proof:

Suppose there is a splitting. Then from (3.10) we know that

$$\omega_{q_1} \circ \omega_{q_2} = \omega_{q_1 q_2} \tag{3.11}$$

and hence $q \mapsto \omega_q$ defines a homomorphism $\omega : Q \to \mathrm{Aut}(N)$. Therefore, we can aim to prove that there is an isomorphism of G with $N \rtimes_\omega Q$.

Note that for any $g \in G$ and any section (not necessarily a splitting):

$$g(s(\pi(g)))^{-1} \tag{3.12}$$

maps to 1 under π (check this: it does not use the fact that s is a homomorphism). Therefore, since the sequence is exact

$$g(s(\pi(g)))^{-1} = \iota(n) \tag{3.13}$$

for some $n \in N$. That is, every $g \in G$ can be written as

$$g = \iota(n)s(q) \tag{3.14}$$

for $n \in N$ and $q \in Q$.

In general if s is just a section the image $s(Q) \subset G$ is not a subgroup. But if the sequence splits, then it is a subgroup. Moreover, when the sequence splits the decomposition is *unique:*

$$\iota(n_1)s(q_1) = \iota(n_2)s(q_2) \Rightarrow \iota(n_2^{-1}n_1) = s(q_2)s(q_1)^{-1} = s(q_2 q_1^{-1}) \tag{3.15}$$

Now, applying π we learn that $q_1 = q_2$, but that implies $n_1 = n_2$.

How does the group law look like in this decomposition? Write

$$\iota(n_1)s(q_1)\iota(n_2)s(q_2) = \iota(n_1)\left(s(q_1)\iota(n_2)s(q_1)^{-1}\right)s(q_1 q_2) \tag{3.16}$$

Note that

$$s(q_1)\iota(n_2)s(q_1)^{-1} = \iota(\omega_{q_1}(n_2)) \tag{3.17}$$

so

$$\iota(n_1)s(q_1)\iota(n_2)s(q_2) = \iota\left(n_1\omega_{q_1}(n_2)\right)s(q_1 q_2) \tag{3.18}$$

eq:SNET

But this just means that

$$\Psi(n,q) = \iota(n)s(q) \tag{3.19}$$

is in fact an isomorphism $\Psi : N \rtimes_\omega Q \to G$. Indeed equation (3.18) just says that:

$$\Psi(n_1,q_1)\Psi(n_2,q_2) = \Psi((n_1,q_1)\cdot_\omega(n_2,q_2)) \tag{3.20}$$

where $\cdot_\omega$ stresses that we are multiplying with the semidirect product rule.

Thus, we have shown that a split extension is isomorphic to a semidirect product $G \cong N \rtimes Q$. The converse is straightforward. ♠

Remark/Definition: In general, when N is abelian it does not follow that $\iota(N)$ is in the center of G. However, very nice things happen when this is true. These are called *central extensions*.

Exercise
If $s : Q \to G$ is any section of π show that for all $q \in Q$,

$$s(q^{-1}) = s(q)^{-1} n = n' s(q)^{-1} \tag{3.21}$$

for some $n, n' \in N$.

3.1 Example 1: $SU(2)$ and $SO(3)$

eq:SU2-to-SO3
Returning to (3.23) there is a standard homomorphism

$$\pi : SU(2) \to SO(3) \tag{3.22}$$

defined by $\pi(u) = R$ where

$$u\vec{x} \cdot \vec{\sigma} u^{-1} = (R\vec{x}) \cdot \vec{\sigma} \tag{3.23}$$ eq:SU2-to-SO3

Note that:

1. Every proper rotation R comes from some $u \in SU(2)$: This follows from the Euler angle parametrization.

2. $\ker(\pi) = \{\pm 1\}$. To prove this we write the general $SU(2)$ element as $\cos\chi + \sin\chi \vec{n}\cdot\vec{\sigma}$. This only commutes with all the σ^i if $\sin\chi = 0$ so $\cos\chi = \pm 1$.

Thus we have the extremely important extension:

$$1 \to \mathbb{Z}_2 \overset{\iota}{\to} SU(2) \overset{\pi}{\to} SO(3) \to 1 \tag{3.24}$$ eq:central

The $\mathbb{Z}_2$ is embedded as the subgroup $\{\pm 1\} \subset SU(2)$, so this is a central extension. Note that there is no continuous splitting. Such a splitting $\pi s = Id$ would imply that $\pi_* s_* = 1$ on the first homotopy group of $SO(3)$. But that is impossible since it would have factor through $\pi_1(SU(2)) = 1$.

Remarks

1. As a manifold $H_1(SO(3); \mathbb{Z}_2) \cong \mathbb{Z}_2$ so there are two double covers of $SO(3)$ and $SU(2)$ is the nontrivial double cover.

2. eq:central
   The extension (3.27) generalizes to

   $$1 \to \mathbb{Z}_2 \overset{\iota}{\to} \mathrm{Spin}(d) \overset{\pi}{\to} SO(d) \to 1 \tag{3.25}$$ eq:SpinCover

   as well as the two Pin groups which extend $O(d)$:

   $$1 \to \mathbb{Z}_2 \overset{\iota}{\to} \mathrm{Pin}^{\pm}(d) \overset{\pi}{\to} O(d) \to 1 \tag{3.26}$$ eq:PinCover

   we discuss these in Section *** below.

3.2 Example 2a: Extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$

Now let us ask which groups G can fit into

$$1 \to \mathbb{Z}_2 \quad \stackrel{\iota}{\to} \quad G \quad \stackrel{\pi}{\to} \quad \mathbb{Z}_2 \to 1 \tag{3.27}$$

eq:central

One obvious possibility is

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \sigma_1, \sigma_2 | \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^2 = 1 \rangle \tag{3.28}$$

We could take $\iota(\sigma_1) = \sigma_1$ and $\pi(\sigma_1) = 1$ and $\pi(\sigma_2) = \sigma_2$. In this case there is an obvious splitting $\pi(\sigma_2) = \sigma_2$.

On the other hand, let us consider the group of complex numbers generated by $\omega = i$. Then $G = \{\pm 1, \pm i\} \cong \mathbb{Z}_4$. Define $\pi : G \to \{\pm 1\}$ by $\pi(\omega) = \omega^2$ and extending so it is a homomorphism. Then $\mathrm{ker}\pi = \{1, \omega^2\} \cong \mathbb{Z}_2$. Therefore G is also an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Yet, G cannot be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ because G has an element of order four. There is clearly no splitting: If $s(\sigma) = \omega^j$ then $\pi \circ s(\sigma) = \sigma$ implies that $\omega^{2j} = -1$ but then

$$1 = s(1) = s(\sigma^2) = s(\sigma)s(\sigma) = \omega^{2j} = -1. \tag{3.29}$$

which is a contradiction.

Remarks

1. It turns out that these are the only extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$, up to isomorphism.

2. Warning: If p is prime there are only two groups of order p^2, up to isomorphism. These can be taken to be $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. Nevertheless, there are p distinct isomorphism classes of extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$.

3.3 Example 2b: Extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$

In instructive example arises by considering an odd prime p and the extensions

$$1 \to \mathbb{Z}_p \to G \to \mathbb{Z}_p \to 1 \tag{3.30}$$

where we will write our groups multiplicatively. Now, using methods of topology one can show that [7]

$$H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p. \tag{3.31}$$

eq:H2Zp

On the other hand, we know from the class equation and Sylow's theorems that there are exactly two groups of order p^2, up to isomorphism! How is that compatible with (3.31)? The answer is that there can be nonisomorphic extensions involving the same central group. To see this, let us examine in detail the standard extension:

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 1 \tag{3.32}$$

eq:StandardExtens

[7] You can also show it by examining the cocycle equation directly.

We write the first, second and third groups in this sequence as

$$\begin{aligned} \mathbb{Z}_p &= \langle \sigma_1 | \sigma_1^p = 1 \rangle \\ \mathbb{Z}_{p^2} &= \langle \alpha | \alpha^{p^2} = 1 \rangle \\ \mathbb{Z}_p &= \langle \sigma_2 | \sigma_2^p = 1 \rangle \end{aligned} \tag{3.33}$$

respectively. Then the first injection must take

$$\iota(\sigma_1) = \alpha^p \tag{3.34}$$

since it must be an injection and it must take an element of order p to an element of order p. The standard sequence then takes the second arrow to be reduction modulo p, so

$$\pi(\alpha) = \sigma_2 \tag{3.35}$$

eq:pi-standard

Now, we try to choose a section of π. Let us try to make it a homomorphism. Therefore we must take $s(1) = 1$. What about $s(\sigma_2)$? Since $\pi(s(\sigma_2)) = \sigma_2$ we have a choice: $s(\sigma_2)$ could be any of

$$\alpha, \alpha^{p+1}, \alpha^{2p+1}, \dots, \alpha^{(p-1)p+1} \tag{3.36}$$

Here we will make the simplest choice $s(\sigma_2) = \alpha$. The reader can check that the discussion is not essentially changed if we make one of the other choices. (After all, this will just change our cocycle by a coboundary!)

Now that we have chosen $s(\sigma_2) = \alpha$, if s were a homomorphism then we would be forced to take:

$$\begin{aligned} s(1) &= 1 \\ s(\sigma_2) &= \alpha \\ s(\sigma_2^2) &= \alpha^2 \\ \vdots \quad & \quad \vdots \\ s(\sigma_2^{p-1}) &= \alpha^{p-1} \end{aligned} \tag{3.37}$$

But now we are stuck! The property that s is a homomorphism requires two contradictory things. On the one hand, we must have $s(1) = 1$ for any homomorphism. On the other hand, from the above equations we also must have $s(\sigma_2^p) = \alpha^p$. But because $\sigma_2^p = 1$ these requirements are incompatible. Therefore, with this choice of section we find a nontrivial cocycle as follows:

$$s(\sigma_2^k)s(\sigma_2^\ell)s(\sigma_2^{k+\ell})^{-1} = \begin{cases} 1 & k+\ell < p \\ \alpha^p & p \le k+\ell \end{cases} \tag{3.38}$$

and therefore,

$$f(\sigma_2^k, \sigma_2^\ell) = \begin{cases} 1 & k+\ell < p \\ \sigma_1 & p \le k+\ell \end{cases} \tag{3.39}$$

In these equations we assume $1 \le k, \ell \le p-1$. We know the cocycle is nontrivial because $\mathbb{Z}_p \times \mathbb{Z}_p$ is not isomorphic to $\mathbb{Z}_{p^2}$.

But now let us use the group structure on the group cohomology. $[f]^r$ is the cohomology class represented by

$$f^r(\sigma_2^k, \sigma_2^\ell) = \begin{cases} 1 & k + \ell < p \\ \sigma_1^r & p \le k + \ell \end{cases} \tag{3.40}$$

This corresponds to replacing (eq:pi-standard 3.35) by

$$\pi_r(\alpha) = (\sigma_2)^r \tag{3.41}$$

eq:pi-standardp

and the sequence will still be exact, i.e. $\ker(\pi_r) = \mathrm{Im}\,(\iota)$, if $(r, p) = 1$, that is, if we compose the standard projection with an automorphism of $\mathbb{Z}_p$. Thus π_r also defines an extension of the form (eq:StandardExtension 3.32). But we claim that it is not isomorphic to the standard extension! To see this let us try to construct ψ so that

$$\begin{tikzcd} & & \langle\alpha\rangle \arrow[dd, "\psi"] \arrow[dr, "\pi_r"] & & \\ 1 \arrow[r] & \langle\sigma_1\rangle \arrow[ur, "\iota"] \arrow[dr, "\iota"'] & & \langle\sigma_2\rangle \arrow[r] & 1 \\ & & \langle\alpha\rangle \arrow[ur, "\pi"'] & & \end{tikzcd} \tag{3.42}$$

eq:19

In order for the right triangle to commute we must have $\psi(\alpha) = \alpha^r$, but then the left triangle will not commute. Thus the extensions $\pi_1, \dots, \pi_{p-1}$, all related by outer automorphisms of the quotient group $\mathbb{Z}_p = \langle\sigma_2\rangle$, define inequivalent extensions with the same group $\mathbb{Z}_{p^2}$ in the middle.

In conclusion, we describe the *group* of isomorphism classes of central extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$ as follows: The identity element is the trivial extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \to 1 \tag{3.43}$$

and then there is an orbit of $(p-1)$ nontrivial extensions of the form

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 1 \tag{3.44}$$

acted on by $\mathrm{Aut}(\mathbb{Z}_p)$.

3.4 Example 3: The isometry group of affine Euclidean space $\mathbb{E}^d$

Definition Let V be a vector space. Then an *affine space modeled on* V is a principal homogeneous space for V. That is, a space with a transitive action of V (as an abelian group) with trivial stabilizer.

The point of the notion of an affine space is that it has no natural origin. A good example is the space of connections on a topologically nontrivial principal bundle.

Let $\mathbb{E}^d$ be the affine space modeled on $\mathbb{R}^d$ with Euclidean metric. The isometries are the 1-1 transformations $f : \mathbb{E}^d \to \mathbb{E}^d$ such that

$$\parallel f(p_1) - f(p_2) \parallel = \parallel p_1 - p_2 \parallel \tag{3.45}$$

for all $p_1, p_2 \in \mathbb{E}^d$. These transformations form a group $\mathrm{Euc}(d)$.

The translations act naturally on the affine space. Given $v \in \mathbb{R}^d$ we define the isometry:

$$T_v(p) := p + v \tag{3.46}$$

so $T_{v_1+v_2} = T_{v_1} + T_{v_2}$ and hence $v \mapsto T_v$ defines a subgroup of $\mathrm{Euc}(d)$ isomorphic to $\mathbb{R}^d$.

One can show that there is a short exact sequence:

$$1 \to \mathbb{R}^d \to \mathrm{Euc}(d) \to O(d) \to 1 \tag{3.47}$$

The rotation-reflections $O(d)$ do *not* act naturally on affine space. In order to define such an action one needs to *choose an origin* of the affine space.

If we do choose an origin then we can identify $\mathbb{E}^d \cong \mathbb{R}^d$ and then to a pair $R \in O(d)$ and $v \in \mathbb{R}^d$ we can associate the isometry: [8]

$$\{R|v\} : x \mapsto Rx + v \tag{3.48}$$

In this notation -known as the *Seitz notation* - the group multiplication law is

$$\{R_1|v_1\}\{R_2|v_2\} = \{R_1R_2|v_1 + R_1v_2\} \tag{3.49}$$

which makes clear that

1. There is a nontrivial automorphism used to construct the semidirect product: $O(d)$:

$$\{R|v\}\{1|w\}\{R|v\}^{-1} = \{1|Rw\} \tag{3.50}$$

and $\pi : \{R|v\} \to R$ is a surjective homomorphism $\mathrm{Euc}(d) \to O(d)$.

2. Thus, although $\mathbb{R}^d$ is abelian, the extension is *not* a central extension.

3. On the other hand, having chosen an origin, the sequence is split. We can choose a splitting $s : O(d) \to \mathrm{Euc}(d)$ by

$$s : R \mapsto \{R|0\} \tag{3.51}$$

Exercise *Manipulating the Seitz notation*

a.) Show that:

$$\begin{aligned}
\{R|v\}^{-1} &= \{R^{-1}| - R^{-1}v\} \\
\{R|0\}\{1|v\} &= \{R|Rv\} \\
\{1|v\}\{R|0\} &= \{R|v\} \\
\{1|w\}\{R|v\} &= \{R|w + v\} \\
\{R_1|v_1\}\{R_2|v_2\}\{R_1|v_1\}^{-1} &= \{R_1R_2R_1^{-1}|R_1v_2 + (1 - R_1R_2R_1^{-1})v_1\} \\
[\{R_1|v_1\}, \{R_2|v_2\}] &= \{R_1R_2R_1^{-1}R_2^{-1}|(1 - R_1R_2R_1^{-1})v_1 - R_1R_2R_1^{-1}R_2^{-1}(1 - R_2R_1R_2^{-1})v_2\}
\end{aligned} \tag{3.52}$$

[8] Logically, since we operate with R first and then translate by v the notation should have been $\{v|R\}$, but unfortunately the notation used here is the standard one.

b.) Using some of these identities check the statements made above.

c.) We stressed that the splitting depends on a choice of origin. Show that another choice of origin leads to the splitting $R \mapsto \{R|(1-R)v\}$, and verify that this is a splitting.

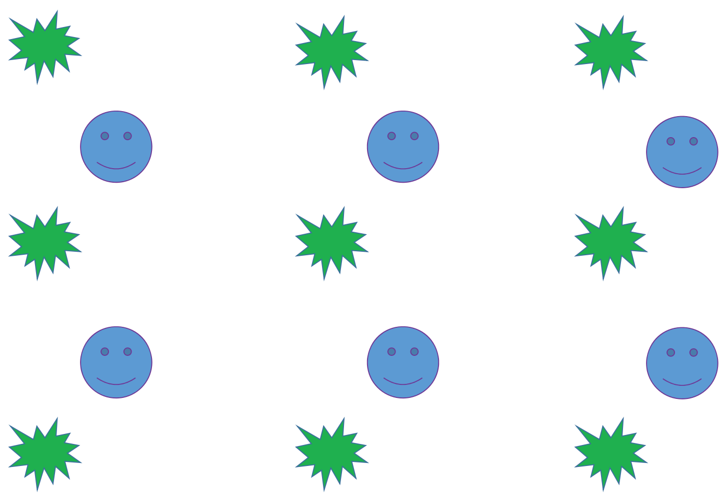

Figure 2: A portion of a crystal in the two-dimensional plane. fig:Crystal

4. A little bit about crystallography

Crytallography

4.1 Crystals and Lattices

A crystal should be distinguished from a lattice. The term "lattice" has several related but slightly different meanings in the literature.

Definition A *lattice* Λ is a free abelian group equipped with a nondegenerate, symmetric bilinear quadratic form:

$$\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to R \tag{4.1}$$

where R is a $\mathbb{Z}$-module.

The natural notion of equivalence is the following: Two lattices $(\Lambda_1, \langle \cdot, \cdot \rangle_1)$ and $(\Lambda_2, \langle \cdot, \cdot \rangle_2)$ are *equivalent* if there is a group isomorphism $\phi : \Lambda_1 \to \Lambda_2$ so that $\phi^*(\langle \cdot, \cdot \rangle_2) = \langle \cdot, \cdot \rangle_1$.

However, we usually think of lattices as actual subsets of some vector space or affine space. If an origin of the lattice has been chosen then we can define:

Definition An *embedded lattice* is a subgroup $L \subset V$ where V is a vector space with a nondegenerate symmetric bilinear quadratic form b. The induced form on Λ defines a lattice in the previous sense.

Now there are several notions of equivalence, discussed briefly in §4.5 below. The most obvious one is that L_1 is equivalent to L_2 if there is an element of the orthogonal group $O(b)$ of V taking L_1 to L_2.

Sometimes it is important not to choose an origin, so we can also have the definition:

Definition An *affine Euclidean lattice* is a subset L of an affine Euclidean space $\mathbb{E}^n$ which is a principal homogeneous space for a free abelian group (i.e. $\mathbb{Z}^n$). If we choose a point as an origin we obtain an embedded lattice in real Euclidean space $\mathbb{R}^n$.

Again, there are several notions of equivalence, discussed below.

Definitions Let L be an embedded lattice in Euclidean space $\mathbb{R}^n$. Then:

a.) A *crystal* is a subset $C \subset \mathbb{E}^n$ invariant under translations by a rank n lattice $L(C) \subset \mathbb{R}^n \subset \mathrm{Euc}(n)$.

b.) The *space group* $G(C)$ of a crystal C is the subgroup of $\mathrm{Euc}(n)$ taking $C \to C$.

c.) The *point group* $P(C)$ of $G(C)$ is the projection of $G(C)$ to $O(n)$. Thus, $G(C)$ sits in a group extension:

$$1 \to L(C) \to G(C) \to P(C) \to 1 \tag{4.2}$$

and $P(C) \cong G(C)/L(C)$.

d.) A *crystallographic group* is a discrete subgroup of $\mathrm{Euc}(n)$ which acts properly discontinuously on $\mathbb{E}^n$ and has a subgroup isomorphic to an embedded rank n-dimensional lattice in the translation subgroup. It therefore sits in a sequence of the form (4.2).

e.) If the group extension (4.2) splits the crystal is said to be *symmorphic*. Similarly, for a crystallographic group G if the corresponding sequence splits it is said to be a *symmorphic group*.

An example of a two-dimensional crystal is shown in Figure 2. The point group is trivial. If we replace the starbursts and smiley faces by points then the point group is a subgroup of $O(2)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4.2 Examples in one dimension

Choose a real number $0 < \delta < 1$ and consider the set

$$C = \mathbb{Z} \amalg (\mathbb{Z} + \delta) \tag{4.3}$$

In this case $G(C)$ contains the translation group $\mathbb{Z}$ whose typical element is $\{1|n\}$. It also contains $\{-1|\delta\}$, which exchanges the two summands in the above disjoint union. So

$$1 \to \mathbb{Z} \to G(C) \to \mathbb{Z}_2 \to 1 \tag{4.4}$$

However, note that

$$\{-1|\delta\}^2 = 1 \tag{4.5}$$

and therefore the sequence splits. This is a symmorphic crystal. Indeed, $G(C) = \mathbb{Z} \rtimes \mathbb{Z}_2$ is the infinite dihedral group. If we move on to consider

$$C = \mathbb{Z} \amalg (\mathbb{Z} + \delta_1) \amalg (\mathbb{Z} + \delta_2) \tag{4.6}$$

with $2\delta_1 - \delta_2 \neq 0 \mathrm{mod} \mathbb{Z}$ and $2\delta_2 - \delta_1 \neq 0 \mathrm{mod} \mathbb{Z}$ and $0 < \delta_1, \delta_2 < \frac{1}{2}$ then there is no point group symmetry and $G(C) \cong \mathbb{Z}$.

4.3 Examples in two dimensions

In a manner similar to our one-dimensional example, if we consider $\mathbb{Z}^2 \amalg (\mathbb{Z}^2 + \vec{\delta})$ for a generic vector δ the symmetry group will be isomorphic to the infinite dihedral group $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$, where we can lift the $\mathbb{Z}_2$ to, for example $\{-1|\vec{\delta}\}$.

Now let $0 < \delta < \frac{1}{2}$ and $\vec{\delta} = (\delta, \frac{1}{2})$. Consider the crystal in two dimensions

$$C = \mathbb{Z}^2 \amalg (\mathbb{Z}^2 + \vec{\delta}) \tag{4.7}$$

Now

$$1 \to \mathbb{Z}^2 \to G(C) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1 \tag{4.8}$$

If we let σ_1, σ_2 be generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ then they have lifts:

$$\hat{\sigma}_1 : (x_1, x_2) \mapsto (-x_1 + \delta, x_2 + \frac{1}{2}) \tag{4.9}$$

$$\hat{\sigma}_2 : (x_1, x_2) \mapsto (x_1, -x_2) \tag{4.10}$$

That is, in Seitz notation:

$$\hat{\sigma}_1 = \{ \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} |(\delta, \frac{1}{2})\} \tag{4.11}$$

$$\hat{\sigma}_2 = \{ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} |0\} \tag{4.12}$$

Note that the square of the lift $\hat{\sigma}_1^2 = \{1|(0,1)\}$ is a nontrivial translation. Thus $\sigma_i \to \hat{\sigma}_i$ is *not* a splitting, and in fact this crystallagraphic group is nonsymmorphic.

4.4 Examples in three dimensions: cubic symmetry and diamond structure

A nice example of the distinction between split and non-split groups in nature are the crystallographic groups of the cubic lattice and of the diamond structure. These are manifested by several materials in nature.

We begin with the hypercubic lattice, considered as the embedded lattice $L = \mathbb{Z}^n \subset \mathbb{R}^n$. The automorphisms must be given by integer matrices which are simultaneously in $O(n)$. Since the rows and columns must square to 1 and be orthogonal these are signed permutation matrices. Therefore

$$\mathrm{Aut}(\mathbb{Z}^n) = \mathbb{Z}_2^n \rtimes S_n \tag{4.13}$$

eq:AutZn

where S_n acts by permuting the coordinates $(x_1, \ldots, x_n)$ and $\mathbb{Z}_2^n$ acts by changing signs $x_i \to \epsilon_i x_i$, $\epsilon_i \in \{\pm 1\}$.

Now, an important sublattice is the fcc lattice, defined to be

$$D_n := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n | x_1 + \cdots + x_n = 0 \mod 2\} \tag{4.14}$$

It is called fcc because the vectors $2e_i$ form an n-dimensional cubic lattice (of side length 2!) but then, for the case of $n = 3$, the vectors $(1,1,0)$, $(0,1,1)$ and $(1,0,1)$ and their translates by $2e_i$ form the midpoints of the faces of the cube.

The dual lattice $D_n^* = \mathrm{Hom}(D_n, \mathbb{Z})$ is

$$D_n^* = \frac{1}{2} BCC_n \tag{4.15}$$

where where *bcc* stands for "body-centered cubic." The lattice BCC_n is the sublattice of $\mathbb{Z}^n$ consisting of $(x_1, \ldots, x_n)$ so that the x_i are either all even or all odd. Note that if all the x_i are even (odd) then adding $\vec{e}$ produces a vector with all x_i odd (even), where $\vec{e} = (1, 1, \ldots, 1) = \vec{e}_1 + \cdots + \vec{e}_n$. Therefore, we can write:

$$BCC_n = 2\mathbb{Z}^n \cup (2\mathbb{Z}^n + \vec{e}) \tag{4.16}$$

Clearly $2\mathbb{Z}^n$ is proportional to the "cubic" lattice. Adding in the orbit of $\vec{e}$ produces one extra lattice vector inside the center of each n-cube of side length 2, hence the name bcc.

Since D_n is an integral lattice it is a sublattice of D_n^*, and it is interesting to show how D_n^* is constructed from D_n. We have

$$D_n^* = D_n \cup (D_n + s) \cup (D_n + v) \cup (D_n + s') \tag{4.17}$$

The vectors s, v, s' are known as "glue vectors" and are given by

$$\begin{aligned}
[s] &= [(\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{n \quad \text{times}})] \\
[v] &= [(\underbrace{0, \ldots, 0}_{n-1 \quad \text{times}}, 1)] \\
[s'] &= [(\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{n-1 \quad \text{times}}, -\frac{1}{2})]
\end{aligned} \tag{4.18}$$

where the square brackets refer to the equivalence class under translation by D_n. ♣Explain this ♣

Remark: These lattices have a nice interpretation in the theory of simple Lie groups: The fcc lattice is the root lattice of $D_n = so(2n)$. The dual bcc lattice is the weight lattice and s and s' are spinor weights. The "glue group" or "disciminant group" is

$$D_n^*/D_n \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & n = 0 \mathrm{mod} 2 \\ \mathbb{Z}_4 & n = 2 \mathrm{mod} 2 \end{cases} \tag{4.19}$$

This is easily verified by noting that $2[s]=[0]$ for n even and $2[s]=[v]$ for n odd. Note that $[s]+[s']=[v]$. Related to this the center of $Spin(N)$ is $\mathbb{Z}_4$ for $N=2\text{mod}4$ and $\mathbb{Z}_2\times\mathbb{Z}_2$ for $N=0\text{mod}4$.

♣DO AUT AND COMMENT ON RELATION TO WEYL GROUPS ♣

Now let us specialize to the case $n=3$, most relevant in the current story to 3 physical dimensions.

Exercise

Show that if we multiply D_3 by 2 then the sum of the coordinate values x_i is 0mod4. Then the three cosets are characterized by the other residues mod 4: $2(D_3+s)$ has $\sum x_i = 3\text{mod}4$, but $2(D_3+s')$ has $\sum x_i = 1\text{mod}4$, and finally $2(D_3+v)=2\text{mod}4$.

Let us consider the point group of D_3. This turns out to be the cubic group O_h, which is also the point group of the cubic lattice. As we showed above, $O_h \cong \mathbb{Z}_2^3 \ltimes S_3$ where S_3 acts as a group of automorphisms on $\mathbb{Z}_2^3$ by permutation. It acts on $\mathbb{R}^3$ by permuting the coordinates (x_1,x_2,x_3) and by sign flips of the coordinates. Let us denote sign flips by ϵ_i. Elements of S_3 are denoted by (ab) and (abc).

The cubic group is a 48 element group. As an abstract group

$$O_h \cong S_4 \times \mathbb{Z}_2 \tag{4.20}$$

The $\mathbb{Z}_2$ factor corresponds to inversion I. [9]

A much more geometrical way to think about the group elements is to think about symmetries of the cube. The S_4 factor can be thought of as permutations of the axes through antipodal vertices of the cube. Then we can organize the elements as follows:

1. Identity.

2. 6 elements of order 4: These are order 4 rotations about an axis through two antipodal midpoints of faces. These are denoted C_4. They correspond to $\epsilon_i(ij)$.

3. 3 elements of order 2: These are the squares C_4^2. These correspond to $\epsilon_i\epsilon_j$.

4. 6 elements of order 2: These are rotations by π about an axis which goes through the midpoint of two opposite edges. These are denoted C_2. They correspond to $\epsilon_i(jk)$ and $I(ij)$.

5. 8 elements which are 3-fold rotations around axes through opposite vertices. They correspond to (ijk) and $\epsilon_i\epsilon_j(ijk)$.

6. Then we have inversion I times the above 24 elements.

See the table below for the explicit transformations.

The space group of D_3 is split since the transformations $\{R|0\}$ where $R\in O_h$ clearly preserves D_3.

Now let us turn to the diamond structure which is, by definition, $D_3 \cup (D_3+s)$.

Note

a.) $4s \in D_3$

[9] The S_4 is the Weyl group of $so(6)=su(4)$.

b.) Diamond structure is not a lattice.

c.) The space group of the diamond structure is non-split, i.e., non-symmorphic. Half of the elements of the point group O_h take $D_3+s \to D_3+s'$ and hence must be accompanied by a translation by an element of $D_3 + s$ in order to preserve the diamond structure.

For the diamond structure a natural lift of ϵ_i is $\{\epsilon_i|s\}$ which exchanges D_3 and $D_3 + s$. Note that the crystal group is non-symmorphic: This lift does *not* square to one, and in fact, there is no lift which will square to one. A lift of $\epsilon_i\epsilon_j$ is $\{\epsilon_i\epsilon_j|0\}$. A lift of $\epsilon_1\epsilon_2\epsilon_3$ is $\{\epsilon_1\epsilon_2\epsilon_3|s\}$

Since s is invariant under the permutations the lift of any element $\sigma \in S_3$ is simply $\{\sigma|0\}$.

The following canonical lifts have half of the group elements lifting with no translation and half lifting with a translation by s.

1. $\{1|0\}$
2. $\{I|s\}$
3. $\{C_4^2|0\}$
4. $\{IC_4^2|s\}$
5. $\{C_2|s\}$
6. $\{IC_2|0\}$
7. $\{C_3|s\}$
8. $\{IC_3|0\}$
9. $\{C_4|s\}$
10. $\{IC_4|0\}$

Cube symmetry	$\mathbb{Z}_2^3 \ltimes S_3$	(x_1, x_2, x_3)	$(T^3)^g$	Lift to $G(C)$
1	1	(x_1, x_2, x_3)	(y, y', y'')	$\{1\|0\}$
I	$\epsilon_1\epsilon_2\epsilon_3$	$(\bar{x}_1, \bar{x}_2, \bar{x}_3)$	$(\varepsilon, \varepsilon', \varepsilon'')$	$\{I\|s\}$
C_4^2	$\epsilon_i\epsilon_j$	$(\bar{x}_1, \bar{x}_2, x_3)$	$(\varepsilon, \varepsilon', y)$	$\{C_4^2\|0\}$
		$(\bar{x}_1, x_2, \bar{x}_3)$	$(\varepsilon, y, \varepsilon')$	
		$(x_1, \bar{x}_2, \bar{x}_3)$	$(y, \varepsilon, \varepsilon')$	
IC_4^2	ϵ_i	$(x_1, x_2, \bar{x}_3)$	$(y, y'\varepsilon)$	$\{IC_4^2\|s\}$
		$(x_1, \bar{x}_2, x_3)$	(y, ε, y')	
		$(\bar{x}_1, x_2, x_3)$	(ε, y, y')	
C_2	$\epsilon_i(jk)$	$(\bar{x}_1, x_3, x_2)$	(ε, y, y)	$\{C_2\|s\}$
		$(x_3, \bar{x}_2, x_1)$	(y, ε, y)	
		$(x_2, x_1, \bar{x}_3)$	(y, y, ε)	
	$I(ij)$	$(\bar{x}_1, \bar{x}_3, \bar{x}_2)$	$(\varepsilon, y, \bar{y})$	
		$(\bar{x}_3, \bar{x}_2, \bar{x}_1)$	$(y, \varepsilon, \bar{y})$	
		$(\bar{x}_2, \bar{x}_1, \bar{x}_3)$	$(y, \bar{y}, \varepsilon)$	
IC_2	$\epsilon_i\epsilon_j(ij)$	$(x_1, \bar{x}_3, \bar{x}_2)$	$(y', y, \bar{y})$	$\{IC_2\|0\}$
		$(\bar{x}_3, x_2, \bar{x}_1)$	$(y, y', \bar{y})$	
		$(\bar{x}_2, \bar{x}_1, x_3)$	$(y, \bar{y}, y')$	
	(ij)	(x_1, x_3, x_2)	(y', y, y)	
		(x_3, x_2, x_1)	(y, y', y)	
		(x_2, x_1, x_3)	(y, y, y')	
C_3	(ijk)	(x_2, x_3, x_1)	(y, y, y)	$\{C_3\|0\}$
		(x_3, x_1, x_2)	(y, y, y)	
	$\epsilon_i\epsilon_j(ijk)$	$(x_2, \bar{x}_3, \bar{x}_1)$	$(y, y, \bar{y})$	
		$(\bar{x}_2, x_3, \bar{x}_1)$	$(\bar{y}, y, y)$	
		$(\bar{x}_2, \bar{x}_3, x_1)$	$(y, \bar{y}, y)$	
		$(x_3, \bar{x}_1, \bar{x}_2)$	$(y, \bar{y}, y)$	
		$(\bar{x}_3, x_1, \bar{x}_2)$	$(y, y, \bar{y})$	
		$(\bar{x}_3, \bar{x}_1, x_2)$	$(\bar{y}, y, y)$	
IC_3	$I(ijk)$	$(\bar{x}_2, \bar{x}_3, \bar{x}_1)$	$(\varepsilon, \varepsilon, \varepsilon)$	$\{IC_3\|s\}$
		$(\bar{x}_3, \bar{x}_1, \bar{x}_2)$	$(\varepsilon, \varepsilon, \varepsilon)$	
	$\epsilon_i(ijk)$	$(\bar{x}_2, x_3, x_1)$	$(\varepsilon, \varepsilon, \varepsilon)$	
		+ 5 more		
C_4	$\epsilon_i(ij)$	$(\bar{x}_2, x_1, x_3)$	$(\varepsilon, \varepsilon, y)$	$\{C_4\|s\}$
		$(x_2, \bar{x}_1, x_3)$	$(\varepsilon, \varepsilon, y)$	
		$(\bar{x}_3, x_2, x_1)$	$(\varepsilon, y, \varepsilon)$	
		$(x_3, x_2, \bar{x}_1)$	$(\varepsilon, y, \varepsilon)$	
		$(x_1, \bar{x}_3, x_2)$	$(y, \varepsilon, \varepsilon)$	
		$(x_1, x_3, \bar{x}_2)$	$(y, \varepsilon, \varepsilon)$	
IC_4	$\epsilon_j\epsilon_k(ij)$	$(x_2, \bar{x}_1, \bar{x}_3)$	$(\varepsilon, \varepsilon, \varepsilon')$	$\{IC_4\|0\}$
		$(\bar{x}_2, x_1, \bar{x}_3)$	$(\varepsilon, \varepsilon, \varepsilon')$	
		$(x_3, \bar{x}_2, \bar{x}_1)$	$(\varepsilon, \varepsilon', \varepsilon)$	
		$(\bar{x}_3, \bar{x}_2, x_1)$	$(\varepsilon, \varepsilon', \varepsilon)$	
		$(\bar{x}_1, x_3, \bar{x}_2)$	$(\varepsilon', \varepsilon, \varepsilon)$	
		$(\bar{x}_1, \bar{x}_3, x_2)$	$(\varepsilon', \varepsilon, \varepsilon)$	

Notation: $\varepsilon, \varepsilon', \varepsilon''$ stands for 0 or $\frac{1}{2}$. x_i stand for real numbers modulo 1. (x_1, x_2, x_3) is a coordinate system on the Brillouin torus of the cubic lattice. Recall that it must be quotiented by $x \to x + s$. y, y', y'' stand for real numbers modulo 1. The primes here indicate that ε and ε' *might* be different, although they need not be different. Similarly for the y, y'. A bar $\bar{x}$ means $-x$. It is standard CM notation. A blank means the entry is identical to the one above it.

Classification

4.5 A word about classification of lattices and crystallographic groups

This is an enormous subject, but perhaps a few words would help put some of the material into context.

When classifying lattices or crystallographic groups we need to be careful about the notion of equivalence.

If we want to speak of the classification of integral lattices that amounts to the classification of positive definite matrices Q over $\mathbb{Z}$ under the equivalence

$$Q \sim SQS^{tr} \qquad S \in GL(n, \mathbb{Z}) \tag{4.21}$$

eq:QuadFormClass

This is an extremely difficult and subtle problem with lots of nontrivial number theory - already for the case $n = 2$.

Let us turn to the classification of embedded lattices in Euclidean $\mathbb{R}^n$. First, note that the set of bases for a vector space V is a principal homogeneous space for $GL(n, \mathbb{R})$: Any two bases are related by such a transformation. If we choose one basis and identify $V \cong \mathbb{R}^n$ then we can choose the standard ordered ON basis $\{e_i\}$

$$\sum_i x_i e_i = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{4.22}$$

Then, given any ordered basis $\{b^{(1)}, \dots, b^{(n)}\}$ of $\mathbb{R}^n$ we can form a matrix B whose columns are the components $b_i^{(\alpha)}$ of those vectors. The change of basis formula for a linear transformation is $\tilde{b}^{(\beta)} = \sum_\alpha T_{\alpha\beta} b^{(\alpha)}$ which acts on B on the right: $B \to BT$.

Now, consider an embedded lattice $L \subset \mathbb{R}^n$. Then if we choose one basis $B \in GL(n, \mathbb{R})$ for L any other basis is related by right-multiplication by an element $T \in GL(n, \mathbb{Z})$. Note well that T must be an integral matrix invertible over the integers! Therefore, we can identify a lattice in a basis-independent way with a single coset of $GL(n, \mathbb{Z})$ in $GL(n, \mathbb{R})$ and *the set of lattices* is in one-one correspondence with the set of orbits

$$GL(n, \mathbb{R})/GL(n, \mathbb{Z}) \tag{4.23}$$

We have not quite characterized the set of lattices intrinsically because our construction made a choice of basis $\{e_i\}$. We can eliminate this dependence by *left-multiplication* of b by elements of $O(n)$. Or - to take an active viewpoint - we can naturally identify two embedded lattices L and L' if one can be brought to the other through an (active) orthogonal transformation. Thus, the set of lattices in $\mathbb{R}^n$ is canonically identified with

$$O(n)\backslash GL(n, \mathbb{R})/GL(n, \mathbb{Z}) \tag{4.24}$$

eq:Lattices

To make a connection with the kind of classification discussed around (4.21) note that if we are given a basis B of L then $Q = B^{\mathrm{tr}} B$ is a symmetric positive definite matrix of inner products, invariant under $B \to OB$, $O \in O(n)$. Under change of basis for L, Q is transformed as in (4.21).

Now (4.24) is an interesting manifold, but for many purposes it is far too fine a classification to be useful. For example $L = \mathbb{Z}^n$ and $L = \lambda\mathbb{Z}^n$ are considered different for any nonzero real number $\lambda \neq \pm 1$.

A courser - but more useful - classification is obtained by the general notion of strata of a group action: [28]

Definition If G acts on a set M then a *stratum* is the set of G-orbits whose stabilizer groups are conjugate in G. The set of strata is denoted $M \parallel G$.

As an example, consider the Lorentz group acting on a vector space with Minkowskian signature. There are four strata (if we consider all four components of the Lorentz group) corresponding to spacelike, lightlike, timelike orbits and the origin.

If we consider the set of strata of $O(n)$ acting on the set of embedded lattices then we will find a finite set. And, for dimension $n = 3$ we get the 7 crystal classes [30], named: Triclinic, Monoclinic,Orthorhombic, Tetragonal, Trigonal, Hexagonal, Cubic. (There is a partial order on this set so they are almost always listed in this order.)

When we consider classification of crystallographic groups $G \subset \mathrm{Euc}(n)$ we again must consider the proper notion of equivalence. The set of conjugacy classes within $\mathrm{Euc}(n)$ is continuously infinite. Again this is related to the fact that continuous deformations of lattices might change their "symmetries." The standard notion of equivalence then is to consider G and G' equivalent if, as subgroups of $\mathrm{Aff}(n)$ there is an element $s \in \mathrm{Aff}(n)$ such that $G' = sGs^{-1}$.

Warning! $\mathrm{Euc}(n) \subset \mathrm{Aff}(n)$ is not a normal subgroup. Similarly, $O(n) \subset GL(n, \mathbb{R})$ is not a normal subgroup. Therefore, we are *not* saying that any affine transformation deforming a crystal leads to a crystal with the "same" symmetry.

Before stating the classification result it is important to distinguish between $\mathrm{Aff}(n)$ and its orientation-preserving subgroup $\mathrm{Aff}^+(n)$. This is the subgroup which projects to $GL^+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$, the subgroup of invertible matrices with positive determinant.

The result of Fedorov and Schoenfliess from 1892 is that in 3 dimensions if we use conjugacy in $\mathrm{Aff}^+(3)$ then there are 230 types of crystallographic group. There are 11 types which can be related to each other by an improper, but not by a proper affine transformation, and hence there are 219 types related by conjugacy in $\mathrm{Aff}^+(3)$

If we view the space group as an extension of a finite subgroup by a lattice then the finite subgroup acts as a group of automorphisms of the lattice and hence has an representation by integral matrices. The pair (P, ρ) where P is a point group and ρ is an integral representation up to conjugacy in $GL(n, \mathbb{Z})$ is called an *arithmetic type*. There are 73 such types in $n = 3$ dimensions. Of the 230 space groups 73 are split and the remaining 157 are nonsplit.

In his famous list of problems for the 20th century Hilbert's 18th problem (part of it) asked whether there were a finite set of space groups in n dimensions for all n. This was answered in the affirmative by Bieberbach in 1910. Such groups do in fact have physical applications. For example, they are very useful in orbifold constructions of conformal field theories.

5. Restatement of Wigner's theorem

WignerRestated

Now that we have the language of group extensions it is instructive to give simple and concise formulation of Wigner's theorem.

Let us begin by introducing a new group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. This is the group whose elements are unitary *and* anti-unitary transformations on $\mathcal{H}$. The unitary operators $U(\mathcal{H})$ form a subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. If u is unitary and a is anti-unitary then ua and au are also anti-unitary, but if a_1, a_2 are antiunitary, then $a_1 a_2$ is unitary. Thus the set of all unitary and anti-unitary operators on $\mathcal{H}$ form a group, which we will denote as $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. Thus we have the exact sequence

$$1 \to U(\mathcal{H}) \quad \stackrel{\iota}{\to} \quad \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \quad \stackrel{\phi}{\to} \quad \mathbb{Z}_2 \to 1 \tag{5.1}$$

eq:RealAutHilb

where ϕ is the homomorphism:

$$\phi(S) := \begin{cases} +1 & S \quad \text{unitary} \\ -1 & S \quad \text{anti}-\text{unitary} \end{cases} \tag{5.2}$$

Now, in Section *** above we defined a homomorphism $\pi : \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \to \mathrm{Aut}_{\rm qtm}(\mathbb{P}\mathcal{H})$ by $\pi(S)(\ell) = \ell_{S(\psi)}$ if $\ell = \ell_\psi$. (Check it is indeed a homomorphism.) Now we recognize the state of Wigner's theorem as the simple statement that π is surjective. What is the kernel? We also showed that $\ker(\pi) \cong U(1)$ where $U(1)$ is the group of unitary transformations:

$$\psi \mapsto z\psi \tag{5.3}$$

with $|z| = 1$. We will often denote this unitary transformation simply by z. Thus, we have the exact sequence

$$1 \to U(1) \quad \stackrel{\iota}{\to} \quad \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \quad \stackrel{\pi}{\to} \quad \mathrm{Aut}_{\rm qtm}(\mathbb{P}\mathcal{H}) \to 1 \tag{5.4}$$

eq:WigSeq

Remarks:

1. For $S \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}))$ we have

$$Sz = z^{\phi(S)} S = \begin{cases} zS & \phi(S) = +1 \\ \bar{z}S & \phi(S) = -1 \end{cases} \tag{5.5}$$

   So the sequence (eq:WigSeq 5.4) is not central!

2. If we restrict the sequence (eq:WigSeq 5.4) to $\ker(\phi)$ then we get (taking $\dim\mathcal{H} = N$ here, but it also holds in infinite dimensions):

$$1 \to U(1) \quad \stackrel{\iota}{\to} \quad U(N) \quad \stackrel{\pi}{\to} \quad PU(N) \to 1 \tag{5.6}$$

   eq:UN-PUN

   which is a central extension, but it is not split. This is in fact the source of interesting things like anomalies in quantum mechanics.

3. The group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ has two connected components, measured by the homomorphism ϕ used in (5.1). This homomorphism "factors through" a homomorphism $\phi' : \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \to \mathbb{Z}_2$ which likewise detects the connected component of this two-component group. The phrase "factors through" means that ϕ and ϕ' fit into the diagram:

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] \arrow[dr, "\phi"'] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] \arrow[d, "\phi'"] & 1 \\ & & & \mathbb{Z}_2 & \end{tikzcd} \tag{5.7}$$

eq:WigPhi

Example: Again let us take $\mathcal{H} \cong \mathbb{C}^2$. As we saw,

$$\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) = O(3) = SO(3) \amalg P \cdot SO(3), \tag{5.8}$$

where P is any reflection. [10] Similarly, if we choose a basis for $\mathcal{H}$ then we can identify

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \cong U(2) \amalg \mathcal{C} \cdot U(2) \tag{5.9}$$

where $\mathcal{C}$ is complex conjugation with respect to that basis so that $\mathcal{C}u = u^*\mathcal{C}$. (Note that $\mathcal{C}$ does not have a 2×2 matrix representation.) Now

$$PU(2) := U(2)/U(1) \cong SU(2)/\mathbb{Z}_2 \cong SO(3) \tag{5.10}$$

Again, there is no continuous cross-section $s : SO(3) \to U(2)$ because such a continuous map would induce

$$s_* : \pi_1(SO(3)) \to \pi_1(U(2)) \tag{5.11}$$

but this would be a homomorphism $s_* : \mathbb{Z}_2 \to \mathbb{Z}$ and the only such homomorphism is zero. But that is incompatible with $\pi \circ s = Id$ which implies $\pi_* s_* = Id$. A splitting of (5.6) would restrict to one for $N = 2$, so there is also no splitting for $N > 2$.

Exercise
Show that the sequence (5.1) splits.

---

6. ϕ-twisted extensions

PhiTwistedExts

So far we have discussed the group of all potential automorphisms of a quantum system $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. However, when we include dynamics, and hence Hamiltonians, a given quantum system will in general only have a subgroup of symmetries. If a physical system has a symmetry group G then we should have a homomorphism $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

[10] Please do not confuse this with the notation $PGL(n)$, $PU(n)$ etc!

In terms of diagrams we have

$$\begin{tikzcd} & & & G \arrow[d, "\rho"] & \\ 1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] & 1 \end{tikzcd} \tag{6.1}$$

eq:G-AutPH

The question we now want to address is:

How are G-symmetries represented on Hilbert space $\mathcal{H}$?

Note that each operation $\rho(g)$ in the group of quantum automorphisms has an entire circle of possible lifts in $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. These operators will form a group of operators which is a certain extension of G. What extension to we get?

To answer this we need the "pullback construction."

6.1 The pullback construction

There is one general construction with extensions which is useful when discussing symmetries in quantum mechanics. This is the notion of *pullback extension*. Suppose we are given both an extension

$$1 \longrightarrow H' \xrightarrow{\iota} H \xrightarrow{\pi} H'' \longrightarrow 1 \tag{6.2}$$

and a homomorphism

$$\rho : G'' \to H'' \tag{6.3}$$

Then the *pullback extension* is defined by a subgroup of the Cartesian product $G \subset H \times G''$:

$$G := \{(h, g'')|\pi(h) = \rho(g'')\} \subset H \times G'' \tag{6.4}$$

and is an extension of the form

$$1 \longrightarrow H' \xrightarrow{\iota} G \xrightarrow{\tilde{\pi}} G'' \longrightarrow 1 \tag{6.5}$$

where $\tilde{\pi}(h, g'') := g''$. It is easy to see that this extension fits in the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H' \arrow[r] \arrow[d, equal] & G \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\rho}"] & G'' \arrow[r] \arrow[d, "\rho"] & 1 \\ 1 \arrow[r] & H' \arrow[r] & H \arrow[r, "\pi"] & H'' \arrow[r] & 1 \end{tikzcd} \tag{6.6}$$

eq:pullback

Moreover, show that this diagram can be used to define the pullback extension.

Remark: In terms of principal bundles, this coincides with the pullback of a principal H' bundle over H'' via the map $\rho : G'' \to H''$.

6.2 ϕ-twisted extensions

Now, let us return to the situation of (eq:G-AutPH 6.1) and apply the pullback construction to define a group G^{tw} that fits in the diagram:

$$
\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{\mathrm{tw}} \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho^{\mathrm{tw}}"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] & 1
\end{tikzcd}
\tag{6.7}
$$

eq:Gtau

That is, the group of operators representing the G-symmetries of a quantum system form an extension of G by $U(1)$.

This motivates two definitions. First

Definition: A $\mathbb{Z}_2$-*graded group* is a pair (G,ϕ) where G is a group and $\phi : G \to \mathbb{Z}_2$ is a homomorphism.

When we have such a group of course we have an extension of $\mathbb{Z}_2$ by G. Our examples above show that in general it does not split. The group is a disjoint union $G_0 \amalg G_1$ of elements which are even and odd under ϕ and we have the $\mathbb{Z}_2$-graded multiplications:

$$
\begin{aligned}
G_0 \times G_0 &\to G_0 \\
G_0 \times G_1 &\to G_1 \\
G_1 \times G_0 &\to G_1 \\
G_1 \times G_1 &\to G_0
\end{aligned}
\tag{6.8}
$$

This is just saying that ϕ is a homomorphism.

Next we have the

Definition Given a $\mathbb{Z}_2$-graded group (G,ϕ) we define a *ϕ-twisted extension of* G to be an extension of the form

$$
1 \longrightarrow U(1) \longrightarrow G^{\mathrm{tw}} \xrightarrow{\pi} G \longrightarrow 1
\tag{6.9}
$$

eq:Gtau

where G^{tw} is a group such that

$$
\tilde{g}z = z^{\phi(g)}\tilde{g} = \begin{cases} z\tilde{g} & \phi(g) = 1 \\ \bar{z}\tilde{g} & \phi(g) = -1 \end{cases}
\tag{6.10}
$$

where $\tilde{g}$ is any lift of $g \in G$, and $|z| = 1$ is any phase. Put differently, if we define $\phi^{\mathrm{tw}} := \phi \circ \pi$ then

$$
\tilde{g}z = z^{\phi^{\mathrm{tw}}(\tilde{g})}\tilde{g} \qquad\qquad \forall \tilde{g} \in G^{\mathrm{tw}}
\tag{6.11}
$$

Example

Take $G = \mathbb{Z}_2$ It will be convenient to denote $M_2 = \{1, \bar{T}\}$, with $\bar{T}^2 = 1$. Of course, $M_2 \cong \mathbb{Z}_2$. We take the $\mathbb{Z}_2$ grading to be $\phi(\bar{T}) = -1$, that is, $\phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity homomorphism. There are two inequivalent ϕ-twisted extensions:

$$
1 \longrightarrow U(1) \longrightarrow M_2^{\mathrm{tw}} \xrightarrow{\tilde{\pi}} M_2 \longrightarrow 1
\tag{6.12}
$$

eq:Gtau

Choose a lift T of $\bar{T}$. Then $\pi(T^2) = 1$, so $T^2 = z \in U(1)$. But, then

$$Tz = TT^2 = T^2T = zT \tag{6.13}$$

on the other hand, $\phi(\bar{T}) = -1$ so

$$Tz = z^{-1}T \tag{6.14}$$

Therefore $z^2 = 1$, so $z = \pm 1$, and therefore $T^2 = \pm 1$. Thus the two groups are

$$M_2^{\pm} = \{zT | zT = Tz^{-1} \qquad \& \qquad T^2 = \pm 1\} \tag{6.15}$$

These possibilities are really distinct: If T' is another lift of $\bar{T}$ then $T' = \mu T$ for some $\mu \in U(1)$ and so

$$(T')^2 = (\mu T)^2 = \mu\bar{\mu}T^2 = T^2 \tag{6.16}$$

Remarks

1. For $\phi = 1$ a ϕ-twisted extension is a *central extension*.
2. For a given $\mathbb{Z}_2$-graded group (G, ϕ) there can be several non-isomorphic ϕ-twisted extensions. These isomorphism classes can be classified by (twisted) group cohomology.
3. It turns out that $M_2^{\pm}$ is also a double cover of $O(2)$ and in fact these turn out to be isomorphic to the Pin-groups $\mathrm{Pin}^{\pm}(2)$.
4. The representation $(G^{\mathrm{tw}}, \rho^{\mathrm{tw}})$ is always guaranteed to act on the Hilbert space, but in a particular situation it might well happen that a set of lifts of $\rho(g)$ generates a smaller group. For example, suppose that $G = M_2$. We therefore have M_2^+ or M_2^- acting on $\mathcal{H}$. If M_2^+ acts then in fact $s : \bar{T} \to T$ is a splitting and a $\mathbb{Z}_2$ group acts on $\mathcal{H}$. On the other hand, if M_2^- acts then T itself generates a $\mathbb{Z}_4$ subgroup of M_2^-. So, $\mathbb{Z}_2$ does not act on the Hilbert space, but a double cover of it does.
5. The above mechanism is the basic origin of *anomalies* in quantum systems: One expects a G symmetry but in fact only a ϕ-twisted extension G^{tw} acts on $\mathcal{H}$. Thus, in the example of M_2^- the fact that T generates a $\mathbb{Z}_4$ group of operators acting on $\mathcal{H}$ rather than a $\mathbb{Z}_2$ group of operators may be regarded as a kind of "anomaly."

7. Real, complex, and quaternionic vector spaces

sec:RCH-VS

7.1 Complex structure on a real vector space

:CplxStrRealVS

Definition Let V be a real vector space. A *complex structure* on V is a linear map $I : V \to V$ such that $I^2 = -1$.

Choose a squareroot of -1 and denote it i. If V is a real vector space with a complex structure I, then we can define an associated complex vector space (V, I). We take (V, I)

to be identical with V, *as sets*, but define the scalar multiplication of a complex number $z \in \mathbb{C}$ on a vector v by

$$z \cdot v := x \cdot v + I(y \cdot v) = x \cdot v + y \cdot I(v) \tag{7.1}$$

where $z = x + iy$ with $x, y \in \mathbb{R}$.

If V is finite dimensional and has a complex structure its dimension (as a real vector space) is even. The dimension of (V, I) as a complex vector space is

$$\dim_{\mathbb{C}}(V, I) = \frac{1}{2}\dim_{\mathbb{R}} V \tag{7.2}$$

eq:half-dim

We will prove this as follows. First note that if v is any nonzero vector in V then v and Iv are clearly linearly independent over $\mathbb{R}$. Linear independence is equivalent to the statement that

$$v = \alpha I v \tag{7.3}$$

for a *real* number α. But then, acting with I we get

$$Iv = -\alpha v \tag{7.4}$$

and hence $\alpha^2 = -1$, which is not possible. Now, suppose that there is a set of linearly independent vectors $v_1, \dots, v_n$ in V with

$$\mathcal{S} = \{v_1, Iv_1, v_2, Iv_2, \dots, v_n, Iv_n\} \tag{7.5}$$

linearly independent over $\mathbb{R}$. Suppose that w is a vector not in the linear span of $\mathcal{S}$. Then we claim that

$$\{w, Iw\} \cup \mathcal{S} \tag{7.6}$$

is also linearly independent over $\mathbb{R}$. A linear dependence would have to take the form

$$\alpha w + \beta I w + \sum_i (\gamma_i v_i + \delta_i I v_i) = 0 \tag{7.7}$$

Acting on this equation by I, and then taking a suitable combination of the two equations gives

$$(\alpha^2 + \beta^2) w + \sum_i \left((\alpha\gamma_i + \beta\delta_i) v_i + (\alpha\delta_i - \beta\gamma_i) I v_i \right) = 0 \tag{7.8}$$

But α and β cannot be both zero since $\mathcal{S}$ was a linearly independent set, and since they are real $\alpha^2 + \beta^2 \neq 0$. But this means that w is in the linear span of $\mathcal{S}$, which is a contradiction. It then follows that the maximal set of the form $\mathcal{S}$ must be a basis for V, which therefore must have a basis of the form $\mathcal{S}$ for some n.

Note that we have proven a nice lemma:

♣Note that in several of the later chapters our basepoint complex structure is $-I_0$. Need to straighten out this convention! ♣

Lemma If I is any $2n \times 2n$ *real* matrix which squares to -1_{2n} then there is $S \in GL(2n, \mathbb{R})$ such that

$$SIS^{-1} = I_0 := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \tag{7.9}$$

eq:CanonCS

Remarks

1. Using the Jordan canonical form theorem we learn that $SIS^{-1} = I_0$ for some complex matrix $S \in GL(2n, \mathbb{C})$, but we proved something stronger above because our matrix S was real.

2. While v and $I(v)$ are linearly independent in the real vector space V they are *linearly dependent* in the complex vector space (V, I). The very definition $i \cdot v := I(v)$ expresses this linear dependence!

Example Consider the real vector space $V = \mathbb{R}^2$. Let us choose

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.10}$$

Then multiplication of the complex scalar $z = x + iy$, with $x, y \in \mathbb{R}$ on a vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ can be defined by:

$$(x + iy) \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} := \begin{pmatrix} a_1 x - a_2 y \\ a_1 y + a_2 x \end{pmatrix} \tag{7.11}$$

eq:cplxstrone

By equation (eq:half-dim 7.2) this must be a one-complex dimensional vector space, so it should be isomorphic to $\mathbb{C}$ as a complex vector space. Indeed this is the case. Define $\Psi : (V, I) \to \mathbb{C}$ by

$$\Psi : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1 + i a_2 \tag{7.12}$$

Then one can check (exercise!) that this is an isomorphism of complex vector spaces.

Quite generally, if I is a complex structure then so is $\tilde{I} = -I$. So what happens if we take our complex structure to be instead:

$$\tilde{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.13}$$

Now the rule for multiplication by a complex number in $(V, \tilde{I})$ is

$$(x + iy) \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} := \begin{pmatrix} a_1 x + a_2 y \\ -a_1 y + a_2 x \end{pmatrix} \tag{7.14}$$

eq:cplxstronep

Now one can check that $\tilde{\Psi} : (V, \tilde{I}) \to \mathbb{C}$ defined by

$$\tilde{\Psi} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1 - i a_2 \tag{7.15}$$

is also an isomorphism of complex vector spaces. (Check carefully that $\tilde{\Psi}(z\vec{a}) = z\tilde{\Psi}(\vec{a})$.)

How are these two constructions related? Note that if we introduce the real linear operator

$$C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.16}$$

then $C^2 = 1$ and

$$CIC^{-1} = CIC = -I \tag{7.17}$$

We see from the above example that a real vector space can have more than one complex structure. Indeed, it follows from our Lemma above that the space of all complex structures on $\mathbb{R}^{2n}$ is a homogeneous space for $GL(2n,\mathbb{R})$. The stabilizer of I_0 is the set of $GL(2n,\mathbb{R})$ matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = A \otimes 1_2 + iB \otimes \sigma^2 \tag{7.18}$$

eq:Stab-I0

and since σ^2 is conjugate to σ^3, over the complex numbers this can be conjugated to

$$\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \tag{7.19}$$

eq:Stab-I0-p

The determinant is clearly $|\det(A+iB)|^2$ and hence $A+iB \in GL(n,\mathbb{C})$. Therefore, the stabilizer of I_0 is a group isomorphic to $GL(n,\mathbb{C})$ and hence we have proven:

Proposition: *The space of complex structures on $\mathbb{R}^{2n}$ is:*

$$\boxed{\mathrm{CplxStr}(\mathbb{R}^{2n}) = GL(2n,\mathbb{R})/GL(n,\mathbb{C})} \tag{7.20}$$

eq:Space-Cplx

If we introduce a metric g on V then we can say that *a complex structure I is compatible with g if*

$$g(Iv, Iv') = g(v, v') \tag{7.21}$$

So, when expressed relative to an ON basis for g the matrix I is orthogonal: $I^{tr} = I^{-1}$. But $I^{-1} = -I$, and hence I is anti-symmetric. Then it is well known that there is a matrix $S \in O(2n)$ so that

$$SIS^{-1} = I_0 \tag{7.22}$$

Now the stabilizer of I_0 in $O(2n)$ is of the form (7.18) and can therefore be conjugated to (7.19). But now $A + iB$ must be a unitary matrix so

The space of complex structures on $\mathbb{R}^{2n}$ compatible with the Euclidean metric a homogeneous space isomorphic to

$$\boxed{\mathrm{CmptCplxStr}(\mathbb{R}^{2n}) \cong O(2n)/U(n)} \tag{7.23}$$

eq:Cplx-Compat

where $A + iB \in U(n)$ with A, B real is embedded into $O(2n)$ as in (7.18).

♣Put here a discussion of the two components and the relation between I and orientation. ♣

7.2 Real structure on a complex vector space

Given a complex vector space V can we produce a real vector space? Of course, by restriction of scalars, if V is complex, then it is also a real vector space, which we can call $V_{\mathbb{R}}$. V and $V_{\mathbb{R}}$ are the same as *sets* but in $V_{\mathbb{R}}$ the vectors v and iv, are linearly independent (they are *not* linearly independent in V!). Thus:

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = 2\dim_{\mathbb{C}} V. \tag{7.24}$$

There is another way we can get real vector spaces out of complex vector spaces. A *real structure* on a complex vector V space produces a different real vector space of half the real dimension of $V_{\mathbb{R}}$, that is, a vector space of real dimension equal to the complex dimension of V.

Definition An *antilinear* map $\mathcal{T} : V \to V$ on a complex vector space V satisfies

1. $\mathcal{T}(v + v') = \mathcal{T}(v) + \mathcal{T}(v')$,
2. $\mathcal{T}(\alpha v) = \alpha^* \mathcal{T}(v)$ where $\alpha \in \mathbb{C}$ and $v \in V$.

Note that $\mathcal{T}$ is a linear map on the underlying real vector space $V_{\mathbb{R}}$.

Definition Suppose V is a complex vector space. Then a *real structure* on V is an antilinear map $\mathcal{C} : V \to V$ such that $\mathcal{C}^2 = +1$.

If $\mathcal{C}$ is a real structure on a complex vector space V then we can define *real vectors* to be those such that

$$\mathcal{C}(v) = v \tag{7.25}$$

Let us call the set of such real vectors V_+. This set is a real vector space, but it is not a complex vector space, because $\mathcal{C}$ is antilinear. Indeed, if $\mathcal{C}(v) = +v$ then $\mathcal{C}(iv) = -iv$. If we let V_- be the *imaginary vectors*, for which $\mathcal{C}(v) = -v$ then we claim

$$V_{\mathbb{R}} = V_+ \oplus V_- \tag{7.26}$$

The proof is simply the isomorphism

$$v \mapsto \left(\frac{v + \mathcal{C}(v)}{2}\right) \oplus \left(\frac{v - \mathcal{C}(v)}{2}\right) \tag{7.27}$$

Moreover multiplication by i defines an isomorphism of real vector spaces: $V_+ \cong V_-$. Thus we have

$$\dim_{\mathbb{R}} V_+ = \dim_{\mathbb{C}} V \tag{7.28}$$

Example $V = \mathbb{C}$,

$$\mathcal{C} : x + iy \to e^{i\varphi}(x - iy) \tag{7.29}$$

The fixed vectors under $\mathcal{C}$ consist of the real line at angle $\varphi/2$ to the x-axis as shown in Figure fig:REALVECTORS 3.

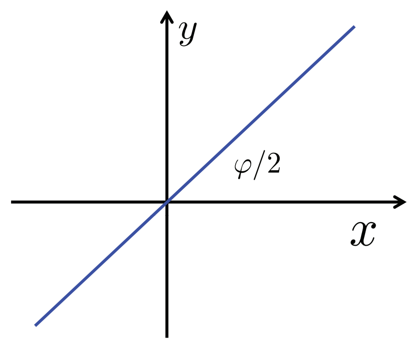

Figure 3: The real structure $\mathcal{C}$ has fixed vectors given by the blue line. This is a real vector space determined by the real structure $\mathcal{C}$. fig:REALVECTORS

In general, if V is a finite dimensional complex vector space, if we choose any basis (over $\mathbb{C}$) $\{v_i\}$ for V then we can define a real structure:

$$\mathcal{C}(\sum_i z_i v_i) = \sum_i \bar{z}_i v_i \tag{7.30}$$

and thus

$$V_+ = \{\sum a_i v_i | a_i \in \mathbb{R}\} \tag{7.31}$$

The space of real structures on $\mathbb{C}^n$ is $GL(n,\mathbb{C})/GL(n,\mathbb{R})$. ♣EXPLAIN THIS ♣

Remark: We introduced a group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. This is the automorphisms of $\mathcal{H}$ as a Hilbert space which are real-linear. It should be distinguished from $\mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$ which would be a much larger group of automorphisms of a real inner product space $\mathcal{H}_{\mathbb{R}}$.

Exercise *Antilinear maps from the real point of view*

Suppose W is a real vector space with complex structure I giving us a complex vector space (W,I).

Show that an antilinear map $\mathcal{T} : (W,I) \to (W,I)$ is the same thing as a real linear transformation $T : W \to W$ such that

$$TI + IT = 0 \tag{7.32}$$

7.2.1 Complex conjugate of a complex vector space

There is another viewpoint on what a real structure is which can be very useful. If V is a complex vector space then we can, canonically, define another complex vector space $\bar{V}$. We begin by declaring $\bar{V}$ to be the same *set*. Thus, for every vector $v \in V$, the same vector, regarded as an element of $\bar{V}$ is simply written $\bar{v}$. However, $\bar{V}$ is different from V *as* *a complex vector space* because we alter the vector space structure by altering the rule for scalar multiplication by $\alpha \in \mathbb{C}$:

$$\alpha \cdot \bar{v} := \overline{\alpha^* \cdot v} \tag{7.33}$$ eq:conjvs

where α^* is the complex conjugate in $\mathbb{C}$.

Of course $\bar{\bar{V}} = V$.

Note that, given any $\mathbb{C}$-linear map $T : V \to W$ between complex vector spaces there is, canonically, a $\mathbb{C}$-linear map

$$\bar{T} : \bar{V} \to \bar{W} \tag{7.34}$$

defined by

$$\bar{T}(\bar{v}) := \overline{T(v)} \tag{7.35}$$

With the notion of $\bar{V}$ we can give an alternative definition of an *anti-linear map*: An anti-linear map $\mathcal{T} : V \to V$ is the same as a $\mathbb{C}$-linear map $T : V \to \bar{V}$, related by

$$\mathcal{T}(v) = \overline{T(v)} \tag{7.36}$$

Similarly, we can give an alternative definition of a real structure on a complex vector space V as a $\mathbb{C}$- *linear* map

$$C : V \to \bar{V} \tag{7.37}$$

such that $C\bar{C} = 1$ and $\bar{C}C = 1$, where $\bar{C} : \bar{V} \to V$ is canonically determined by C as above. In order to relate this to the previous viewpoint note that $\mathcal{C} : v \mapsto \bar{C}(\bar{v})$ is an antilinear transformation $V \to V$ which squares to 1.

Remark: Real structures always exist and therefore V and $\bar{V}$ are isomorphic complex vector spaces, but not canonically isomorphic.

Exercise

A linear transformation $T : V \to W$ between two complex vector spaces with real structures C_V and C_W commutes with the real structures if the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow C_V & & \downarrow C_W \\ \bar{V} & \xrightarrow{\bar{T}} & \bar{W} \end{array} \tag{7.38}$$

commutes.

Show that in this situation T defines an $\mathbb{R}$-linear transformation on the underlying real vector spaces: $T_+ : V_+ \to W_+$.

Exercise *Complex conjugate from the real point of view*

Suppose W is a real vector space with complex structure I so that we can form the complex vector space (W, I). Show that

$$\overline{(W, I)} = (W, -I) \tag{7.39}$$

7.3 Complexification

If V is a real vector space then we can define its *complexification* $V_{\mathbb{C}}$ by putting a complex structure on $V \oplus V$. This is simply the real linear transformation

$$I : (v_1, v_2) \mapsto (-v_2, v_1) \tag{7.40}$$ eq:cplx-def-1

and clearly $I^2 = -1$. Another way to define the complexification of V is to take

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}. \tag{7.41}$$ eq:cplx-def-2

Note that we are taking a tensor product of vector spaces over $\mathbb{R}$ to get a real vector space, but there is a natural action of the complex numbers on these vectors:

$$z \cdot (v \otimes z') := v \otimes zz' \tag{7.42}$$

making $V_{\mathbb{C}}$ into a complex vector space. In an exercise below you show that these two definitions are equivalent.

Note that

$$\dim_{\mathbb{C}} V_{\mathbb{C}} = \dim_{\mathbb{R}} V \tag{7.43}$$

Note that $V_{\mathbb{C}}$ has a canonical real structure. Indeed

$$\overline{V_{\mathbb{C}}} = V \otimes_{\mathbb{R}} \overline{\mathbb{C}} \tag{7.44}$$

and we can define $C : V_{\mathbb{C}} \to \overline{V_{\mathbb{C}}}$ by setting

$$C : v \otimes 1 \mapsto v \otimes \bar{1} \tag{7.45}$$

and extending by $\mathbb{C}$-linearity. Thus

$$\begin{aligned} C(v \otimes z) &= C(z \cdot (v \otimes 1)) && \text{def of } V_{\mathbb{C}} \\ &= z \cdot C((v \otimes 1)) && \mathbb{C} - \text{linear extension} \\ &= z \cdot (v \otimes \bar{1}) \\ &= v \otimes \overline{z^*} && \text{definition of scalar action on} \bar{V}_{\mathbb{C}} \end{aligned} \tag{7.46}$$

Finally, it is interesting to ask what happens when one begins with a *complex* vector space V and then complexifies the underlying real space $V_{\mathbb{R}}$. If V is complex then we claim there is an isomorphism of complex vector spaces:

$$(V_{\mathbb{R}})_{\mathbb{C}} \cong V \oplus \bar{V} \tag{7.47}$$ eq:complxi

Proof: The vector space $(V_{\mathbb{R}})_{\mathbb{C}}$ is, by definition the space of pairs (v_1, v_2), $v_i \in V_{\mathbb{R}}$ with complex structure defined by $I : (v_1, v_2) \to (-v_2, v_1)$. Now we map:

$$\psi : (v_1, v_2) \mapsto (v_1 + iv_2) \oplus (v_1 - iv_2) \tag{7.48}$$

and compute

$$(x + Iy) \cdot (v_1, v_2) = (xv_1 - yv_2, xv_2 + yv_1) \tag{7.49}$$

so

$$\psi : z \cdot (v_1, v_2) \mapsto (x + iy) \cdot (v_1 + iv_2) \oplus (x - iy) \cdot (v_1 - iv_2) = z \cdot v + \bar{z} \cdot \bar{v} \tag{7.50}$$

Another way to look at (eq:complxi 7.47) is as follows. Let V be a real vector space with complex structure I. Now consider $V \otimes_{\mathbb{R}} \mathbb{C}$. There are now two ways of multiplying by a complex number $z = x + iy$: We can multiply the second factor $\mathbb{C}$ by z or we could operate on the first factor with $x + Iy$. We can decompose our space $V \otimes_{\mathbb{R}} \mathbb{C}$ into eigenspaces where $Iv = +iv$ and $Iv = -iv$ using the projection operators

$$P_{\pm} = \frac{1}{2}(1 \mp I \otimes i) \tag{7.51}$$

The image of P_+ is the vector space V of vectors with $Iv = iv$ and the image of P_- is the vector space $\bar{V}$ of vectors with $Iv = -iv$.

Exercise *Equivalence of two definitions*

a.) Suppose V is a real vector space. Show that the two definitions (eq:cplx-def-1 7.40) and (eq:cplx-def-2 7.41) define canonically isomorphic complex vector spaces. [11]

b.) If V is a real vector space write the canonical real structure of $V_{\mathbb{C}}$ in terms of pairs (v_1, v_2) in $V \oplus V$. [12]

Exercise

Show that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C} \tag{7.52}$$

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \tag{7.53}$$

as algebras.

---

[11] *Answer:* $(v_1, v_2) \to v_1 \otimes 1 + v_2 \otimes i$.

[12] *Answer:* $\mathcal{C} : (v_1, v_2) \to (v_1, -v_2)$. Check that this anticommutes with I.

Exercise

Suppose V is a complex vector space with a real structure C and that V_+ is the real vector space of fixed points of C.

Show that, as complex vector spaces

$$V \cong V_+ \otimes_{\mathbb{R}} \mathbb{C}. \tag{7.54}$$

7.4 The quaternions and quaternionic vector spaces

If V is a complex vector space then the complex vector space

$$V \oplus \bar{V} \tag{7.55}$$

has some interesting extra structure. Of course, it is a complex vector space, so it has multiplication by I:

$$I : (v_1, \overline{v_2}) \mapsto (iv_1, i\overline{v_2}) = (iv_1, -\overline{iv_2}) \tag{7.56}$$

But now, let us introduce another operator J

$$J : (v_1, \overline{v_2}) \mapsto (-v_2, \overline{v_1}) \tag{7.57}$$

Note that

1. $J^2 = -1$
2. $IJ + JI = 0$. So J is $\mathbb{C}$-anti-linear.

Whenever we have a vector space with two independent operators I and J with

$$I^2 = -1 \qquad J^2 = -1 \qquad IJ + JI = 0 \tag{7.58}$$

we get a third: $K := IJ$. Note that

$$I^2 = -1 \qquad J^2 = -1 \qquad K^2 = -1 \tag{7.59}$$

$$IJ + JI = JK + KJ = KI + IK = 0 \tag{7.60}$$

These are the abstract relations of the quaternions. To put this in proper context recall the definition:

Definition An algebra $\mathcal{A}$ over a field κ is a κ-vector space together with a κ-bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Concretely, this means that there is a multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ written $a \cdot b$ for $a, b \in \mathcal{A}$ such that

1. $a \cdot (b + c) = a \cdot b + a \cdot c$,
2. $(b + c) \cdot a = b \cdot a + c \cdot a$,

3. $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$, for $\alpha \in \kappa$.

If there is a multiplicative unit $\mathcal{A}$ is called *unital*. If $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ then $\mathcal{A}$ is called associative.

A good example of an algebra over κ is End(V) where V is a vector space over κ. Choosing a basis we can identify this with the set of $n \times n$ matrices over κ. We will see many more examples.

Definition The *quaternion algebra* $\mathbb{H}$ is the algebra over $\mathbb{R}$ with generators $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfying the relations

$$\mathfrak{i}^2 = -1 \qquad \mathfrak{j}^2 = -1 \qquad \mathfrak{k}^2 = -1 \tag{7.61}$$

$$\mathfrak{i}\mathfrak{j} + \mathfrak{j}\mathfrak{i} = \mathfrak{i}\mathfrak{k} + \mathfrak{k}\mathfrak{i} = \mathfrak{j}\mathfrak{k} + \mathfrak{k}\mathfrak{j} = 0 \tag{7.62}$$

The quaternions form a four-dimensional algebra over $\mathbb{R}$, as a vector space we can write

$$\mathbb{H} = \mathbb{R}\mathfrak{i} \oplus \mathbb{R}\mathfrak{j} \oplus \mathbb{R}\mathfrak{k} \oplus \mathbb{R} \cong \mathbb{R}^4 \tag{7.63}$$

The algebra is associative, but noncommutative. It has a rich and colorful history, which we will not recount here. Note that if we denote a generic quaternion by

$$q = x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} + x_4 \tag{7.64}$$

then we can define the conjugate quaternion by the equation

$$\bar{q} := -x_1\mathfrak{i} - x_2\mathfrak{j} - x_3\mathfrak{k} + x_4 \tag{7.65}$$

and

$$q\bar{q} = \bar{q}q = x_\mu x_\mu \tag{7.66}$$

Definition: A *quaternionic vector space* is a vector space V over $\kappa = \mathbb{R}$ together with three real linear operators $I, J, K \in \mathrm{End}(V)$ satisfying the quaternion relations. In other words, it is a real vector space which is a module for the quaternion algebra.

Just as we can have a complex structure on a real vector space, so we can have a *quaternionic structure* on a complex vector space V. This is a $\mathbb{C}$-anti-linear operator K on V which squares to -1. Once we have $K^2 = -1$ we can combine with the operator I which is just multiplication by $\sqrt{-1}$, to produce $J = KI$ and then we can check the quaternion relations. The underlying real space $V_{\mathbb{R}}$ is then a quaternionic vector space.

It is possible to put a quaternionic Hermitian structure on a quaternionic vector space and thereby define the quaternionic unitary group. Alternatively, we can define $U(n, \mathbb{H})$ as the group of $n \times n$ matrices over $\mathbb{H}$ such that $uu^\dagger = u^\dagger u = 1$. In order to define the conjugate-transpose matrix we use the quaternionic conjugation $q \to \bar{q}$ defined above.

Exercise
Show that $U(1, \mathbb{H}) \cong SU(2)$

Exercise

a.) Show that a

$$\mathfrak{i} \to \sqrt{-1}\sigma^1 \qquad \mathfrak{j} \to -\sqrt{-1}\sigma^2 \qquad \mathfrak{k} \to \sqrt{-1}\sigma^3 \tag{7.67}$$

defines a set of 2×2 complex matrices satisfying the quaternion algebra. Under this mapping a quaternion q is identified with a 2×2 complex matrix

$$q \to \rho(q) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \tag{7.68}$$

with $z = x_4 + ix_3$ and $w = x_2 + ix_1$.

b.) Show that $\det(\rho(q)) = q\bar{q} = x_\mu x_\mu$ and use this to define a homomorphism $SU(2) \times SU(2) \to SO(4)$.

Exercise *Complex structures on* $\mathbb{R}^4$

a.) Show that the complex structures on $\mathbb{R}^4$ compatible with the Euclidean metric can be identified as the maps

$$q \mapsto nq \qquad\qquad n^2 = -1 \tag{7.69}$$

OR

$$q \mapsto qn \qquad\qquad n^2 = -1 \tag{7.70}$$

b.) Use this to show that the space of such complex structures is $S^2 \amalg S^2$.

c.) Explain the relation to $O(4)/U(2)$.

Exercise *A natural sphere of complex structures*

Show that if V is a quaternionic vector space with complex structures I, J, K then there is a natural sphere of complex structures give by

$$\mathcal{I} = x_1 I + x_2 J + x_3 K \qquad x_1^2 + x_2^2 + x_3^2 = 1 \tag{7.71}$$

Exercise *Regular representation*

♣This is too important to be an exercise, and is used heavily later. ♣

Compute the left and right regular representations of $\mathbb{H}$ on itself Choose a real basis for $\mathbb{H}$ with $v_1 = \mathfrak{i}, v_2 = \mathfrak{j}, v_3 = \mathfrak{k}, v_4 = 1$. Let $L(q)$ denote left-multiplication by a quaternion q and $R(q)$ right-multiplciation by q. Then the representation matrices are:

$$L(\mathfrak{q})v_a := \mathfrak{q} \cdot v_a := L(\mathfrak{q})_{ba} v_b \tag{7.72}$$

$$R(\mathfrak{q})v_a := v_a \cdot \mathfrak{q} := R(\mathfrak{q})_{ba} v_b \tag{7.73}$$

a.) Show that:

$$L(\mathfrak{i}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{7.74}$$

$$L(\mathfrak{j}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{7.75}$$

$$L(\mathfrak{k}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.76}$$

$$R(\mathfrak{i}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{7.77}$$

$$R(\mathfrak{j}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{7.78}$$

$$R(\mathfrak{k}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.79}$$

b.) Show that these matrices generate the full 16-dimensional algebra $M_4(\mathbb{R})$.

Exercise *'t Hooft symbols and the regular representation of* $\mathbb{H}$

The famous 't Hooft symbols, introduced by 't Hooft in his work on instantons in gauge theory are defined by

$$\alpha^{\pm,i}_{\mu\nu} := \frac{1}{2}(\pm\delta_{i\mu}\delta_{\nu 4} \mp \delta_{i\nu}\delta_{\mu 4} + \epsilon_{i\mu\nu}) \tag{7.80}$$

where $1 \leq \mu, \nu \leq 4$

a.) Show that

$$\alpha^{+,1} = \frac{1}{2}R(\mathfrak{i}) \qquad \alpha^{+,2} = \frac{1}{2}R(\mathfrak{j}) \qquad \alpha^{+,3} = \frac{1}{2}R(\mathfrak{k}) \tag{7.81}$$

$$\alpha^{-,1} = -\frac{1}{2}L(\mathfrak{i}) \qquad \alpha^{-,2} = -\frac{1}{2}L(\mathfrak{j}) \qquad \alpha^{-,3} = -\frac{1}{2}L(\mathfrak{k}) \tag{7.82}$$

b.) Verify the relations

$$\begin{aligned} [\alpha^{\pm,i}, \alpha^{\pm,j}] &= -\epsilon^{ijk}\alpha^{\pm,k} \\ [\alpha^{\pm,i}, \alpha^{\mp,j}] &= 0 \\ \{\alpha^{\pm,i}, \alpha^{\pm,j}\} &= -\frac{1}{2}\delta^{ij} \end{aligned} \tag{7.83}$$

So

$$\begin{aligned} \alpha^{+,i}\alpha^{+,j} &= -\frac{1}{4}\delta^{ij} - \frac{1}{2}\epsilon^{ijk}\alpha^{+,k} \\ \alpha^{-,i}\alpha^{-,j} &= -\frac{1}{4}\delta^{ij} - \frac{1}{2}\epsilon^{ijk}\alpha^{-,k} \end{aligned} \tag{7.84}$$

Exercise

It is also sometimes useful to identify $\mathbb{H} \cong \mathbb{C}^2$ by choosing the complex structure to be $L(\mathfrak{i})$. Thus we can write $\mathfrak{q} = z_1 + z_2\mathfrak{j}$ where $z_1 = x_1 + \mathfrak{i}y_1$ and $z_2 = x_2 + \mathfrak{i}y_2$ with x_i, y_i real.

a.) Show that $L(\mathfrak{j})$ acts by

$$L(\mathfrak{j}) : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{7.85}$$

b.) Show that $R(\mathfrak{q})$ act $\mathbb{C}$-linearly, and hence can be represented as 2×2 matrices acting from the left:

$$R(\mathfrak{i}) = \begin{pmatrix} \mathfrak{i} & 0 \\ 0 & -\mathfrak{i} \end{pmatrix} \tag{7.86}$$

$$R(\mathfrak{j}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.87}$$

$$R(\mathfrak{k}) = \begin{pmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{pmatrix} \tag{7.88}$$

7.5 Summary

To summarize we have described three basic structures we can put on vector spaces:

♣Actually, there are four. We can have a complex structure on a quaternionic space. We should also derive the the moduli spaces of all four cases as eq:ClassCar recorded in (C.14) eq:ClassCartSpace to (C.17). These are used later. ♣

1. A *complex structure* on a real vector space W is a real linear map $I : W \to W$ with $I^2 = -1$.

2. A *real structure* on a complex vector space V is a $\mathbb{C}$-anti-linear map $K : V \to V$ with $K^2 = +1$.

3. A *quaternionic structure* on a complex vector space V is a $\mathbb{C}$-anti-linear map $K : V \to V$ with $K^2 = -1$.

Exercise *Tensor algebras and real and quaternionic structures*

Suppose V is a complex vector space.

a.) Show that if V has a real structure then it induces a natural real structure on $V^{\otimes n}$. Moreover, each of the fixed symmetry types under S_n (i.e. the isotypical subspaces under the symmetric group) have a real structure.

b.) Show that if V has a quaternionic structure then it naturally induces a real structure on $V^{\otimes n}$ for n even and a quaternionic structure on $V^{\otimes n}$ for n odd.

---

8. ϕ-twisted representations

c:PhiTwistReps

Wigner's theorem is the source of the importance of group representation theory in physics. In these notes we are emphasizing the extra details coming from the fact that in general some symmetry operators are represented as $\mathbb{C}$-antilinear operators. In this section we summarize a few of the differences from standard representation theory.

8.1 Some definitions

c:PhiRepBasics

There are some fairly straightforward definitions generalizing the usual definitions of group representation theory.

Definitions:

1. A *ϕ-representation* (or *ϕ-rep* for short) of a $\mathbb{Z}_2$-graded group (G, ϕ) is a complex vector space V together with a homomorphism

$$\rho : G \to \mathrm{End}(V_{\mathbb{R}}) \tag{8.1}$$

such that

$$\rho(g) = \begin{cases} \mathbb{C} - \text{linear} & \phi(g) = +1 \\ \mathbb{C} - \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \tag{8.2}$$

2. An *intertwiner* or *morphism* between two ϕ-reps (ρ_1, V_1) and (ρ_2, V_2) is a $\mathbb{C}$-linear map $T : V_1 \to V_2$, i.e., $T \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, which commutes with the G-action:

$$T\rho_1(g) = \rho_2(g)T \qquad \forall g \in G \tag{8.3}$$

   We write $\mathrm{Hom}_{\mathbb{C}}^G(V_1, V_2)$ for the set of all intertwiners.

3. An *isomorphism of ϕ-reps* is an intertwiner T which is an isomorphism of complex vector spaces.

4. A ϕ-rep is said to be *ϕ-unitary* if V has a nondegenerate sesquilinear pairing such that $\rho(g)$ is an isometry for all g. That is, it is unitary or anti-unitary according to whether $\phi(g) = +1$ or $\phi(g) = -1$, respectively.

5. A ϕ-rep (ρ, V) is said to be *reducible* if there is a proper (i.e. nontrivial) ϕ-sub-representation. That is, if there is a complex vector subspace $W \subset V$, with W not $\{0\}$ or V which is G-invariant. If it is not reducible it is said to be *irreducible*.

Remarks:

1. In our language, then, what we learn from Wigner's theorem is that if we have a quantum symmetry group $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ then there is a $\mathbb{Z}_2$-graded extension (G^{tw}, ϕ) and the Hilbert space is a ϕ-representation of (G^{tw}, ϕ). In general we will refer to a ϕ-representation of some extension (G^{tw}, ϕ) of (G, ϕ) as a *ϕ-twisted representation of G*.

2. In the older literature of Wigner and Dyson the term "corepresentation" for a ϕ-unitary representation is used, but in modern parlance the name "corepresentation" has several inappropriate connotations, so we avoid it. The term "ϕ-representation" is not standard, but it should be.

3. If G is a compact group it has a left- and right-invariant Haar measure. Using this one can show that any ϕ-rep on an inner product space is unitarizable. That is, by choosing an appropriate basis one can make all the operators $\rho(g)$ unitary or anti-unitary. The way to show this is that if $h^{(1)}$ is the original inner product on V then we define a new inner product by

$$h^{(2)}(v_1, v_2) := \int_G [dg] h^{(1)}(\rho(g)v_1, \rho(g)v_2) \tag{8.4}$$

   and it is straightforward to see that the rep is ϕ-unitary with respect to $h^{(2)}$.

4. An important point below will be that $\mathrm{Hom}_{\mathbb{C}}^G(V_1, V_2)$ is, *a priori* only a *real* vector space. If T is an intertwiner the iT certainly makes sense as a linear map from V_1 to V_2 but if any of the $\rho(g)$ are anti-linear then iT will *not* be an intertwiner. Of course, if the $\mathbb{Z}_2$-grading ϕ of G is trivial and $\phi(g) = 1$ for all g then $\mathrm{Hom}_{\mathbb{C}}^G(V_1, V_2)$ admits a natural complex structure, namely $T \to iT$.

Example: Let us consider the ϕ-twisted representations of $M_2 = \{1, \bar{T}\}$ where $\phi(\bar{T}) = -1$. We showed above that there are precisely two ϕ-twisted extensions $M_2^{\pm}$. First, let us suppose $\mathcal{H}$ is a ϕ-rep of M_2^+. Then set

$$K = \rho(T). \tag{8.5}$$

This operator is anti-linear and squares to $+1$. Therefore K is a real structure on $\mathcal{H}$. On the other hand, if the ϕ-twisted extension of M_2 is M_2^- then $K^2 = -1$. Therefore we have a quaternionic structure on $\mathcal{H}$. Thus we conclude: *The ϕ-twisted representations of (M_2, ϕ), with $\phi(\bar{T}) = -1$ are the complex vector spaces with a real structure (for M_2^+) union the complex vector spaces with a quaternionic structure (for M_2^-).*

Exercise *ϕ-reps and $\mathbb{Z}_2$-gradings*

a.) Show that a ϕ-representation of (G, ϕ) can be defined as a real vector space W with a complex structure I and a homomorphism

$$\rho : G \to \mathrm{End}(W) \tag{8.6}$$

such that

$$\rho(g)I = \phi(g)I\rho(g) \tag{8.7}$$

b.) Show that if (W, I) is a real vector space with a complex structure then conjugation by I defines a $\mathbb{Z}_2$-grading on $\mathrm{End}(W)$ and on the group $\mathrm{Aut}(G)$ so that a ϕ-rep is a homomorphism of $\mathbb{Z}_2$-graded groups. This leads to a mathematically more sophisticated viewpoint on ϕ-reps.

---

subsec:ShurPhi

8.2 Schur's Lemma for ϕ-reps

While many of the standard notions and constructions of representation theory carry over straightforwardly to the theory of ϕ-reps, sometimes they come with very interesting new twists. A good example of this is Schur's lemma.

One very important fact for us below will be the analog of Schur's lemma. To state it correctly we recall a basic definition:

Definition An *associative division algebra* over a field κ is an associative unital algebra $\mathcal{A}$ over κ such that for every nonzero $a \in \mathcal{A}$ there is a multiplicative inverse $a^{-1} \in \mathcal{A}$, i.e. $aa^{-1} = a^{-1}a = 1$.

Then we have

Theorem [Schur's Lemma].

a.) If A is an intertwiner between two irreducible ϕ-reps (ρ, V) and (ρ', V') then either $A = 0$ or A is an isomorphism.

b.) Suppose (ρ, V) is an irreducible ϕ-representation of (G, ϕ). Then the commutant, that is, the set of all intertwiners A of (ρ, V) with itself:

$$Z(\rho, V) := \{A \in \mathrm{End}_{\mathbb{C}}(V) | \forall g \in G \qquad A\rho(g) = \rho(g)A\} \tag{8.8}$$

is a real associative division algebra.

Proof:

Part a: Suppose $A \in \mathrm{Hom}_{\mathbb{C}}^{G}(V,V')$. Then $\ker(A) \subset V$ is a sub-ϕ-representation of V and also Im $(A) \subset V'$ is a sub-ϕ-rep of V'. Since V is irreducible it must be that one of the following is true:

- $\ker(A) = 0$
- $\ker(A) = V$

If $\ker(A) = V$ then $A = 0$. So, if $A \neq 0$ then $\ker(A) = 0$. Moreover Im $(A) \subset V'$ is nonzero. Since V' is irreducible it follows that Im $(A) = V'$. Therefore A is an isomorphism of ϕ-reps.

Part b: Now suppose that A is an interwiner of (ρ, V) with itself. If $A \neq 0$ then $\ker(A) = 0$, which means that A is invertible. Since $Z(\rho, V)$ is a subalgebra of an associative algebra it is also associative. Therefore $Z(\rho, V)$ is an associative division algebra over the field $\kappa = \mathbb{R}$. As we remarked above, even though A is $\mathbb{C}$-linear the ground field must be considered to be $\mathbb{R}$ and not $\mathbb{C}$ because some elements $\rho(g)$ might be $\mathbb{C}$-anti-linear, so if $A \in Z(\rho, V)$ it does *not* follow that $iA \in Z(\rho, V)$. $\diamondsuit$

Schur's lemma for ϕ-representations naturally raises the question of finding examples of real division algebras. In fact, there are only three. This is the very beautiful theorem of Frobenius:

Theorem: If $\mathcal{A}$ is a finite dimensional [13] real associative division algebra then one of three possibilities holds:

- $\mathcal{A} \cong \mathbb{R}$
- $\mathcal{A} \cong \mathbb{C}$
- $\mathcal{A} \cong \mathbb{H}$

Proof: Let D be a real, associative division algebra. Given $a \in D$ we can form $L(a) \in \mathrm{End}(D)$, defined by

$$L(a) : b \mapsto a \cdot b \tag{8.9}$$

Let $V := \{a | \mathrm{Tr}(L(a)) = 0\}$. Then $D \cong \mathbb{R} \oplus V$, separates D into the traceless and trace parts. Now we need a little

Lemma: $V = \{a \in D | a^2 \leq 0\}$.

Proof of Lemma: If $a \neq 0$ consider the characteristic polynomial of $L(a)$

$$p_a(x) := \det(x - L(a)). \tag{8.10}$$

This polynomial has real coefficients and therefore has a factorization over $\mathbb{C}$ which we can write as

$$p_a(x) = \prod_i (x - r_i) \prod_\alpha (x - z_\alpha)(x - \bar{z}_\alpha) \tag{8.11}$$

[13] The Gelfand-Mazur theorem asserts that *any* unital Banach algebra over $\mathbb{R}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

where r_i are the real roots and z_α are a collection of roots which are not real, so that all non-real roots can be arranged in complex conjugate pairs. Thanks to the Cayley-Hamilton theorem we know that $p_a(a) = 0$. But since D is a division algebra this must mean that:

$$a - r_i = 0 \tag{8.12}$$

eq:root-r

for some i, *OR*

$$a^2 - 2\mathrm{Re}(z_\alpha)a + |z_\alpha|^2 = 0 \tag{8.13}$$

eq:root-cc

for some α.

Note that in the case (8.13) we must use the second-order polynomial with real coefficients rather than the first-order polynomial with complex coefficients since the division algebra is over the real numbers. Now, if we are in the case (8.12) then $\mathrm{Tr}(L(a)) \neq 0$ so to prove the Lemma we assume we are in the case (8.13). Moreover, this equation cannot hold for two different values of α, otherwise we would subtract the two equations and reduce to the case of (8.12). Therefore, the characteristic polynomial of $L(a)$ is of the form:

$$p_a(x) = (x^2 - 2\mathrm{Re}(z)x + |z|^2)^m \tag{8.14}$$

for some non-real complex number z and some positive integer m. Now, recall that the coefficient of x^{2m-1} must be $-\mathrm{Tr}(L(a))$. Since we are assuming this is zero we must have $\mathrm{Re}(z) = 0$ and hence $a^2 = -|z|^2 < 0$. This proves the Lemma $\diamondsuit$.

Now, note that $Q(a, b) := -ab - ba$ is a positive definite form on V since $Q(a, b) = a^2 + b^2 - (a + b)^2$ and hence $Q(a, a) = -2a^2 \geq 0$ on V. If $D \neq \mathbb{R}$ so that V is nonzero then the quadratic form $Q(a, a)$ on V is positive definite over $\mathbb{R}$ we can diagonalize it to the form $2\delta_{ij}$. Therefore, we can choose a basis $\{e_i\}_{i=1,\dots,N}$ for V such that

$$e_i e_j + e_j e_i = -2\delta_{ij} \qquad\qquad 1 \leq i, j \leq n \tag{8.15}$$

eq:NegDefCliff

Now, we can choose a *minimal* set of generators of the algebra from the set $\{e_i\}_{i=1,\dots,N}$. (The trace part is generated by squaring any e_i so we do not need to include any element of $\mathbb{R}$ to generate the algebra.) Without loss of generality we can say that the first n elements of the basis constitute a *minimal* set of generators. Thus, we have algebraically independent elements $e_i \in V$ with $1 \leq i \leq n$, $n \leq N$, satisfying (8.15). These are the defining relations of the real *Clifford algebra*, $C\ell_{-n}$, something we will study at length later on.

For $n > 2$ we note that [14]

$$(1 + e_1e_2e_3)(1 - e_1e_2e_3) = 0 \tag{8.16}$$

Since D is a division algebra this means we must have $e_1e_2e_3 = \pm 1$, and hence $e_3 = \pm e_1e_2$. But we assumed we had a minimal set of generators. So we have reached a contradiction and hence $n = 1, 2$ are the only possibilities other than $D = \mathbb{R}$.

For $n = 1, 2$ we can check explicitly that $D \cong \mathbb{C}$ or $D \cong \mathbb{H}$ as real algebras: For $n = 1$ the general element is $x_1 + ex_2$ where x_1, x_2 are real. The identification with $x_1 + \sqrt{-1}x_2$

[14] It is precisely at this point that we use the hypothesis that D is associative.

is an isomorphism with $\mathbb{C}$. Similarly, the generators e_1, e_2 and $e_1 e_2$ can be mapped to $\mathfrak{i}, \mathfrak{j}$ and $\mathfrak{k}$, respectively to define an isomorphism of the case $n = 2$ with $\mathbb{H}$. ♢♢

Examples

1. Let $G = M_2$ with $\phi(\bar{T}) = -1$. Take $V = \mathbb{C}$, $\rho(\bar{T}) = \mathcal{C} \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ given by complex conjugation $\mathcal{C}(z) = \bar{z}$. Then $Z(\rho, V) = \mathbb{R}$.

2. Let $G = U(1)$ with $\phi = 1$, so the grading is trivial (all even). Let $V = \mathbb{C}$ and $\rho(z)v = zv$. Then $Z(\rho, V) = \mathbb{C}$. Notice we could replace G with any subgroup of multiplicative n^{th} roots of 1 in this example, so long as $n > 2$.

3. Let $G = M_2^-$, with $\phi(T) = -1$. Take $V = \mathbb{C}^2$ and represent

$$\rho(e^{i\theta}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i\theta} z_1 \\ e^{i\theta} z_2 \end{pmatrix} \tag{8.17}$$

$$\rho(T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{8.18}$$

One checks that these indeed define a ϕ-representation of M_2^-. We claim that in this case $Z(\rho, V) \cong \mathbb{H}$.

To prove the claim let us map $\mathbb{C}^2 \to \mathbb{H}$ by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_1 + z_2 \mathfrak{j} = (x_1 + \mathfrak{i} y_1) + (x_2 + \mathfrak{i} y_2)\mathfrak{j} \tag{8.19}$$

Thus, if we think of the ϕ-twisted representation as acting on the quaternions then we have:

$$\rho(e^{i\theta}) = \cos\theta + \sin\theta L(\mathfrak{i}) \tag{8.20}$$

and T is represented by

$$\rho(T) = L(\mathfrak{j}) \tag{8.21}$$

Of course, the commutant algebra $Z(\rho, V)$ must commute with the real algebra generated by all the elements $\rho(g)$. Therefore, it must commute with left-multiplication by arbitrary quaternions. From this it easily follows that $Z(\rho, V)$ is the algebra of *right-multiplication* by arbitrary quaternions. In fact this identifies $Z(\rho, V) \cong \mathbb{H}^{\mathrm{opp}}$ but as you show in an exercise below $\mathbb{H}^{\mathrm{opp}} \cong \mathbb{H}$ as a real algebra. ♣Perhaps put this general remark earlier. ♣

so the algebra of operators generated by $\rho(g)$ is the quaternion algebra acting on $V = \mathbb{R}^4$ as the left regular representation. The commutant of these operations is therefore *right*-multiplication by any quaterion $R(q)$, and hence $Z \cong \mathbb{H}$.

Remarks:

1. The above argument shows that the only division algebras over the complex numbers is $\mathbb{C}$ itself. The only change in the proof is that the characteristic polynomial $p_a(x)$ factorizes and $a - z_\alpha = 0$ for some root. Therefore, the traceless part of the algebra vanishes and hence $D \cong \mathbb{C}$.

2. One can drop the associativity condition in the definition of a division algebra by modifying the defining property to the statement that if $a \neq 0$ then the equation $ax = y$ for any y has a unique solution $x = by$ for some b. Then a theorem from topology (due to Kervaire and Bott-Milnor) says that the only division algebras over $\mathbb{R}$ have real dimensions $1, 2, 4, 8$. Moreover, a theorem of Hurwitz says that the normed division algebras over $\mathbb{R}$ (i.e. those with a norm so that $\parallel ab \parallel = \parallel a \parallel \parallel b \parallel$) are precisely $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and just one more finite dimensional division algebra over $\mathbb{R}$, namely the octonions $\mathbb{O}$. This has dimension 8 and can be constructed from a kind of doubling of the quaternions. The dimensions of the divison algebras $1, 2, 4, 8$ are related to the dimensions in which minimal supersymmetric Yang-Mills theory can exist: $3, 4, 6, 10$.

3. Note that the $\mathbb{C}$-linear map $v \to iv$ is in $Z(\rho, V)$ iff $\phi(g) = 1$ for all g. If $i \in Z(\rho, V)$ then $Z(\rho, V)$ is in fact a division algebra over $\mathbb{C}$, and hence must be isomorphic to $\mathbb{C}$. Thus, we recover Schur's lemma for ordinary irreps of G over $\mathbb{C}$ as a special case: If $\phi = 1$ then $Z(\rho, V) \cong \mathbb{C}$ given by $v \to zv$. Warning: It is possible to have $Z(\rho, V) \cong \mathbb{C}$ even when $\phi \neq 1$ and hence $i \notin Z(\rho, V)$.

Exercise

If e_i satisfy the defining relations (eq:NegDefCliff 8.15) of $C\ell_{-n}$ show that

$$(e_i e_j)^2 = -1 \qquad \forall i \neq j \tag{8.22}$$

$$(e_i e_j e_k)^2 = +1 \qquad \forall i \neq j \neq k \neq i \tag{8.23}$$

Exercise

The quaternions $\mathbb{H}$ form a division algebra over $\mathbb{R}$. Therefore $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ is an algebra over $\mathbb{C}$. It has 4 dimensions as a complex algebra. Why is it not a division algebra?

Exercise *Opposite algebra*

If $\mathcal{A}$ is an algebra then we define the opposite algebra $\mathcal{A}^{\rm opp}$ to be the same vector space as $\mathcal{A}$ over the field κ but the multiplication $m^{\rm opp} : \mathcal{A}^{\rm opp} \times \mathcal{A}^{\rm opp} \to \mathcal{A}^{\rm opp}$ is related to the multiplication $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$m^{\rm opp}(a,b) := m(b,a) \tag{8.24}$$

Show that $\mathbb{H}^{\rm opp}$ is isomorphic to $\mathbb{H}$. [15]

Exercise

Consider $G = \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^2)$ acting on $\mathbb{C}^2$ in the standard way. Show that $Z \cong \mathbb{R}$ if $\phi = \phi_{\mathcal{H}}$ is the canonical $\mathbb{Z}_2$-grading while $Z \cong \mathbb{C}$ if $\phi = 1$.

8.3 Complete Reducibility

A very important theorem in ordinary representation theory is the complete reducibility of representations of compact groups. This extends more or less directly to ϕ-reps.

If (G,ϕ) is a $\mathbb{Z}_2$-graded group then $(G,\phi)^\vee$, known as the "dual," is the set of inequivalent irreducible ϕ-representations of G. For each element of $\lambda \in (G,\phi)^\vee$ we select a representative irrep V_λ. Thanks to Schur's lemma it is unique up to isomorphism.

Theorem: If (ρ, V) is a finite-dimensional ϕ-unitary rep of (G,ϕ) then V is isomorphic to a representation of the form

$$\oplus_{\lambda \in (G,\phi)^\vee} W_\lambda \tag{8.25}$$

eq:CompleteReduci

where, for each λ, W_λ is itself (noncanonically) isomorphic to a direct sum of representations V_λ:

$$W_\lambda \cong \underbrace{V_\lambda \oplus \cdots \oplus V_\lambda}_{s_\lambda \quad \text{times}} \tag{8.26}$$

eq:CompleteReduci

(If $s_\lambda = 0$ this is the zero vector space.)

Proof: The proof is a simple consequence of the following lemma: Suppose that $W \subset V$ is a ϕ-sub-rep of V. Then we claim that

$$W^\perp = \{w' | (w,w') = 0 \qquad \forall w \in W\} \tag{8.27}$$

is also a ϕ-sub-rep of V. This is simple because if $w' \in W^\perp$ then for all $g \in G$ and all $w \in W$

$$(w, \rho(g)w') = (\rho(g^{-1})w, w') = 0 \tag{8.28}$$

Therefore, choose any nonzero vector $v \in V$ and let $W(v)$ be the smallest G-invariant subspace containing v. This must be an irrep. Now consider $W(v)^\perp$ and choose a nonzero

[15] *Answer*: Take $q \to \bar{q}$

vector in that space (if it exists) and repeat. Because V is finite-dimensional, after some number of steps the subspace

$$(W(v_1) \oplus W(v_2) \oplus \cdots \oplus W(v_n))^{\perp} \tag{8.29}$$

must in fact be zero and the procedure stops. By arranging the summands into subsets corresponding to the isomorphism class λ we arrive at (8.25), (8.26). $\diamondsuit$

Remarks

1. The isomorphism of a representation with (8.25), (8.26) is known as an *isotypical decomposition.* The nonnegative integers s_λ are known as degeneracies.

2. Concretely the theorem means that we can choose a "block-diagonal" basis for V so that relative to this basis the matrix representation of $\rho(g)$ has the form

$$\rho(g) \sim \begin{pmatrix} \ddots & & \\ & 1_{s_\lambda} \otimes \rho_\lambda(g) & \\ & & \ddots \end{pmatrix} \tag{8.30}$$

   We need to be careful about how to interpret $\rho_\lambda(g)$ because anti-linear operators don't have a matrix representation over the complex numbers. If we are working with ordinary representations over $\mathbb{C}$ and $\dim_{\mathbb{C}} V_\lambda = t_\lambda$ then $1_{s_\lambda} \otimes \rho_\lambda(g)$ means a matrix of the form

$$1_{s_\lambda} \otimes \rho_\lambda(g) = \begin{pmatrix} \rho_\lambda(g) & 0 & 0 \cdots & 0 \\ 0 & \rho_\lambda(g) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & \rho_\lambda(g) \end{pmatrix} \tag{8.31}$$

eq:BlockMatrix

   where $\rho_\lambda(g)$ and each of the 0's above is a $t_\lambda \times t_\lambda$ matrix and there is an $s_\lambda \times s_\lambda$ matrix of such blocks. On the other hand, if $\rho(g)$ is anti-linear then it does not have a matrix representation over the complex numbers. If we wish to work with matrix representations what we must do is work with $(V_{\mathbb{R}}, I)$ where I is a complex structure on $V_{\mathbb{R}}$, and similarly for the irreps $(V_{\lambda,\mathbb{R}}, I_\lambda)$. Then $\rho_\lambda(g)$ means a real representation matrix which is $2t_\lambda \times 2t_\lambda$ and anticommutes with I_λ. See the beginning of §8.4 for a specific way to do this.

3. Equation (8.26) is noncanonical. What this means is that in the isomorphism (8.26) one could compose with an isomorphism that mixes the summands. Put differently, one could change basis in (8.31) by a matrix of the form $S \otimes 1_{t_\lambda}$ with S an invertible $s_\lambda \times s_\lambda$ matrix. However, we would like to stress that the decomposition (8.25) is completely canonical. We can define W_λ to be the image of the map

$$\mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, V) \otimes V_\lambda \to V \tag{8.32}$$

given by the evaluation map

$$T \otimes v \mapsto T(v) \tag{8.33}$$

Note that the G-action on the left-hand side is $g : T \otimes v \mapsto T \otimes \rho(g)v$ and on the right-hand side $g : T(v) \mapsto g \cdot T(v)$. Hence the evaluation map is an intertwiner. Therefore, the canonical way to write the isotypical decomposition is

$$V \cong \oplus_\lambda \mathrm{Hom}_{\mathbb{C}}^G(V_\lambda, V) \otimes_{\mathbb{R}} V_\lambda \tag{8.34}$$

Recall that $\mathrm{Hom}_{\mathbb{C}}^G(V_\lambda, V)$ is a real vector space, while V_λ is a complex vector space. We therefore take the tensor product over $\mathbb{R}$ regarding V_λ as a real vector space but the result of the tensor product is naturally a complex vector space.

4. Now if we combine this canonical formulation of the isotypical decomposition with the second part of Schur's lemma to compute the real algebra of self-endomorphisms $\mathrm{End}_{\mathbb{C}}^G(V)$. To lighten the notation let $S_\lambda := \mathrm{Hom}_{\mathbb{C}}^G(V_\lambda, V)$ and let D_λ be the real division algebra over $\mathbb{R}$ of self-intertwiners of V_λ. Then we compute:

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(V, V) &\cong V^* \otimes_{\mathbb{C}} V \\
&\cong \oplus_{\lambda,\lambda'} (S_\lambda^* \otimes_{\mathbb{R}} S_{\lambda'}) \otimes_{\mathbb{R}} (V_\lambda^* \otimes_{\mathbb{C}} V_{\lambda'}) \\
&\cong \oplus_{\lambda,\lambda'} \mathrm{Hom}(S_\lambda, S_{\lambda'}) \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\lambda'})
\end{aligned} \tag{8.35}$$

Now G acts trivially on the $\mathrm{Hom}(S_\lambda, S_{\lambda'})$ factors and in the natural way on $\mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\lambda'})$. Therefore, taking the G-invariant part to get the intertwiners we invoke Schur's lemma

$$\mathrm{Hom}_{\mathbb{C}}^G(V_\lambda, V_{\lambda'}) = \delta_{\lambda,\lambda'} D_\lambda \tag{8.36}$$

and hence

$$\mathrm{Hom}_{\mathbb{C}}^G(V, V) \cong \oplus_\lambda \mathrm{End}(S_\lambda) \otimes_{\mathbb{R}} D_\lambda \tag{8.37}$$

Of course, $\mathrm{End}(S_\lambda)$ is isomorphic to the algebra of real matrices $Mat_{s_\lambda}(\mathbb{R})$ upon choosing a basis and therefore

$$\mathrm{End}_{\mathbb{C}}^G(V) \cong \oplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda) \tag{8.38}$$

is a direct sum of matrix algebras over real division algebras.

5. We proved complete reducibility for finite-dimensional ϕ-unitary reps. For G which is compact the result extends to infinite-dimensional representations. In fact, this is equivalent to the Peter-Weyl theorem. For a nice discussion see SegalLectures [36]. For noncompact groups the theorem can fail. For example the representation of $\mathbb{Z}$ or $\mathbb{R}$ on $\mathbb{R}^2$ given by matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{8.39}$$

is reducible but not completely reducible. The subspace W of vectors of the form

$$\begin{pmatrix} r \\ 0 \end{pmatrix} \tag{8.40}$$

is a nontrivial invariant subspace, but there is no complementary invariant subspace in $\mathbb{R}^2$.

bsec:ComRedAlg

8.4 Complete Reducibility in terms of algebras

The complete reducibility and commutant subalgebra can also be expressed nicely in terms of the group algebra $\mathbb{R}[G]$. We work with $V_{\mathbb{R}}$ with complex structure I with operators $\rho_{\mathbb{R}}(g)$ commuting or anticommuting with I according to $\phi(g)$. This defines a subalgebra of $\mathrm{End}(V_{\mathbb{R}})$. If G is compact this algebra can be shown to be semisimple and therefore, by a theorem of Wedderburn all representations are matrix representations by matrices over a division algebra over $\mathbb{R}$. See Appendix app:CentralSimple A for background on semisimple algebras.

It is useful to be explicit and make a choice of basis. Therefore, we choose a basis to identify $V \cong \mathbb{C}^N$. Then we identify $V_{\mathbb{R}} \cong \mathbb{R}^{2N}$ by mapping each coordinate

$$z \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.41}$$

The complex structure on $\mathbb{R}^{2N}$ is therefore

$$I_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8.42}$$

While the real structure of conjugation with respect to this basis is the operation

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.43}$$

Having chosen a basis $V \cong \mathbb{C}^N$ the $\mathbb{C}$-linear operators $\rho(g)$ with $\phi(g) = 1$ can identified with $N \times N$ complex matrices and then they are promoted to $2N \times 2N$ real matrices by replacing each complex matrix element by

$$z_{ij} \rightarrow \begin{pmatrix} x_{ij} & -y_{ij} \\ y_{ij} & x_{ij} \end{pmatrix} \tag{8.44}$$

The operators with $\phi(g) = -1$ must be represented by $\mathcal{C}$ times a matrix of the above type.

Now we want to describe the algebra $\rho(G)$ over $\mathbb{R}$ generated by the real $2N \times 2N$ matrices $\rho(g)$ together with I_0. To do this let us introduce some notation: If K is any algebra then mK will denote the algebra of $m \times m$ matrices over K of the specific form $\mathrm{Diag}\{k, k, \dots, k\}$. Thus mK and K are isomorphic as abstract algebras. Similarly, if K is any algebra we denote by $K[m]$ the algebra of all $m \times m$ matrices whose elements are in K. Note that $m(K[n])$ and $(mK)[n]$ are canonically isomorphic so we just write $mK[n]$ when we combine the two constructions. Finally, with this notation we can state the:

Theorem The algebra $\mathcal{A}(\rho(G), I) \subset \mathrm{End}(V_{\mathbb{R}})$ generated over $\mathbb{R}$ by the operators $\rho(g)$ and I is equivalent to

$$\mathcal{A}(\rho(G), I) \cong \oplus_{\lambda} s_{\lambda} D_{\lambda}[\tau_{\lambda}] \tag{8.45}$$

eq:WeylThm-1

and the commutant $Z(\rho, V)$ is equivalent to

$$Z(\rho, V) \cong \oplus_{\lambda} \tau_{\lambda} D_{\lambda}^{\mathrm{opp}}[s_{\lambda}] \tag{8.46}$$

eq:WeylThm-2

Note that the dimensions τ_λ are slightly different from the complex dimensions t_λ of V_λ in general. Let us denote the real dimension of D_λ by $d_\lambda = 1, 2, 4$ according to $D_\lambda = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then

$$\tau_\lambda = \frac{2}{d_\lambda} t_\lambda \tag{8.47}$$

Recall that when $D_\lambda \cong \mathbb{H}$ there must be an action of $\mathbb{H}$ on V and hence t_λ must be even, so τ_λ is always an integer, as it must be.

Remarks

1. We omit the proof, which may be found in Weyl's book. It amounts to the statement that $\mathbb{R}[G]$ is a semisimple algebra over $\mathbb{R}$ together with Wedderburn's theorem that any representation of a semisimple algebra over $\mathbb{R}$ is a direct sum of matrix algebras over a division algebra over $\mathbb{R}$.

2. To illustrate the reason that $D^{\rm opp}$ appears in the commutant consider the following representative example. Suppose we have an algebra such as $\mathbb{H}[m]$. If we represent this as real matrices then we must represent the quaternions $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ as real matrices. We do this using - say - the left regular representation. Hence each of the matrix elements is promoted to a 4×4 real matrix to make a $4m \times 4m$ real matrix. Thus we regard $\mathbb{H}[m] \subset \mathbb{R}[4m]$ as a subalgebra of matrices. We ask: What is the commutant of $\mathbb{H}[m]$ within $\mathbb{R}[4m]$? Some elements of the commutant are obvious, namely the matrices of the form ${\rm Diag}\{R(q), \dots, R(q)\}$ where $R(q)$ is the 4×4 matrix representing right-multiplication of q on quaternions. This represents rightmultiplication of an $m \times m$ matrix of quaternions by q. The theorem says that this is the full commutant. Note that since $R(q_1)R(q_2) = R(q_2 q_1)$ so that the commutant is more naturally regarded as $\mathbb{H}^{\rm opp}$.

8.5 Application: Classification of Irreps of G on a complex vector space

subsec:ComRedAlg

As an application of §8.4 we rederive the standard trichotomy of complex irreducible representations of a group. The question we want to address is this:

Suppose $\rho : G \to {\rm Aut}(V)$ is an ordinary irreducible representation of V. (That is, an irreducible unitary ϕ-rep with $\phi = 1$.) Then there is canonically a complex conjugate representation $(\bar\rho, \bar V)$. If we choose a basis for V so that $\rho(g)$ are complex matrices then $\rho(g)^*$ is also a representation. The conjugate representation $(\bar\rho, \bar V)$ is easily seen to be irreducible and the question is: *What is the relation between the original rep and its conjugate?*

To answer this we consider two real algebras. This first, denoted by A is the real algebra generated by the set of operators $\rho(G) \subset {\rm End}(V_{\mathbb{R}})$. The second, denoted by B is the algebra generated by A and I, the complex structure on $V_{\mathbb{R}}$. Both of these algebras are semisimple and hence the above theorem applies.

Because we have an irreducible representation Schur's lemma tells us that

$$B \cong \mathbb{C}[n] \tag{8.48}$$

and hence

$$Z(B) \cong n\mathbb{C} \tag{8.49}$$

Let us now consider A. There are two cases: $I \in A$ and $I \notin A$. If $I \in A$ then $A = B$ and so A has real dimension $2n^2$. If $I \notin A$ then B must have twice the dimension of A and hence A has real dimension n^2. The only possibilities compatible with Weyl's theorem above are

- $A = 2\mathbb{R}[n]$ and $Z(A) = n\mathbb{R}[2]$
- $A = \mathbb{C}[n]$ and $Z(A) = n\mathbb{C}$
- $A = \mathbb{H}[\frac{n}{2}]$ and $Z(A) = \frac{n}{2}\mathbb{H}^{\rm opp}$

We call the three cases above as type $\mathbb{R}, \mathbb{C}, \mathbb{H}$. In the cases where $Z(A)$ is of type $\mathbb{R}$ or $\mathbb{H}$ we can check by hand that there is an operator $P \in Z(A)$ with $PI = -IP$. Of course, $P \notin Z(B)$. On the other hand, since P is in $Z(A)$ we know that

$$P\rho(g)_{\mathbb{R}} = \rho(g)_{\mathbb{R}} P \tag{8.50}$$

But this means that in terms of complex matrices there is an invertible matrix S such that

$$\rho(g)^* = S\rho(g)S^{-1} \tag{8.51}$$

eq:UnitaryEquiv

Moreover, we can take S to be unitary. [16] So the representation (ρ, V) and its complex conjugate $(\bar{\rho}, \bar{V})$ are unitarily equivalent. Moreover, compatibility of (8.51) with the complex conjugate equation shows that

$$\rho(g) = S^* S\rho(g)(S^* S)^{-1} \tag{8.52}$$

and hence, by Schur's lemma $S^*S = z1$ for a complex number z. Since S is unitary, the determinant of this equation shows that z is a root of unity. On the other hand, conjugating the equation show that z is real. Therefore, z must be ± 1. Moreover, again since S is unitary, the equation implies that $S^{tr} = zS$ is symmetric or antisymmetric. So $P = \mathcal{C}S$ where $\mathcal{C}$ is complex conjugation and $P^2 = z$, is ± 1. We check that in case $\mathbb{R}$ we have $P^2 = +1$ and in case $\mathbb{H}$ we have $P^2 = -1$. Conversely, if there is an invertible matrix satisfying (8.51) it follows that we can take S to be unitary and we can construct a $P \in Z(A)$ but $P \notin Z(B)$ with $P^2 = \pm 1$. Therefore, the above trichotomy is equivalent to the following statement:

If (ρ, V) is an irrep of G then one of the following holds:

- *Potentially Real Representations*: (ρ, V) is equivalent to its conjugate and there exists an S with $S^{tr} = S$. In this case we can find a basis of V where the representation matrices are real.
- *Complex Representations*: (ρ, V) is *not equivalent* to its conjugate.

[16] To prove this show that $S^\dagger S$ is in the commutant over $\mathbb{C}$. Therefore, by irreducibility $S^\dagger S = z1$ for some complex number z. By a suitable rescaling of S we can then make it unitary.

- *Pesudoreal Representations* or, equivalently, *Quaternionic Representations*: (ρ, V) is equivalent to its conjugate and there exists an S with $S^{tr} = -S$. In this case V has a quaternionic structure commuting with $\rho(g)$. Thus, we can identify the representation with a quaternionic matrix representation.

♣Interpret S as a sesquilinear form which is orthogonal or symplectic. ♣

Examples

1. Consider the irreducible representations of $SU(2)$. Using the fact that $USp(2) = SU(2) = U(1, \mathbb{H})$ we see that there is a canonical representation of the $SU(2)$ on $\mathbb{H}$ by left-multiplication by unit quaternions. Identifying $\mathbb{H} \cong \mathbb{C}^2$ this becomes the standard fundamental two-dimensional representation of $SU(2)$ on $\mathbb{C}^2$. The standard identity on Pauli matrices:
$$(\sigma^\ell)^* = -\sigma^2\sigma^\ell\sigma^2 \tag{8.53}$$
means that the generators of the representation transform as
$$(\sqrt{-1}\sigma^\ell)^* = S(\sqrt{-1}\sigma^\ell)S^{-1} \tag{8.54}$$
where $S = \sqrt{-1}\sigma^2$ is antisymmetric. The isomorphism of the representation with its complex conjugate is $v \to \sqrt{-1}\sigma^2 v^*$ where $v \in \mathbb{C}^2$. Taking symmetric tensor products $\mathrm{Sym}^n\mathbb{C}^2$ will have a real structure commuting with $SU(2)$ for n even and a quaternionic structure for n. This is the familiar rule that integer spin has a representation by real matrices and half-integer spin is pseudoreal and does not.

2. For $G = U(1)$ the representations $\rho_n(e^{\mathrm{i}\theta}) = e^{\mathrm{i}n\theta}$ are complex for $n \neq 0$.

3. For $G = SU(n)$ with $n > 2$ the fundamental representation of dimension n is complex. A quick way to prove this is to note that the characters of a real or pseudoreal representation must be real functions on the Cartan torus. This is clearly not the case for the characters of the n-dimensional representation, when $n > 2$.

4. A beautiful result of Frobenius and Schur is the following. Let $[dg]$ be an invariant measure on G of weight 1. Then if (ρ, V) is an irreducible representation of G on a complex vector space V then
$$\int_G [dg]\mathrm{Tr}_V(\rho(g))^2 = \begin{cases} +1 & \text{type} \quad \mathbb{R} \\ 0 & \text{type} \quad \mathbb{C} \\ -1 & \text{type} \quad \mathbb{H} \end{cases} \tag{8.55}$$
For a proof see Dyson3fold [18]. There is an analog for ϕ-reps which we give below.

Exercise

Write the representation of a unit quaternion $u \in U(1, \mathbb{H})$ in the spin-3/2 representation of $SU(2)$ as a 2×2 matrix of quaternions.

---

9. Symmetry of the dynamics

sec:SymmDyn

With the possible exception of exotic situations in which quantum gravity is important, physics takes place in space and time, and time evolution is described, in quantum mechanics, by unitary evolution of states.

That is, there should be a family of unitary operators $U(t_1, t_2)$, strongly continuous in both variables and satisfying composition laws $U(t_1, t_3) = U(t_1, t_2)U(t_2, t_3)$ so that

$$\rho(t_1) = U(t_1, t_2)\rho(t_2)U(t_2, t_1) \tag{9.1}$$

Let us - for simplicity - make the assumption that our physical system has time-translation invariance so that $U(t_1, t_2) = U(t_1 - t_2)$ is a strongly continuous group of unitary transformations.

Again, except in unusual situations associated with nontrivial gravitational fields we can assume our spacetime is time-orientable. Then, any physical symmetry group G must be equipped with a homomorphism

$$\tau : G \to \mathbb{Z}_2 \tag{9.2}$$

telling us whether the symmetry operations preserve or reverse the orientation of time. That is $\tau(g) = +1$ are symmetries which preserve the orientation of time while $\tau(g) = -1$ are symmetries which reverse it.

On the other hand, Wigner's theorem also provides us with an intrinsic homomorphism $\phi : G \to \mathbb{Z}_2$ and it is natural to ask how these two homomorphisms are related.

By Stone's theorem, $U(t)$ has a self-adjoint generator H, the Hamiltonian, so that we may write

♣There is an obvious generalization of this statement for $U(t_1, t_2)$. Is it proved rigorously somewhere? ♣

$$U(t) = \exp\left(-\frac{it}{\hbar}H\right) \tag{9.3}$$

eq:Hamiltonian-Ev

Now, we say a quantum symmetry $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ lifting to $\rho^{\mathrm{tw}} : G^{\mathrm{tw}} \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ is a *symmetry of the dynamics* if for all $g \in G^{\mathrm{tw}}$:

$$\rho^{\mathrm{tw}}(g)U(t)\rho^{\mathrm{tw}}(g)^{-1} = U(\tau(g)t) \tag{9.4}$$

eq:Symm-Dyn

where $\tau : G^{\mathrm{tw}} \to \mathbb{Z}_2$ is inherited from the analogous homomorphism on G.

Now, substituting (9.3) and paying proper attention to ϕ we learn that the condition for a symmetry of the dynamics (9.4) is equivalent to

$$\phi(g)\rho^{\mathrm{tw}}(g)H\rho^{\mathrm{tw}}(g)^{-1} = \tau(g)H \tag{9.5}$$

in other words,

$$\rho^{\mathrm{tw}}(g)H\rho^{\mathrm{tw}}(g)^{-1} = \phi(g)\tau(g)H \tag{9.6}$$

Thus, the answer to our question is that ϕ and τ are *unrelated* in general. We should therefore define a third homomorphism $\chi : G \to \mathbb{Z}_2$

$$\chi(g) := \phi(g)\tau(g) \in \{\pm 1\} \tag{9.7}$$

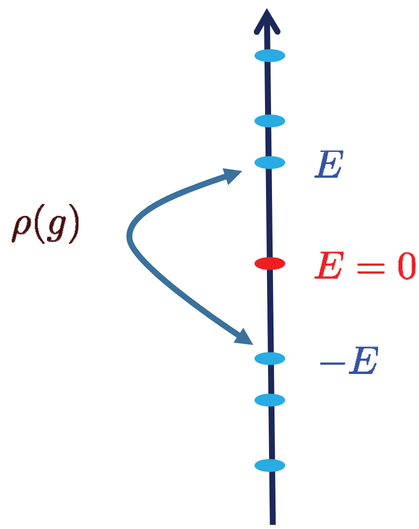

Figure 4: If a symmetry operation has $\chi(g) = -1$ then the spectrum of the Hamiltonian must be symmetric around zero. fig:SymmetricSpec

Note that

$$\phi \cdot \tau \cdot \chi = 1 \tag{9.8}$$

Remarks

1. We should stress that in general a system can have time-orientation reversing symmetries but the simple transformation $t \to -t$ is not a symmetry. Rather, it must be accompanied by other transformations. Put differently, the exact sequence

$$1 \to \ker(\tau) \to G \to \mathbb{Z}_2 \to 1 \tag{9.9}$$

in general *does not split*. Many authors assume it does, and that we can always write $G = G_0 \times \mathbb{Z}_2$ where G_0 is a group of time-orientation-preserving symmetries. However, when considering, for example, the magnetic space groups the sequence typically does not split. As a simple example consider a crystal

$$C = \left(\mathbb{Z}^2 + (\delta_1, \delta_2)\right) \amalg \left(\mathbb{Z}^2 + (-\delta_2, \delta_1)\right) \amalg \left(\mathbb{Z}^2 + (-\delta_1, -\delta_2)\right) \amalg \left(\mathbb{Z}^2 + (\delta_2, -\delta_1)\right) \tag{9.10}$$

and suppose there is a dipole moment, or spin S on points in the sub-crystal

$$C_+ = (\mathbb{Z} + (\delta_1, \delta_2)) \amalg (\mathbb{Z} + (-\delta_1, -\delta_2)) \tag{9.11}$$

but a spin $-S$ at the complementary sub-crystal

$$C_- = (\mathbb{Z} + (-\delta_2, \delta_1)) \amalg (\mathbb{Z} + (\delta_2, -\delta_1)) \tag{9.12}$$

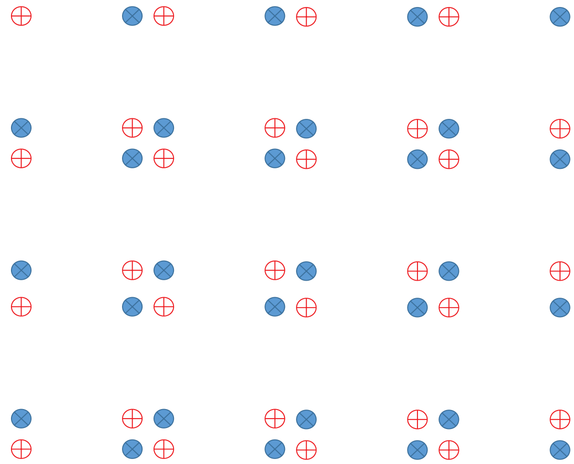

Figure 5: In this figure the blue crosses represent an atom with a local magnetic moment pointing up while the red crosses represent an atom with a local magnetic moment pointing down. The magnetic point group is isomorphic to D_4 but the homomorphism τ to $\mathbb{Z}_2$ has a kernel $\mathbb{Z}_2 \times \mathbb{Z}_2$ (generated by π rotation around a lattice point together with a reflection in a diagonal). Since D_4 is nonabelian the sequence $1 \to \widehat{P}_0 \to \widehat{P} \xrightarrow{\tau} \mathbb{Z}_2 \to 1$ plainly does not split. fig:TimeNoSplit

such that reversal of time orientation exchanges S with $-S$. Then the time-orientation-reversing symmetries must be accompanied by a $\pi/2$ or $3\pi/2$ rotation around some integer point or a reflection in some diagonal. See Figure fig:TimeNoSplit 5. Therefore, the extension of the point group is our friend:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1 \tag{9.13}$$

which does not split.

2. It is very unusual to have a nontrivial homomorphism χ. Note that

$$\rho^{\mathrm{tw}}(g) H \rho^{\mathrm{tw}}(g)^{-1} = \chi(g) H \tag{9.14}$$

implies that if any group element has $\chi(g) = -1$ then the spectrum of H must be symmetric around zero as shown in Figure fig:SymmetricSpectrum 4. In many problems, e.g. in the standard Schrödinger problem with potentials which are bounded below, or in relativistic QFT with H bounded below we must have $\chi(g) = 1$ for all g and hence $\phi(g) = \tau(g)$, which is what one reads in virtually every physics textbook: A symmetry is anti-unitary iff it reverses the orientation of time.

3. However, there *are* physical examples where $\chi(g)$ can be non-trivial, that is, there can be symmetries which are both anti-unitary and time-orientation preserving. An example are the so-called "particle-hole" symmetries in free fermion systems. We will discuss those later.

4. The transformations with $\chi(g) = -1$ are sometimes called "charge-conjugation symmetries" and are sometimes called "particle-hole symmetries." The CMT literature is inconsistent about whether we should allow "symmetry groups" with $\chi \neq 1$ and about whether "particle-hole symmetry" should be a $\mathbb{C}$-linear or a $\mathbb{C}$-anti-linear operation. So we have deliberately avoided using the term "particle-hole symmetry" and "charge conjugation" associated with $\chi(g)$.

9.1 A degeneracy threorem

Suppose that $\chi = 1$ and there is a time-orientation-reversing symmetry with $\rho(g)^2 = -1$. Then since $\rho(g)$ is anti-unitary $\mathcal{H}$ has a quaternionic structure which commutes with H. It follows that the H-eigenspaces have a quaternionic structure which means that their complex dimension must be *even.* That is, the eigenvalues of the Hamiltonian must have even degeneracy.

One important example where this comes up is systems with a rotational symmetry together with a time-reversal symmetry T which takes $(x,t) \to (x,-t)$. Then it follows that the Hermitian generators of rotations must satisfy $T\vec{J}T^{-1} = -\vec{J}$ so T must be an antilinear operator that *commutes* with the $SU(2)$ representation. We have seen that the natural quaternionic structure on the fundamental induces an antilinear operator commuting with $SU(2)$ which satisfies

$$T^2 = (-1)^{2j} \tag{9.15}$$

and hence for half-integer spin T defines a quaternionic structure, whereas for integer spin it defines a real structure. If we are working with a Hamiltonian for a half-integer spin particle then it follows that the energy eigenvalues have even degeneracy. This is sometimes referred to as "Kramer's theorem."

10. Dyson's 3-fold way

DysonThreeFold

Often in physics we begin with a Hamiltonian (or action) and then find the symmetries of the physical system in question. However there are cases when the dynamics are very complicated. A good example is in the theory of nuclear interactions. The basic idea has been applied to many physical systems in which one can identify a set of quantum states corresponding to a large but finite-dimensional Hilbert space. Wigner had the beautiful idea that one could understand much about such a physical system by assuming the Hamiltonian of the system is randomly selected from an *ensemble* of Hamiltonians with a probability distribution on the ensemble. In particular one could still make useful predictions of expected results based on averages over the ensemble.

So, suppose $\mathcal{E}$ is an ensemble of Hamiltonians with a probability measure $d\mu$. Then if $\mathcal{O}$ is some attribute of the Hamiltonians (such as the lowest eigenvalue, or the typical eigenvalue spacing) then we might *expect* our complicated system to have the attribute $\mathcal{O}$ close to the expectation value:

$$\langle \mathcal{O} \rangle := \int_{\mathcal{E}} d\mu \mathcal{O}. \tag{10.1}$$

Of course, for this approach to be sensible there should be some natural or canonical measure on the ensemble $\mathcal{E}$, justified by some *a priori* physically reasonable principles. For example, if we take the space of *all* Hermitian operators on some (say, finite-dimensional) Hilbert space $\mathbb{C}^N$ then any probability distribution which is

- Invariant under unitary transformation.
- Statistically independent for H_{ii} and $\mathrm{Re}(H_{ij})$ and $\mathrm{Im}(H_{ij})$ for $i < j$

can be shown Mehta [27] to be of the form

$$d\mu = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-a\mathrm{Tr}(H^2)+b\mathrm{Tr}(H)+c} \tag{10.2}$$

The specific choice

$$d\mu = \frac{1}{Z} \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-\frac{N}{2}\mathrm{Tr} H^2} \tag{10.3}$$

where Z is a constant chosen so that $\int d\mu = 1$ defines what is known as the *Gaussian unitary ensemble*.

Now sometimes we know *a priori* that the system under study has a certain kind of symmetry. Dyson pointed out in Dyson3fold [18] that such symmetries can constrain the ensemble in ways that affect the probability distribution $d\mu$ in important ways.

10.1 The Dyson problem

Now we can formulate the main problem which was addressed in Dyson3fold [18]:

Given a $\mathbb{Z}_2$-graded group (G, ϕ) and a ϕ-unitary rep $(\rho, \mathcal{H})$, what is the ensemble of commuting Hamiltonians? That is: What is the set of self-adjoint operators commuting with $\rho(g)$ for all g?

Note that the statement of the problem presumes that $\chi(g) = 1$. In Section §sec:10FoldWay 15 below we generalize the problem to allow for $\chi \neq 1$.

The solution to Dyson's problem follows readily from the machinery we have developed. We assume that we can write the isotypical decomposition of $\mathcal{H}$ as

$$\mathcal{H} \cong \oplus_\lambda S_\lambda \otimes_{\mathbb{R}} V_\lambda \tag{10.4}$$

This will always be correct if G is compact. Moreover, $\mathcal{H}$ is a Hilbert space and there are Hermitian structures on S_λ and V_λ so that V_λ a ϕ-unitary rep and we have an isomorphism of ϕ-unitary reps.

Now, if $\chi(g) = 1$ then any Hamiltonian H on $\mathcal{H}$ must commute with the symmetry operators $\rho(g)$ and hence must be in $\mathrm{End}_{\mathbb{C}}^{G}(\mathcal{H})$. But we have computed this commutant above. Choosing an ON basis for S_λ we have

$$Z(\rho, \mathcal{H}) \cong \oplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda) \tag{10.5}$$

The subset of matrices $\mathrm{Mat}_{s_\lambda}(D_\lambda)$ which are Hermitian is

$$\mathrm{Herm}_{s_\lambda}(D_\lambda) = \begin{cases} \text{Real} \quad \text{symmetric} & D_\lambda = \mathbb{R} \\ \text{Complex} \quad \text{Hermitian} & D_\lambda = \mathbb{C} \\ \text{Quaternion} \quad \text{Hermitian} & D_\lambda = \mathbb{H} \end{cases} \tag{10.6}$$

where quaternion Hermitian means that the matrix elements H_{ij} of H are quaternions and $\overline{H_{ij}} = H_{ji}$. (In particular, the diagonal elements are real.)

In conclusion, the answer to the Dyson problem is the ensemble:

$$\mathcal{E} = \prod_\lambda \mathrm{Herm}_{s_\lambda}(D_\lambda) \tag{10.7}$$

Each ensemble $\mathrm{Herm}_N(D)$ has a natural probability measure invariant under the unitary groups

$$U(N, D) := \begin{cases} O(N; \mathbb{R}) & D = \mathbb{R} \\ U(N) & D = \mathbb{C} \\ Sp(N) \cong USp(2N; \mathbb{C}) & D = \mathbb{H} \end{cases} \tag{10.8}$$

such that the matrix elements (not related by symmetry) are statistically independent. These are:

$$d\mu_{GOE} = \frac{1}{Z_{GOE}} \prod_{i=1}^{N} dH_{ii} \prod_{i<j} dH_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr}H^2} \tag{10.9}$$

where $H \in \mathrm{Herm}_N(\mathbb{R})$ is real symmetric.

$$d\mu_{GUE} = \frac{1}{Z_{GUE}} \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr}H^2} \tag{10.10}$$

where $H \in \mathrm{Herm}_N(\mathbb{C})$ is complex Hermitian.

$$d\mu_{GSE} = \frac{1}{Z_{GSE}} \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^4 H_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr}H^2} \tag{10.11}$$

where $H \in \mathrm{Herm}_N(\mathbb{H})$ is quaternionic Hermitian.

Remarks: Examples of physical systems exhibiting the different ensembles are discussed in [43, 44]. Zirnbauer1,Zirnbauer2

- GOE (Type $\mathbb{R}$): Highly excited levels of atomic nuclei, as probed by scattering with low energy neutrons. Since the strong force is both parity and time-reversal invariant here $G = O(3) \times \mathbb{Z}_2$ with the $\mathbb{Z}_2$ factor coming from time-reversal. This was the original context for the Wigner hypothesis. Conjecturally, the large energy levels of a Schrödinger Hamiltonian with classical chaotic dynamics with time-reversal invariance obey GOE statistics.
- GUE (Type $\mathbb{C}$): Similarly, conjecturally, the large energy levels of the quantization of Schrödinger Hamiltonian with chaotic dynamics and no time-reversal invariance. Here $G = \mathbb{Z}_2$. A very interesting aspect of the Riemann zeta function is that the zeroes on the critical line with large imaginary part appear to exhibit GUE statistics. This in fact generalizes to other L-functions of analytic number theory [Katz-Sarnak, Keating-Snaith].
- GSE (Type $\mathbb{H}$): Electrons in disordered metals. In the single electron approximation

$$H = \frac{p^2}{2m} + U(x) + \vec{V}_{SO}(x) \cdot (\vec{\sigma} \times \vec{p}) \tag{10.12}$$

  where $U(x)$ and $V_{SO}(x)$ are drawn from a statistical ensemble. This has time reversal invariance so we can take $G = \mathbb{Z}_2$.

Exercise
Show that
a.) $Z_{GOE} = 2^{\frac{N}{2}} \left(\frac{\pi\sigma^2}{2N}\right)^{N(N+1)/4}$.
b.) $Z_{GUE} =$
c.) $Z_{GSE} =$

10.2 Eigenvalue distributions

The space of Hermitian matrices is a cone so we could rescale H by any real number and hence change the variance of the distribution. The reason we chose the factor N above is that with this normalization the eigenvalue distribution has a good large N limit known as Wigner's semicircle law. Indeed, by making a change of variables

$$H = U\Lambda U^\dagger \tag{10.13}$$

where $\Lambda = \mathrm{Diag}\{\lambda_1, \dots, \lambda_N\}$ is a diagonal matrix of real eigenvalues we get a joint probability distribution for the eigenvalues. To find it we use the map

$$\mathbb{R}^N \times U(N, D) \to \mathrm{Herm}_N(D) \tag{10.14}$$

given by $(\Lambda, U) \to U\Lambda U^\dagger$. This factors through to a map

$$\pi : \mathbb{R}^N \times U(N, D)/U(1, D)^N \to \mathrm{Herm}_N(D) \tag{10.15}$$

Near the origin of $U(N,D)$ we parametrize the group elements by the Lie algebra using the exponential map. So $U = e^{\epsilon} = 1 + \epsilon + \cdots$ where $\epsilon = \sum_{i,j} \epsilon_{ij} e_{ij}$ with $\overline{\epsilon_{ij}} = -\epsilon_{ji}$. Then the group invariant measure on $U(N,D)/U(1,D)^N$ at the origin is just $\prod_{i<j} d^{\beta}\epsilon_{ij}$ with

$$\beta = \begin{cases} 1 & \mathbb{R} \\ 2 & \mathbb{C} \\ 4 & \mathbb{H} \end{cases} \tag{10.16}$$

Now note that

$$\begin{aligned} H = \sum_{i,j} H_{ij} e_{ij} &= (1 + \epsilon + \cdots) \sum_k \lambda_k e_{kk} (1 - \epsilon + \cdots) \\ &= \Lambda + [\epsilon, \Lambda] + \cdots \\ &= \Lambda + \left[\sum_{i<j} (\lambda_i - \lambda_j) \epsilon_{ij} e_{ij} + h.c. \right] + \cdots \end{aligned} \tag{10.17}$$

so that, the measure $\prod_k dH_{kk} \prod_{i<j} d^{\beta} H_{ij}$ pulls back under π^* to

$$\prod_k d\lambda_k \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} d^{\beta} \epsilon_{ij} (1 + \mathcal{O}(\epsilon)^2) \tag{10.18}$$

Now we use group translation invariance to conclude that

$$\int_{U(N,D)/U(1,D)^N} \pi^* \left(\prod_k dH_{kk} \prod_{i<j} d^{\beta} H_{ij} \right) = const. \prod_{1 \le i<j \le N} |\lambda_i - \lambda_j|^{\beta} \prod_k d\lambda_k \tag{10.19}$$

and hence the joint probability distribution for the eigenvalues is

$$d\mu(\Lambda) = \frac{1}{Z_{\Lambda,\beta}} \prod_{1 \le i<j \le N} |\lambda_i - \lambda_j|^{\beta} \exp \left(-\frac{N}{2\sigma^2} \sum_{i=1}^{N} \lambda_i^2 \right) \tag{10.20}$$

eq:JointProb

From the joint probability distribution of eigenvalues we can determine the probability distribution for one eigenvalue $\rho_N(\lambda) d\lambda$. With the above normalization of the variance $\rho_N(\lambda)$ has a good limit for $N \to \infty$ which can be shown by saddle-point methods to be

$$\lim_{N \to \infty} \rho_N(\lambda) d\lambda = \frac{2}{\pi} \sqrt{1 - x^2} \theta(1 - x^2) dx \tag{10.21}$$

where $\theta(\alpha)$ is the Heaviside step function ($= 0$ for $\alpha < 0$ and $= 1$ for $\alpha > 0$) and

$$x = \frac{\lambda}{\lambda_0} \qquad \lambda_0 = \sigma^{-1} \sqrt{\frac{\beta}{2}}. \tag{10.22}$$

This is known as *Wigner's semicircle law*. For much more about this see [27]. Note that the single-eigenvalue distribution is essentially independent of symmetry type. However, the joint probability distribution (10.20) is clearly strongly β-dependent. ♣Say more. ♣

11. Gapped systems and the notion of phases

:GappedSystems

An active area of current [17] research in condensed matter theory is the "classification of phases of matter." There are physical systems, such as the quantum Hall states, "topological insulators" and "topological superconductors" which are thought to be "topologically distinct" from "ordinary phases of matter." We put quotation marks around all these phrases because they are never defined with any great precision, although it is quite clear that precise definitions in principle must exist.

One way to define a "phase of matter" is to consider *gapped systems*.

Definition By a *gapped system* we mean a pair of a Hilbert space $\mathcal{H}$ with a self-adjoint Hamiltonian H where 0 is not in the spectrum of H and $1/H$ is a bounded operator.

Remarks

1. Except in quantum theories of gravity one is always free to add a constant to the Hamiltonian of any closed quantum system. Typically, though not always, the constant is chosen so that $E = 0$ lies between the ground state and the first excited state. For example, if we were studying the Schrodinger Hamiltonian for a single electron in the Hydrogen atom instead of the usual operator $H_a = \frac{p^2}{2m} - \frac{Ze^2}{r}$ we might choose $H_a + 12eV$ so that the groundstate would be at $-1.6eV$ and the continuum would begin at $E_c = 12eV$.

Now suppose we have a *continuous family* of quantum systems. Defining this notion precisely is not completely trivial. See Appendix D of Freed:2012uu [22] for details. Roughly speaking, we have a family of Hilbert spaces $\mathcal{H}_s$ and Hamiltonians H_s varying continuously with parameters s in some topological space $\mathcal{S}$. [18]

Suppose we are given a continuous family of quantum systems $(\mathcal{H}_s, H_s)_{s\in\mathcal{S}}$. Then a subspace $\mathcal{D} \subset \mathcal{S}$ of Hamiltonians for which $0 \in \mathrm{Spec}(H)$ is a generically real codimension one subset of $\mathcal{S}$. It could be very complicated and very singular in places.

Definition Given a continuous family of quantum systems $(\mathcal{H}_s, H_s)_{s\in\mathcal{S}}$ we define a *phase* of the system to be a connected component of $\mathcal{S} - \mathcal{D}$.

Another way to define the same thing is to say that two quantum systems $(\mathcal{H}_0, H_0)$ and $(\mathcal{H}_1, H_1)$ are *homotopic* if there is a continuous family of systems $(\mathcal{H}_s, H_s)$ interpolating between them. [19] Phases are then homotopy classes of quantum systems in the set of all gapped systems.

[17] 2007-2013

[18] To be slightly more precise: We use the compact-open topology to define a bundle of Hilbert spaces over $\mathcal{S}$ and we use this topology for the representations of topological groups. The map $s \to H_s$ should be such that $(t, s) \to \exp[-itH_s]$ is continuous from $\mathbb{R} \times \mathcal{S} \to U(\mathcal{H})_{\mathrm{c.o.}}$ where we use the compact-open topology on the unitary group.

[19] Strictly speaking, we should allow for an isomorphism between the endpoint systems and the given $(\mathcal{H}_0, H_0)$ and $(\mathcal{H}_1, H_1)$ so that homotopy is an equivalence relation on isomorphism classes of quantum systems.

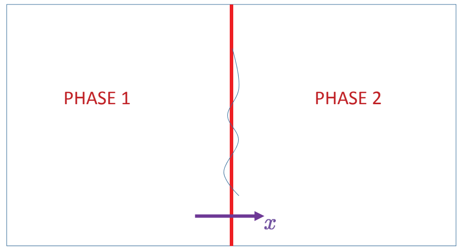

Figure 6: A domain wall between two phases. The wavy line is meant to suggest a localized low energy mode trapped on the domain wall. fig:DomainWall

Remark: A common construction in this subject is to consider a domain wall between two phases as shown in Figure 6. The domain wall has a thickness and the Hamiltonian is presumed to be sufficiently local that we can choose a transverse coordinate x to the domain wall and the Hamiltonian for the local degrees of freedom is a family H_x. (Thus, x serves both as a coordinate in space and as a parameter for a family of Hamiltonians.) Then if the domain wall separates two phases by definition the Hamiltonian must fail to be gapped for at least one value $x = x_0$ within the domain wall. This suggests that there will be massless degrees of freedom confined to the wall. That indeed happens in some nice examples of domain walls between phases of gapped systems.

The focus of these notes is on the generalization of this classification idea to continuous families of quantum systems with a symmetry. Thus we assume now that there is a group G acting as a symmetry group of the quantum system: $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. As we have seen that G is naturally $\mathbb{Z}_2$-graded by a homomorphism ϕ, there is a ϕ-twisted extension G^{tw} and a ϕ-representation of G^{tw} on $\mathcal{H}$. Now, as we have also seen, if we have a symmetry of the dynamics then there is are also homomorphisms $\tau : G^{\mathrm{tw}} \to \mathbb{Z}_2$ and $\chi : G^{\mathrm{tw}} \to \mathbb{Z}_2$ with $\phi(g)\tau(g)\chi(g) = 1$. When we combine this with the assumption that H is gapped we see that we can define a $\mathbb{Z}_2$-grading on the Hilbert space given by the sign of the Hamiltonian. That is, we can decompose:

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \tag{11.1}$$

eq:Z2grad

where $\mathcal{H}^0$ subspace on which $H > 0$ and $\mathcal{H}^1$ is the subspace on which $H < 0$. Put differently, since H is gapped we can define $\Pi = \mathrm{sign}(H)$. Then $\Pi^2 = 1$ and Π serves as the grading operator defining the $\mathbb{Z}_2$ grading (11.1). From this viewpoint the equation (9.4), written as

$$\rho^{\mathrm{tw}}(g)H = \chi(g)H\rho^{\mathrm{tw}}(g) \tag{11.2}$$

means that the operators $\rho^{\mathrm{tw}}(g)$ have a definite $\mathbb{Z}_2$-grading: They are *even* if $\chi(g) = +1$. That means they preserve the sign of the energy and hence take $\mathcal{H}^0 \to \mathcal{H}^0$ and $\mathcal{H}^1 \to \mathcal{H}^1$ while they are *odd* if $\chi(g) = -1$ and exchange $\mathcal{H}^0$ with $\mathcal{H}^1$. See §12 below for a summary of $\mathbb{Z}_2$-graded linear algebra.

This motivates the following definition:

Definition Suppose G is a bigraded group, that is, it has a homomorphism $G \to \mathbb{Z}_2 \times \mathbb{Z}_2$ or, what is the same thing, a pair of homomorphisms (ϕ, χ) from G to $\mathbb{Z}_2$. Then we define a *(ϕ, χ)-representation of G* to be a complex $\mathbb{Z}_2$-graded vector space $V = V^0 \oplus V^1$ and a homomorphism $\rho : G \to \mathrm{End}(V_{\mathbb{R}})$ such that

$$\rho(g) = \begin{cases} \mathbb{C} - \text{linear} & \phi(g) = +1 \\ \mathbb{C} - \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \qquad \textit{and} \qquad \rho(g) = \begin{cases} \text{even} & \chi(g) = +1 \\ \text{odd} & \chi(g) = -1 \end{cases} \tag{11.3}$$

eq:PhiChiReps

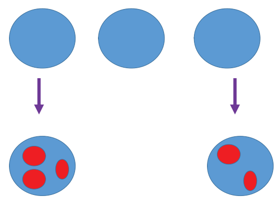

Figure 7: The blue regions in the top row represent different phases of a family of gapped Hamiltonians. The red regions in the bottom row represent different phases with a specified symmetry. Some of the original phases might not have the symmetry at all. Some of the connected components of the original phases might break up into several components with a fixed symmetry. fig:SymmetryPhase

In terms of this concept we see that if G is a symmetry of a gapped quantum system then there is a (ϕ, χ)-representation of G^{tw}. We can again speak of continuous families of quantum systems with G-symmetry. This means that we have $(\mathcal{H}_s, H_s, \rho_s)$ where the representation ρ_s is a symmetry of the dynamics of H_s which also varies continuously with $s \in \mathcal{S}$. If we have a continuous family of gapped systems then we have a continuous family of (ϕ, χ)-representations. Again we can define phases with G-symmetry to be the connected components of $\mathcal{S} - \mathcal{D}$. This can lead to an interesting refinement of the classification of phases without symmetry, as explained in Figure 7. We will denote the set of phases by

$$\mathcal{TP}(G^{\mathrm{tw}}, \phi, \chi, \mathcal{S}) \tag{11.4}$$

In general, this is just a set. In some nice examples that set turns out to be related to an abelian group which in turn ends up being a twisted equivariant K-theory group.

An example of how this refinement is relevant to condensed matter physics is that in topological band structure we can consider families of one-electron Hamiltonians which respect a given (magnetic) space-group. Then there is an interesting refinement of the usual K-theoretic classification of band structures [22] which will be discussed in Chapter 25.

We have been led rather naturally to the notion of $\mathbb{Z}_2$-graded linear algebra. Therefore in the next section §12 we very briefly recall a few relevant facts and definitions. ♣Probably better to make super-linear algebra review an appendix ♣

12. $\mathbb{Z}_2$-graded, or super-, linear algebra

rLinearAlgebra

In this section "super" is merely a synonym for "$\mathbb{Z}_2$-graded." Super linear algebra is extremely useful in studying supersymmetry and supersymmetric quantum theories, but its applications are much broader than that and the name is thus a little unfortunate.

Superlinear algebra is very similar to linear algebra, but there are some crucial differences: *It's all about signs.*

For a longer version of this chapter see my notes, *Linear Algebra User's Manual*, section 23.

12.1 Super vector spaces

It is often useful to add the structure of a $\mathbb{Z}_2$*-grading* to a vector space. A $\mathbb{Z}_2$-graded vector space over a field κ is a vector space over κ which, moreover, is written as a direct sum

$$V = V^0 \oplus V^1. \tag{12.1}$$ eq:zeet

The vector spaces V^0, V^1 are called the even and the odd subspaces, respectively. We may think of these as eigenspaces of a "parity operator" P_V which satisfies $P_V^2 = 1$ and is $+1$ on V^0 and -1 on V^1. If V^0 and V^1 are finite dimensional, of dimensions m, n respectively we say the super-vector space has graded-dimension or superdimension $(m|n)$.

A vector $v \in V$ is called *homogeneous* if it is an eigenvector of P_V. If $v \in V^0$ it is called *even* and if $v \in V^1$ it is called *odd*. We may define a *degree* or *parity* of homogeneous vectors by setting $\deg(v) = \bar{0}$ if v is even and $\deg(v) = \bar{1}$ if v is odd. Here we regard $\bar{0}, \bar{1}$ in the additive abelian group $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Note that if v, v' are homogeneous vectors of the same degree then

$$\deg(\alpha v + \beta v') = \deg(v) = \deg(v') \tag{12.2}$$ eq:dnge

for all $\alpha, \beta \in \kappa$. We can also say that $P_V v = (-1)^{\deg(v)} v$ acting on homogeneous vectors. For brevity we will also use the notation $|v| := \deg(v)$. Note that $\deg(v)$ is *not defined* for general vectors in V.

Mathematicians define the *category of super vector spaces* so that a *morphism* from $V \to W$ is a linear transformation which *preserves grading*. We will denote the space of morphisms from V to W by $\underline{\mathrm{Hom}}(V, W)$. The underline is there to distinguish from the space of linear transformations from V to W discussed below. The space of morphisms $\underline{\mathrm{Hom}}(V, W)$ is just the set of ungraded linear transformations of ungraded vector spaces, $T : V \to W$, which commute with the parity operator $TP_V = P_W T$.

So far, there is no big difference from, say, a $\mathbb{Z}$-graded vector space. However, important differences arise when we consider *tensor products.*

Put differently: we defined a category of supervector spaces, and now we will make it into a *tensor category.* (See definition below.)

The tensor product of two $\mathbb{Z}_2$ graded spaces V and W is $V \otimes W$ as vector spaces over κ, but the $\mathbb{Z}_2$-grading is defined by the rule:

$$\begin{aligned} (V \otimes W)^0 &:= V^0 \otimes W^0 \oplus V^1 \otimes W^1 \\ (V \otimes W)^1 &:= V^1 \otimes W^0 \oplus V^0 \otimes W^1 \end{aligned} \tag{12.3}$$ eq:tsnp

Thus, under tensor product the degree is additive on homogeneous vectors:

$$\deg(v \otimes w) = \deg(v) + \deg(w) \tag{12.4}$$

eq:tnesvct

If κ is any field we let $\kappa^{p|q}$ denote the supervector space:

$$\kappa^{p|q} = \underbrace{\kappa^p}_{\text{even}} \oplus \underbrace{\kappa^q}_{\text{odd}} \tag{12.5}$$

Thus, for example:

$$\mathbb{R}^{n_e|n_o} \otimes \mathbb{R}^{n'_e|n'_o} \cong \mathbb{R}^{n_e n'_e + n_o n'_o | n_e n'_o + n_o n'_e} \tag{12.6}$$

and in particular:

$$\mathbb{R}^{1|1} \otimes \mathbb{R}^{1|1} = \mathbb{R}^{2|2} \tag{12.7}$$

$$\mathbb{R}^{2|2} \otimes \mathbb{R}^{2|2} = \mathbb{R}^{8|8} \tag{12.8}$$

$$\mathbb{R}^{8|8} \otimes \mathbb{R}^{8|8} = \mathbb{R}^{128|128} \tag{12.9}$$

Now, in fact we have a *braided tensor category*:

In ordinary linear algebra there is an isomorphism of tensor products

$$c_{V,W} : V \otimes W \to W \otimes V \tag{12.10}$$

eq:BrdIso

given by $c_{V,W} : v \otimes w \mapsto w \otimes v$. In the category of super vector spaces there is also an isomorphism (12.10) defined by taking

$$c_{V,W} : v \otimes w \to (-1)^{|v|\cdot|w|} w \otimes v \tag{12.11}$$

eq:SuperBraid

on homogeneous objects, and extending by linearity.

Let us pause to make two remarks:

1. Note that in (12.11) we are now viewing $\mathbb{Z}/2\mathbb{Z}$ as a ring, not just as an abelian group. Do not confuse $\deg v + \deg w$ with $\deg v \deg w$! In computer science language $\deg v + \deg w$ corresponds to *XOR*, while $\deg v \deg w$ corresponds to *AND*.

2. It is useful to make a general rule: In equations where the degree appears it is understood that all quantities are homogeneous. Then we extend the formula to general elements by linearity. Equation (12.11) is our first example of another general rule: In the super world, commuting any object of homogeneous degree A with any object of homogeneous degree B results in an "extra" sign $(-1)^{AB}$. This is sometimes called the "*Koszul sign rule.*"

With this rule the tensor product of a collection $\{V_i\}_{i \in I}$ of super vector spaces

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}, \tag{12.12}$$

eq:TensSupVect

is well-defined and *independent of the ordering of the factors*. This is a slightly nontrivial fact. See the remarks below.

We define the $\mathbb{Z}_2$-graded-symmetric and $\mathbb{Z}_2$-graded-antisymmetric products to be the images of the projection operators

$$P = \frac{1}{2}\left(1 \pm c_{V,V}\right) \tag{12.13}$$

Therefore the $\mathbb{Z}_2$-graded-symmetric product of a supervector space is the $\mathbb{Z}_2$-graded vector space with components:

$$\begin{aligned} S^2(V)^0 &\cong S^2(V^0) \oplus \Lambda^2(V^1) \\ S^2(V)^1 &\cong V^0 \otimes V^1 \end{aligned} \tag{12.14}$$

and the $\mathbb{Z}_2$-graded-antisymmetric product is

$$\begin{aligned} \Lambda^2(V)^0 &\cong \Lambda^2(V^0) \oplus S^2(V^1) \\ \Lambda^2(V)^1 &\cong V^0 \otimes V^1 \end{aligned} \tag{12.15}$$

Remarks

1. In this section we are stressing the differences between superlinear algebra and ordinary linear algebra. These differences are due to important signs. If the characteristic of the field κ is 2 then ± 1 are the same. Therefore, in the remainder of this section we assume κ is a field of characteristic different from 2.

2. Since the transformation $c_{V,W}$ is nontrivial in the $\mathbb{Z}_2$-graded case the fact that (12.12) eq:TensSupVect is well-defined is actually slightly nontrivial. To see the issue consider the tensor product $V_1 \otimes V_2 \otimes V_3$ of three super vector spaces. Recall the relation $(12)(23)(12) = (23)(12)(23)$ of the symmetric group. Therefore, we should have "coherent" isomorphisms:

$$(c_{V_2,V_3} \otimes 1)(1 \otimes c_{V_1,V_3})(c_{V_1,V_2} \otimes 1) = (1 \otimes c_{V_1,V_2})(c_{V_1,V_3} \otimes 1)(1 \otimes c_{V_2,V_3}) \tag{12.16}$$

eq:z2yb

and this is easily checked.

In general a *tensor category* is a category with a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ denoted $(X,Y) \to X \otimes Y$ with an associativity isomorphism $F_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ satisfying the pentagon coherence relation. A braiding is an isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$. The associativity and braiding isomorphisms must satisfy "coherence equations." The category of supervector spaces is perhaps the simplest example of a braided tensor category going beyond the category of vector spaces.

3. Note well that $S^2(V)$ as a supervector space does *not* even have the same dimension as $S^2(V)$ in the ungraded sense! Moreover, if V has a nonzero odd-dimensional summand then $\Lambda^n(V)$ does not vanish no matter how large n is.

Exercise

a.) Show that $c_{V,W}c_{W,V} = 1$.
b.) Check (12.16).

eq:z2yb

Exercise *Reversal of parity*

a.) Introduce an operation which switches the parity of a supervector space: $(\Pi V)^0 = V^1$ and $(\Pi V)^1 = V^0$. Show that Π defines a functor of the category of supervector spaces to itself which squares to one.

b.) In the category of finite-dimensional supervector spaces when are V and ΠV isomorphic? [20]

c.) Show that one can identify ΠV as the functor defined by tensoring V with the canonical odd one-dimensional vector space $\kappa^{0|1}$.

12.2 Linear transformations between supervector spaces

If the ground field κ is taken to have degree 0 then the dual space $V^\vee$ in the category of supervector spaces consists of the morphisms $V \to \kappa^{1|0}$. Note that $V^\vee$ inherits a natural $\mathbb{Z}_2$ grading:

♣Not true. Hom$(V, \kappa^{1|0})$ is purely even. Just define superlinear transformations as below. ♣

$$\begin{aligned}(V^\vee)^0 &:= (V^0)^\vee \\ (V^\vee)^1 &:= (V^1)^\vee\end{aligned} \tag{12.17}$$

eq:duals

Thus, we can say that $(V^\vee)^\epsilon$ are the linear functionals $V \to \kappa$ which vanish on $V^{1+\epsilon}$.

Taking our cue from the natural isomorphism in the ungraded theory:

$$\mathrm{Hom}(V,W) \cong V^\vee \otimes W \tag{12.18}$$

we use the same definition so that the space of linear transformations between two $\mathbb{Z}_2$-graded spaces becomes $\mathbb{Z}_2$ graded. We also write $\mathrm{End}(V) = \mathrm{Hom}(V,V)$.

In particular, a linear transformation is an even linear transformation between two $\mathbb{Z}_2$-graded spaces iff $T : V^0 \to W^0$ and $V^1 \to W^1$, and it is odd iff $T : V^0 \to W^1$ and $V^1 \to W^0$. Put differently:

$$\begin{aligned}\mathrm{Hom}(V,W)^0 &\cong \mathrm{Hom}(V^0,W^0) \oplus \mathrm{Hom}(V^1,W^1) \\ \mathrm{Hom}(V,W)^1 &\cong \mathrm{Hom}(V^0,W^1) \oplus \mathrm{Hom}(V^1,W^0)\end{aligned} \tag{12.19}$$

The general linear transformation is neither even nor odd.

If we choose a basis for V made of vectors of homogeneous degree and order it so that the even degree vectors come first then with respect to such a basis even transformations have block diagonal form

---

[20] *Answer*: An isomorphism is a degree-preserving isomorphism of vector spaces. Therefore if V has graded dimension $(m|n)$ then ΠV has graded dimension $(n|m)$ so they are isomorphic in the category of supervector spaces iff $n = m$.

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{12.20}$$ eq:matev

while odd transformations have block diagonal form

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{12.21}$$ eq:matevp

Remarks

1. Note well! There is a difference between $\mathrm{Hom}(V,W)$ and $\underline{\mathrm{Hom}}(V,W)$. The latter is the space of morphisms from V to W in the category of supervector spaces. They consist of just the even linear transformations: [21]

   $$\underline{\mathrm{Hom}}(V,W) = \mathrm{Hom}(V,W)^0 \tag{12.22}$$

   One reason for this definition is that otherwise the graded dimension $(n_e|n_o)$ is not an invariant of a super-vector-space.

2. If $T : V \to W$ and $T' : V' \to W'$ are linear operators on super-vector-spaces then we can define the $\mathbb{Z}_2$ graded tensor product $T \otimes T'$. Note that $\deg(T \otimes T') = \deg(T) + \deg(T')$, and on homogeneous vectors we have

   $$(T \otimes T')(v \otimes v') = (-1)^{\deg(T')\deg(v)} T(v) \otimes T'(v') \tag{12.23}$$ eq:tensortmsn

As in the ungraded case, $\mathrm{End}(V)$ is a ring, but now it is a $\mathbb{Z}_2$-graded ring under composition: $T_1 T_2 := T_1 \circ T_2$. That is if $T_1, T_2 \in \mathrm{End}(V)$ are homogeneous then $\deg(T_1 T_2) = \deg(T_1) + \deg(T_2)$, as one can easily check using the above block matrices. These operators are said to *graded-commute*, or *supercommute* if

$$T_1 T_2 = (-1)^{\deg T_1 \deg T_2} T_2 T_1 \tag{12.24}$$ eq:KRule

---

Exercise

Show that if $T : V \to W$ is a linear transformation between two super-vector spaces then

a.) T is even iff $TP_V = P_W T$

b.) T is odd iff $TP_V = -P_W T$.

---

[21] Warning! Some authors use the opposite notation Hom vs. $\underline{\mathrm{Hom}}$ for distinguishing hom in the category of supervector spaces from "internal hom." In particular, see §1.6 of [15]. IAS-VOL1

12.3 Superalgebras

The set of linear transformations $\mathrm{End}(V)$ of a supervector space is an example of a superalgebra. In general we have:

Definition

a.) A *superalgebra* $\mathcal{A}$ is a supervector space over a field κ together with a morphism

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{12.25}$$

of supervector spaces. We denote the product as $a \otimes a' \mapsto aa'$. Note this implies that

$$\deg(aa') = \deg(a) + \deg(a'). \tag{12.26}$$

We assume our superalgebras to be unital so there is a $1_{\mathcal{A}}$ with $1_{\mathcal{A}} a = a 1_{\mathcal{A}} = a$. Henceforth we simply write 1 for $1_{\mathcal{A}}$.

b.) The superalgebra is *associative* if $(aa')a'' = a(a'a'')$.

c.) Two elements a, a' in a superalgebra are said to *graded-commute*, or *super-commute* provided

$$aa' = (-1)^{|a||a'|} a'a \tag{12.27}$$

If every pair of elements a, a' in a superalgebra graded-commmute then the superalgebra is called *graded-commutative* or *supercommutative.*

d.) The *supercenter*, or $\mathbb{Z}_2$*-graded center* of an algebra, denoted $Z_s(\mathcal{A})$, is the subsuperalgebra of $\mathcal{A}$ such that all homogeneous elements $a \in Z_s(\mathcal{A})$ satisfy

$$ab = (-1)^{|a||b|} ba \tag{12.28}$$

for all homogeneous $b \in \mathcal{A}$.

Example 1: *Matrix superalgebras.* If V is a supervector space then $\mathrm{End}(V)$ as described above is a matrix superalgebra. As an exercise, show that the supercenter is isomorphic to κ, consisting of the transformations $v \to \alpha v$, for $\alpha \in \kappa$. So in this case the center and super-center coincide.

Example 2: *Grassmann algebras.* The Grassmann algebra of an ordinary vector space W is just the exterior algebra of W considered as a $\mathbb{Z}_2$-graded algebra. We will denote it as $\mathrm{Grass}[W]$. In plain English, we take vectors in W to be odd and use them to generate a superalgebra with the rule that

$$w_1 w_2 + w_2 w_1 = 0 \tag{12.29}$$

for all w_1, w_2. In particular (provided the characteristic of κ is not two) we have $w^2 = 0$ for all w. Thus, if we choose basis vectors $\theta^1, \dots, \theta^n$ for W then we can view $\mathrm{Grass}(W)$ as the quotient of the supercommutative polynomial superalgebra $\kappa[\theta^1, \dots, \theta^n]/I$ where the relations in I are:

$$\theta^i \theta^j + \theta^j \theta^i = 0 \qquad\qquad (\theta^i)^2 = 0 \tag{12.30}$$

The typical element then is

$$a = x + x_i\theta^i + \frac{1}{2!}x_{ij}\theta^i\theta^j + \cdots + \frac{1}{n!}x_{i_1,\dots,i_n}\theta^{i_1}\cdots\theta^{i_n} \tag{12.31}$$

The coefficients $x_{i_1,\dots,i_m}$ are m^{th}-rank totally antisymmetric tensors in $\kappa^{\otimes m}$. We will sometimes also use the notation $\mathrm{Grass}[\theta^1,\dots,\theta^n]$.

Definition Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. The *graded tensor product* $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is the superalgebra which is the graded tensor product as a vector space and the multiplication of homogeneous elements satisfies

$$(a_1\widehat{\otimes}b_1)\cdot(a_2\widehat{\otimes}b_2) = (-1)^{|b_1||a_2|}(a_1a_2)\widehat{\otimes}(b_1b_2) \tag{12.32}$$

eq:GradedTensor

Example For matrix superalgebras we have $\mathrm{End}(V)\widehat{\otimes}\mathrm{End}(V') \cong \mathrm{End}(V\otimes V')$, and in particular:

$$\mathrm{End}(\mathbb{C}^{n_e|n_o})\widehat{\otimes}\mathrm{End}(\mathbb{C}^{n'_e|n'_o}) \cong \mathrm{End}(\mathbb{C}^{n_e|n_o}\otimes\mathbb{C}^{n'_e|n'_o}) \cong \mathrm{End}(\mathbb{C}^{n_en'_e+n_on'_o|n_en'_o+n_on'_e}) \tag{12.33}$$

eq:TensorMSA

Remarks

1. Every $\mathbb{Z}_2$-graded algebra is also an ungraded algebra: We just forget the grading. However this can lead to some confusions:

2. An algebra can be $\mathbb{Z}_2$-graded-commutative and not ungraded-commutative: The Grassmann algebras are an example of that. We can also have algebras which are ungraded commutative but not $\mathbb{Z}_2$-graded commutative. The Clifford algebras $C\ell_{\pm1}$ described below provide examples of that.

3. The $\mathbb{Z}_2$-graded-center of an algebra can be different from the center of an algebra as an ungraded algebra. Again, the Clifford algebras $C\ell_{\pm1}$ described below provide examples.

4. One implication of (eq:GradedTensor 12.32) is that when writing matrix representations of graded algebras we do *not* get a matrix representation of the graded tensor product just by taking the tensor product of the matrix representations. This is important when discussing reps of Clifford algebras, as we will stress below.

5. As for ungraded algebras, there is a notion of *simple, semi-simple, and central superalgebras*. These are discussed in the Appendix app:CentralSimple A.

Exercise

If V is a supervector space show that the super-center of $\mathrm{End}(V)$ consists of scalar multiples of the identity.

Exercise *The opposite algebra*

a.) For any ungraded algebra A we can define the *opposite algebra* A^{opp} by the rule

$$a \cdot^{\mathrm{opp}} b := ba \tag{12.34}$$

Show that A^{opp} is still an algebra.

♣We already said this above. ♣

b.) Show that there is natural morphism of algebras: [22] $A \otimes A^{\mathrm{opp}} \to \mathrm{End}(A)$.

c.) For any superalgebra A we can define the *opposite superalgebra* A^{opp} by the rule

$$a \cdot^{\mathrm{opp}} b := (-1)^{|a||b|} ba \tag{12.35}$$

Show that A^{opp} is still an superalgebra.

d.) Show that A is supercommutative iff $A = A^{\mathrm{opp}}$.

e.) Show that there is natural morphism of super-algebras: $A \widehat{\otimes} A^{\mathrm{opp}} \to \mathrm{End}(A)$.

♣Check it out for Grassmann algebras ♣

12.4 Modules over superalgebras

Definition A *super-module M over a super-algebra $\mathcal{A}$* (where $\mathcal{A}$ is itself a superalgebra over a field κ) is a supervector space M over κ together with a κ-linear map $\mathcal{A} \times M \to M$ defining a left-action or a right-action. That is, it is a left-module if, denoting the map by $L : \mathcal{A} \times M \to M$ we have

$$L(a, L(b, m)) = L(ab, m) \tag{12.36}$$

and it is a right-module if, denoting the map by $R : \mathcal{A} \times M \to M$ we have

$$R(a, R(b, m)) = R(ba, m) \tag{12.37}$$

In either case:

$$\deg(R(a, m)) = \deg(L(a, m)) = \deg(a) + \deg(m) \tag{12.38}$$

The notations $L(a, m)$ and $R(a, m)$ are somewhat cumbersome and instead we write $L(a, m) = am$ and $R(a, m) = ma$ so that $(ab)m = a(bm)$ and $m(ab) = (ma)b$. We also sometimes refer to a super-module over a super-algebra $\mathcal{A}$ just as a *representation of* $\mathcal{A}$.

Definition A *linear transformation* between two super-modules M, N over $\mathcal{A}$ is a κ-linear transformation of supervector spaces such that if T is homogeneous and M is a left $\mathcal{A}$-module then $T(am) = (-1)^{|T||a|} a T(m)$ while if M is a right $\mathcal{A}$-module then $T(ma) = T(m)a$. We denote the space of such linear transformations by $\mathrm{Hom}_{\mathcal{A}}(M, N)$. If N is a left

[22] *Answer*: Given $a \otimes b$ we consider the linear transformation $x \mapsto axb$.

$\mathcal{A}$-module then $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is a left $\mathcal{A}$-module with $(a\cdot T)(m) := a\cdot(T(m))$. If N is a right $\mathcal{A}$-module then $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is a right $\mathcal{A}$-module with $(T\cdot a)(m) := (-1)^{|a||m|}T(m)a$. When $M=N$ we denote the module of linear transformations by $\mathrm{End}_{\mathcal{A}}(M)$.

Just as in the case of supervector spaces, we must be careful about the definition of a morphism:

Definition A *morphism* in the category of $\mathcal{A}$-modules is a morphism T of supervector spaces which commutes with the $\mathcal{A}$-action.

Example *Matrix superalgebras.* In the ungraded world a matrix algebra $\mathrm{End}(V)$ for a finite dimensional vector space, say, over $\mathbb{C}$, has a unique irreducible representation, up to isomorphism. This is just the space V itself. A rather tricky point is that if V is a supervector space $V=\mathbb{C}^{p|q}$ then V and ΠV are *inequivalent* representations of $\mathrm{End}(V)$. One way to see this is that if η is a generator of $\Pi=\mathbb{C}^{0|1}$ then $T(\eta v)=(-1)^{|T|}\eta T(v)$ is a priori a different module. In terms of matrices

$$\begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12.39}$$

So the LHS gives a representation of the matrix superalgebra, but it is not related by an invertible element of $\underline{\mathrm{End}}(\mathbb{C}^{p|q})$. The even subalgebra $\mathrm{End}(\mathbb{C}^p)\oplus\mathrm{End}(\mathbb{C}^q)$ has a unique faithful representation $\mathbb{C}^p\oplus\mathbb{C}^q$ and hence the matrix superalgebra $\mathrm{End}(\mathbb{C}^{p|q})$ has exactly two irreducible modules.

Exercise *Tensor product of modules*

Let $\mathcal{A}$ and $\mathcal{B}$ be superalgebras with modules M and N, respectively. Show that the rule

$$(a\otimes b)\cdot(m\otimes n) := (-1)^{|b||m|}(am)\otimes(bn) \tag{12.40}$$

does indeed define $M\otimes N$ as an $\mathcal{A}\widehat{\otimes}\mathcal{B}$ module. Be careful with the signs!

Exercise *Left modules vs. right modules*

a.) Show that if $(a,m)\to L(a,m)$ defines the structure of a left-$\mathcal{A}$-module on M then the new product $R:\mathcal{A}^{\rm opp}\times M\to M$ defined by

$$R(a,m) := (-1)^{|a||m|}L(a,m) \tag{12.41}$$

defines M as a *right* $\mathcal{A}^{\rm opp}$-module. That is, show that

$$R(a_1,R(a_2,m)) = R(a_2\cdot^{\rm opp} a_1, m) \tag{12.42}$$

b.) Similarly, show that if M is a right $\mathcal{A}$-module then it can be canonically considered also to be a left- $\mathcal{A}^{\rm opp}$ module.

c.) Show that if M is a module for a supercommutative algebra $\mathcal{A}$ then it can be considered either as a left- or right- $\mathcal{A}$-module. Because of this, when $\mathcal{A}$ is supercommutative, we will sometimes write the module multiplication on the left or the right, depending on which order is more convenient to keep the signs down.

---

c:SuperHilbert

12.5 Star-structures and super-Hilbert spaces

There are at least *three* notions of a real structure on a complex superalgebra which one will encounter in the literature:

1. It is a $\mathbb{C}$-antilinear involutive *automorphism* $a \mapsto a^{\bigstar}$. Hence $\deg(a^{\bigstar}) = \deg(a)$ and $(ab)^{\bigstar} = a^{\bigstar} b^{\bigstar}$.

2. It is a $\mathbb{C}$-antilinear involutive *anti-automorphism.* Thus $\deg(a^*) = \deg(a)$ but

$$(ab)^* = (-1)^{|a||b|} b^* a^* \tag{12.43}$$

3. It is a $\mathbb{C}$-antilinear involutive *anti-automorphism.* Thus $\deg(a^\star) = \deg(a)$ but

$$(ab)^\star = b^\star a^\star \tag{12.44}$$

If $\mathcal{A}$ is a supercommutative complex superalgebra then structures 1 and 2 coincide: $a \to a^{\bigstar}$ is the same as $a \to a^*$. See remarks below for the relation of 2 and 3.

Definition A sesquilinear form h on a complex supervector space $\mathcal{H}$ is a map $h : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

1. It is even, so that $h(v, w) = 0$ if v and w have opposite parity

2. It is $\mathbb{C}$-linear in the second variable and $\mathbb{C}$-antilinear in the first variable

3. An *Hermitian form* on a supervector space is a sesquilinear form which moreover satisfies the symmetry property:

$$(h(v, w))^* = (-1)^{|v||w|} h(w, v) \tag{12.45}$$

4. If in addition for all nonzero $v \in \mathcal{H}^0$

$$h(v, v) > 0 \tag{12.46}$$

while for all nonzero $v \in \mathcal{H}^1$

$$i^{-1} h(v, v) > 0, \tag{12.47}$$

then $\mathcal{H}$ endowed with the form h is a *super-Hilbert space*.

For bounded operators we define the adjoint of a homogeneous linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$h(T^* v, w) = (-1)^{|T||v|} h(v, Tw) \tag{12.48}$$

The spectral theorem is essentially the same as in the ungraded case with one strange modification. For even Hermitian operators the spectrum is real. However, for odd Hermitian operators the point spectrum sits in a real subspace of the complex plane which is *not* the real line! If T is odd then an eigenvector v such that $Tv = \lambda v$ must have even and odd parts $v = v_e + v_o$. Then the eigenvalue equation becomes

$$\begin{aligned} Tv_e &= \lambda v_o \\ Tv_o &= \lambda v_e \end{aligned} \tag{12.49}$$

Now the usual proof that the point spectrum is real is modified to:

$$\begin{aligned} \lambda^* h(v_o, v_o) &= h(\lambda v_o, v_o) = h(Tv_e, v_o) = h(v_e, Tv_o) = \lambda h(v_e, v_e) \\ \lambda^* h(v_e, v_e) &= h(\lambda v_e, v_e) = h(Tv_o, v_e) = -h(v_o, Tv_e) = -\lambda h(v_o, v_o) \end{aligned} \tag{12.50}$$

These two equations have the same content: Since $v \neq 0$ and we are in a superHilbert space it must be that

$$h(v_e, v_e) = i^{-1} h(v_o, v_o) > 0 \tag{12.51}$$

and therefore the phase of λ is determined. It lies on the line passing through $e^{i\pi/4} = (1+i)/\sqrt{2}$ in the complex plane, as shown in Figure fig:SUPERHERMITIAN 8

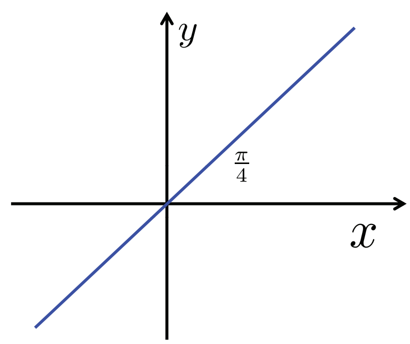

Figure 8: When the Koszul rule is consistently implemented odd super-Hermitian operators have a spectrum which lies along the line through the origin which runs through $1 + i$. fig:SUPERHERMITIA

Example: An example of a natural super-Hilbert space is the Hilbert space of L^2-spinors on an even-dimensional manifold with $(-1)^F$ given by the chirality operator. An odd self-adjoint operator which will have nonhomogeneous eigenvectors is the Dirac operator on an even-dimensional manifold. One usually thinks of the eigenvalues as real for this operator and that is indeed the case if we use the star-structure $\star$, number 3 above. See the exercise below.

Remarks

1. In general star-structures 2 and 3 above are actually closely related. Indeed, given a structure $a \to a^*$ of type 2 we can define a structure of type 3 by defining either

$$a^\star = \begin{cases} a^* & |a| = 0 \\ ia^* & |a| = 1 \end{cases} \tag{12.52}$$

or

$$a^\star = \begin{cases} a^* & |a| = 0 \\ -ia^* & |a| = 1 \end{cases} \tag{12.53}$$

eq:relstar

It is very unfortunate that in most of the physics literature the definition of a star structure is that used in item 3 above. For example a typical formula used in manipulations in superspace is

$$\overline{\theta_1 \theta_2} = \bar{\theta}_2 \bar{\theta}_1 \tag{12.54}$$

and the fermion kinetic energy

$$\int dt i \bar{\psi} \frac{d}{dt} \psi \tag{12.55}$$

is only "real" with the third convention. The rationale for this convention, especially for fermionic fields, is that they will eventually be quantized as operators on a Hilbert space. Physicists find it much more natural to have a standard Hilbert space structure, even if it is $\mathbb{Z}_2$-graded. On the other hand, item 2 implements the Koszul rule consistently and makes the analogy to classical physics as close as possible. So, for example, the fermionic kinetic term is

$$\int dt \bar{\psi} \frac{d}{dt} \psi \tag{12.56}$$

and is "manifestly real."

Fortunately, as we have just noted one convention can be converted to the other, but the difference will, for example, show up as factors of i in comparing supersymmetric Lagrangians in the different conventions, as the above examples show.

Exercise

a.) Show that a super-Hermitian form h on a super-Hilbert space can be used to define an ordinary Hilbert space structure on $\mathcal{H}$ by taking $\mathcal{H}^0 \perp \mathcal{H}^1$ and taking

$$\begin{aligned} (v,w) &:= h(v,w) & v,w \in \mathcal{H}^0 \\ (v,w) &:= i^{-1} h(v,w) & v,w \in \mathcal{H}^1 \end{aligned} \tag{12.57}$$

eq:UnbradedHermit

b.) Show that if T is an operator on a super-Hilbert-space then the super-adjoint T^* and the ordinary adjoint $T^\dagger$, the latter defined with respect to (eq:UnbradedHermitian 12.57), are related by

$$T^* = \begin{cases} T^\dagger & |T| = 0 \\ iT^\dagger & |T| = 1 \end{cases} \tag{12.58}$$

c.) Show that $T \to T^\dagger$ is a star-structure on the superalgebra of operators on superspace which is of type 3 above.

d.) Show that if T is an odd self-adjoint operator with respect to $*$ then $e^{-i\pi/4}T$ is an odd self-adjoint operator with respect to $\dagger$. In particular $e^{-i\pi/4}T$ has a point spectrum in the real line.

e.) More generally, show that if a is odd and real with respect to $*$ then $e^{-i\pi/4}a$ is real with respect to $\star$ defined by (eq:relstar 12.53).

---

13. Clifford Algebras and Their Modules

lgebrasModules

Some references for this section are:

1. E. Cartan, *The theory of Spinors*
2. Chevalley,

2'. P. Deligne, "Notes on spinors," in *Quantum Fields and Strings: A Course for Mathematicians*

3. One of the best treatments is in Atiyah, Bott, and Shapiro, "Clifford Modules"
4. A textbook version of the ABS paper can be found in Lawson and Michelson, *Spin Geometry*, ch.1
5. Freund, Introduction to Supersymmetry
6. M. Sohnius, "Introducing Supersymmetry" Phys. Rept.
7. T. Kugo and P. Townsend, "Supersymmetry and the division algebras," Nuc. Phys. **B221** (1983)357.
8. M. Rausch de Traubenberg, "Clifford Algebras in Physics," arXiv:hep-th/0506011.
9. Freedman and van Proeyen, *Supergravity*

13.1 The real and complex Clifford algebras

13.1.1 Definitions

Clifford algebras are defined for a general nondegenerate symmetric quadratic form Q on a vector space V over κ. They are officially defined as a quotient of the tensor algebra of V by the ideal generated by the set of elements of TV of the form $v_1 \otimes v_2 + v_2 \otimes v_1 - 2Q(v_1, v_2) \cdot 1$ for any $v_1, v_2 \in V$. A more intuitive definition is that $C\ell(Q)$ is the $\mathbb{Z}_2$ graded algebra over κ which has a set of *odd* generators $\{e_i\}$ in one-one correspondence with a basis, also denoted $\{e_i\}$, for the vector space V. The only relations on the generators are given by

♣Poor choice of notation since e_{ij} is an element of the Clifford algebra but also our notation for matrix units... ♣

$$\{e_i, e_j\} = 2Q_{ij} \cdot 1 \qquad (13.1)$$

where $Q_{ij} \in \kappa$ is the matrix of Q with respect to a basis $\{e_i\}$ of V, and $1 \in C\ell(Q)$ is the multiplicative identity. Henceforth we will usually identify κ with $\kappa \cdot 1$ and drop the explicit 1.

Because e_i are odd and 1 is even, the algebra $C\ell(Q)$ does not admit a $\mathbb{Z}$-grading. However, every expression in the relations on the generators is even so the algebra admits a $\mathbb{Z}_2$ grading:

$$C\ell(Q) = C\ell(Q)^0 \oplus C\ell(Q)^1 \qquad (13.2)$$

Of course, one is always free to regard $C\ell(Q)$ as an ordinary ungraded algebra, and this is what is done in much of the physics literature. However, as we will show below, comparing the graded and ungraded algebras leads to a lot of insight.

Incidentally, it turns out that $C\ell(Q)^0$ is isomorphic to an *ungraded* Clifford algebra: See Section §subsubsec:EvenSubAlg 13.1.2 below.

Suppose we can choose a basis $\{e_i\}$ for V so that Q_{ij} is diagonal. Then $e_i^2 = q_i \neq 0$. It follows that $C\ell(Q)$ is *not supercommutative*, because an odd element must square to zero in a supercommutative algebra. Henceforth we assume Q_{ij} has been diagonalized, so that e_i anticommutes with e_j for $i \neq j$. Thus, we have the basic Clifford relations:

$$e_i e_j + e_j e_i = 2 q_i \delta_{ij} \tag{13.3}$$

eq:CliffRels

When $\{i_1, \dots, i_p\}$ are all distinct is useful to define the notation

$$e_{i_1 \cdots i_p} := e_{i_1} \cdots e_{i_p} \tag{13.4}$$

Of course, this expression is totally antisymmetric in the indices, and a moment's thought shows that it forms a basis for $C\ell(Q)$ as a vector space and so we have

$$C\ell(Q) \cong \Lambda^* V \tag{13.5}$$

eq:VSISO

We stress that (eq:VSISO 13.5) is only an isomorphism of *vector spaces*. If V is finite-dimensional with $d = \dim_\kappa V$ then we conclude that

$$\dim_\kappa C\ell(Q) = \sum_{p=0}^{d} \binom{d}{p} = 2^d \tag{13.6}$$

We must also stress that while the left and right hand sides (eq:VSISO 13.5) are both algebras over κ the equation is completely false as an isomorphism of algebras. The right hand side of (eq:VSISO 13.5) is a Grassmann algebra, which is supercommutative and as we have noted $C\ell(Q)$ is not supercommutative.

If we take the case of a real vector space $\mathbb{R}^d$ then WLOG we can diagonalize Q to the form

$$Q = \begin{pmatrix} +1_r & 0 \\ 0 & -1_s \end{pmatrix} \tag{13.7}$$

For such a quadratic form on a real vector space we denote the real Clifford algebra $C\ell(Q)$ by $C\ell_{r+,s-}$. [23]

We can similarly discuss the complex Clifford algebras $\mathbb{C}\ell_n$. Note that over the complex numbers if $e^2 = +1$ then $(ie)^2 = -1$ so we do not need to account for the signature, and WLOG we can just consider $\mathbb{C}\ell_n$ for $n \geq 0$.

♣It is probably better to use the notation $C\ell_{r,-s}$ where r, s are always understood to be nonnegative integers. ♣

♣Nevertheless, it is sometimes useful to consider the algebras with the extra data of a basis so that e_i^2 have definite signs, and then speak of $C\ell_n$ with $n \in \mathbb{Z}$. Should we discuss this? ♣

[23] The notation $C\ell_{r,s}$ is used in different ways by different authors. Some have r generators squaring to $+1$ and s generators squaring to -1, and and some have the opposite convention. It is impossible to remember this convention so we always explicitly write which are $+$ and which are $-$, when it matters, *except* when one of then is negative. For an integer n we denote $C\ell_n := C\ell_{n+,0-}$ when n is nonnegative and we denote $C\ell_n := C\ell_{|n|-,0+}$ for n nonpositive. So we have a single notation $C\ell_n$ and the sign of n tells us the sign of e_i^2.

Exercise

a.) Show that $v \in V$ is considered as an element of $C\ell(Q)$ then

$$v \cdot v = Q(v) \cdot 1 \tag{13.8}$$

eq:QuadRel

b.) Show that (13.8) could in fact be taken as a definition of the generating relations of the Clifford algebra $C\ell(Q)$.

Remark: In physics we often distinguish $v \in V$ from $v \in C\ell(Q)$ by the notation $\not{v}$. Thus, for example, if $p = p^i e_i$ is a vector on the pseudo-sphere

$$p^i p^j Q_{ij} = R^2. \tag{13.9}$$

then $\not{p}^2 = R^2 \cdot 1$.

Exercise *Opposite Clifford algebra*

Show that if $\mathcal{A} = C\ell_{r+,s-}$ then $\mathcal{A}^{\rm opp} = C\ell_{s+,r-}$.

Since $\mathcal{A}$ is not supercommutative we cannot conclude that these are isomorphic, and, in general, they are not.

13.1.2 The even subalgebra

sec:EvenSubAlg

The even subalgebra is an ungraded algebra and is isomorphic, as an ungraded algebra, to another Clifford algebra.

For example, if $d \geq 1$ then

$$\mathbb{C}\ell_d^0 \cong \mathbb{C}\ell_{d-1} \qquad \textit{ungraded algebras.} \tag{13.10}$$

The proof is straightfoward. For $d = 1$ the statement is obvious. If $d > 1$ then choose some basis vector, say e_1 and let

$$\tilde{e}_j := e_1 e_{j+1} \qquad j = 1, \dots, d-1 \tag{13.11}$$

Then one easily checks that the $\tilde{e}_j$ satisfy the standard Clifford relations defining $\mathbb{C}\ell_{d-1}$, albeit with quadratic form $-2\delta_{ij}$. However, as we have remarked, over the complex numbers one can always change the signature. Note that there is no *canonical* isomorphism - we made a choice of a basis vector in our construction.

When working over the real numbers we must be more careful about signs. choose any basis element e_{i_0} and consider the algebra generated by

$$\tilde{e}_j = e_{i_0 j} \qquad j \neq i_0 \tag{13.12}$$

Note that

$$\tilde{e}_j\tilde{e}_k + \tilde{e}_k\tilde{e}_j = -2q_{i_0 i_0}q_{jk} \qquad j,k \neq i_0 \tag{13.13}$$

and therefore [24]

$$\begin{aligned} C\ell^0(r_+, s_-) &\cong C\ell(r_+, (s-1)_-) \qquad s \geq 1 \\ C\ell^0(r_+, s_-) &\cong C\ell(s_+, (r-1)_-) \qquad r \geq 1 \end{aligned} \tag{13.14}$$

eq:twoevens

Exercise
Show that when both $r \geq 1$ and $s \geq 1$ then the two equations in (eq:twoevens 13.14) are compatible.

Exercise
Show that

$$(C\ell(r_+, s_-))^0 \cong (C\ell(s_+, r_-))^0 \tag{13.15}$$

13.1.3 Relations by tensor products

One important advantage of regarding $C\ell(Q)$ as a superalgebra, rather than just an algebra is that if $Q_1 \oplus Q_2$ is a quadratic form on $V_1 \oplus V_2$ then

$$C\ell(Q_1 \oplus Q_2) \cong C\ell(Q_1)\widehat{\otimes}C\ell(Q_2) \tag{13.16}$$

eq:CA-GTP

As we will see below, this is completely false if we regard the Clifford algebras as ungraded algebras. Since equation (eq:CA-GTP 13.16) is crucially important below let us give a proof: Let $\{e_i\}$ and $\{f_\alpha\}$ be bases for V_1 and V_2 respectively. In the Clifford algebra the corresponding generators anticommute:

$$e_i f_\alpha + f_\alpha e_i = 0 \tag{13.17}$$

Now in an ordinary tensor product we have

$$(e_i \otimes 1)\cdot(1 \otimes f_\alpha) = (1 \otimes f_\alpha)\cdot(e_i \otimes 1) \tag{13.18}$$

but in a graded tensor product we get an extra sign, since e_i and f_α are both odd:

$$(e_i\widehat{\otimes}1)\cdot(1\widehat{\otimes}f_\alpha) = -(1\widehat{\otimes}f_\alpha)\cdot(e_i\widehat{\otimes}1) \tag{13.19}$$

[24] Note that this implies that we must have $C\ell((r+1)_+, s_-) \cong C\ell((s+1)_+, r_-)$ for all $r, s \geq 0$. One can indeed prove this is so using the periodicity isomorphisms and the observation that $C\ell(2_+) \cong C\ell(1_+, 1_-) \cong \mathbb{R}(2)$. Nevertheless, at first site this might seem to be very unlikely since the transverse dimensions are $r-s-1$ and $s-r-1$ and in general are *not* equal modulo 8. Note that the sum of the transverse dimensions is $-2 = 6\mathrm{mod}8$. Thus, we have the pairs $(0,6)$, $(1,5)$, $(2,4)$, and $(3,3)$. One can check from the table that these all do in fact have the same Morita type! Of course, the dimensions are the same, so they must in fact be isomorphic.

Therefore we must use the graded tensor product in (eq:CA-GTP 13.16).

From (eq:CA-GTP 13.16) we have some useful identities: First, note that for $n > 0$:

$$C\ell_n \cong \underbrace{C\ell_1 \widehat{\otimes} \cdots \widehat{\otimes} C\ell_1}_{\text{n times}} \tag{13.20}$$

eq:RCliff-Tens1

$$C\ell_{-n} \cong \underbrace{C\ell_{-1} \widehat{\otimes} \cdots \widehat{\otimes} C\ell_{-1}}_{\text{n times}} \tag{13.21}$$

eq:RCliff-Tens2

More generally we have

$$C\ell_{r+,s-} = \underbrace{C\ell_1 \widehat{\otimes} \cdots \widehat{\otimes} C\ell_1}_{\text{r times}} \widehat{\otimes} \underbrace{C\ell_{-1} \widehat{\otimes} \cdots \widehat{\otimes} C\ell_{-1}}_{\text{s times}} \tag{13.22}$$

eq:RCliff-Tens3

Moreover

$$\mathbb{C}\ell_n \cong \underbrace{\mathbb{C}\ell_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathbb{C}\ell_1}_{\text{n times}} \tag{13.23}$$

eq:CplxCliff-Tens

If we view the Clifford algebras as ungraded algebras then the tensor product relations are a bit more complicated:

Lemma: As ungraded algebras we have the following isomorphisms:

$$C\ell(r_+, s_-) \otimes C\ell(2_+) \cong C\ell((s+2)_+, r_-) \tag{13.24}$$

eq:SkipTwo-1

$$C\ell(r_+, s_-) \otimes C\ell(2_-) \cong C\ell(s_+, (r+2)_-) \tag{13.25}$$

eq:SkipTwo-3

$$C\ell(r_+, s_-) \otimes C\ell(1_+, 1_-) \cong C\ell((r+1)_+, (s+1)_-) \tag{13.26}$$

eq:SkipTwo-2

$$\mathbb{C}\ell_n \otimes \mathbb{C}\ell_2 \cong \mathbb{C}\ell_{n+2} \tag{13.27}$$

eq:SkipTwo-4

Proofs:

• Let e_i be generators of $C\ell_{r_+,s_-}$, f_α, $\alpha = 1, 2$ be generators of $C\ell_2$. Note that the obvious set of generators $e_i \otimes 1$ and $1 \otimes f_\alpha$, do not satisfy the relations of the Clifford algebra, because they do not anticommute. On the other hand if we take

$$\tilde{e}_i := e_i \otimes f_{12} \qquad \tilde{e}_{d+\alpha} := 1 \otimes f_\alpha \tag{13.28}$$

where $f_{12} = f_1 f_2$, then $\tilde{e}_M$, $M = 1 \ldots, d+2$ satisfy the Clifford algebra relations and also generate the tensor product. Now note that $(f_{12})^2 = -1$ and hence:

$$(e_i \otimes f_{12})^2 = -(e_i)^2 \tag{13.29}$$

(no sum on i).

An almost identical proof works for tensoring with $C\ell_{-2}$. Similarly, in the case $C\ell_{1+,1-}$ we have $(f_{12})^2 = +1$ and hence:

$$(e_i \otimes f_{12})^2 = +(e_i)^2 \tag{13.30}$$

(no sum on i).

Complexifying any of the above identities yields the last one. ♢

Remarks These isomorphisms, and the consequences below are very useful in physics because they relate Clifford algebras and spinors in different dimensions. Notice in particular, item 2, which relates the Clifford algebra in a spacetime to that on the transverse space to the lightcone. Since they are relations of ungraded tensor products they can be used to build up (ungraded) representations of larger algebras from smaller algebras. For the complex case see **** below.

Exercise
Show that $C\ell((s+1)_+, r_-) \cong C\ell((r+1)_+, s_-)$ as ungraded algebras.

---

c:CliffVolElmt

13.1.4 The Clifford volume element

A key object in discussing the structure of Clifford algebras is the *Clifford volume element*. When V is provided with an orientation this is the canonical element in $C\ell(Q)$ defined by

$$\omega := e_1 \cdots e_d \tag{13.31}$$

where $d = \dim_\kappa V$ and $e_1 \wedge \cdots \wedge e_d$ is the orientation of V. Since there are two orientations there are really two volume elements.

Note that:

Remarks

1. The Clifford volume element ω or ω_c in the complex case (see below) is often referred to as the *chirality operator* in physics, or sometimes as γ_5.

2. For d even, ω is even and anti-commutes with the generators $e_i\omega = -\omega e_i$. Therefore it is neither in the center nor in the ungraded center of $C\ell(Q)$. It is in the ungraded center of the ungraded algebra $C\ell(Q)^0$.

3. For d odd, ω is odd and $e_i\omega = +\omega e_i$. Therefore it is in the *ungraded* center $Z(C\ell(Q))$ but, because it is odd, it is not in the graded center $Z_s(C\ell(Q))$.

4. Thus, ω is never in the supercenter of $C\ell(Q)$. In fact, we will see that the super-center of $C\ell_{r,s}$ is $\mathbb{R}$ and the super-center of $\mathbb{C}\ell_d$ is $\mathbb{C}$.

5. ω^2 is always ± 1 (independent of the orientation). The precise rule is worked out in equation (eq:omsq 13.37) below. Here is the way to remember the result: The sign only depends on the value of $r_+ - s_-$ modulo 4. Therefore we can reduce the question to $C\ell_n$ and the result only depends on n modulo four. For $n = 0\text{mod}4$ the sign is clearly $+1$. For $n = 2\text{mod}4$ it is clearly -1, because $(e_1e_2)^2 = -e_1^2e_2^2 = -1$ as long as e_1^2 and e_2^2 have the same sign. For $C\ell_{+1}$ and $C\ell_{-1}$ it is obviously $+1$ and -1, respectively.

Exercise *The transpose anti-automorphism*

An important anti-automorphism, the *transpose* $\beta : C\ell(Q) \to C\ell(Q)$ is defined as follows: $\beta(1) = 1$ and $\beta(v) = v$ for $v \in V$. Now we extend this to be an anti-automorphism so that $\beta(\phi_1\phi_2) = \phi_2\phi_1$. In particular:

$$\beta(e_{i_1}e_{i_2}\cdots e_{i_k}) = e_{i_k}e_{i_{k-1}}\cdots e_{i_2}e_{i_1} \tag{13.32}$$

Show that:

$$\beta(e_{i_1\cdots i_k}) = e_{i_k\cdots i_1} = (-1)^{\frac{1}{2}k(k-1)}e_{i_1\cdots i_k} \tag{13.33}$$

Note: The functions $f(k) = (-1)^{\frac{1}{2}k(k-1)}$ and $g(k) = (-1)^{\frac{1}{2}k(k+1)}$ appear frequently when doing computation with Clifford algebras. Note that $f(k)$ and $g(k)$ only depend on $k$mod4, $f(k) = g(-k)$ and

$$(-1)^{\frac{1}{2}k(k-1)} = \begin{cases} +1 & k = 0, 1\text{mod}4 \\ -1 & k = 2, 3\text{mod}4 \end{cases} \tag{13.34}$$

eq:qnfen

$$(-1)^{\frac{1}{2}k(k+1)} = \begin{cases} +1 & k = 0, 3\text{mod}4 \\ -1 & k = 1, 2\text{mod}4 \end{cases} \tag{13.35}$$

eq:qnfena

Exercise *The Clifford volume element*

a.) Show that the volume element in $C\ell(r_+, s_-)$

$$\omega = e_1e_2\cdots e_d \tag{13.36}$$

$d = r_+ + s_-$, satisfies

$$\omega^2 = (-1)^{\frac{1}{2}(s_- - r_+)(s_- - r_+ + 1)} = \begin{cases} +1 & for(s_- - r_+) = 0, 3\text{mod}4 \\ -1 & for(s_- - r_+) = 1, 2\text{mod}4 \end{cases} \tag{13.37}$$

eq:omsq

[Answer: The easiest way to compute is to write

$$\omega\cdot\omega = (-1)^{\frac{1}{2}d(d-1)}\omega\beta(\omega) = (-1)^{\frac{1}{2}d(d-1)+s} = (-1)^{\frac{1}{2}(s-r)(s-r+1)}] \tag{13.38}$$

b.) Show that under a change of basis $e^\mu \to f^\mu = \sum g^{\nu\mu}e^\nu$ where $g \in O(Q)$ we have $\omega' = \det g\ \omega$, so that ω indeed transforms as the volume element.

d.) $\omega e^\mu = (-1)^{d+1}e^\mu\omega$. Thus ω is central for d odd and is not central for d even.

Note:

1. $d_T = s_- - r_+$ generalizes the number of dimensions *transverse* to the light cone in Lorentzian geometry.

2. $\omega^2 = 1$ and ω is central only for $d_T = s - r = 3\text{mod}4$.

Exercise *Clifford volume element and Hodge star*

If V is a real vector space with nondegenerate metric then given an orientation we can define a Hodge $*$. This is a linear operator on $\Lambda^* V$ which exchanges $\Lambda^k V$ with $\Lambda^{d-k} V$ such that, on differential forms

$$\omega * \omega = \| \omega \|^2 \operatorname{vol}(g) \tag{13.39}$$

Under the isomorphism (eq:VSISO 13.5) the Hodge $*$ must correspond to a linear operator on $C\ell_{r,-s}$. Find this operator.

♣Give the answer in a footnote. ♣

13.2 Clifford algebras and modules over $\kappa = \mathbb{C}$

13.2.1 Structure of the (graded and ungraded) algebras and modules

Let us begin by considering the low-dimensional examples. We will contrast both the graded and ungraded structures, to highlight the differences.

Of course $\mathbb{C}\ell_0 \cong \mathbb{C}$ is purely even. Nevertheless, as a superalgebra it has two inequivalent irreducible graded modules $M_0^+ \cong \mathbb{C}^{1|0}$ and $M_0^- \cong \mathbb{C}^{0|1}$. As an ungraded algebra it has one irreducible module - the regular representation $N_0 \cong \mathbb{C}$.

Moving on to $\mathbb{C}\ell_1$. As a vector space it is isomorphic to $\mathbb{C}^2$ with the natural basis $\{1, e\}$, so the general element is $z_1 + z_2 e$ with multiplication

$$(z_1 + z_2 e)(z_1' + z_2' e) = (z_1 z_1' + z_2 z_2') + (z_1 z_2' + z_2 z_1') e \tag{13.40}$$

As an exercise the reader should show that this algebra is a simple superalgebra: There are no proper graded ideals. It is also central: The *graded* center is $Z_s(\mathbb{C}\ell_1) \cong \mathbb{C}$. Thus, it is a central simple superalgebra over $\mathbb{C}$. (See the Appendix app:CentralSimple A for the basic definitions of *central* and *simple* superalgebras.)

The algebra $\mathbb{C}\ell_1$ is a two-dimensional vector space and thus cannot be a matrix superalgebra! The latter would be $\mathrm{End}(\mathbb{C}^{n|m})$ and would have complex dimension $(n+m)^2$, but 2 is not a perfect square.

As an *ungraded* algebra

$$\mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C} \qquad\qquad \textit{ungraded}! \tag{13.41}$$

where the RHS is the algebra with multiplication

$$(z_1 \oplus z_2)(z_1' \oplus z_2') = z_1 z_1' \oplus z_2 z_2' \tag{13.42}$$

This is not a simple algebra and $P_\pm = \frac{1}{2}(1 \pm e)$ are orthogonal projectors to the two ideals given by the two summands. Moreover, the ungraded center is the whole algebra. Thus it is *not* an central algebra: Its center contains the ground field as a proper subalgebra.

Let us consider the representations of $\mathbb{C}\ell_1$:

As a superalgebra $\mathbb{C}\ell_1$ has a unique irrep $M_1 \cong \mathbb{C}^{1|1}$. We must represent $\rho(e)$ by an *odd* operator which squares to $+1$. The most general such operator is

$$\rho(e) = x\sigma^1 + y\sigma^2 \qquad\qquad x^2 + y^2 = 1, \qquad x, y \in \mathbb{C} \tag{13.43}$$

But all these choices are equivalent by an even automorphism, hence an invertible element of $\underline{\mathrm{End}}(\mathbb{C}^{1|1})$. Indeed, conjugation by the even transformation $\cos\theta + i\sin\theta\sigma^3$ rotates (x,y) by 2θ. However, we cannot represent $\rho(e)$ by σ^3 because this would not be odd.

On the other hand, as an ungraded algebra $\mathbb{C}\ell_1$ has two inequivalent one-dimensional representations $N_1^{\pm} \cong \mathbb{C}$ with $\rho_{\pm}(e) = \pm 1$.

Now let us move on to $\mathbb{C}\ell_2$. As a superalgebra we can write two inequivalent irreducible modules $M_2^{\pm}$ for $\mathbb{C}\ell_2$ with $M_2^{\pm} \cong \mathbb{C}^{1|1}$ as supervector spaces. Therefore $\mathbb{C}\ell_2 \cong \mathrm{End}(\mathbb{C}^{1|1})$ as a superalgebra. This superalgebra has super-center $Z_s(\mathbb{C}\ell_2) \cong \mathbb{C}$ and is graded-simple. Thus it is a central simple superalgebra. For M_2^- we can take, for example,

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(e_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{13.44}$$ eq:M2-min

and for M_2^+

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(e_2) = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{13.45}$$ eq:M2-plus

The invariant distinction between these is apparent when we look at the volume form: $\omega = e_1 e_2$. This is even, is in the center of $\mathbb{C}\ell_2^0$, and squares to -1. Therefore, in an irrep it should act as a scalar $\pm i$ on the even subspace. That cannot be changed by a superisomorphism.

The above matrix representations also show that, as an ungraded algebra $\mathbb{C}\ell_2$ is isomorphic to $M_2(\mathbb{C}) = \mathbb{C}(2)$. This is a simple algebra with ungraded center $Z \cong \mathbb{C}$. It has a unique irrep $N_2 \cong \mathbb{C}^2$. Note that this simple example already shows us the failure of the identity (13.23) eq:CplxCliff-Tens for the ordinary tensor product. Indeed, with the ordinary tensor product we have

$$\mathbb{C}\ell_1 \otimes_{\mathbb{C}} \mathbb{C}\ell_1 \cong (\mathbb{C} \oplus \mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \tag{13.46}$$ eq:ungradtens

which is an abelian, nonsimple algebra of dimension four.

At this point we have established:

Clifford Algebra	$\mathbb{C}\ell_0$	$\mathbb{C}\ell_{+1}$	$\mathbb{C}\ell_{+2}$
Graded algebra	$\mathbb{C}$	$\mathbb{C}[e], e^2 = 1$	$\mathrm{End}(\mathbb{C}^{1\|1})$
Ungraded algebra	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$M_2(\mathbb{C})$
Graded irreps	$M_0^{\pm} \cong \mathbb{C}^{1\|0}, \mathbb{C}^{0\|1}$	$M_1 \cong \mathbb{C}^{1\|1}$	$M_2^{\pm} \cong \mathbb{C}^{1\|1}, \rho(e_1 e_2)\|_{M_2^{\pm,0}} = \pm i$
Ungraded irreps	$N_0 \cong \mathbb{C}$	$N_1^{\pm} \cong \mathbb{C}, \rho(e) = \pm 1$	$N_2 \cong \mathbb{C}^2$

What about dimensons $n > 2$? Now we can use tensor products to get the general structure.

First, in the superalgebra case we can invoke (13.23). We know how to multiply matrix superalgebras by (12.33). As we have stressed, $\mathbb{C}\ell_1$ is not a matrix superalgebra, but $\mathbb{C}\ell_2$ is. Therefore, since $\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \widehat{\otimes} \mathbb{C}\ell_2$ we have the key fact

$$\boxed{\mathbb{C}\ell_{n+2} \cong \mathrm{End}(\mathbb{C}^{1|1}) \widehat{\otimes} \mathbb{C}\ell_n} \tag{13.47}$$

eq:ComplexPeriodi

Therefore, one can show inductively that

$$\mathbb{C}\ell_{2k} \cong \mathrm{End}(\mathbb{C}^{2^{k-1}|2^{k-1}}) \tag{13.48}$$

eq:CClffd-even

and hence

$$\mathbb{C}\ell_{2k+1} \cong \mathrm{End}(\mathbb{C}^{2^{k-1}|2^{k-1}}) \widehat{\otimes} \mathbb{C}\ell_1 \tag{13.49}$$

eq:CClffd-odd

In both cases they are central simple superalgebras.

Note that

$$\omega^2 = \begin{cases} +1 & n = 0, 1 \mathrm{mod} 4 \\ -1 & n = 2, 3 \mathrm{mod} 4 \end{cases} \tag{13.50}$$

so if we define

$$\omega_c := \begin{cases} \omega & n = 0, 1 \mathrm{mod} 4 \\ i\omega & n = 2, 3 \mathrm{mod} 4 \end{cases} \tag{13.51}$$

then $\omega_c^2 = 1$

For n even there are two irreducible modules $M_n^{\pm} \cong \mathbb{C}^{2^{[\frac{n}{2}]-1}|2^{[\frac{n}{2}]-1}}$. The volume element ω_c is even and therefore can be restricted to the even subspace M^0 of any $\mathbb{C}\ell_n$ module (for n even). Moreover ω_c is central in the even algebra and will therefore be a scalar in an irreducible module. The two irreducible modules are then distinguished by the sign of the volume element ω_c restricted to the even subspace $(M_n^{\pm})^0$. For n odd there is a unique irreducible module $M_n \cong \mathbb{C}^{2^{[\frac{n}{2}]}|2^{[\frac{n}{2}]}}$.

We can use the above results to derive the ungraded algebras. For $n = 2k$ even

$$\mathbb{C}\ell_n \cong \mathrm{End}(\mathbb{C}^{2^k}) \qquad n = 0(2), \qquad \textit{ungraded} \tag{13.52}$$

This is a central simple algebra with a unique irrep $N_{2k} \cong \mathbb{C}^{2^k}$. For $n = 2k+1$ odd

$$\mathbb{C}\ell_n \cong \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}) \qquad n = 1(2), \qquad \textit{ungraded} \tag{13.53}$$

eq:ungradedodd

Now ω_c is (ungraded) central and we can make orthogonal projection operators $P_{\pm} = \frac{1}{2}(1 \pm \omega_c)$ onto the two simple ideals (i.e. the two summands in (13.53). There are two inequivalent representations on $\mathbb{C}^{2^k}$ according to whether ω_c is represented as ± 1.

To summarize, we have:

Clifford Algebra	$\mathbb{C}\ell_{2k}$	$\mathbb{C}\ell_{2k+1}$
Graded algebra	$\mathrm{End}(\mathbb{C}^{2^{k-1}\vert 2^{k-1}})$	$\mathrm{End}(\mathbb{C}^{2^{k-1}\vert 2^{k-1}})\widehat{\otimes}\mathbb{C}\ell_1$
Ungraded algebra	$\mathbb{C}(2^k)$	$\mathbb{C}(2^k)\oplus\mathbb{C}(2^k)$
Graded irreps	$M^{\pm}_{2k}\cong\mathbb{C}^{2^{k-1}\vert 2^{k-1}}$, $\rho(\omega_c)\vert_{M^{\pm,0}_{2k}}=\pm 1$	$M_{2k+1}\cong\mathbb{C}^{2^k\vert 2^k}$
Ungraded irreps	$N_{2k}\cong\mathbb{C}^{2^k}$	$N^{\pm}_{2k+1}\cong\mathbb{C}^{2^k}$, $\rho(\omega_c)=\pm 1$

Remarks

1. The irreducible representation N_{2k} is often called the "Dirac representation."

2. Although $\mathbb{C}\ell_n$ is not supercommutative, it is nevertheless true that $\mathbb{C}\ell_n^{\rm opp}\cong\mathbb{C}\ell_{-n}\cong\mathbb{C}\ell_n$. Therefore we do not need to distinguish left-modules from right-modules. (See exercise ***** above.) In our discussion above we have always implicitly worked with left-modules.

3. A more conceptual way to explain the relation between the ungraded and graded modules is that there is an equivalence of categories between the category $\mathcal{N}_n$ of ungraded modules of $\mathbb{C}\ell_n$ and the category $\mathcal{M}_{n+1}$ of graded modules of $\mathbb{C}\ell_{n+1}$. To go in one direction, if we have $M\in\mathcal{M}_{n+1}$ then M^0 is a module for $\mathbb{C}\ell^0_{n+1}$. But $\mathbb{C}\ell^0_{n+1}\cong\mathbb{C}\ell_n$ as an ungraded module, so we can regard M^0 as an object in $\mathcal{N}_n$. In the other direction we note that $\mathbb{C}\ell_{n+1}$ is a right $\mathbb{C}\ell^0_{n+1}$-module, so given $N\in\mathcal{N}_n$ we can produce a *graded* $\mathbb{C}\ell_{n+1}$-module via

$$\mathbb{C}\ell_{n+1}\otimes_{\mathbb{C}\ell^0_{n+1}}N \tag{13.54}$$

These are inverse functors.

♣Should explain why. ♣

4. It is also interesting to compare the irreducible modules in different dimensions. We can do this by embedding $\iota:\mathbb{C}^k\hookrightarrow\mathbb{C}^{k+1}$ say, by $(z_1,\dots,z_k)\to(z_1,\dots,z_k,0)$. Then it is not hard to show that

$$\iota^*(M^{\pm}_{2k})\cong M_{2k-1}\qquad\qquad \iota^*(M_{2k+1})\cong M^+_{2k}\oplus M^-_{2k} \tag{13.55}$$

eq:RestrictGraded

$$\iota^*(N^{\pm}_{2k+1})\cong N_{2k}\qquad\qquad \iota^*(N_{2k})\cong N^+_{2k-1}\oplus N^-_{2k-1} \tag{13.56}$$

eq:RestrictUnGrad

5. We are now in a position to write explicit matrix representations for the Clifford modules. We use (13.27) as follows. Suppose that γ^i, $i=1,\dots,2k-1$ is an irreducible representation of $\mathbb{C}_{2k-1}$ by complex $2^{k-1}\times 2^{k-1}$ matrices with, say, $(\gamma^i)^2=1$. Since it is irreducible we must have

eq:SkipTwo-4

♣Line this up with the explicit rep from harmonic oscillators given in Section 18 below. ♣ sec:FermionsSp

$$\gamma^{123\cdots(2k-1)}=\gamma^1\gamma^2\cdots\gamma^{2k-1}=z_k 1_{2^{k-1}} \tag{13.57}$$

where z_k is a complex number in $\{\pm 1, \pm i\}$. Now we can produce an irrep of $\mathbb{C}\ell_{2k+1}$ by

$$\begin{aligned}
\Gamma^i &= \gamma^i \otimes \sigma^1 = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} \qquad i = 1, \dots, 2k-1 \\
\Gamma^{2k} &= 1_{2^{k-1}} \otimes \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\Gamma^{2k+1} &= 1_{2^{k-1}} \otimes \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{13.58}$$

eq:evenodd

In this way we build up an explicit irrep in two higher dimensions. For example, if we start out with $\gamma^1 = 1$ in one dimension we build the three Pauli matrices in three dimensions. In general we have

$$\Gamma^{123\cdots(2k+1)} = \Gamma^1\Gamma^2 \cdots \Gamma^{2k+1} = iz_k 1_{2^k} \tag{13.59}$$

In particular, choosing the two values for z_k produces the two irreps of $\mathbb{C}\ell_{2k+1}$ as an ungraded algebra. Starting with $\gamma^1 = z_1 = \pm 1$ one obtains

$$z_k = i^{k-1} z_1 \tag{13.60}$$

As an illustration of (eq:RestrictUnGraded 13.56) the matrices $\Gamma^1, \dots, \Gamma^{2k}$ provide an explicit irreducible representation of $\mathbb{C}\ell_{2k}$. (In fact, it can be also be taken as a graded representation.) [25]

13.2.2 Morita equivalence and the complex K-theory of a point

subsec:Kpoint

Equation (eq:ComplexPeriodicity 13.47) shows that the Morita equivalence classes of complex Clifford algebras have a *mod two periodicity*:

$$[\mathbb{C}\ell_{n+2}] = [\mathbb{C}\ell_n] \tag{13.61}$$

eq:ModTwoMorita

As explained in Appendix app:CentralSimple A there is a group structure on Morita equivalence classes

$$[\mathbb{C}\ell_n] \cdot [\mathbb{C}\ell_m] := [\mathbb{C}\ell_n \widehat{\otimes} \mathbb{C}\ell_m] = [\mathbb{C}\ell_{n+m}] = [\mathbb{C}\ell_{(n+m)\mathrm{mod}2}] \tag{13.62}$$

Therefore, the graded Brauer group of $\mathbb{C}$ is the group $\mathbb{Z}_2$.

♣Need to distinguish Morita equivalence classes from notation for K-theory classes below. ♣

At this point we are at the threshhold of the subject of K-theory. This is a generalization of the cohomology groups of topological spaces. At this point we are only equipped to discuss the "cohomology groups" of a point, but even this involves some interesting ideas.

Let $\mathcal{M}_0$ be the abelian monoid of finite-dimensional complex super-vector-spaces. This is in harmony with our notation above because a finite-dimensional complex supervector space is the same thing as a graded module for $\mathbb{C}\ell_0 = \mathbb{C}$. The monoid operation is direct sum and the identity is the 0 vector space. We consider a submonoid $\mathcal{M}_0^{\mathrm{triv}}$ of supervector

[25]In particle physics courses the logic is exactly the reverse of what we said here. Usually one finds an irreducible representation of γ^μ, with $\mu = 1, 2, 3, 4$ and then discovers that one can introduce $\gamma^5 = \gamma^{1234}$ to give an irreducible representation in five dimensions.

spaces for which there exists an odd invertible operator T. That is, $T \in \mathrm{End}(V)^1$ so that $T : V^0 \to V^1$ is an isomorphism. This is a submonoid because if (V_1, T_1) and (V_2, T_2) are "trivial" then $T_1 \oplus T_2$ "trivializes" $V_1 \oplus V_2$. Now we consider the quotient monoid $\mathcal{M}_0/\mathcal{M}_0^{\mathrm{triv}}$. There is a well-defined sum on equivalence classes:

$$[M_1] \oplus [M_2] := [M_1 \oplus M_2] \tag{13.63}$$

and in the quotient monoid there are additive inverses. The reason is that

$$[M] \oplus [\Pi M] = [M \oplus \Pi M] = 0 \tag{13.64}$$

The second equality holds because the super-linear transformation of $M \oplus \Pi M$ given by $v_1 \oplus v_2 \mapsto v_2 \oplus v_1$ is odd (why ?!?) and obviously invertible. The abelian group $K^0(pt)$ is, by definition,

$$K^0(pt) := \mathcal{M}_0/\mathcal{M}_0^{\mathrm{triv}} \tag{13.65}$$

with the above abelian group structure. Indeed $K^0(pt) \cong \mathbb{Z}$. One way to see that is to define a linear map $\mathcal{M}_0 \to \mathbb{Z}$ via

$$V \mapsto n_e - n_o \tag{13.66}$$

if $V \cong \mathbb{C}^{n_e|n_o}$. Clearly the kernel of this map are supervector spaces isomorphic to $\mathbb{C}^{r|r}$ for some $r \geq 0$. But these are precisely the super-vector spaces in $\mathcal{M}_0^{\mathrm{triv}}$.

Now let us similarly define, for $n > 0$,

$$K^{-n}(pt) := \mathcal{M}_n/\mathcal{M}_n^{\mathrm{triv}}. \tag{13.67}$$

eq:CplxKn-Point

Here $\mathcal{M}_n$ is the monoid of finite-dimensional complex graded modules for $\mathbb{C}\ell_n$. Meanwhile $\mathcal{M}_n^{\mathrm{triv}}$ is the submonoid of $\mathbb{C}\ell_n$-modules M such that there exists an invertible odd operator $T \in \mathrm{End}(M)$ such that T graded-commutes with the $\mathbb{C}\ell_n$-action. The choice of superscript $-n$ instead of $+n$ in (13.67) is related to the connection to algebraic topology, a connection which is far from obvious at this point!

Let us work out some examples of $K^{-n}(pt)$ with $n > 0$.

Consider $K^{-1}(pt)$. Then there is a unique irreducible module M_1 for $\mathbb{C}\ell_1$. We can take $M_1 \cong \mathbb{C}^{1|1}$ with, say, $\rho(e) = \sigma^1$. Then we can introduce the odd invertible operator $T = \sigma^2$ which graded commutes with $\rho(e)$. Therefore $M_1 \in \mathcal{M}_1^{\mathrm{triv}}$ and since $\mathbb{C}\ell_1$ is a super-simple algebra all the modules are direct sums of M_1. Therefore $\mathcal{M}_1^{\mathrm{triv}} = \mathcal{M}_1$ and hence $K^{-1}(pt) \cong 0$.

Next consider $K^{-2}(pt)$. Then there are two irreducible modules $M_2^{\pm}$ for $\mathbb{C}\ell_2$. We can represent $M_2^{\pm}$ as $M_2^{\pm} \cong \mathbb{C}^{1|1}$ together with $\rho(e_1) = \pm\sigma^1$ and $\rho(e_2) = \sigma^2$. Any module will be a direct sum of copies of $M_2^{\pm}$. Now, any odd operator T on $\mathbb{C}^{1|1}$ which anticommutes with σ^1 and σ^2 must vanish. Therefore neither M_2^+ nor M_2^- are in $\mathcal{M}_0^{\mathrm{triv}}$. We should not hastily conclude that $\mathcal{M}_0^{\mathrm{triv}}$ is the zero monoid! Indeed, consider $M_2^+ \oplus M_2^- \cong \mathbb{C}^{2|2}$. Let v_0, v_1 be an ordered basis for M_2^+, with v_0 even and v_1 odd, and similarly let w_0, w_1 be an ordered basis for M_2^- and consider the ordered basis v_0, w_0 for the even subspace of

$M_2^+ \oplus M_2^-$ and v_1, w_1 for the odd subspace of $M_2^+ \oplus M_2^-$. Then in this basis, as a $\mathbb{C}\ell_2$ module we have

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^3 \qquad \rho(e_2) = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = -\sigma^2 \otimes 1 \tag{13.68}$$

Having made these choices notice that we can introduce

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^1 \tag{13.69}$$

which is plainly odd, invertible, and anticommutes with $\rho(e_1)$ and $\rho(e_2)$. (Note that it also cannot be written as a direct sum of operators on M_2^+ and M_2^-, respectively.) Therefore, in $K^{-2}(pt)$ we have $[M_2^-] = -[M_2^+]$. From this it is clearly that, as abelian monoids

$$\mathcal{M}_2 \cong \mathbb{Z}_+ \oplus \mathbb{Z}_+ \tag{13.70}$$

(generated by $M_2^\pm$) while

$$\mathcal{M}_2^{\rm triv} \cong \mathbb{Z}_+ \tag{13.71}$$

(generated by $M_2^+ \oplus M_2^-$). Therefore

$$K^{-2}(pt) \cong \mathbb{Z} \tag{13.72}$$

the isomorphism being given by $[n_+ M_2^+ \oplus n_- M_2^-] \mapsto n_+ - n_-$.

We have gone through this in excruciating detail, but now, thanks to the mod-two periodicity it should be clear that for $n \geq 0$

$$K^{-n}(pt) \cong \begin{cases} \mathbb{Z} & n = 0(2) \\ 0 & n = 1(2) \end{cases} \tag{13.73}$$

eq:KnPoint

Remarks

1. Equation (13.73) should be contrasted with the more familiar (co)homology theory of singular, Cech, or DeRham cohomology. The cohomology groups $H^p(X)$ of a topological space X can be defined for all integers p, but for $X = pt$ only one group is nonzero:

$$H^p(pt) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & \text{else} \end{cases} \tag{13.74}$$

2. There are very many ways to introduce and discuss K-theory. In the original approach of Atiyah and Hirzebruch [8], $K^{-n}(pt)$ was defined in terms of stable isomorphism classes of complex vector bundles on S^n. One of the main points of [7] was the reformulation in terms of Clifford modules, an approach which culminated in the beautiful paper of Atiyah and Singer [9]. We have chosen this approach because it is the one closest to the way K-theory appears in physics. In string theory, T turns out to be the classical value of a tachyon field [42]. In the applications to topological phases of matter T is related to "topologically trivial pairing of particles and holes". See, e.g. [29, 38].

3. In general, given an abelian monoid $\mathcal{M}$ there are two ways to produce an associated abelian group. One, the method adopted here, is to define a submonoid $\mathcal{M}^{\rm triv}$ so that the quotient $\mathcal{M}/\mathcal{M}^{\rm triv}$ admits inverses and hence is a group. A second method, known as the *Grothendieck group* is to consider the produce $\mathcal{M} \times \mathcal{M}$ and divide by an equivalence relation. We say that (a,b) is equivalent to (c,d) if there is an $e \in \mathcal{M}$ with

$$a + d + e = c + b + e \tag{13.75}$$

eq:GrothGrpRel

The idea is that if we could cancel then this would say $a - b = c - d$. Now it is easy to see that the set of equivalence classes $[(a,b)]$ is an abelian group, with $[(a,b)] = -[(b,a)]$. A standard example is that the Grothendieck group of $\mathcal{M} = \mathbb{Z}_+$ produces the integers. Note that if we took $\mathcal{M} = \mathbb{Z}_+ \cup \{\infty\}$ then the Grothendieck group would be the trivial group. This idea actually generalizes to additive categories where we have a notion of sum of objects. In that case (13.75) should be understood to mean that there exists an isomorphism between $a + d + e$ and $c + b + e$. Then one takes the monoid of isomorphism classes of objects to the *Grothendieck group of the category.*

♣Explain the Grothendieck group approach to $K^{-n}(pt)$? ♣

4. In fact, there is more mathematical structure here because we can take graded tensor products of Clifford modules. These induce a product structure on the equivalence classes:

$$[M_1] \cdot [M_2] := [M_1 \widehat{\otimes} M_2] \tag{13.76}$$

This is well-defined because if $M \in \mathcal{M}_n^{\rm triv}$ then $M \widehat{\otimes} M'$ has an odd invertible linear transformation $T \widehat{\otimes} 1$ and hence $M \widehat{\otimes} M' \in \mathcal{M}_n^{\rm triv}$. This allows us to define a graded ring:

$$\oplus_{n \geq 0} K^{-n}(pt) \cong \mathbb{Z}[u] \tag{13.77}$$

where u, known as the *Bott element* can be take to be $u = [M_2^+]$. Note that it has degree two.

13.2.3 Digression: A hint of the relation to topology

Consider a representation of $\mathbb{C}\ell_d$ by anti-Hermitian gamma matrices on a vector space (with basis) V where Γ^μ are such that $\{\Gamma^\mu, \Gamma^\nu\} = -2\delta^{\mu\nu}$, where $\mu = 1, \ldots, d$. Let $\dim_{\mathbb{C}} V = L$.

Suppose $x_0, x_\mu,\ \mu = 1, \dots, d$ are functions on the unit sphere S^d embedded in $\mathbb{R}^{d+1}$, so

$$x_0^2 + x_\mu x_\mu = 1 \tag{13.78}$$

Consider the matrix-valued function

$$T(x) := x_0 1 + x_\mu \Gamma^\mu \tag{13.79}$$ eq:tachfld

Note that

$$T(x)T(x)^\dagger = 1 \tag{13.80}$$

and therefore $T(x)$ is a unitary matrix for every $(x_0, x_\mu) \in S^d$. We can view $T(x)$ as describing a continuous map $T : S^d \to U(L)$. Therefore it defines an element of the homotopy group $[T] \in \pi_d(U(L))$. The following examples show that the homotopy class of the map can be nontrivial:

Example 1: If $d = 1$ then we could take either of the ungraded irreducible representations $V = \mathbb{C}$ and $\Gamma = \pm i$. If $x_0^2 + x_1^2 = 1$ then

$$T^\pm(x) = x_0 \pm i x_1 \tag{13.81}$$

and, for either choice of sign, $[T^\pm]$ is a generator of $\pi_1(U(1)) = \mathbb{Z}$.

Example 2: If $d = 3$ then we may choose either of the ungraded representations $V = \mathbb{C}^2$ and $\Gamma^i = \pm\sqrt{-1}\sigma^i$ and then

$$T(x) = x_0 + x_i \Gamma^i \tag{13.82}$$

is one way to parametrize $SU(2)$. Thus the map $T : S^3 \to SU(2)$ is the identity map (with the appropriate orientation on S^3). If we fix a an orientation on S^3 we get winding number ± 1 and hence $[T^\pm]$ is a generator of $\pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Here is one easy criterion for triviality of $[T]$: Suppose we can introduce another anti-Hermitian $L \times L$ gamma matrix on V, call it Γ, so that $\Gamma^2 = -1$ and $\{\Gamma, \Gamma^\mu\} = 0$. Now consider the unit sphere

$$S^{d+1} = \{(x_0, x_\mu, y) | x_0^2 + \sum_{\mu=1}^{d} x_\mu x_\mu + y^2 = 1\} \subset \mathbb{R}^{d+2} \tag{13.83}$$

Then we can define

$$\tilde{T}(x, y) = x_0 + x_\mu \Gamma^\mu + y\Gamma \tag{13.84}$$

When restricted to $S^{d+1} \subset \mathbb{R}^{d+2}$, $\tilde{T}$ is also unitary and maps $S^{d+1} \to U(L)$. Moreover $\tilde{T}(x, 0) = T(x)$ while $\tilde{T}(0, 1) = \Gamma$. Thus $\tilde{T}(x, y)$ provides an explicit homotopy of $T(x)$ to the constant map.

Thus, *if the representation V of $\mathbb{C}\ell_d$ is the restriction of a representation of $\mathbb{C}\ell_{d+1}$ then $T(x)$ is automatically homotopically trivial.*

Let us see what this means if we combine it with what we learned above about the irreducible ungraded representations of $\mathbb{C}\ell_d$.

Figure 9: The map on the equator extends to the northern hemisphere, and is therefore homotopically trivial. fig:clffstpa

1. If $d = 2p$ we have irrep $N_{2p} \cong \mathbb{C}^{2^p}$. It is indeed the restriction of $N^{\pm}_{2p+1} \cong \mathbb{C}^{2^p}$ and hence, $T(x)$ must define a trivial element of $\pi_{2p}(U(L))$, with $L = 2^p$.

2. On the other hand, if $d = 2p+1$ then $N^{\pm}_{2p+1} \cong \mathbb{C}^{2^p}$ is *not* the restriction of $N_{2p+2} \cong \mathbb{C}^{2^{p+1}}$. All we can conclude from what we have said above is that $T(x)$ *might* define a homotopically nontrivial element of $\pi_{2p+1}(U(L))$ with $L = 2^p$. On the other hand, if we had used $V = N^{+}_{2p+1} \oplus N^{-}_{2p+1}$ then since V is the restriction of the representation N_{2p+2} and $T = T^+ \oplus T^-$, it follows that the homotopy classes satisfy $[T^-] = -[T^+]$.

Now, a nontrivial result of ABS [7] is:[26]

Theorem[Atiyah, Bott, Shapiro]. If V is an irreducible representation of $\mathbb{C}\ell_d$ then then $[T]$ generates $\pi_d(U(L))$.

It therefore follows that $\pi_{2p}(U(L)) = 0$ and $\pi_{2p+1}(U(L)) \cong \mathbb{Z}$, with generator $[T^+]$ or $[T^-]$.

These facts are compatible with the statement in topology that

$$\pi_{2p-1}(U(N)) = \mathbb{Z} \qquad N \geq p \tag{13.85}$$ eq:piodd

$$\pi_{2p}(U(N)) = 0 \qquad N > p \tag{13.86}$$ eq:pieven

Note that these equations say that for N sufficiently large, the homotopy groups do not depend on N. These are called the *stable homotopy groups* of the unitary groups and can be denoted $\pi_k(\mathbf{U})$. The mod two periodicity of $\pi_k(\mathbf{U})$ as a function of k is known as Bott periodicity.

To make the connection to vector bundles on spheres we use the above matrix-valued functions as transition functions in the clutching construction.

Now we recall from the theory of fiber bundles the following

Theorem. If $d > 1$ and G is connected then principal G-bundles on S^d are topologically classified by $\pi_{d-1}(G)$, i.e. there is an isomorphism of sets:

$$\mathrm{Prin}_G(S^d) \cong \pi_{d-1}(G)$$

[26]More precisely, they used the above $T(x)$ to define a K-theoretic Thom class. Then the result we have stated follows from the relation of K-theory to homotopy theory.

It follows from this theorem that, for $N > d/2$ we have

$$\mathrm{Vect}_N(S^d) \cong \begin{cases} \mathbb{Z} & d = 0(2) \\ 0 & d = 1(2) \end{cases} \tag{13.87}$$

where $\mathrm{Vect}_N(S^d)$ is the set of isomorphism classes of rank N complex vector bundles over S^d.

One way to measure the integer is via a characteristic class known as the Chern character $\mathrm{ch}(E) \in H^{2*}(X;\mathbb{Q})$. If we put a connection on the bundle then we can write an explicit representative for the image of $\mathrm{ch}(E)$ in DeRham cohomology. Locally the connection is an anti-hermitian matrix-valued 1-form A. It transforms under gauge transformations like

$$(d + A) \to g^{-1}(d + A)g \tag{13.88}$$

The fieldstrength is

$$F = dA + A^2 \tag{13.89}$$

and is locally an anti-hermitian matrix-valued 2-form transforming as $F \to g^{-1}Fg$. Then, in DeRham cohomology

$$\mathrm{ch}(E) = [\mathrm{Trexp}\left(\frac{F}{2\pi \mathrm{i}}\right)] \tag{13.90}$$

and the topological invariant is measured by

$$\int_{S^d} \mathrm{ch}(E) \tag{13.91}$$

Note that since $\mathrm{ch}(E)$ has even degree this only has a chance of being nonzero for d even. On a bundle with transition function g on the equator we can take $A = rg^{-1}dg$ on the northern hemisphere, where $g(x)$ is a function only of the "angular coordinates" on the hemisphere and $A = 0$ on the southern hemisphere. Note that thanks to the factor of r, which vanishes at the north pole this defines a first-order differentiable connection. For this connection the fieldstrength is

$$F = drg^{-1}dg - r(1-r)(g^{-1}dg)^2 \tag{13.92}$$

and hence if $d = 2\ell$

$$\begin{aligned} \int_{S^{2\ell}} \mathrm{ch}(E) &= (-1)^{\ell-1}\frac{1}{(2\pi \mathrm{i})^\ell(\ell-1)!}\int_0^1 (r(1-r))^{\ell-1}dr \int_{S^{2\ell-1}} \mathrm{Tr}(g^{-1}dg)^{2\ell-1} \\ &= (-1)^{\ell-1}\frac{(\ell-1)!}{(2\pi \mathrm{i})^\ell(2\ell-1)!}\int_{S^{2\ell-1}} \mathrm{Tr}(g^{-1}dg)^{2\ell-1} \end{aligned} \tag{13.93}$$

The integral of the Maurer-Cartan form over the equator measures the homotopy class of the transition function g. It is not at all obvious that this integral will be an integer, but for $U(N)$ and the trace in the N it is. This is a consequence of the Atiyah-Singer index theorem.

Note that from the viewpoint of vector bundles there is no obvious abelian group operation on $\mathrm{Vect}_N(S^d)$, despite the fact that in this isomorphism of sets the RHS has a structure of an abelian group. We can of course take direct sum, but this operation changes the rank.

It is fruitful to consider the abelian monoid obtained by taking the direct sum

$$\mathrm{Vect}(S^d) := \oplus_{N\geq 0}\mathrm{Vect}_N(S^d). \tag{13.94}$$

As mentioned above, we can immediately obtain an abelian group by using the Grothendieck construction. More generally, consider the Grothendieck construction applied to $\mathrm{Vect}(X)$ for any topological space X. We consider equivalence classes $[(E_1, E_2)]$ where $[(E_1, E_2)] = [(F_1, F_2)]$ if there exists a G with

$$E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G \tag{13.95}$$

Intuitively, we think of $[(E_1, E_2)]$ as a difference $E_1 - E_2$. The Grothendieck group of $\mathrm{Vect}(X)$ is the original Atiyah-Hirzebruch definition of $K^0(X)$.

Example: If we consider from this viewpoint the K-theory of a point $K^0(pt)$ then we obtain the abelian group $\mathbb{Z}$, the isomorphism being $[(E_1, E_2)] \to \mathrm{dim}E_1 - \mathrm{dim}E_2$.

For vector bundles the Grothendieck construction can be considerably simplified thanks to the Serre-Swan theorem:

Theorem[Serre; Swan] Any vector bundle [27] has a complementary bundle so that $E \oplus E^\perp \cong \theta_N$ is a trivial rank N bundle for some N. Equivalently, every bundle is a subbundle of a trivial bundle defined by a continuous family of projection operators.

This leads to the notion of *stable equivalence* of vector bundles: Two bundles E_1, E_2 are *stably equivalent* if there exist trivial bundles θ_s of rank s so that

$$E_1 \oplus \theta_{s_1} \cong E_2 \oplus \theta_{s_2} \tag{13.96}$$

Example: A very nice example, in the category of real bundles is the tangent bundle of S^2. The real rank two bundle TS^2 is topologically nontrivial. You can't comb the hair on a sphere. However, if we consider $S^2 \subset \mathbb{R}^3$ the normal bundle is a real rank one bundle and is trivial. But that means $TS^2 \oplus \theta_1 \cong \theta_3$. So TS^2 is stably trivial.

Returning to the general discussion. In the difference $E_1 - E_2$ we can add and subtract the complementary bundle to get $(E_1 \oplus E_2^\perp) - \theta_N$ for some N. If we restrict the bundle to any point we get an element of $K^0(pt)$. By continuity, it does not matter what point we choose, provided X is connected.

In other words, there is a homomorphism

$$K^0(X) \to K^0(pt) \tag{13.97}$$

[27] over a suitable nice topological space

The kernel of this homomorphism is, by definition, $\tilde{K}^0(X)$. We can represent it by formal differences of the form $E - \theta_N$ where $N = \mathrm{rank}(E)$.

For spheres, we have

$$\tilde{K}^0(S^d) = \begin{cases} \mathbb{Z} & d = 0(2) \\ 0 & d = 1(2) \end{cases} \tag{13.98}$$

and this is the abelian group which is to be compared with the group (eq:KnPoint 13.73) defined above.

♣Explain a little more that ch is a homomorphism to deduce this isomorphism of groups. ♣

Remarks

1. We can nicely tie together the relation to projected bundles by noting that if Γ^i are *Hermitian* matrices then $P^{\pm} = \frac{1}{2}(1 \pm x^i\Gamma^i)$ are projection operators on spheres. Therefore, consider the relation between irreducible representations of $\mathbb{C}\ell_{2k-1}$ and $\mathbb{C}\ell_{2k+1}$ given in (eq:evenodd 13.58). Let $\mu = 1, \dots, 2k+1$ and consider the projection operators

$$P_{\pm}(X_\mu) := \frac{1}{2}(1 + X_\mu \Gamma^\mu) \tag{13.99}$$

acting on the trivial bundle $S^{2k} \times V$ where $V = \mathbb{C}^{2^k}$.

We now define two bundles $V_{\pm} \to S^{2k}$ of rank 2^{k-1} which are the images of the projection operators $P_{\pm}$, respectively.

Focus on V_+ which is the image of P_+. Let us compute a trivialization on the two hemispheres and compute the transition function. Write the coordinates as

$$X^\mu = (x^i, x^{2k}, y) \tag{13.100}$$

Choose a basis v_α, $\alpha = 1, \dots, 2^{k-1}$ for the irrep of $\mathbb{C}\ell_{2k-1}$. Then

$$\begin{pmatrix} v_\alpha \\ 0 \end{pmatrix} \tag{13.101}$$

is a trivialization of the bundle V_+ at the north pole $y = 1$. Indeed:

$$P_+ \begin{pmatrix} v_\alpha \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+y)v_\alpha \\ (\gamma^i x^i + i x^{2k}) v_\alpha \end{pmatrix} \tag{13.102}$$

Similarly,

$$\begin{pmatrix} 0 \\ v_\alpha \end{pmatrix} \tag{13.103}$$

provides a trivialization at the south pole $y = -1$ and

$$P_+ \begin{pmatrix} 0 \\ v_\alpha \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\gamma^i x^i - i x^{2k}) v_\alpha \\ (1-y)v_\alpha \end{pmatrix} \tag{13.104}$$

The transition function at $y = 0$ is, essentially, the unitary matrix

$$T(x) = x^{2k} + i\gamma^i x^i \tag{13.105}$$

which is where we began our discussion above. This construction generalizes the standard constructions of the magnetic monopole and instanton bundles on S^2 and S^4, respectively. Indeed, the projected connections on $V_\pm$ define the basic (anti)monopole and (anti)instanton connections.

♣Need to improve notation here! ♣

2. Now, as in our discussion using Clifford modules, there is another approach where we consider an abelian monoid and divide by a submonoid of "trivial" elements. As we mentioned, the latter viewpoint is closer to the physics. The abelian monoid consists of isomoprhism classes of $\mathbb{Z}_2$-graded bundles equipped with odd operators. The trivial submonoid are those with invertible odd operators. Very roughly speaking the difference $E^0 - E^1$ in the Grothendieck construction is to be compared with a $\mathbb{Z}_2$-graded bundle $\mathcal{E}$ with an odd operator $T \in \mathrm{End}(\mathcal{E})$ so that $E^0 \cong \mathrm{ker}T$ and $E^1 \cong \mathrm{cok}T$. Introducing Hermitian structures we have $E^1 \cong \mathrm{ker}T^\dagger$ so the picture is that

♣Should tie this into Bott periodicity: A loop of projectors is a unitary and a loop of unitaries is a projector. ♣

$$\mathcal{E}^0 - \mathcal{E}^1 \cong (\mathrm{ker}T|_{\mathcal{E}^0} \oplus (\mathrm{ker}T|_{\mathcal{E}^0})^\perp) - (\mathrm{ker}T^\dagger|_{\mathcal{E}^1} \oplus (\mathrm{ker}T^\dagger|_{\mathcal{E}^1})^\perp). \tag{13.106}$$

Now T provides a bundle isomorphism between $\mathrm{ker}T|_{\mathcal{E}^0})^\perp$ and $\mathrm{ker}T^\dagger|_{\mathcal{E}^1})^\perp$, so these can be canceled.

13.3 Real Clifford algebras and Clifford modules of low dimension

In this section we consider the real Clifford algebras $C\ell_n$ for $|n| \le 4$. We also describe their irreducible modules and hence the abelian monoid $\mathcal{M}_n$ of isomorphism classes of $\mathbb{Z}_2$-graded representations.

♣Should go back to previous section and change notation to $\mathcal{M}_n^c$. etc. ♣

13.3.1 dim$V = 0$

Already for $C\ell_0 \cong \mathbb{R}$ there is a difference between graded and ungraded modules. There is a unique irreducible ungraded module, namely $\mathbb{R}$ acting on itself. But there are two inequivalent graded modules, $\mathbb{R}^{1|0}$ and $\mathbb{R}^{0|1}$.

13.3.2 dim$V = 1$

As in the complex case, $C\ell_{\pm 1}$ cannot be a matrix superalgebra for simple dimensional reasons. It therefore defines a new Morita equivalence class. Unlike the complex case we need to distinguish the cases where $e^2 = \pm 1$.

As a vector space $C\ell_{+1}$ is

$$C\ell_{+1} = \mathbb{R} \oplus \mathbb{R}e \tag{13.107}$$

the algebra structure is:

$$(a \oplus be)(c \oplus de) = (ac + bd) \oplus (bc + ad)e. \tag{13.108}$$

As an ungraded algebra this is sometimes known as the "double numbers." As an ungraded algebra $C\ell_{+1} \cong \mathbb{R} \oplus \mathbb{R}$ because we can introduce projection operators $P_\pm = \frac{1}{2}(1 \pm e)$, so

$$C\ell_{+1} \cong \mathbb{R}P_+ \oplus \mathbb{R}P_- \qquad \text{ungraded!} \tag{13.109}$$

However, as a graded algebra there is a unique irreducible representation, $\tilde{\eta}$. As a graded vector space $\tilde{\eta} = \mathbb{R}^{1|1}$ and, WLOG, we can take

$$\rho(e) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13.110}$$

Note that e is odd and squares to 1. Since the irrep is unique up to isomorphism ♣Really should prove this. ♣

$$\mathcal{M}_1 \cong \mathbb{Z}_+ \tilde{\eta} \tag{13.111}$$

In the ungraded case there are *two* inequivalent ungraded irreducible representations $N_1^{\pm} \cong \mathbb{R}$ with $\rho(e) = \pm 1$.

Similarly, $C\ell_{-1}$ has a single generator e with relation $e^2 = -1$. Therefore

$$C\ell_{-1} = \mathbb{R} \oplus \mathbb{R}e \tag{13.112}$$

as a vector space. The multiplication is

$$(a \oplus be)(c \oplus de) = (ac - bd) \oplus (bc + ad)e \tag{13.113}$$

so $C\ell_{-1}$ is isomorphic to the complex numbers $\mathbb{C}$ as an *ungraded algebra*, although not as a graded algebra.

As a graded algebra $C\ell_{-1}$ has a unique irreducible representation η which is, as a super-vector space is $\eta = \mathbb{R}^{1|1}$, but now with

$$\rho(e) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := \epsilon = -i\sigma^2 \tag{13.114}$$

We therefore have:

$$\mathcal{M}_{-1} \cong \mathbb{Z}_+ \eta \tag{13.115}$$

As an ungraded algebra $C\ell_{-1}$ has a unique ungraded irreducible representation: $N_{-1} = \mathbb{C}$ acts on itself. (As representations of a real algebra $\rho(e) = \pm i$ are equivalent.)

Remark: As with $\mathbb{C}\ell_1$, both $C\ell_{-1}$ and $C\ell_{+1}$ are *commutative* as ungraded algebras but *noncommutative* as superalgebras. Thus the centers of these as ungraded algebras are $C\ell_{\pm 1}$ but the supercenter of $C\ell_{\pm 1}$ as graded algebras are $Z_s(C\ell_{\pm 1}) \cong \mathbb{R}$.

13.3.3 dim$V = 2$

Now, $C\ell_{1,-1}$ has two irreducible graded representations $\mathbb{R}_{\pm}^{1|1}$ with

$$\rho(e_1) = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := \epsilon \tag{13.116}$$

Note that these are both odd, they anticommute, and they square to ± 1, respectively. Moreover, they generate all linear transformations on $\mathbb{R}^{1|1}$:

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+d & b+c \\ b-c & a-d \end{pmatrix} \tag{13.117}$$

and the algebra is that of $\mathbb{R}(2)$. Therefore, $C\ell_{1,-1}$ is a supermatrix algebra:

$$C\ell_{1,-1} \cong \mathrm{End}(\mathbb{R}^{1|1}) \tag{13.118}$$

It is interesting to compare this with $C\ell_{+2}$. We claim that $C\ell_{+2}$ is *not* equivalent to a matrix superalgebra. This is no longer immediately clear from dimensional reasons. However, the only possibility would be $\mathrm{End}(\mathbb{R}^{1|1})$ for dimensional reasons. Now WLOG we can take

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13.119}$$

But then what do we take for $\rho(e_2)$? It must be odd, and it must anticommute with $\rho(e_1)$. The only possibility is a real multiple of the matrix $\epsilon = -i\sigma^2$. But this matrix squares to -1. Hence $C\ell_{+2}$ is not a matrix superalgebra.

As an *ungraded* algebra we can write a faithful representation of $C\ell_{+2}$:

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \rho(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.120}$$

eq:epsdef

since these matrices anticommute and both square to $+1$. These generate the full matrix algebra $M_2(\mathbb{R})$ as an ungraded algebra: Note that

$$e^1e^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{13.121}$$

Now we can write an arbitrary 2×2 real matrix as a linear combination of $1, \sigma_1, \sigma_3, -i\sigma_2$:

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+c & b-d \\ b+d & a-c \end{pmatrix} \tag{13.122}$$

However, if we try to use the operators (eq:epsdef 13.120) on $\mathbb{R}^{1|1}$ this is *not* a representation of $C\ell_{+2}$ as a graded algebra because $\rho(e_2)$ is not odd.

One can show that there is a unique irreducible representation of $C\ell_{+2}$ as a superalgebra. One way to construct it is to take the graded tensor product $\tilde{\eta}^2 := \tilde{\eta}\widehat{\otimes}\tilde{\eta}$. As a vector space this is $\mathbb{R}^{2|2}$. Since we take the graded tensor product we must be careful about signs, and we cannot just take the usual tensor product of the matrix representations. Thus let v_0, v_1 be even, odd basis elements of η with

$$\begin{aligned} \rho(e)v_0 &= v_1 \\ \rho(e)v_1 &= v_0 \end{aligned} \tag{13.123}$$

Now to take the graded tensor product let w_0, w_1 be a corresponding basis for the second factor. Then choose an ordered basis for $\tilde{\eta}^2$ to be

$$\{v_0\widehat{\otimes}w_0, v_1\widehat{\otimes}w_1, v_0\widehat{\otimes}w_1, v_1\widehat{\otimes}w_0\} \tag{13.124}$$

A little computation shows that

$$\rho(e_1) = \rho(e)\widehat{\otimes}1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^1 \tag{13.125}$$

$$\rho(e_2) = 1\widehat{\otimes}\rho(e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^3 \tag{13.126}$$

One can show that

$$\mathcal{M}_{+2} \cong \mathbb{Z}_+\tilde{\eta}^2 \tag{13.127}$$

Now consider the opposite algebra $C\ell_{-2}$. Again, one can show it is not a matrix superalgebraby an argument analogous to that we gave for $C\ell_{+2}$. As an ungraded algebra we can write an isomorphism with $\mathbb{H}$ by sending the generators to imaginary unit quaternions:

$$\begin{aligned} e^1 &\to \mathfrak{i} \\ e^2 &\to \mathfrak{j} \\ e^1e^2 &\to \mathfrak{k} \end{aligned} \tag{13.128}$$

Therefore, as an ungraded algebra

$$C\ell_{-2} \cong \mathbb{H} \tag{13.129}$$

Once again there is a unique irreducible graded module up to isomorphism which we can identify with $\eta^2 := \eta\widehat{\otimes}\eta$:

$$\mathcal{M}_{-2} \cong \mathbb{Z}_+\eta^2 \tag{13.130}$$

The reader should do the analogous computation to what we did for $\tilde{\eta}^2$ and show that for η^2 the representation is

$$\rho(e_1) = \rho(e)\widehat{\otimes}1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma^1 \otimes \epsilon \tag{13.131}$$

$$\rho(e_2) = 1\widehat{\otimes}\rho(e) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \epsilon \otimes 1 \tag{13.132}$$

Remarks

1. When we complexify there is no distinction between the signatures. Any of the above three algebras can be used to show that

$$\mathbb{C}\ell(2) \cong \mathbb{C}(2) \tag{13.133}$$

`eq:ccxclff`

2. The above representation of $C\ell_{1,-1}$ is useful for describing a Majorana-Weyl fermion in $1+1$ dimensions. Let us modify it slightly and write

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.134}$$

For example, the $1+1$ Dirac equation on $1+1$ dimensional Minkowski space $\mathbb{M}^{1,1}$

$$(\gamma^0\partial_0 + \gamma^1\partial_1)\Psi = 0 \tag{13.135}$$

can be rewritten as

$$\begin{pmatrix} \partial_0 - \partial_1 & 0 \\ 0 & \partial_0 + \partial_1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0 \tag{13.136}$$

so that ψ_+ is a left-mover and $\psi_=$ is a right-mover. This explains why the volume element $\gamma^0\gamma^1$ is called the "chirality operator."

Exercise *Even subalgebra*

a.) Using the ungraded representation of $C\ell_{-2}$ in (eq:epsdef 13.120) show that the even subalgebra of $C\ell(2,0)$ is the algebra of matrices:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{13.137}$$

and is isomorphic to $\mathbb{C}$.

b.) Show that the even subalgebra of $C\ell_{\pm 2}$ is isomorphic to $\mathbb{C}$.

c.) Show that we can identify $C\ell_{\pm 2}$ as $\mathbb{C}[\varepsilon_\pm]$ where ε is odd, $\varepsilon_\pm^2 = \pm 1$ and $z\varepsilon = \varepsilon\bar{z}$.

Exercise

We have now obtained two algebra structures on the vector space $\mathbb{R}^4$: $\mathbb{R}(2)$ and $\mathbb{H}$. Are they isomorphic? (Hint: Is $\mathbb{R}(2)$ a division algebra?)

Exercise *Representations of Clifford algebras*

Show that

$$\rho(e_1) = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \qquad \rho(e_2) = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \tag{13.138}$$

is a graded representation of $C\ell_{+2}$ on $\mathbb{R}^{2|2}$. Show that it is equivalent to the one given above.

Exercise
Show that η^2 gives an irreducible graded representation of $C\ell_{-2}$.

13.3.4 $\dim V = 3$

Again, as graded algebras $C\ell_{\pm 3}$ are not matrix superalgebras, simply for dimensionful reasons. We might ask whether they are matrix algebras over simpler superalgebras. For dimensional reasons the only possibility would be $\mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes} C\ell_{\pm 1}$. However, the latter contain the element

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{13.139}$$

which is nonzero, even, and noninvertible. However, an argument along the lines of (14.1) eq:INV-cl2-1 below shows that in $C\ell_{\pm 3}$ any nonzero even element is in fact invertible.

Nevertheless, $C\ell_{\pm 3}$ can be expressed more simply as follows. We claim that, as graded algebras:

$$C\ell_{\pm 3} \cong \mathbb{H}\widehat{\otimes} C\ell_{\mp 1} \tag{13.140}$$ eq:Cl13H

where $\mathbb{H}$ is purely even. This is easily proved by mapping the generators according to

$$\begin{aligned} e_1 &\to \mathfrak{i} \otimes e \\ e_2 &\to \mathfrak{j} \otimes e \\ e_3 &\to \mathfrak{k} \otimes e \end{aligned} \tag{13.141}$$

Note that since $\mathbb{H}$ is purely even this map of generators is even, as it must be, and moreover preserves the Clifford relations.

As ungraded algebras we can use the tensor product rules and the isomorphisms already established to conclude

$$\begin{aligned} C\ell_3 &\cong C\ell_{-1} \otimes C\ell_2 \\ &\cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}(2) \\ &\cong \mathbb{C}(2) \end{aligned} \tag{13.142}$$ eq:clthrp

$$\begin{aligned} C\ell_{-3} &\cong C\ell_1 \otimes C\ell_{-2} \\ &\cong (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \\ &\cong \mathbb{H} \oplus \mathbb{H} \end{aligned} \tag{13.143}$$ eq:clthrm

Note particularly, that, as an ungraded algebra $C\ell_{-3}$ is not a simple algebra. The reason is that we can introduce projection operators

$$P_{\pm} = \frac{1}{2}(1 \pm e_1 e_2 e_3) \tag{13.144}$$

One can show that once again

$$\mathcal{M}_{+3} \cong \mathbb{Z}_{+}\tilde{\eta}^3 \tag{13.145}$$

$$\mathcal{M}_{-3} \cong \mathbb{Z}_{+}\eta^3 \tag{13.146}$$

Note that, as a vector space $\eta^3 \cong \mathbb{R}^{4|4}$, as is $\tilde{\eta}^3$.

Remark: Again we can illustrate the failure of the ungraded version of (13.16) (a failure which we already pointed out in (13.46)). Note that as ungraded algebras with ungraded tensor product

$$C\ell_1 \otimes_{\mathbb{R}} C\ell_{-1} \cong \mathbb{C} \oplus \mathbb{C} \tag{13.147}$$

but $C\ell_{1,-1} \cong \mathbb{R}(2)$, as an ungraded algebra. Moreover

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{13.148}$$

To prove this, simply note that we have projection operators $P_{\pm} = \frac{1}{2}(1\otimes 1 \pm i \otimes i)$. Therefore, as ungraded algebras

$$C\ell_{-1} \otimes_{\mathbb{R}} C\ell_{-1} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{13.149}$$

but in fact $C\ell_{-2} \cong \mathbb{H}$ as an ungraded algebra. Finally,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \tag{13.150}$$

This follows since the usual Pauli matrices (together with the identity) form a basis of all 2×2 matrices over the complex numbers. Therefore

$$C\ell_{-1} \otimes_{\mathbb{R}} C\ell_{-2} \cong \mathbb{C}(2) \tag{13.151}$$

as an ungraded algebra, but we just showed above that $C\ell_{-3} \cong \mathbb{H} \oplus \mathbb{H}$, as an ungraded algebra. I hope at this point that it is clear that the graded viewpoint is both useful and more elegant.

Exercise

Write $C\ell_{1_+,2_-}$ and $C\ell_{1_-,2_+}$ in terms of simpler superalgebras.

13.3.5 dim$V = 4$

Now, something important and interesting happens when we reach $C\ell_{\pm 4}$.

Now we can show that, as graded algebras $C\ell_{\pm 4} \cong \mathrm{End}(\mathbb{R}^{1|1}) \otimes \mathbb{H}$ with $\mathbb{H}$ purely even. We do this by exhibiting explicit graded isomorphisms as follows:

For $C\ell_{+4}$ we use:

$$\begin{aligned} e_1 &\to \kappa \begin{pmatrix} 0 & \mathfrak{i} \\ -\mathfrak{i} & 0 \end{pmatrix} \\ e_2 &\to \kappa \begin{pmatrix} 0 & \mathfrak{j} \\ -\mathfrak{j} & 0 \end{pmatrix} \\ e_3 &\to \kappa \begin{pmatrix} 0 & \mathfrak{k} \\ -\mathfrak{k} & 0 \end{pmatrix} \\ e_4 &\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{13.152}$$

eq:cliff4+

where $\kappa \in \{\pm 1\}$. This defines two irreducible graded modules $\tilde{\mu}^{\pm}$ which are isomorphic to $\mathbb{R}^{4|4}$ as supervector spaces. (One should think of them as $\mathbb{R}^{1|1}\widehat{\otimes}\mathbb{H}$.) The invariant distinction is that $\rho(e_1e_2e_3e_4) = \kappa = \pm 1$ on the even subspace.

Similarly, for $C\ell_{-4}$:

$$\begin{aligned} e_1 &\to \kappa \begin{pmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{pmatrix} \\ e_2 &\to \kappa \begin{pmatrix} 0 & \mathfrak{j} \\ \mathfrak{j} & 0 \end{pmatrix} \\ e_3 &\to \kappa \begin{pmatrix} 0 & \mathfrak{k} \\ \mathfrak{k} & 0 \end{pmatrix} \\ e_4 &\to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{13.153}$$

eq:cliff4-

The two cases $\kappa = \pm 1$ define two modules $\mu^{\pm}$.

One can show that

$$\mathcal{M}_4 \cong \mathbb{Z}_+\tilde{\mu}^+ \oplus \mathbb{Z}_+\tilde{\mu}^- \tag{13.154}$$

$$\mathcal{M}_{-4} \cong \mathbb{Z}_+\mu^+ \oplus \mathbb{Z}_+\mu^- \tag{13.155}$$

lCliffIrrepSum

13.3.6 Summary

It is time to summarize what we have learned about the graded and ungraded irreps of the low dimensional real Clifford algebras. (Here $\varepsilon_{\pm}$ is odd and $\varepsilon_{\pm}^2 = \pm 1$. See exercise above):

Clifford Algebra	Ungraded algebra	Graded algebra	Ungraded irreps	Graded irreps
$C\ell_{+4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\vert 1})\otimes\mathbb{H}$	$\mathbb{H}^2$	$\tilde{\mu}^{\pm}$
$C\ell_{+3}$	$\mathbb{C}(2)$	$\mathbb{H}\widehat{\otimes}\mathbb{R}[\varepsilon_-]$	$\mathbb{C}^2$	$\tilde{\eta}^3$
$C\ell_{+2}$	$\mathbb{R}(2)$	$\mathbb{C}[\varepsilon_+]$, $z\varepsilon_+ = \varepsilon_+\bar{z}$	$\mathbb{R}^2$	$\tilde{\eta}^2$
$C\ell_{+1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}[\varepsilon_+]$	$\mathbb{R}_{\pm}, \rho(e) = \pm 1$	$\tilde{\eta}$
$C\ell_0$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^{1\vert 0}, \mathbb{R}^{0\vert 1}$
$C\ell_{-1}$	$\mathbb{C}$	$\mathbb{R}[\varepsilon_-]$	$\mathbb{C}$	η
$C\ell_{-2}$	$\mathbb{H}$	$\mathbb{C}[\varepsilon_-]$, $z\varepsilon_- = \varepsilon_-\bar{z}$	$\mathbb{H}$	η^2
$C\ell_{-3}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}\widehat{\otimes}\mathbb{R}[\varepsilon_+]$	$\mathbb{H}_{\pm}$, $\rho(e_1e_2e_3) = \pm 1$	η^3
$C\ell_{-4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\vert 1})\otimes\mathbb{H}$	$\mathbb{H}^2$	$\mu^{\pm}$

:PeriodTheorem

13.4 The periodicity theorem

The fact that $C\ell_{\pm 4}$ are the same, and are matrix superalgebras over the even division algebra $\mathbb{H}$ is very significant. Together with the tensor product rules (13.22) we can derive a beautiful periodicity structure. It is the analog of (13.47) for Clifford algebras over the real numbers.

We can use the tensor product rule (13.20) to build up $C\ell_n$ from $C\ell_1$, for $n > 0$. When we get to $n = 8$ something special happens. We can

$$\begin{aligned} C\ell_8 &\cong C\ell_4 \widehat{\otimes} C\ell_4 \\ &\cong \mathrm{End}(\mathbb{R}^{2|2})\widehat{\otimes}(\mathbb{H}\otimes\mathbb{H}) \end{aligned} \tag{13.156}$$

But now we can use

$$\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} \cong \mathrm{End}(\mathbb{R}^{4|0}). \tag{13.157}$$ eq:HHid

We have already proven (13.157) above: Recall that $\mathbb{H}^{\mathrm{opp}} \cong \mathbb{H}$ and the operators $L(q)$ and $R(q)$ of left- and right-multiplication of quaternions in the regular representation generate the most general linear transformation on $\mathbb{H}$. Therefore,

$$C\ell_8 \cong \mathrm{End}(\mathbb{R}^{8|8}) \tag{13.158}$$

and hence we have a mod-eight periodicity of Morita classes:

$$C\ell_{n+8} \cong \mathrm{End}(\mathbb{R}^{8|8})\widehat{\otimes} C\ell_n \qquad n \geq 0 \tag{13.159}$$

By induction we conclude that

$$C\ell_{8k+r} \cong \mathrm{End}(\mathbb{R}^{2^{4k-1}|2^{4k-1}})\otimes C\ell_r \tag{13.160}$$

with $k, r \geq 0$.

We can of course do something similar with the negative signature algebras $C\ell_{-n}$.

But now, thanks to eq:Cl13H (13.140), which we repeat:

$$C\ell_{\pm 3} \cong \mathbb{H}\widehat{\otimes}C\ell_{\mp 1} \tag{13.161}$$

we can relate $C\ell_n$ for n's which are negative and positive to each other at the level of Morita equivalence. For example note that

$$\begin{aligned} C\ell_5 &\cong C\ell_4\widehat{\otimes}C\ell_1 \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}\mathbb{H}\otimes C\ell_1 \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}C\ell_{-3} \end{aligned} \tag{13.162}$$

and therefore

$$\begin{aligned} C\ell_6 &\cong C\ell_5\widehat{\otimes}C\ell_1 \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}C\ell_{-3}\widehat{\otimes}C\ell_1 \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}(C\ell_1\widehat{\otimes}C\ell_{-1})\widehat{\otimes}C\ell_{-2} \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}\mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}C\ell_{-2} \\ &\cong \mathrm{End}(\mathbb{R}^{2|2})\widehat{\otimes}C\ell_{-2} \end{aligned} \tag{13.163}$$

and similarly,

$$C\ell_7 \cong \mathrm{End}(\mathbb{R}^{4|4})\widehat{\otimes}C\ell_{-1} \tag{13.164}$$

There is an entirely analogous set of formulae relating $C\ell_n$ for $n = -5, -6, -7$ to matrix superalgebras over the smaller Clifford algebras $C\ell_3, C\ell_2, C\ell_1$, respectively.

The upshot is that if we define the following 8 basic superalgebras:

$$\begin{aligned} D_0^s &:= \mathbb{R} \\ D_{\pm 1}^s &:= C\ell_{\pm 1} \\ D_{\pm 2}^s &:= C\ell_{\pm 2} \\ D_{\pm 3}^s &:= C\ell_{\pm 3} \\ D_4^s = D_{-4}^s &:= \mathbb{H} \end{aligned} \tag{13.165}$$

eq:Dsalpha

where D_0^s and D_4^s are purely even, then all the Clifford algebras are matrix superalgebras over the D_α^s:

Clifford Algebra	Ungraded Algebra	$M_{r\vert s} \otimes D^s_\alpha$
$C\ell_{+8}$	$\mathbb{R}(16)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{8\vert 8})\hat{\otimes}D^s_0$
$C\ell_{+7}$	$\mathbb{C}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{4\vert 4})\hat{\otimes}D^s_{-1}$
$C\ell_{+6}$	$\mathbb{H}(4)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2\vert 2})\hat{\otimes}D^s_{-2}$
$C\ell_{+5}$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\vert 1})\hat{\otimes}D^s_{-3}$
$C\ell_{+4}$	$\mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\vert 1})\hat{\otimes}D^s_4$
$C\ell_{+3}$	$\mathbb{C}(2)$	D^s_3
$C\ell_{+2}$	$\mathbb{R}(2)$	D^s_2
$C\ell_{+1}$	$\mathbb{R} \oplus \mathbb{R}$	D^s_1
$C\ell_0$	$\mathbb{R}$	D^s_0
$C\ell_{-1}$	$\mathbb{C}$	D^s_{-1}
$C\ell_{-2}$	$\mathbb{H}$	D^s_{-2}
$C\ell_{-3}$	$\mathbb{H} \oplus \mathbb{H}$	D^s_{-3}
$C\ell_{-4}$	$\mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\vert 1})\hat{\otimes}D^s_4$
$C\ell_{-5}$	$\mathbb{C}(4)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\vert 1})\hat{\otimes}D^s_{+3}$
$C\ell_{-6}$	$\mathbb{R}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2\vert 2})\hat{\otimes}D^s_{+2}$
$C\ell_{-7}$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{4\vert 4})\hat{\otimes}D^s_{+1}$
$C\ell_{-8}$	$\mathbb{R}(16)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{8\vert 8})\hat{\otimes}D^s_0$

Note particularly that, at the level of Morita equivalence we have

$$[C\ell_{\pm 1}] = [C\ell_{\mp 7}] \qquad [C\ell_{\pm 2}] = [C\ell_{\mp 6}] \qquad [C\ell_{\pm 3}] = [C\ell_{\mp 5}] \tag{13.166}$$

Therefore, the graded Morita equivalence class of $[C\ell_n]$ where $n \in \mathbb{Z}$ is positive or negative is determined by the residue $\alpha = n \bmod 8$, and we have:

$$[C\ell_n] = [D^s_\alpha] \tag{13.167}$$

and moreover, the multiplication on Morita equivalence classes is just given by

$$[D^s_\alpha] \cdot [D^s_\beta] = [D^s_{\alpha+\beta}] \tag{13.168}$$

Thus the real graded Brauer group over $\mathbb{R}$ is $\mathbb{Z}/8\mathbb{Z}$.

The Wedderburn type of the ungraded algebras is now easily determined from the graded ones by using the explicit determination we gave above for the basic cases $C\ell_n$ with $|n| \leq 4$. Notice that there is a basic genetic code in this subject

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{R}, \dots \tag{13.169}$$

We will meet it again and again. One would do well to memorize this sequence. It is illustrated in Figure fig:BottClock 10.

Finally, we can now easily determine the structure of $C\ell_{r_+,s_-}$ for all r, s. The Morita class is determined by:

$$[C\ell_{r_+,s_-}] = [D^s_{r-s}] \tag{13.170}$$

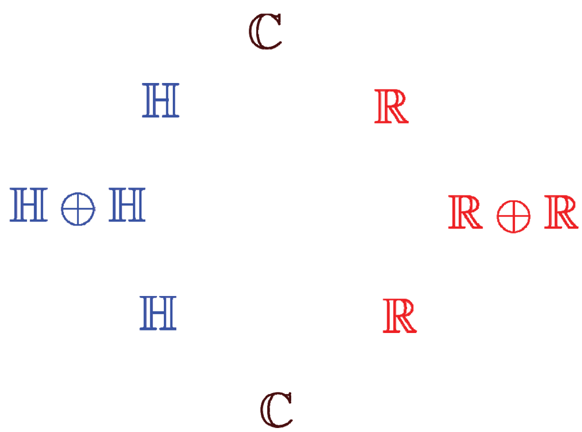

Figure 10: An illustration of the "Bott clock": For $C\ell_n$ with decreasing n read it clockwise (= decreasing phase) and with increasing n read it counterclockwise (= increasing phase). `fig:BottClock`

and hence, lifting $\alpha = (r - s)\text{mod}8$ to $|\alpha| \leq 4$

$$C\ell_{r_+, s_-} \cong \text{End}(\mathbb{R}^{2^n|2^n})\widehat{\otimes} C\ell_\alpha \tag{13.171}$$ `eq:gencliff`

for an n which can be computed by matching dimensions (see exercise below).

Exercise
Show that the nonnegative integer n in (13.171) `eq:gencliff` is given by

$$n = \frac{r + s - |\alpha|}{2} - 1 \tag{13.172}$$

Exercise
Show that if n, m are *any* integers, then

$$C\ell_n \widehat{\otimes} C\ell_m \cong C\ell_{n+m} \widehat{\otimes} M \tag{13.173}$$

where M is a matrix superalgebra $\text{End}(\mathbb{R}^{\ell|\ell})$ and find a formula for ℓ.

Exercise
Show that

$$\mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}C\ell_{\mp 2} \cong \mathbb{H}\widehat{\otimes}C\ell_{\pm 2} \tag{13.174}$$ eq:Cl22H

One way to answer:

$$\begin{aligned} H\widehat{\otimes}C\ell_2 &\cong H\widehat{\otimes}C\ell_1\widehat{\otimes}C\ell_1 \\ &\cong C\ell_{-3}\widehat{\otimes}C\ell_1 \\ &\cong C\ell_1\widehat{\otimes}C\ell_{-1}\widehat{\otimes}C\ell_{-2} \\ &\cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}C\ell_{-2} \end{aligned} \tag{13.175}$$

Exercise *The real Clifford algebras in Lorentzian signature*

Using the above results compute the real Clifford algebras in Lorentzian signature as ungraded algebras:

$d = s+1$	$C\ell(s_+, 1_-)$	$C\ell(1_+, s_-)$
0+1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$
1 +1	$\mathbb{R}(2)$	$\mathbb{R}(2)$
2 +1	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{C}(2)$
3 +1	$\mathbb{R}(4)$	$\mathbb{H}(2)$
4 +1	$\mathbb{C}(4)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$
5 +1	$\mathbb{H}(4)$	$\mathbb{H}(4)$
6 +1	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{C}(8)$
7 +1	$\mathbb{H}(8)$	$\mathbb{R}(16)$
8 +1	$\mathbb{C}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
9 +1	$\mathbb{R}(32)$	$\mathbb{R}(32)$
10 +1	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{C}(32)$
11 +1	$\mathbb{R}(64)$	$\mathbb{H}(32)$

13.5 KO-theory of a point

subsec:KO-point

Now in this section we describe the real KO-theory ring of a point along the lines we discussed in Section §subsubsec:Kpoint 13.2.2. In order to complete the story we need to name the irreducible representations of $C\ell_{\pm 8}$. These are supermatrix algebras and so we have simply $\lambda^{\pm} \cong \mathbb{R}^{8|8}$ for $C\ell_{-8}$ and $\tilde{\lambda}^{\pm} \cong \mathbb{R}^{8|8}$ for $C\ell_{+8}$. The superscript $\pm$ refers to the sign of the volume form on the even subspace.

One can construct very nice explicit modules for $\lambda^{\pm}$ and $\tilde{\lambda}^{\pm}$. See Section **** below.

Now let us consider $KO^{-n}(pt)$ along the above lines. A useful viewpoint is that we are considering real algebras and modules as fixed points of a real structure on the complex modules and algebras. Recall we described $\mathcal{M}_n^{\text{triv,c}}$ (where the extra c in the superscript reminds us that we are talking about complex modules of complex Clifford algebras) as those modules which admit an odd invertible operator which graded commutes with the Clifford action. In order to speak of real structures we can take our complex modules to have an Hermitian structure. Then the conjugation will act as $T \to \pm T^{\dagger}$ where the $\pm$ is a choice of convention. We will choose the convention $T \to -T^{\dagger}$. The other convention leads to an equivalent ring, after switching signs on the degrees. ♣check ♣

Note that we have introduced an Hermitian structure into this discussion. If one strictly applies the the Koszul rule to the definition of Hermitian structures and adjoints in the $\mathbb{Z}_2$-graded case then some unusual signs and factors of $\sqrt{-1}$ appear. See Section §subsec:SuperHilbert 12.5 above. We will use a standard Hermitian structure on $\mathbb{R}^{n|m}$ and $\mathbb{C}^{n|m}$ such that the even and odd subspaces are orthogonal and the standard notion of adjoint. Since we introduce the structure the question arises whether the groups we define below depend on that choice. It can be shown that these groups do not depend on that choice, and the main ingredient in the proof is the fact that the space of Hermitian structures is a contractible space.

This motivates the following definitions:

Definition

a.) For $n \in \mathbb{Z}$, $\mathcal{M}_n$ is the abelian monoid of modules for $C\ell_n$ under direct sum.

b.) For $n \in \mathbb{Z}$, $\mathcal{M}_n^{\text{triv}}$ is the submonoid of $\mathcal{M}_n$ consisting of those modules which admit an odd invertible anti-hermitian operator T which graded-commutes with the $C\ell_n$ action.

c.)

$$KO^n(pt) := \mathcal{M}_n / \mathcal{M}_n^{\text{triv}} \tag{13.176}$$

We now compute the $KO^n(pt)$ groups for low values of n:

1. Of course $KO^0(pt) \cong \mathbb{Z}$, with the isomorphism given by the superdimension.

2. Now consider $KO^1(pt)$. In our model for $\tilde{\eta}$ we had $\rho(e) = \sigma^1$. Therefore we could introduce $T = \epsilon$. Thus $[\tilde{\eta}] = 0$ in KO-theory and $KO^1(pt) = 0$.

3. Next consider $KO^{-1}(pt)$. In our model for η we had $\rho(e) = \epsilon$. Now we cannot introduce an antisymmetric operator which graded commutes with ϵ. Thus, η is a

nontrivial class. However, we encounter a new phenomenon relative to the complex case. Consider $2\eta = \eta \oplus \eta$. As a vector space this is $\mathbb{R}^{2|2}$ and as usual taking an ordered bases with even elements first we have

$$\rho(e) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \epsilon \otimes 1 \tag{13.177}$$

We can therefore introduce $T = \sigma^1 \otimes \epsilon$ which is odd, anticommutes with $\rho(e)$, and squares to -1. Therefore, $KO^{-1}(pt) \cong \mathbb{Z}_2$ with generator $[\eta]$.

4. Moving on to $KO^{-2}(pt)$. One can use our explicit model for η^2 to show that there is no odd operator T of the required type. Of course $2\eta^2$ will again be a trivial module in KO. Thus $KO^{-2}(pt) \cong \mathbb{Z}_2$. ♣Give more details on this point. ♣

5. Now consider $KO^{+2}(pt)$. The monoid $\mathcal{M}_{+2}$ is generated by $\tilde{\eta}^2$ and since $\tilde{\eta}$ is trivial in the KO-group, so is $\tilde{\eta}^2$. Therefore $KO^{+2}(pt) = 0$. Exactly the same reasoning shows that $KO^{+3}(pt) = 0$.

6. Next we consider $KO^{-3}(pt)$. $\mathcal{M}_{-3}$ is generated by η^3. However, as a vector space $\eta^3 \cong \mathbb{R}^{4|4}$. But this space supports the representations $\mu^\pm$ of $C\ell_{-4}$. Therefore, the fourth Clifford generator can serve as T and we learn that η^3 is trivial in the KO group. Thus $KO^{-3}(pt) = 0$.

7. Next we consider $KO^{-4}(pt)$. Of course $\mu^\pm$ descend to nontrivial elements in the KO-group. On the other hand, $C\ell_{-5} \cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes} C\ell_{+3}$ so we can construct a graded irrep of $C\ell_{-5}$ on $\mathbb{R}^{1|1}\widehat{\otimes}\tilde{\eta}^3$ and this generates $\mathcal{M}_{-5}$. One can check that the restriction of $\mathbb{R}^{1|1}\widehat{\otimes}\tilde{\eta}^3$ is just $\mu^+ \oplus \mu^-$. Therefore, we can take $T = \rho(e_5)$ and hence $[\mu^-] = -[\mu^+]$ in the quotient $\mathcal{M}_{-4}/\mathcal{M}_{-4}^{\mathrm{triv}}$. Thus, $KO^{-4}(pt) \cong \mathbb{Z}$. Again, since $\mathcal{M}_{-5}$ is generated by $\mathbb{R}^{1|1}\widehat{\otimes}\tilde{\eta}^3$ we have $KO^{-5}(pt) = 0$

8. Similarly, from our table above we read off that $\mathcal{M}_{-6}$ is generated by $\mathbb{R}^{2|2}\widehat{\otimes}\tilde{\eta}^2$ and $\mathcal{M}_{-7}$ is generated by $\mathbb{R}^{3|3}\widehat{\otimes}\tilde{\eta}$. Hence $KO^{-6}(pt) = KO^{-7}(pt) = 0$.

9. Reasoning as above from the table we learn that $KO^4(pt) \cong \mathbb{Z}$ generated by either $[\tilde{\mu}^+]$ or $[\tilde{\mu}^-] = -[\tilde{\mu}^+]$, while $KO^5(pt) = 0$, (because $[\eta^3] = 0$), $KO^6(pt) \cong \mathbb{Z}_2$ (generated by $\mathbb{R}^{2|2}\widehat{\otimes}\eta^2$) and $KO^7(pt) \cong \mathbb{Z}_2$ (generated by $\mathbb{R}^{3|3}\widehat{\otimes}\eta$)

10. Finally, $KO^{+8}(pt)$ is generated by $[\tilde{\lambda}^\pm]$, with $[\tilde{\lambda}^-] = -[\tilde{\lambda}^+]$, while $KO^{-8}(pt)$ is generated by $[\lambda^\pm]$. with $[\lambda^-] = -[\lambda^+]$.

Now, using the periodicity of the Clifford algebras we conclude that:

Theorem

$KO^n(pt)$ is mod-eight periodic in n and the groups $KO^{-n}(pt)$ for $1 \le n \le 8$ are given by [28]

$$\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \tag{13.178}$$

Once again, one can introduce an interesting ring structure in the KO-group. This ring structure has not yet played any significant role in the physical applications of K-theory, neither to string theory nor to condensed matter physics. But we explain it anyway for its mathematical virtue.

As in the complex case we define the product on equivalence classes of modules by

$$[M_1] \cdot [M_2] \cong [M_1 \widehat{\otimes} M_2] \tag{13.179}$$

(As before, check that the multiplication is well-defined on KO-theory.) If we consider

$$KO^{\le 0}(pt) := \oplus_{n \ge 0} KO^{-n}(pt) \tag{13.180}$$

Then the graded ring is given by

$$KO^{\le 0}(pt) = \mathbb{Z}[\eta, \mu, \lambda]/I \tag{13.181}$$

eq:KO-ring-1

where η, μ, λ are generators [29] of degree

$$\deg(\eta) = -1 \qquad \deg(\mu) = -4 \qquad \deg(\lambda) = -8 \tag{13.182}$$

and the relations are given by the ideal:

$$I = \langle 2\eta, \eta^3, \eta\mu, \mu^2 - 4\lambda \rangle \tag{13.183}$$

eq:KO-ring-2

We have already checked the relations $2\eta = 0$ and $\eta^3 = 0$. Then $\eta\mu$ is a module for $C\ell_{-5}$ but we have already shown this KO-group is zero. Consider μ^2. As a vector space this is $\mathbb{R}^{32|32}$. The volume form on the even subspace is $+1$. Therefore μ^2 is a nonzero multiple of $\lambda^+ \cong \mathbb{R}^{8|8}$, and by dimensions that multiple must be four. Thus

$$\mu^2 \cong 4\lambda \tag{13.184}$$

♣Need to argue there are no further relations. ♣

There is a similar result for $KO^{\ge 0}(pt)$: We introduce generators $\deg(\tilde{\mu}) = +4$ and $\deg(\tilde{\lambda}) = +8$, while KO^7 and KO^6 are generated by $\tilde{\lambda}\eta$ and $\tilde{\lambda}\eta^2$ respectively. [30]

Finally, using Morita equivalence we can define a ring structure on

$$KO^*(pt) = \oplus_{n \in \mathbb{Z}} KO^n(pt) \tag{13.185}$$

When multiplying modules for $C\ell_n$ with $C\ell_m$ with n and m of different sign we identify with the Morita equivalent module for $C\ell_{n+m}$.

[28] This is easily memorized using the "Bott song." Sing the names of the groups to the tune of "Ah! vous dirai-je, Maman," aka "Twinkle, twinkle, little star."

[29] Here $\mu = \mu^+$ and $\lambda = \lambda^+$.

[30] Given our table above we know these are not minimal dimensional representations but by Morita equivalence they generate the same KO group.

The net result is then

$$KO^*(pt) = \mathbb{Z}[\eta, \mu, \tilde{\mu}, \lambda, \tilde{\lambda}]/I \quad (13.186)$$

eq:KO-ring-11

$$I = \langle 2\eta, \eta^3, \eta\mu, \eta\tilde{\mu}, \mu\tilde{\mu} - 4, \lambda\tilde{\lambda} - 1, \mu^2 - 4\lambda, \tilde{\mu}^2 - 4\tilde{\lambda} \rangle \quad (13.187)$$

eq:KO-ring-22

where $\tilde{\mu} = \tilde{\mu}^+$ and $\tilde{\lambda} = \tilde{\lambda}^+$.

As an example consider $\tilde{\lambda}\lambda$. This is a module for $C\ell_{+8,-8} \cong \mathrm{End}(\mathbb{R}^{128|128})$. But $\tilde{\lambda}\lambda \cong \mathbb{R}^{128|128}$ as supervector spaces. Therefore $\lambda\tilde{\lambda} = 1$ in the KO-ring.

Remarks:

1. The discussion we gave in subsubsec:HintTopology 13.2.3 about the relation to the topology of complex vector bundles has a direct analog for real vector bundles. The upshot is that the stable homotopy groups of the orthogonal groups $O(N)$ are given by the Bott song:

p	0	1	2	3	4	5	6	7
$\pi_p(\mathbf{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

   Note, for example, that indeed $\pi_0(\mathbf{O}) = \mathbb{Z}_2$, because $O(N)$ has two components. (In this case, it is true for all N and there is no need to take the stable limit). The other homotopy groups require a choice of a basepoint, and a natural choice of basepoint is the identity element of the group

2. Since $C\ell_{\pm 4} \cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}\mathbb{H}$, the category of quaternionic vector spaces is equivalent to the category of $C\ell_4$-modules. This implies that if we look at quaternionic vector bundles, whose transition functions can be reduced to the compact symplectic group $USp(2n)$, then we learn about the stable homotopy groups of the symplectic groups. The interpretation in terms of $C\ell_4$-modules shifts the degree by 4 and so we have

p	0	1	2	3	4	5	6	7
$\pi_p(\mathbf{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

13.6 Digression: A model for λ using the octonions

mbda-Octonions

A beautiful model for $\lambda^{\pm}$ and $\tilde{\lambda}^{\pm}$ can be constructed using a nonassociative division algebra known as the *octonions*. We recommend the engaging review article by Baez Baez [10] on this subject.

The octonions $\mathbb{O}$ is a nonassociative real division algebra of dimension 8. One way to define the multiplication - via the "Cayley-Dickson process" - is to identify $\mathbb{O} \cong \mathbb{H} \oplus \mathbb{H}$ and define the multiplication as: [31]

$$(a, b) \cdot (c, d) \equiv (ac - \bar{d}b, da + b\bar{c}) \tag{13.188}$$ eq:octmult

In the double-quaternion notation $x = (q_1, q_2)$ we have $\bar{x} \equiv (\bar{q}_1, -q_2)$ and $Re(x) \equiv Re(q_1) = \frac{1}{2}(x + \bar{x})$.

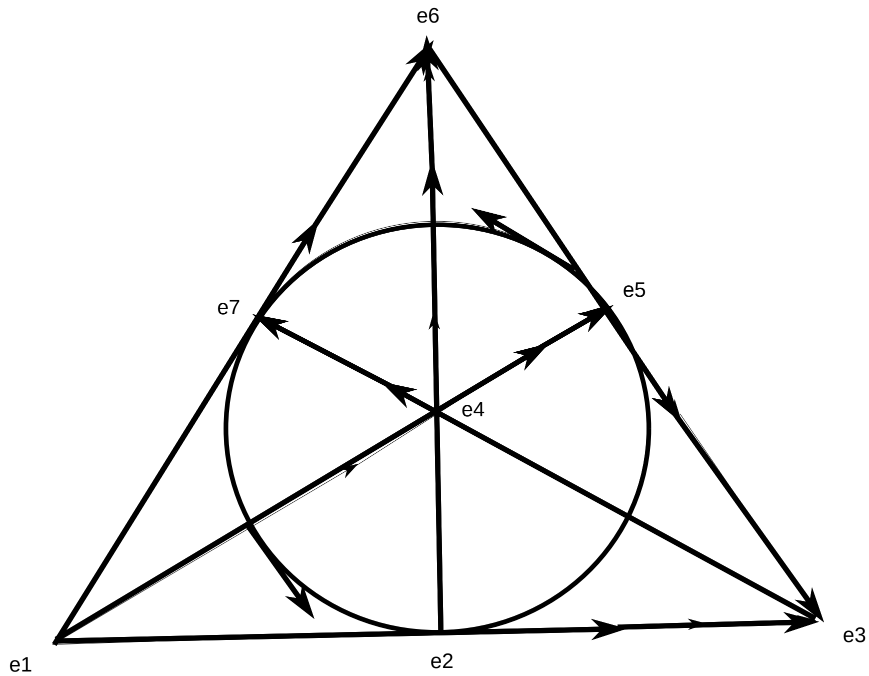

Figure 11: The multiplication law of the imaginary unit octonions. The arrow encodes the sign of the nonzero structure constants. Thus $e_1 e_2 = e_3$, etc. There are 7 points and 7 lines in this figure. fig:octonions

Let e_a, $a = 1, \dots, 7$ be the ordered basis of imaginary octonions given by

$$(\mathfrak{i}, 0), (\mathfrak{j}, 0), (\mathfrak{k}, 0), (0, 1), (0, \mathfrak{i}), (0, \mathfrak{j}), (0, \mathfrak{k}) \tag{13.189}$$

Now make a basis v_α, $\alpha = 0, \dots, 7$ for $\mathbb{O}$ by taking $v_0 = (1, 0)$ and $v_\alpha = e_\alpha$, $1 \le \alpha \le 7$.

Applying the rule (13.188) one finds that v_0 is the identity and

$$e_\alpha e_\beta + e_\beta e_\alpha = -\delta_{\alpha,\beta} \tag{13.190}$$

Therefore if we define 7 real 8×8 matrices:

$$e_a \cdot v_\alpha := \sum_{\beta=0}^{7} (\gamma_a)_{\beta\alpha} v_\beta \tag{13.191}$$

they will be antisymmetric and will give an ungraded irrep of $C\ell_{-7}$. It turns out that $\omega = +1$. The matrix elements are always in $\{0, \pm 1\}$. Indeed, if $\alpha \neq \beta$ then $e_\alpha \cdot e_\beta$ is $\pm e_\gamma$ ♣CHECK! ♣

[31] A multiplication table is in Jacobsen, *Basic Algebra I*, p. 426. Note he has a sign mistake for $i_7 \times i_3$.

for some γ and the precise rule for multiplication is given in Figure fig:octonions 11. In other words, the matrix elements of the gamma matrices are just the structure constants of the octonions! ♣CHECK! ♣

Now, using these matrices we can give an explicit model for $\lambda^{\pm}$:

$$\begin{aligned}\rho(e_i) &= \kappa \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix} = \sigma^1 \otimes \gamma_i \qquad 1 \le i \le 7 \\ \rho(e_8) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon \otimes 1\end{aligned} \tag{13.192}$$

with $\kappa = \pm 1$. Similarly, we can give an explicit model for $\tilde{\lambda}^{\pm}$.

Exercise

Compute the explicit 8-dimensional real representation of $C\ell_{-7}$ defined by the octonions.

---

14. The 10 Real Super-division Algebras

lSuperDivision

Definition An associative unital superalgebra over a field κ is an associative *super-division algebra* if every nonzero *homogeneous* element is invertible.

Example 1: We claim that $\mathbb{C}\ell_1$ is a superdivision algebra over $\kappa = \mathbb{C}$ (and hence a superdivision algebra over $\mathbb{R}$). Elements in this superalgebra are of the form $x + ye$ with $x, y \in \mathbb{C}$. Homogeneous elements are therefore of the form x or ye, and are obviously invertible, if nonzero. Note that it is *not true* that every nonzero element is invertible! For example $1 + e$ is a nontrivial zero-divisor since $(1 + e)(1 - e) = 0$. Thus, $\mathbb{C}\ell_1$ is not a division algebra, as an ungraded algebra.

Example 2: We also claim that the 8 superalgebras D^s_α, with $\alpha \in \mathbb{Z}/8\mathbb{Z}$ defined in (eq:Dsalpha 13.165) are real super-division algebras. The argument of Example 1 show that $C\ell_{\pm 1}$ are super-division algebras. For $C\ell_{\pm 2}$ the even subaglebra is isomorphic to the complex numbers, which is a division algebra. It follows that $C\ell_{\pm 2}$ are superdivision algebras. To spell this out in more detail: For $C\ell_{+2}$ we note that for even elements we can write

$$(x + ye_{12})(x - ye_{12}) = x^2 + y^2 \tag{14.1}$$ eq:INV-cl2-1

and for odd elements we can write

$$(xe_1 + ye_2)^2 = x^2 + y^2 \tag{14.2}$$ eq:INV-cl2-2

where $x, y \in \mathbb{R}$. Thus the nonzero homogeneous elements are invertible. For $C\ell_{-2}$ the equation (eq:INV-cl2-1 14.1) holds and (eq:INV-cl2-2 14.2) simply has a sign change on the RHS. So this too is a superdivision algebra. More conceptually, note that $C\ell^0_{\pm 2}$ is isomorphic to $\mathbb{C}$, which is a division algebra, and $C\ell^1_{\pm 2}$ is related to $C\ell^0_{\pm 2}$ by multiplying with an invertible element. We can now apply this strategy to $C\ell_{\pm 3}$: The even subalgebra is isomorphic to the quaternion algebra, which is a division algebra and the odd subspace is related to the even subspace by multiplication with an invertible odd element. Hence $C\ell_{\pm 3}$ is a superdivision algebra.

Note well that $C\ell_{+1,-1}$ being a matrix superalgebra is definitely *not* a superdivision algebra! For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{14.3}$$

is even and is a nontrivial zerodivisor. By the same token, $C\ell_{\pm 4} \cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}\mathbb{H}$ is also not a superdivision algebra.

The key result we need is really a corollary of Wall's theorem classifying central simple superalgebras. For a summary of Wall's result see Appendix subsec:WallTheorem A.4.

Theorem There are 10 superdivision algebras over the real numbers: The three purely even algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, together with the 7 superalgebras $\mathbb{C}\ell_1, C\ell_{\pm 1}, C\ell_{\pm 2}, C\ell_{\pm 3}$.

Proof: A superdivision algebra D^s over $\mathbb{R}$ must be a simple superalgebra over $\mathbb{R}$. Otherwise some element $a \in D^s$ would have a nontrivial Jordan form for $L(a)$ from which we could construct a nontrivial zero-divisor.

Wall's paper Wall [40] gives a classification of simple superalgebras over a general field κ. The first invariant is the even part of the supercenter $Z^0_s(D^s)$. This must be both a field and a division algebra, and is therefore either $\mathbb{R}$ or $\mathbb{C}$. The algebra D^s is then central simple over $\mathbb{R}$ or $\mathbb{C}$. These we can then classify central simple superalgebras over $\kappa = \mathbb{R}$ and over $\kappa = \mathbb{C}$ using Wall's paper. The central simple superalgebras with nonzero odd part turn out to be all Clifford algebras. From our previous characterizations of these we see that except for $\mathbb{C}\ell_1$ and $C\ell_{\pm 1}, C\ell_{\pm 2}, C\ell_{\pm 3}$, all the Clifford algebras have a factor which is a matrix superalgebra. These cannot be superdivision algebras. On the other hand, we have checked explicitly that $\mathbb{C}\ell_1$ and $C\ell_{\pm 1}, C\ell_{\pm 2}, C\ell_{\pm 3}$ are in fact superdivision algebras. So we have the complete list. $\diamondsuit$

♣Need to explain how that is related to the triple of invariants in Wall's theorem in the appendix. ♣

♣If we drop associativity are there super-analogs of the octonions? ♣

15. The 10-fold way for gapped quantum systems

sec:10FoldWay

We are now in a position to describe the generalization of Dyson's 3-fold way to a 10-fold way, valid for gapped quantum systems.

Recall from our discussion of a general symmetry of dynamics (sec:SymmDyn §9) that if G is a symmetry of the dynamics of a quantum system then there are two independent homomorphisms $(\phi, \chi) : G \to \mathbb{Z}_2$. In the Dyson problem one explicitly assumes that $\chi = 1$. Nevertheless, as we saw when discussing phases of gapped systems in Section sec:GappedSystems §11, there is a natural $\mathbb{Z}_2$-grading of the Hilbert space so that, if $\chi \neq 1$ then the Hilbert space is a (ϕ, χ)-representation of G. (See Definition (eq:PhiChiReps 11.3).) Therefore we can state the

Generalized Dyson Problem: Let G be a bigraded compact group and $\mathcal{H}$ a $\mathbb{Z}_2$-graded (ϕ,χ)-representation $\mathcal{H}$ of G. What is the ensemble of gapped Hamiltonians H such that G is a symmetry of the dynamics and H induces the original $\mathbb{Z}_2$-grading ?

We can proceed to answer this along lines closely analogous to those for Dyson's 3-fold way.

First, we imitate the definitions of Section §8.1 for ϕ-representations:

Definitions:

1. If G is a bigraded group by (ϕ,χ) then a (ϕ,χ)-representation is defined in (11.3).

2. An *intertwiner* or *morphism* between two (ϕ,χ)-reps (ρ_1, V_1) and (ρ_2, V_2) is a $\mathbb{C}$-linear map $T: V_1 \to V_2$, which is a morphism of super-vector spaces: $T \in \underline{\mathrm{Hom}}_{\mathbb{C}}(V_1, V_2)$, which commutes with the G-action:

$$T\rho_1(g) = \rho_2(g)T \qquad \forall g \in G \tag{15.1}$$

   We write $\underline{\mathrm{Hom}}^G_{\mathbb{C}}(V_1, V_2)$ for the set of all intertwiners.

3. An *isomorphism of (ϕ,χ)-reps* is an intertwiner T which is an isomorphism of complex supervector spaces.

4. A (ϕ,χ)-rep is said to be *ϕ-unitary* if V has a nondegenerate even Hermitian structure [32] such that $\rho(g)$ is an isometry for all g. That is, it is unitary or anti-unitary according to whether $\phi(g) = +1$ or $\phi(g) = -1$, respectively.

5. A (ϕ,χ)-rep (ρ, V) is said to be *reducible* if there is a nontrivial proper (ϕ,χ)-sub-representation. That is, if there is a complex super-vector subspace $W \subset V$, (and hence $W^0 \subset V^0$ and $W^1 \subset V^1$) with W not $\{0\}$ or V which is G-invariant. If it is not reducible it is said to be *irreducible*.

As before, if G is compact and (ρ, V) is a (ϕ,χ)-rep then WLOG we can assume that the rep is unitary, by averaging. Then if W is a sub-rep the orthogonal complement is another (ϕ,χ)-rep, and hence we have complete reducibility.

Let $\{V_{\underline{\lambda}}\}$ be a set of representatives of the distinct isomorphism classes of irreducible (ϕ,χ)-representations. We then obtain the isotypical decomposition of (ϕ,χ)-representations:

$$\mathcal{H} \cong \oplus_{\underline{\lambda}} \underline{\mathrm{Hom}}^G_{\mathbb{C}}(V_{\underline{\lambda}}, V) \widehat{\otimes} V_{\underline{\lambda}} \tag{15.2}$$

Now we need to deal with a subtle point. In addition to intertwiners we needed to consider the *graded intertwiners* $\mathrm{Hom}^G_{\mathbb{C}}(V, V')$ between two (ϕ,χ)-representations. These

[32] See Section §12.5 above.

are super-linear transformations T such that if we decompose $T = T^0 + T^1$ into even and odd transformations then $T^0 \in \underline{\mathrm{Hom}}_{\mathbb{C}}^{G}(V, V')$ but T^1 instead satisfies

$$T^1 \rho(g) = \chi(g) \rho'(g) T^1 \qquad \forall g \in G \tag{15.3}$$

Two irreducible representations can be distinct as (ϕ, χ)-representations but can be graded-isomorphic. The simplest example is $G = \{1\}$ which has graded irreps $\mathbb{C}^{1|0}$ and $\mathbb{C}^{0|1}$.

Let $\{V_\lambda\}$ be a set of representatives of the distinct graded-isomorphism classes of irreducible (ϕ, χ)-representations. We then obtain the isotypical decomposition of (ϕ, χ)-representations:

$$\mathcal{H} \cong \oplus_\lambda \mathrm{Hom}_{\mathbb{C}}^{G}(V_\lambda, V) \widehat{\otimes} V_\lambda \tag{15.4}$$

eq:PhiChiIsotyp

Note that $\mathrm{Hom}_{\mathbb{C}}^{G}(V_\lambda, V)$ is no longer an even vector space in general. This will be more convenient to us because of the nature of the super-Schur lemma:

Lemma[Super-Schur] Let G be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded group, graded by the pair of homomorphisms (ϕ, χ).

a.) If T is a graded intertwiner between two irreducible (ϕ, χ)-representations (ρ, V) and (ρ', V') then either $T = 0$ or there is an isomorphism of (ρ, V) and (ρ', V').

b.) If (ρ, V) is an irreducible (ϕ, χ)-representation then the super-commutant $Z_s(\rho, V)$, namely, the set of graded intertwiners of (ρ, V) with itself is a super-division algebra.

Proof: The usual proof of the Schur lemma works, although one should take some care because $\mathbb{Z}_2$-gradings introduce some extra things to check.

a.) If T is nonzero then $T = T^0 + T^1$ and at least one of T^0 or T^1 is nonzero. If T^0 is nonzero then we consider $W = \mathrm{ker} T^0$. Note that W is a $\mathbb{Z}_2$-graded subspace of V since if $T^0(w^0 \oplus w^1) = 0$ then we conclude that both $T^0(w^0) = 0$ and $T^0(w^1) = 0$. Moreover, W is G-invariant. If T^0 is nonzero then $W \neq V$ and hence, by irreducibility $W = \{0\}$. Moreover if $T^0 \neq 0$ then $W' = \mathrm{Im}\ T^0$ is nonzero. Again, we check that W' is a $\mathbb{Z}_2$-graded subspace of V' and is G-invariant and hence $W' = V'$. Similarly, if T^1 is nonzero then we can check that $W = \mathrm{ker} T^1$ is a G-invariant nonzero $\mathbb{Z}_2$-graded subspace of V and hence is $\{0\}$ and $W' = \mathrm{Im}\ T^1$ is a G-invariant nonzero $\mathbb{Z}_2$-graded subspace of V' and hence is $\{V'\}$. Either way, T^0 or T^1 will provide the required (graded) isomorphism.

b.) The argument used in the proof of (a) shows that when $(\rho, V) = (\rho', V')$ if T is homogeneous then it is an isomorphism, and hence invertible. $\diamondsuit$

Now we can now proceed as before to derive the analog of Dyson's ensembles. We consider the isotypical decomposition (15.4) of $\mathcal{H}$. Let $S_\lambda := \mathrm{Hom}_{\mathbb{C}}^{G}(V_\lambda, V)$. It is a real super-vector space of degeneracies. Now we compute the set of superlinear transformations:

$$\mathrm{Hom}_{\mathbb{C}}(V, V) \cong \oplus_{\lambda,\mu} (S_\lambda^* \widehat{\otimes} S_\mu) \widehat{\otimes} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\mu) \tag{15.5}$$

Now we take the graded G-invariants and apply the super-Schur lemma to get

$$\mathrm{Hom}_{\mathbb{C}}^{G}(V, V) \cong \oplus_{\lambda,\mu} \mathrm{End}(S_\lambda) \widehat{\otimes} D_\lambda^s \tag{15.6}$$

where for each isomorphism class of graded-irreducible (ϕ, χ)-rep λ, D^s_λ is one of the 10 real super-division algebras. Since $\mathrm{End}(S_\lambda)$ is a real matrix superalgebra the graded commutant is

$$Z_s(\rho, \mathcal{H}) \cong \oplus_\lambda \mathrm{Mat}_{s_\lambda}(D^s_\lambda) \tag{15.7}$$

Finally, let us apply this to the generalized Dyson problem. If G is a symmetry of the dynamics determined by H then

$$H\rho(g) = \chi(g)\rho(g)H \tag{15.8}$$

and hence the $\mathbb{C}$-linear operator H is in the graded-commutant of the given (ϕ, χ) representation $\mathcal{H}$. Therefore, H is in the space (15.7). For each irreducible representation λ there is a corresponding super-division algebra D^s_λ and this gives the 10-fold classification.

To write the ensemble of Hamiltonians more explicitly we recall that H must be a self-adjoint element of $Z_s(\rho, \mathcal{H})$. There is a natural $*$ structure on the superdivision algebras since the Clifford generators can be represented as Hermitian or anti-Hermitian operators. That is, we take $e_i^* = \pm e_i$ with the sign determined by $e_i^* = e_i^3$. We then extend this to be an anti-automorphism, and for $\mathrm{Mat}_{s_\lambda}(D^s_\lambda)$ we take $*$ to include transposition. H must be a self-adjoint element of this superalgebra.

Moreover, if $\chi(g)$ is nontrivial for any g then H must be in the odd subspace of the superdivision algebra.

Thus, the 10-fold way is the following:

1. If the (ϕ, χ) representation has $\chi = 1$ then the generalized Dyson problem is identical to the original Dyson problem, and there are three possible ensembles.

2. But if χ is nontrivial then there are new ensembles not allowed in the Dyson classification. In these cases, D^s_λ is one of the 7 superalgebras which are not purely even and H is an odd element of the superalgebra $\mathrm{Mat}_{s_\lambda}(D^s_\lambda)$.

Remarks:

1. It was easy to give examples of the three classes in Dyson's 3-fold way. Below we will give examples using the 10 bigraded "CT-groups" discussed in Section §16 below.

2. The above is, strictly speaking, a new result, although it is really a simple corollary of [22]. However, it should be stressed that the result is just a general statement about quantum mechanics. No mention has been made of bosons vs. fermions, or interacting vs. noninteracting.

3. A key point we want to stress is that the 10-fold way is usually viewed as $10 = 2 + 8$, where 2 and 8 are the periodicities in complex and real K-theory. And then the K-theory classification of topological phases is criticized because it only applies to free systems. However, we believe this viewpoint is slightly misguided. The unifying concept is really that of a real super-division algebra, and there are 10 such. They can be parceled into $10 = 8 + 2$ but they can also equally naturally be parceled into $10 = 7 + 3$ (with the 3 referring to the purely even superdivision algebras).

4. The Altland-Zirnbauer classification discussed below makes explicit reference to free fermions.

♣Note: We already had an exercise on parity reversal above. There is some redundancy here. ♣

Exercise *Parity reversal*

If V is a supervector space its parity-reverse ΠV is the supervector space such that $(\Pi V)^0 = V^1$ and $(\Pi V)^1 = V^0$.

a.) Let V be a super-vector space. Under what conditions are V and ΠV isomorphic in the category of super-vector spaces? [33]

b.) Show that there is a canonical super-linear transformation $\pi : V \to \Pi V$ given by $\pi(v^0 \oplus v^1) = v^1 \oplus v^0$. Is it even or odd? [34]

c.) Show that if V is a (ϕ, χ)-representaton of G then ΠV is also a (ϕ, χ) representation. Show that, in the physical context this corresponds to switching the sign of the Hamiltonian.

d.) Is the operator π of part (b) a graded intertwiner?

e.) Can V and ΠV be inequivalent (ϕ, χ) representations?

Exercise

Show that (eq:Cl13H 13.140) is an isomorphism of $*$-structures.

15.1 Digression: Dyson's 10-fold way

As a curious digression we note that in Dyson's original paper on ϕ-representations Dyson3fold [18] he in fact had a 10-fold classification of irreducible ϕ-representations! For completeness we review it here.

Dyson assumes that ϕ is surjective, i.e. ϕ is nontrivial, and considers an irreducible ϕ-representation (ρ, V) of complex dimension n. Let $G_0 = \ker\phi$. One useful approach to Dyson's 10-fold way is to identify V with $(V_{\mathbb{R}}, I)$, where I is a complex structure on $V_{\mathbb{R}}$ and consider certain subalgebras of $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ generated by group representation operators. The algebra generated by $\rho(g)$ for $g \in G_0$ is denoted $\mathcal{A}$. The algebra generated by $\mathcal{A}$ together with I is denoted $\mathcal{B}$. Finally, the algebra generated by I and $\rho(g)$ for all $g \in G$ is denoted $\mathcal{D}$. The *commutants* in $\mathrm{End}_{\mathbb{R}}(V)$ of $\mathcal{A}, \mathcal{B}, \mathcal{D}$ are denoted $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, respectively. Note that $\mathrm{End}_{\mathbb{R}}(V) \cong \mathbb{R}(2n)$. Dyson's 10 cases are then summarized by the table:

[33] *Answer*: If V is finite-dimensional then we must have $\dim V^0 = \dim V^1$. In general there is no canonical isomorphism between V and ΠV.

[34] *Answer*: π is an odd operator.

Dyson Type	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$
$\mathbb{R}\mathbb{R}$	$\mathbb{R}(2n)$	$\mathbb{C}(n)$	$2\mathbb{R}(n)$	$n\mathbb{R}(2)$	$n\mathbb{C}$	$2n\mathbb{R}$
$\mathbb{R}\mathbb{C}$	$\mathbb{R}(2n)$	$\mathbb{C}(n)$	$\mathbb{C}(n)$	$n\mathbb{C}$	$n\mathbb{C}$	$2n\mathbb{R}$
$\mathbb{R}\mathbb{H}$	$\mathbb{R}(4m)$	$\mathbb{C}(2m)$	$\mathbb{H}(m)$	$m\overline{\mathbb{H}}$	$2m\mathbb{C}$	$4m\mathbb{R}$
$\mathbb{C}\mathbb{R}$	$\mathbb{C}(2m)$	$\mathbb{C}(m)\oplus\mathbb{C}(m)$	$2\mathbb{R}(m)\oplus 2\mathbb{R}(m)$	$m\mathbb{R}(2)\oplus m\mathbb{R}(2)$	$m\mathbb{C}\oplus m\mathbb{C}$	$2m\mathbb{C}$
$\mathbb{C}\mathbb{C}_1$	$\mathbb{C}(2m)$	$\mathbb{C}(m)\oplus\mathbb{C}(m)$	$\mathbb{C}(m)\oplus\mathbb{C}(m)$	$m\mathbb{C}\oplus m\mathbb{C}$	$m\mathbb{C}\oplus m\mathbb{C}$	$2m\mathbb{C}$
$\mathbb{C}\mathbb{C}_2$	$\mathbb{C}(2m)$	$\mathbb{C}(m)\oplus\mathbb{C}(m)$	$2\mathbb{C}(m)$	$m\mathbb{C}(2)$	$m\mathbb{C}\oplus m\mathbb{C}$	$2m\mathbb{C}$
$\mathbb{C}\mathbb{H}$	$\mathbb{C}(4p)$	$\mathbb{C}(2p)\oplus\mathbb{C}(2p)$	$\mathbb{H}(p)\oplus\mathbb{H}(p)$	$p\overline{\mathbb{H}}\oplus p\overline{\mathbb{H}}$	$2p\mathbb{C}\oplus 2p\mathbb{C}$	$4p\mathbb{C}$
$\mathbb{H}\mathbb{R}$	$\mathbb{H}(m)$	$2\mathbb{C}(m)$	$4\mathbb{R}(m)$	$m\mathbb{R}(4)$	$m\mathbb{C}(2)$	$m\overline{\mathbb{H}}$
$\mathbb{H}\mathbb{C}$	$\mathbb{H}(m)$	$2\mathbb{C}(m)$	$2\mathbb{C}(m)$	$m\mathbb{C}(2)$	$m\mathbb{C}(2)$	$m\overline{\mathbb{H}}$
$\mathbb{H}\mathbb{H}$	$\mathbb{H}(2p)$	$2\mathbb{C}(2p)$	$2\mathbb{H}(p)$	$p\overline{\mathbb{H}}(2)$	$2p\mathbb{C}(2)$	$2p\overline{\mathbb{H}}$

Recall from Section subsec:ComRedAlg §8.4 the notation: $\mathbb{K}(s)$, for s a positive integer, is the algebra of $s\times s$ matrices over a real division algebra $\mathbb{K}$. Then $\ell\mathbb{K}(s)$ is the algebra of $\ell s\times\ell s$ block diagonal matrices over $\mathbb{K}$ where all ℓ diagonal $s\times s$ blocks are the same. Thus $\ell\mathbb{K}(s)$ is isomorphic to $\mathbb{K}(s)$ as an abstract algebra. On the quaternionic space $\mathbb{H}^s$ there is a left action of $\mathbb{H}(s)$ and a right action of the opposite algebra $\overline{\mathbb{H}}$. Finally, while V always has complex dimension n, in some cases it is useful to define integers $m=n/2$ and $p=n/4$.

The fact that $\mathcal{D}$ and its commutant $\mathcal{Z}$ are matrix algebras over a real division algebra follows (and is equivalent to) the assumption that (ρ,V) is an irreducible ϕ-rep. In general, although V is irreducible it will become reducible when considered as a representation of the index two subgroup G_0 of G. The algebra $\mathcal{A}$ will be semisimple and Dyson proves that when writing it as a direct sum over simple algebras they all have the same Wedderburn type. Thus there is a well-defined pair of Wedderburn types $(\mathbb{K}_1,\mathbb{K}_2)$ of $(\mathcal{D},\mathcal{A})$, or, equivalently, of $(\mathcal{Z},\mathcal{X})$. Dyson shows, by exhibiting examples, that these are uncorrelated: All nine possible combinations do occur for some suitable ϕ-representation. Finally, the case $(\mathbb{C},\mathbb{C})$ usefully splits into two subcases according to whether the two representations of G_0 are equivalent or inequivalent. That gives 10 cases.

♣Is this somehow a simple consequence of G_0 being a normal subgroup of index two? ♣

From the viewpoint of these notes we should remark that there is an *a priori* different 10-fold classification of irreducible ϕ-representations Freed:2012uu [22]. The algebra $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ has an involution

$$T\to ITI^{-1} \tag{15.9}$$

and we can use this to define a $\mathbb{Z}_2$-grading on $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ without choosing any $\mathbb{Z}_2$-grading on $V_{\mathbb{R}}$. The subalgebra $\mathcal{D}$ has graded commutant $Z_s(\rho,V)$ consisting of $A\in\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ so that if we decompose $A=A^0+A^1$ into even and odd pieces then

$$A^0 I = IA^0 \qquad \& \qquad A^0\rho(g)=\rho(g)A^0 \qquad\qquad \forall g\in G \tag{15.10}$$

$$A^1 I = -IA^1 \qquad \& \qquad A^1\rho(g)=\phi(g)\rho(g)A^1 \qquad\qquad \forall g\in G \tag{15.11}$$

Then $Z_s(\rho,V)$ is a real super-division algebra (apply the reasoning of the proof of the Schur lemma), and we have seen that there are ten types, yielding a 10-fold classification of irreducible ϕ-representations.

This raises the obvious question of whether Dyson's old 10-fold classification coincides with the one given by the real superdivision algebras.

In general it is clear that the even part of the superdivision algebra is precisely the same as the algebra $\mathcal{Z}$. On the other hand, the definition of the odd part of the superdivision algebra does not appear in Dyson's discussion so the relation between the two classification schemes is not obvious, even though both are 10-fold ways. The fact that there are 10 distinct cases does not mean that they are the "same" ! [35]

Nevertheless we conjecture that the two classifications are the same. More precisely:

Conjecture: There is a 1-1 correspondence between the 10 Dyson types and the real superdivision algebras so that the classification of irreducible ϕ-representations, for all $\mathbb{Z}_2$-graded groups (G, ϕ) with G compact and ϕ nontrivial, is the same.

Assuming this conjecture, examination of examples leads to the correspondence:

Superdivision Algebra	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}\ell_1$	$C\ell_{-3}$	$C\ell_{-2}$	$C\ell_{-1}$	$C\ell_1$	$C\ell_2$	$C\ell_3$
Dyson Type	$\mathbb{R}\mathbb{C}$	$\mathbb{C}\mathbb{C}_1$	$\mathbb{H}\mathbb{C}$	$\mathbb{C}\mathbb{C}_2$	$\mathbb{H}\mathbb{H}$	$\mathbb{C}\mathbb{H}$	$\mathbb{R}\mathbb{H}$	$\mathbb{R}\mathbb{R}$	$\mathbb{C}\mathbb{R}$	$\mathbb{H}\mathbb{R}$

Example 1: Suppose $G = \mathbb{Z}_2$, $\phi(\sigma) = -1$, where $\sigma \in G$ is the nonidentity element. We take $V = \mathbb{C}$ and $\rho(\sigma)$ acts by complex conjugation. Then $V_{\mathbb{R}} = \mathbb{R}^2$, $I = \epsilon$, $\mathcal{A} = 2\mathbb{R}$, $\mathcal{X} = \mathbb{R}(2)$, and σ acts by σ^3. So $\mathcal{D}$ is generated by ϵ and σ^3 and hence $\mathcal{D} = \mathbb{R}(2)$ so $\mathcal{Z} = 2\mathbb{R}$. Thus, this example is Dyson type $\mathbb{R}\mathbb{R}$. On the other hand, in the supercommutant we search for a T with $T\epsilon = -\epsilon T$ and $T\sigma^3 = -\sigma^3 T$. Such a T is proportional to σ^1 and hence the graded commutant is $C\ell_1 = \mathbb{R} \oplus \mathbb{R}T$.

Example 2: Suppose $G = O(2)$, graded by $\phi(g) = \det g$. Thus $G_0 = SO(2)$. Let $V = \mathbb{C}$ so that $V_{\mathbb{R}} = \mathbb{R}^2$, $I = \epsilon$, and $O(2)$ acts by its defining representation. Now $\mathcal{A} = \mathcal{X} = \{x + y\epsilon | x, y \in \mathbb{R}\} \cong \mathbb{C}$. To compute $\mathcal{D}$ we adjoin any reflection, and then we find $\mathcal{D} = \mathbb{R}(2)$, so $\mathcal{Z} = 2\mathbb{R}$. Thus, this representation is Dyson type $\mathbb{R}\mathbb{C}$. On the other hand, any odd element T in the graded commutant must anticommute with ϵ (so it is odd), and yet commute with $\rho(G_0)$ which consists of matrices of the form $\cos\theta 1 + \sin\theta\epsilon$. This is clearly impossible, so that the graded commutant is just $2\mathbb{R}$, and is thus isomorphic to $C\ell_0 = \mathbb{R}$.

Example 3: Now take $G = \mathbb{Z}_4 \cong \langle\omega\rangle$ where ω is a primitive fourth root of unity. Define the $\mathbb{Z}_2$-grading by $\phi(\omega) = \omega^2 = -1$. Thus $G_0 = \{1, \omega^2\} \cong \mathbb{Z}_2$. Our ϕ-representation

[35] For example, I am fortunate to have all 10 fingers. Which superdivision algebra corresponds to my right thumb?

space will be $V = \mathbb{C}^2$, which we identify with $V_{\mathbb{R}} = \mathbb{R}^4 \cong \mathbb{H}$ and the complex structure is $I = L(\mathfrak{i})$. Then the ϕ-representation is defined by $\rho(\omega) = L(\mathfrak{j})$. (Note that even though G is abelian we have an irreducible ϕ-representation of complex dimension two!) Note that $\rho(\omega)$ is indeed antilinear, and $\rho(\omega^2) = -1$. Thus, the restriction of the representation to G_0 is highly reducible: It is four copies of the sign representation of $\mathbb{Z}_2$. Now the algebra $\mathcal{D}$ generated by $L(\mathfrak{i})$ and $L(\mathfrak{j})$ is the algebra of operators $L(q)$ for $q \in \mathbb{H}$ and hence is isomorphic to $\mathbb{H}$. The commutant, $\mathcal{Z}$, is therefore the algebra of operators $R(q)$ for $q \in \mathbb{H}$ and is therefore isomorphic to $\mathbb{H}^{\mathrm{opp}}$. On the other hand, since $\rho(\omega^2) = -1$ is a multiple of the identity matrix the algebra $\mathcal{A}$ is just $4\mathbb{R}$ and hence the commutant $\mathcal{X}$ is $\mathbb{R}(4)$. Therefore, this example is of Dyson type $\mathbb{H}\mathbb{R}$. Next, to compute the graded commutant we note that the odd operators anticommuting with $L(\mathfrak{j})$ are those of the form $L(\mathfrak{k})R(q)$ for $q \in \mathbb{H}$. That is,

$$D^1 = \{L(\mathfrak{k})R(q) | q \in \mathbb{H}\} \tag{15.12}$$

This means that the superdivision algebra is generated by

$$e_1 = L(\mathfrak{k})R(\mathfrak{i}) \qquad e_2 = L(\mathfrak{k})R(\mathfrak{j}) \qquad e_3 = L(\mathfrak{k})R(\mathfrak{k}) \tag{15.13}$$

and hence the superdivision algebra is $C\ell_{+3}$.

♣Link this to Example 2 of Section 8.2. ♣ subsec:ShurPhi

Exercise *Challenge*

Prove the conjecture. If you succeed, you get an automatic A^+ in the course! [36]

---

16. Realizing the 10 classes using the CT groups

sec:CTgroups

To make contact with some of the literature on topological insulators we describe here the 10 "CT groups." (This is a nonstandard term used in [22].) Freed:2012uu This is a set of 10 bigraded groups which we now define.

To motivate the 10 CT groups note that in some disordered systems, (sometimes well-described by free fermions), the only symmetries we might know about *a priori* are the

[36] *Here is one approach*: For each Dyson type we try to construct a central simple superalgebra in such a way that there is a one-one correspondence between the Dyson type and the Morita equivalence class of the algebra. To this end we first define a ϕ-representation V to be of type p if there exists a $P \in \mathcal{X}$ which anticommutes with I and $P^2 \propto 1$. We say V of type np otherwise. Equivalently, a ϕ-rep is of type p if as representation of G_0 it is either real or quaternionic. Now, in seven out of the ten cases with V irreducible it turns out that the ϕ-rep is of type p. The remaining 3 cases, which are necessarily of type np, are the Dyson types $\mathbb{R}\mathbb{C}$, $\mathbb{H}\mathbb{C}$, and $\mathbb{C}\mathbb{C}_1$. In these cases note that that $V \oplus \bar{V}$ is of type p. Let $U = V$ if V is of type p and $U = V \oplus \bar{V}$ if V is of type np. Consider the sub-algebras $\mathcal{D}, \mathcal{A}, \mathcal{X}, \mathcal{Z}$ of $\mathrm{End}_{\mathbb{R}}(U)$, defined as above. One can check in examples that adjoining P to $\mathcal{Z}$ defines a $\mathbb{Z}_2$-graded Clifford algebra $\mathcal{Z}^+$, (with the sign of the commutation with I defining the grading) and U is a Clifford module for $\mathcal{Z}^+$. The choice of P is not unique, so one must prove that the Morita class of $\mathcal{Z}^+$ is independent of P and that the Morita class only depends on the Dyson type, and not the particular representation.

presence or absence of "time-reversal" and "particle-hole" symmetry. Thus it is interesting to consider the various ϕ-twisted extensions of the group

$$M_{2,2} = \langle \bar{T}, \bar{C} | \bar{T}^2 = \bar{C}^2 = \bar{T}\bar{C}\bar{T}\bar{C} = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{16.1}$$

or of its subgroups. We make this a $\mathbb{Z}_2$-graded group with the choice

$$\phi(\bar{T}) = \phi(\bar{C}) = -1. \tag{16.2}$$

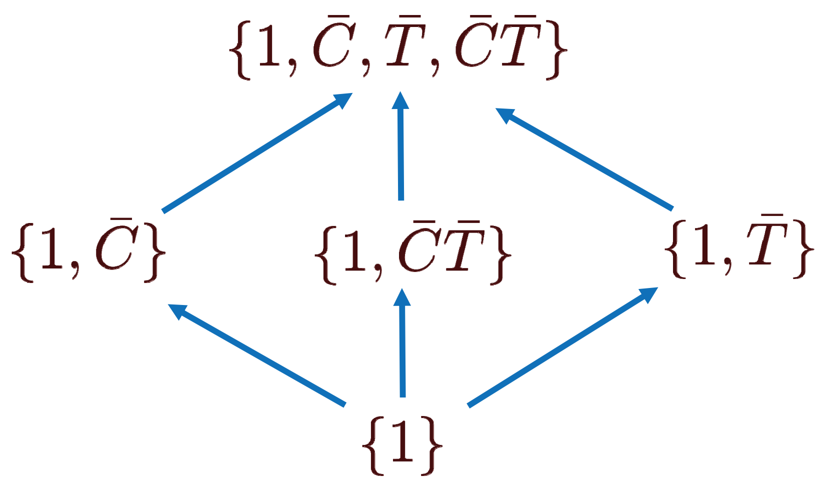

Figure 12: The 5 subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. fig:M22SUBGROUPS

Now let us consider the ϕ-twisted extensions of $M_{2,2}$. This is a simple generalization of the example we discussed in Section §6, equation (6.12). First, let us note that there are 5 subgroups of $M_{2,2}$ depending on whether $\bar{T}, \bar{C}$ or $\bar{T}\bar{C}$ is in the group. See Figure 12.

As in the example (6.12) the isomorphism class of the extension is completely determined by whether the lift T and/or C of $\bar{T}$ and/or $\bar{C}$ squares to ± 1. After a few simple considerations discussed in the exercises below it follows that one has the table of 10 ϕ-twisted extensions of the subgroups of $M_{2,2}$:

Subgroup $U \subset M_{2,2}$	T^2	C^2	[Clifford]
$\{1\}$			$[\mathbb{C}\ell_0] = [\mathbb{C}]$
$\{1, \bar{S}\}$			$[\mathbb{C}\ell_1]$
$\{1, \bar{T}\}$	$+1$		$[C\ell_0] = [\mathbb{R}]$
$M_{2,2}$	$+1$	-1	$[C\ell_{-1}]$
$\{1, \bar{C}\}$		-1	$[C\ell_{-2}]$
$M_{2,2}$	-1	-1	$[C\ell_{-3}]$
$\{1, \bar{T}\}$	-1		$[C\ell_4] = [\mathbb{H}]$
$M_{2,2}$	-1	$+1$	$[C\ell_{+3}]$
$\{1, \bar{C}\}$		$+1$	$[C\ell_{+2}]$
$M_{2,2}$	$+1$	$+1$	$[C\ell_{+1}]$

Now the group $M_{2,2}$ has a natural bigrading, which, WLOG (see the exercise below) we can take to be

$$\begin{aligned} \phi(\bar{T}) &= -1 \qquad & \phi(\bar{C}) &= -1 \\ \chi(\bar{T}) &= +1 \qquad & \chi(\bar{C}) &= -1 \\ \tau(\bar{T}) &= -1 \qquad & \tau(\bar{C}) &= +1 \end{aligned} \tag{16.3}$$

where we have defined τ from ϕ and χ so that $\tau \cdot \phi \cdot \chi = 1$. These can be used to define bigradings of the ten ϕ-twisted extensions of all the subgroups of $M_{2,2}$.

Now, we can generalize the remark near the example of Section subsec:PhiRepBasics §8.1. Recall that we could identify ϕ-representations of ϕ-twisted extensions of M_2 with real and quaternionic vector spaces. If we consider subgroups of M_2 then for the trivial subgroup we also get complex vector spaces. This trichotomy is generalized to a decachotomy for the CT groups:

Theorem There is a one-one correspondence, given in the table above, between the ten CT groups and the ten real super-division algebras (equivalently, the 10 Morita classes of the real and complex Clifford algebras) such that there is an equivalence of categories between the (ϕ, χ)-representations of the CT group and the graded representations of the corresponding Clifford algebra.

Proof:

We systematically consider the ten cases beginning with a (ϕ, χ)-representation of a CT group and producing a corresponding representation of a Clifford algebra. Then we show how the inverse functor is constructed.

1. First, consider the subgroup $U = \{1\}$. A (ϕ, χ) representation W is simply a $\mathbb{Z}_2$-graded complex vector space, so $V = W$ is a graded $\mathbb{C}\ell_0$-module.

2. Now consider $U = \{1, \bar{S}\}$. There is a unique central extension and $S = CT$ acts on W as an odd operator which, WLOG, we can take to square to one. Moreover, S is $\mathbb{C}$-linear. Therefore, we can take $V = W$ and identify S with an odd generator of $\mathbb{C}\ell_1$.

3. Now consider $U = \{1, \bar{C}\}$. On the representation W of $U^{\rm tw}$ we have two odd anti-linear operators C and iC. Note that

$$(iC)^2 = C^2 \qquad\qquad \{iC, C\} = 0 \tag{16.4}$$

since C is antilinear. Therefore, we can define a graded Clifford module $V = W$ with $e_1 = C$ and $e_2 = iC$. It is a Clifford module for a real Clifford algebra, again because C is anti-linear. The Clifford algebra is $C\ell_{+2}$ if $C^2 = +1$ and $C\ell_{-2}$ if $C^2 = -1$.

4. Next, consider $U = \{1, \bar{T}\}$. The lift T to $U^{\rm tw}$ acts on a (ϕ, χ) representation W as an even, $\mathbb{C}$-antilinear operator. It is therefore a real structure if $T^2 = +1$ and a quaternionic structure if $T^2 = -1$. In the first case, the fixed points of T define a real $\mathbb{Z}_2$-graded vector space $V = W|_{T=+1}$ which is thus a graded module for $C\ell_0$. In the second case, T defines a quaternionic structure on $V = W_{\mathbb{R}}$. As we have seen, $C\ell_4$ is Morita equivalent to $\mathbb{H}$, and in fact the $C\ell_4$ module is $V \oplus V$. (Recall equation (13.152) above.)

5. Now consider $U = M_{2,2}$. This breaks up into 4 cases:

6. If $T^2 = +1$ then, as we have just discussed T defines a real structure. As shown in the exercises, WLOG we can choose the lift of C so that $CT = TC$. Therefore, C acts as an odd operator on the real vector space of $T = +1$ eigenstates: $V = W|_{T=+1}$. Then V is the corresponding module for $C\ell_{\pm 1}$ according to whether $C^2 = \pm 1$.

7. If $T^2 = -1$, then, as we just discussed, T defines a quaternionic structure on $V = W_{\mathbb{R}}$. Then C, iC, and iCT are odd endomorphisms of $W_{\mathbb{R}}$ and one checks they generate a $C\ell_{+3}$ action if $C^2 = +1$ and a $C\ell_{-3}$ action if $C^2 = -1$.

To complete the proof we need to describe the inverse functor, namely, given a Clifford module V in each of the 10 cases, how do we produce a (ϕ, χ)-representation for a corresponding CT group?

For $\mathbb{C}\ell_0, \mathbb{C}\ell_1$ we take $W = V$ and e_1 represents S. For $C\ell_{0,\pm 1}$, given a real module V we take $W = V \otimes \mathbb{C}$, and let $T = 1 \otimes \mathcal{C}$ where $\mathcal{C}$ is complex conjugation. Then $C = e_1 \otimes \mathcal{C}$ defines the corresponding CT module. If V is a real $C\ell_{\pm 2}$ module then we make a complex vector space $W = (V, I = e_1 e_2)$. We may take $C = e_1$. This is odd and antilinear. If V is a real $C\ell_{\pm 3}$ module then $W = (V, \mp e_1 e_2)$ and we take $C = e_1$ and $T = -e_2 e_3$.

We leave it to the reader to check that these are indeed inverse functors. $\diamondsuit$

Now, in order to give our application to the generalized Dyson problem we note a key:

Proposition: Let $U^{\rm tw}$ be one of the 10 bigraded CT groups and let D be the associated real superdivision algebra. Let (ρ, W) be an irreducible (ϕ,χ)-rep of $U^{\rm tw}$. Then, the graded commutant $Z_s(\rho, W)$ is a real superdivision algebra isomorphic to $D^{\rm opp}$.

Proof: We consider the 10 cases in succession.

♣There must be a simpler conceptual proof! ♣

1. For $U = \{1\}$ we have $D = \mathbb{C}$ and there are two inequivalent irreducible (ϕ,χ) representations $W = \mathbb{C}^{1|0}$ and $W = \mathbb{C}^{0|1}$. (There is only one graded-irreducible representation.) In either case we clearly have $Z_s = \mathbb{C}$.

2. Now consider $U = \{1, \bar{S}\}$ so $D = \mathbb{C}\ell_1$. There is a unique (up to isomorphism) irreducible representation $W = \mathbb{C}^{1|1}$, and choosing the natural basis we have
$$\rho(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16.5}$$
It follows that the graded commutant $Z_s(\rho, W)$ consists of the $\mathbb{C}$-linear transformations which in this basis have the form
$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C} \tag{16.6}$$
The $\mathbb{Z}_2$-graded algebra of such matrices is isomorphic to $\mathbb{C}\ell_{-1} \cong \mathbb{C}\ell_1^{\rm opp}$. (It is also isomorphic to $\mathbb{C}\ell_1$ in this example.)

3. Now consider $U = \{1, \bar{C}\}$ with $C^2 = \xi$, where $\xi \in \{\pm 1\}$. These correspond to $D = C\ell_{\pm 2}$ for $\xi = \pm 1$, respectively. Then up to isomorphism we can take the irrep to be $W = \mathbb{C}^{1|1}$ and we can take
$$C : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \xi \bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{16.7}$$ eq:C-xi-act
Computing the conditions $A^0 C = C A^0$ and $A^1 C = -C A^1$ reveals that A must be a $\mathbb{C}$-linear transformation which in this basis is
$$A = \begin{pmatrix} \alpha & \beta \\ -\xi\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C} \tag{16.8}$$ eq:grd-int-1
so $Z_s(\rho, W) \cong C\ell_{\mp 2} \cong D^{\rm opp}$.

4. Next, consider $U = \{1, \bar{T}\}$. If $T^2 = +1$ then $D = C\ell_0 = \mathbb{R}$ and there are two inequivalent irreducible (ϕ,χ) representations of $U^{\rm tw}$ namely $W = \mathbb{C}^{1|0}$ or $W = \mathbb{C}^{0|1}$. In both cases in the natural basis $Tz = \bar{z}$. Therefore $Z_s \cong \mathbb{R} \cong C\ell_0^{\rm opp}$. If $T^2 = -1$ then $D = \mathbb{H}$ and there are again two inequivalent irreps $W = \mathbb{C}^{2|0}$ or $W = \mathbb{C}^{0|2}$ and we can take
$$T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{16.9}$$

(Note: This is an *even* transformation!) Now a simple computation shows that if A is a 2×2 complex matrix in this basis then

$$TA = \epsilon \bar{A} \epsilon^{-1} T \tag{16.10}$$

where $\bar{A}$ is simply complex conjugation of the matrix elements of A. The fixed points $A = \epsilon \bar{A} \epsilon^{-1}$ defines a matrix realization of the quaternions:

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C} \tag{16.11}$$

and therefore $Z_s \cong \mathbb{H} \cong D^{\rm opp}$. (An alternative and slicker argument identifies $W \cong \mathbb{H}$ with I given by $L(\mathfrak{i})$ and T given by $L(\mathfrak{j})$. Then it is clear that $Z_s = \{R(q) | q \in \mathbb{H}\} \cong \mathbb{H}^{\rm opp}$.)

5. Now consider $U = M_{2,2}$. This breaks up into 4 cases:

6. If $T^2 = +1$ and $C^2 = \xi$ then $D = C\ell_{\pm 1}$ for $\xi = \pm 1$ and there is a unique irrep isomorphic to $W \cong \mathbb{C}^{1|1}$. We can still take C to act according to (eq:C-xi-act 16.7) but now we must take

$$T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \tag{16.12}$$

eq:T-xi-act

so that T is even, antilinear, and commutes with C. From our computations above we know that graded commutation with C implies that a graded intertwiner A is of the form (eq:grd-int-1 16.8) and commutation with T implies that $\alpha, \beta \in \mathbb{R}$ and hence for $D \cong C\ell_{\pm 1}$ we have $Z_s(\rho, W) \cong C\ell_{\mp 1} \cong D^{\rm opp}$.

7. If $T^2 = -1$, then, up to isomorphism we have $W \cong \mathbb{C}^{2|2}$ and now, up to isomorphism we can take

$$C : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} \xi \bar{z}_3 \\ \xi \bar{z}_4 \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \tag{16.13}$$

eq:c-xi-act-ii

$$T : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \\ -\bar{z}_4 \\ \bar{z}_3 \end{pmatrix} \tag{16.14}$$

eq:T-xi-act-ii

Now write A as a 2×2 block matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha, \beta, \gamma, \delta \in M_2(\mathbb{C}) \tag{16.15}$$

Then $AT = TA$ shows that each 2×2 block satisfies $\alpha = \epsilon \bar{\alpha} \epsilon^{-1}$, and so forth. Then graded commutativity with C shows that $\delta = \bar{\alpha}$ and $\gamma = -\epsilon \bar{\beta}$. Therefore

$$A = \begin{pmatrix} \alpha & \beta \\ -\epsilon \bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{16.16}$$

where α, β are 2×2 complex matrices satisfying the quaternion condition $\alpha = \epsilon \bar{\alpha} \epsilon^{-1}$ and $\beta = \epsilon \bar{\beta} \epsilon^{-1}$. Therefore, for $D = C\ell_{\pm 3}$ we get $Z_s(\rho, W) \cong C\ell_{\mp 3} \cong D^{\rm opp}$. $\diamondsuit$

We can now give examples of all 10 generalized Dyson classes. If $U^{\rm tw}$ corresponds to one of the even superdivision algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ then there are two irreps $W_{\pm}$. The general rep of $U^{\rm tw}$ is isomorphic to $\mathcal{H} = W_+^{\oplus n_+} \oplus W_-^{\oplus n_-}$. Then the graded commutant is

$$Z_s(\rho, \mathcal{H}) = {\rm End}(\mathbb{R}^{n_+|n_-}) \widehat{\otimes} D^{\rm opp} \tag{16.17}$$

In these cases the group $U^{\rm tw}$ (which is isomorphic to ${\rm Pin}^{\pm}(1)$, see below) is purely even so the Hamiltonian can be even or odd or a sum of even and odd. We can therefore forget about the grading and we recover precisely Dyson's 3 cases. If $U^{\rm tw}$ corresponds to one of the remaining 7 superdivision algebras (those which are not even) then there is a unique graded irrep W and up to isomorphism $\mathcal{H} = W^{\oplus n}$ so again

$$Z_s(\rho, \mathcal{H}) = {\rm End}(\mathbb{R}^{n}) \widehat{\otimes} D^{\rm opp} \tag{16.18}$$

As discussed above we can impose Hermiticity conditions on the graded commutant to get the relevant ensembles of Hamiltonians.

♣We should write those ensembles out explicitly. ♣

Remark: We motivated the study of $M_{2,2}$ and its subgroups using the example of disordered systems. Unfortunately, in the literature on this subject it is often assumed that given a pair of homomorphisms

$$(\tau, \chi) : G \to M_{2,2} \tag{16.19}$$

such that $\tau \cdot \chi = \phi$, we will always have $G \cong G_0 \times U$, where U is a subgroup of $M_{2,2}$ and G_0 is $\ker(t) \cap \ker(\chi)$. This is *not true* in general! There is a sequence

$$1 \to G_0 \to G \to U \to 1 \tag{16.20}$$

and in general it will not split, let alone be a direct product.

♣Should mention some examples here. ♣

Exercise

Show that for $M_{2,2}$ one may always choose, (after a possible rescaling by a phase), lifts T and C of $\bar{C}$ and $\bar{T}$, respectively so that $TC = CT$.

Exercise

Show that in the case the subgroup is $U = \{1, \bar{S}\}$ with $\bar{S} = \bar{C}\bar{T}$, one may always choose a lift so that $S^2 = 1$ (or S^2 is any other phase, for that matter.)

Exercise

a.) Consider $M_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and suppose ψ_1, ψ_2 are two distinct homomorphisms to $\mathbb{Z}_2 \cong \{\pm 1\}$. Then WLOG we can choose generators $\bar{T}, \bar{C}$ with

$$\begin{aligned} \psi_1(\bar{T}) &= -1 & \psi_2(\bar{T}) &= +1 \\ \psi_1(\bar{C}) &= +1 & \psi_2(\bar{C}) &= -1 \end{aligned} \tag{16.21}$$

In particular, if τ, χ are distinct and we define $\phi = \tau \cdot \chi$ then $\phi(\bar{T}) = \phi(\bar{C}) = -1$.

---

17. Pin and Spin

sec:PinSpin

17.1 Definitions

The Pin and Spin groups are double covers of orthogonal and special orthogonal groups, respectively. They are best defined as groups of invertible elements inside a Clifford algebra.

To motivate the definition let us recall a few facts about the orthogonal and special orthogonal groups. Let $\mathbb{R}^{t,s}$ be the real vector space of dimension $d = t + s$ with symmetric bilinear form $(x, y) = \eta_{ij} x^i y^j$ where $\eta_{ij} = \mathrm{Diag}\{-1^t, +1^s\}$. By definition, $O(t,s)$ is the group of automorphisms of this bilinear form. If the form is definite we write $\mathbb{R}^d$ and $O(d)$.

Now, if $x \in \mathbb{R}^{t,s}$ is a vector such that $(x, x) \neq 0$ then we can define a transformation $R_x : \mathbb{R}^{t,s} \to \mathbb{R}^{t,s}$:

$$R_x : y \mapsto y - 2\frac{(x,y)}{(x,x)} x \tag{17.1}$$

Note that, $R_x = R_{\alpha x}$ where α is any nonzero real number and hence R_x only depends on the unoriented real line through x, so we could write R_ℓ, where ℓ is the line through x. A short computation, making use of the symmetry of the form, shows that R_x is an orthogonal transformation:

$$(R_x y_1, R_x y_2) = (y_1, y_2) \tag{17.2}$$

In the case of definite signature there is a simple geometric intuition here: A real line in $\mathbb{R}^d$ determines a unique orthogonal plane through the origin and R_x is reflection in that plane. A basic fact of group theory is that the group $O(t,s)$ is generated by the reflections R_x in vectors of nonzero norm.

The group $O(t,s)$ has four connected components when $t > 0$ and $s > 0$ and two components when the form has definite signature. The special orthogonal group $SO(t,s)$ is the subgroup of orientation preserving transformations and has two components. The transformations R_x are orientation reversing and hence $SO(t,s)$ contains products $R_{x_1} R_{x_2}$. In fact, these products generate the group $SO(t,s)$.

Returning to definite signature, the product of two reflections $R_{x_1} R_{x_2}$ is a rotation in the two dimensional plane spanned by the vectors x_1, x_2. See Figure fig:REFLECT-TWO-LINES 13.

Now, to define the Pin and Spin groups we consider the vector space $\mathbb{R}^{t,s}$ as embedded in the real Clifford algebra $\mathrm{Cliff}_{-t,s}$ as the linear span of the generators, and we must make a few definitions:

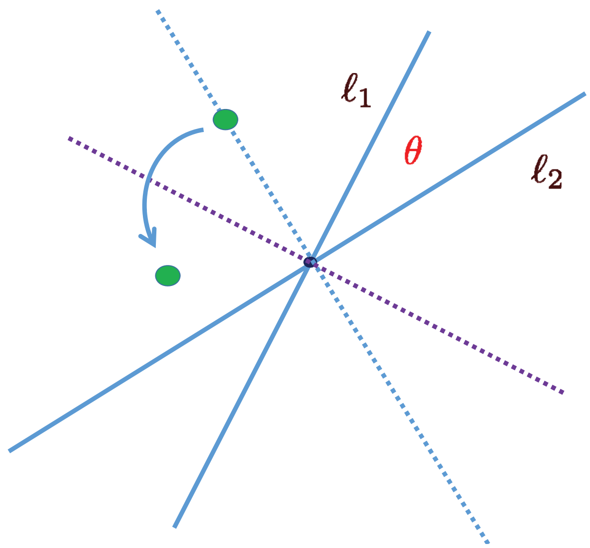

Figure 13: A product of reflections $R_{\ell_1} R_{\ell_2}$ is a rotation by angle 2θ around the point of intersection, where $0 \leq \theta \leq \frac{\pi}{2}$ is the acute angle between ℓ_1 and ℓ_2. The rotation is ccw (cw) if the rotation of ℓ_2 into ℓ_1 by θ is ccw (cw). The easy way to remember this is to consider the image of a point on a plane orthogonal to ℓ_2, as shown. fig:REFLECT-TWO-L

First consider the *group* $C\ell^*_{-t,s}$ of invertible elements of the algebra.
Examples:

1. $C\ell_1^* = \{a+be^1 | a^2-b^2 \neq 0\} \cong \mathbb{R}^* \times \mathbb{R}^*$. Recall that as an ungraded algebra $C\ell_1 \cong \mathbb{R} \oplus \mathbb{R}$ via the projection operators $P_\pm = \frac{1}{2}(1 \pm e)$, from which the group structure above is obvious.
2. $C\ell^*_{-1} = \mathbb{C}^* \cong \mathbb{R}_+ \times U(1)$
3. $C\ell^*_{-2} = \mathbb{H}^* \cong \mathbb{R}_+ \times SU(2)$
4. $C\ell^*_{-3} \cong \mathbb{H}^* \times \mathbb{H}^*$
5. $C\ell^*_{-4}$
6. $C\ell^*_{-5} \cong GL(4, \mathbb{C})$
7. $C\ell^*_{-6} \cong GL(8, \mathbb{R})$
8. $C\ell^*_{-7} \cong GL(8, \mathbb{R}) \times GL(8, \mathbb{R})$
9. $C\ell^*_{-8k} \cong GL(2^{4k}, \mathbb{R})$

♣Are these the best examples?? ♣

Now define the algebra automorphism $\lambda : C\ell_{-t,s} \to C\ell_{-t,s}$ by defining it on the generators to be $\lambda(e_i) = -e_i$ and extending it to be an algebra automorphism. On homogeneous elements it is just the $\mathbb{Z}_2$-grading. If $\phi \in C\ell^*_{-t,s}$ we define the *twisted adjoint action*: It is a homomorphism of groups (not algebras!):

♣The notation ϕ for a general element of Clifford is bad since it is an important homomorphism above and below. ♣

$$\widetilde{\mathrm{Ad}} : C\ell^*_{-t,s} \to GL(C\ell_{-t,s}) \cong GL(2^{t+s}, \mathbb{R}) \tag{17.3}$$

where on the RHS we mean the group of invertible linear transformations of $C\ell_{-t,s}$ as a vector space. It is defined by

$$\widetilde{\mathrm{Ad}}(\phi) : \psi \mapsto \lambda(\phi) \cdot \psi \cdot \phi^{-1} \qquad\qquad \forall \psi \in C\ell_{-t,s} \tag{17.4}$$

One easily checks the homomorphism property: $\widetilde{\mathrm{Ad}}(\phi_1)\widetilde{\mathrm{Ad}}(\phi_2) = \widetilde{\mathrm{Ad}}(\phi_1 \cdot \phi_2)$ and hence $\widetilde{\mathrm{Ad}}$ defines a representation of the the group $C\ell^*_{-t,s}$. The reason we put in the extra twisting by parity, λ, is that we want certain operators of the form $\widetilde{\mathrm{Ad}}(\phi)$ to act as reflection operators on the subspace $\mathbb{R}^{t,s} \subset C\ell_{-t,s}$ spanned by the generators e_i. In particular, $x = x^i e_i \in \mathbb{R}^{t,s}$ is an invertible element of $C\ell_{-t,s}$ iff $(x,x) \neq 0$ and the inverse, in the group $C\ell^*_{-t,s}$, is

$$x^{-1} = \frac{x}{(x,x)} \tag{17.5}$$

Then for any $y = y^i e_i \in \mathbb{R}^{t,s} \subset C\ell_{-t,s}$ (invertible or not) we have

$$\begin{aligned} \widetilde{\mathrm{Ad}}(x)y &= -xyx^{-1} \\ &= -\frac{xyx}{(x,x)} = -\frac{(xy+yx)x}{(x,x)} + y \\ &= y - 2\frac{(y,x)}{(x,x)}x \end{aligned} \tag{17.6}$$

It follows that if we consider the *subgroup* of $C\ell^*_{-t,s}$ generated by x with $(x,x) \neq 0$ then under $\widetilde{\mathrm{Ad}}$ that subgroup covers the entire orthogonal group $O(t,s)$. Moreover, since $\widetilde{\mathrm{Ad}}(\alpha x) = \widetilde{\mathrm{Ad}}(x)$ for α a nonzero scalar we can, WLOG take those vectors to be of norm ± 1. This leads to the definitions:

Definition: $\mathrm{Pin}(t,s)$ is the subgroup of $C\ell^*_{-t,s}$ generated by vectors of norm ± 1. $\mathrm{Spin}(t,s)$ is the subgroup of even elements. In equations: [37]

$$\mathrm{Pin}(t,s) := \{\pm v_1 \cdots v_n \quad | \quad v_s \in \mathbb{R}^{t,s} \quad \& \quad |(v_s,v_s)| = 1 \quad 1 \leq s \leq n\} \tag{17.7}$$

eq:Pints-def

$$\mathrm{Spin}(t,s) := \{\pm v_1 \cdots v_{2n} \quad | \quad v_s \in \mathbb{R}^{t,s} \quad \& \quad |(v_s,v_s)| = 1 \quad 1 \leq s \leq 2n\} = C\ell^0_{-t,s} \cap \mathrm{Pin}(t,s) \tag{17.8}$$

eq:Spints-def

In the case of a definite signature we write $\mathrm{Pin}^+(d) = \mathrm{Pin}(0,d)$ and $\mathrm{Pin}^-(d) = \mathrm{Pin}(d,0)$.

[37] With the exception of $\mathrm{Pin}(0,1) = \mathrm{Pin}^+(1)$ and $\mathrm{Spin}(1) = \{\pm 1\}$ we can drop the $\pm$ in the definition of the Pin and Spin group. If t,s are positive then for an appropriate vector v we can arrange that $v^2 = \pm 1$. In the definite signature case of $\mathrm{Pin}(d)$ or $\mathrm{Spin}(d)$ consider $(e_1e_2)^2 = -1$.

We will see that $\mathrm{Spin}(d,0) \cong \mathrm{Spin}(0,d)$ so we just denote this as $\mathrm{Spin}(d)$.

Note that under the homomorphism $\widetilde{\mathrm{Ad}}$, the group $\mathrm{Pin}(t,s)$ maps to the linear transformations on $C\ell_{-t,s}$ that have the special property that they take the subspace $\mathbb{R}^{t,s}$ to itself and preserve the norm. That is,

$$\widetilde{\mathrm{Ad}} : \mathrm{Pin}(t,s) \to O(t,s) \to 1 \tag{17.9}$$

Moreover, for a single vector v, $\widetilde{\mathrm{Ad}}(v)$ is orientation reversing and hence

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(t,s) \to SO(t,s) \to 1 \tag{17.10}$$

Now we consider the kernel of $\widetilde{\mathrm{Ad}}$. Suppose $\phi \in \mathrm{Pin}(t,s)$ is in the kernel. Decompose ϕ into its even and odd pieces: $\phi = \phi^0 + \phi^1$. Then, for all $y \in \mathbb{R}^{t,s}$ we have $\lambda(\phi)y = y\phi$. Since y is odd this is equivalent to two equations:

$$\begin{aligned} \phi^0 y &= y\phi^0 \\ \phi^1 y &= -y\phi^1 \end{aligned} \tag{17.11}$$

In particular, since the generators of $C\ell_{-t,s}$ are in $\mathbb{R}^{t,s}$ these equations say that ϕ is in the graded center of $C\ell_{-t,s}$. But we know that the Clifford algebra is a central superalgebra, so $\phi \in \mathbb{R}^*$ is an invertible scalar. What scalars can we get? If

$$v_1 \cdots v_n = \alpha \in \mathbb{R}^* \tag{17.12}$$

then we can apply the transpose anti-automorphism β (see exercise 1 in Section §13.1.4 subsubsec:CliffVolElmt above) to this equation and multiply the two equations to get

$$v_1^2 \cdots v_n^2 = \alpha^2 \tag{17.13}$$

and hence $\alpha^2 = \pm 1$. Since α is real, $\alpha^2 = +1$ and hence $\alpha = \pm 1$. Therefore, the kernel is just the group $\{\pm 1\} \cong \mathbb{Z}_2$ and we have the exact sequences:

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(t,s) \stackrel{\widetilde{Ad}}{\to} O(t,s) \to 1 \tag{17.14}$$

eq:pinextseq

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(t,s) \stackrel{\widetilde{Ad}}{\to} SO(t,s) \to 1 \tag{17.15}$$

eq:spinextseq

17.1.1 The norm function

Recall from exercise 1 in Section §13.1.4 subsubsec:CliffVolElmt above that the transpose operation $\phi \to \beta(\phi)$ is defined to be the unique ungraded anti-automorphism that is the identity on $\mathbb{R}^{t,s}$. Thus, [38]

$$\beta(\phi_1\phi_2) = \phi_2\phi_1 \tag{17.16}$$

and $\beta(e_i) = e_i$. In terms of a basis:

$$\beta(e_{i_1} e_{i_2} \cdots e_{i_k}) = e_{i_k} e_{i_{k-1}} \cdots e_{i_2} e_{i_1} \tag{17.17}$$

[38] Note well that this violates the Koszul sign rule! That leads to some awkward signs in some equations with β. It is possible to define a closely related anti-automorphism which respects the Koszul rule.

Definition Define $\bar{\phi} := \lambda \circ \beta(\phi)$, and the *norm function:*

$$N(\phi) := \phi\bar{\phi}. \tag{17.18}$$

The norm function has some nice properties when restricted to the *Clifford group* $\Gamma(t,s)$, namely is the subgroup of $C\ell(t,s)^*$ which preserves the vector space $\mathbb{R}^{t,s}$ generated by e_i under twisted adjoint action. That is, $\phi \in \Gamma(t,s)$ if for all vectors $y = y^i e_i$ (where y^i are real numbers)

$$\lambda(\phi) \cdot y \cdot \phi^{-1} \in \mathbb{R}^{t,s} \tag{17.19}$$

First, we claim that if $\phi \in \Gamma(t,s)$ then $N(\phi) \in \mathbb{R}^*$. To see this let $\lambda(\phi)y\phi^{-1} := y'$. Take the transpose of this equation and solve for y to get $y = \lambda(\bar{\phi})y'\bar{\phi}^{-1}$. Therefore

$$\begin{aligned}
y' &= \lambda(\phi)y\phi^{-1} \\
&= \lambda(\phi)\lambda(\bar{\phi})y'\bar{\phi}^{-1}\phi^{-1} \\
&= \lambda(\phi\bar{\phi})y'(\phi\bar{\phi})^{-1} \\
&= \widetilde{\mathrm{Ad}}(\phi\bar{\phi})y'
\end{aligned} \tag{17.20}$$

is true for all y'. Therefore $\phi\bar{\phi} \in \ker(\widetilde{\mathrm{Ad}})$ and we have already shown [39] that $\ker(\widetilde{\mathrm{Ad}}) \cong \mathbb{R}^*$.

It follows that $N(\phi_1\phi_2) = \phi_1\phi_2\bar{\phi}_2\bar{\phi}_1 = \phi_1 N(\phi_2)\bar{\phi}_1 = \phi_1\bar{\phi}_1 N(\phi_2) = N(\phi_1)N(\phi_2)$ and hence $N : \Gamma(t,s) \to \mathbb{R}^*$ is a homomorphism.

Moreover, we claim that for $\phi \in \Gamma(t,s)$ we have $\widetilde{\mathrm{Ad}}(\phi) \in O(t,s)$. To prove this, note that if $\phi \in \Gamma(t,s)$ and $y \in \mathbb{R}^{t,s}$ then

$$\begin{aligned}
\widetilde{\mathrm{Ad}}(\phi)y \cdot \overline{\widetilde{\mathrm{Ad}}(\phi)y} &= \lambda(\phi)y\phi^{-1}\bar{\phi}^{-1}\bar{y}\lambda(\bar{\phi}) \\
&= (\lambda(\phi)\lambda(\bar{\phi}))(\bar{\phi}\phi)^{-1}y\bar{y} \\
&= y\bar{y}
\end{aligned} \tag{17.21}$$

eq:presnorm

Alternatively, just use the fact that N is a homomorphism: $N(\widetilde{\mathrm{Ad}}(\phi)y) = N(\lambda(\phi)y\phi^{-1}) = N(\phi)N(y)N(\phi)^{-1} = N(y)$.

Therefore we have shown that:

$$1 \to \mathbb{R}^* \to \Gamma(t,s) \stackrel{\widetilde{Ad}}{\to} O(t,s) \to 1 \tag{17.22}$$

eq:pinone

Moreover, it follows that an alternative definition of $\mathrm{Pin}(t,s)$ can be given as

$$\mathrm{Pin}(t,s) := \{\phi \in \Gamma(t,s) : |N(\phi)| = 1\} \subset \Gamma(t,s) \tag{17.23}$$

eq:DefPin-alt

and in fact $\Gamma(t,s) \cong \mathrm{Pin}(t,s) \times \mathbb{R}_+$.

Note that since $\mathrm{Spin}(t,s) := \mathrm{Pin}(t,s) \cap C\ell^0_{-t,s}$ and $C\ell^0_{-t,s} = C\ell^0_{-s,t}$ it follows that $\mathrm{Spin}(t,s) = \mathrm{Spin}(s,t)$. However, the analogous statement for Pin is definitely false.

One useful application of the norm function is that it gives a neat definition of the groups Pin^c and Spin^c which are useful in both geometry and physics. To define these

[39]The same argument works for $\Gamma(t,s)$.

we work with the complexified Clifford algebras. In the complex case we define $x \to \bar{x}$ to include complex conjugation. That is, if x is in a real Clifford algebra then $\overline{(x \otimes z)} = \bar{x} \otimes \bar{z}$. We can again define the Clifford group $\Gamma_c(t,s) \subset \mathbb{C}\ell_d^*$ as the group preserving the subspace $\mathbb{R}^{t,s} \otimes \mathbb{C}$ under $\widetilde{\mathrm{Ad}}$. Now the kernel of $\widetilde{\mathrm{Ad}}$ is $\mathbb{C}^*$ and for $x \in \mathbb{C}^*$ we have $N(x) = 1$ for $|x| = 1$, i.e. for $x \in U(1)$. The same computation (eq:presnorm 17.21) above shows that in the complex case the image of $\widetilde{\mathrm{Ad}}$ is in $U(d) \subset GL(d,\mathbb{C})$, but one can also show that

$$\overline{\widetilde{\mathrm{Ad}}(y)} = \widetilde{\mathrm{Ad}}(\bar{y}) \tag{17.24}$$

and hence the image is in fact in $O(d) \subset U(d)$.

Taking our queue from (eq:DefPin-alt 17.23) we define:

$$\mathrm{Pin}^c(d) := \{\phi \in \Gamma_c(t,s) \subset \mathbb{C}\ell_d : |N(\phi)| = 1\} \tag{17.25}$$

eq:PincDef

The intersection with $(\mathbb{C}\ell(d))^0$ defines $\mathrm{Spin}^c(d)$ so we get

$$1 \to U(1) \to \mathrm{Pin}^c(d) \xrightarrow{\widetilde{Ad}} O(d;\mathbb{R}) \to 1 \tag{17.26}$$

eq:pintwoa

$$1 \to U(1) \to \mathrm{Spin}^c(d) \xrightarrow{\widetilde{Ad}} SO(d;\mathbb{R}) \to 1 \tag{17.27}$$

eq:spintwob

From this one can show

$$\mathrm{Pin}^c(d) = (\mathrm{Pin}^{\pm}(d) \times U(1))/\mathbb{Z}_2 \tag{17.28}$$

$$\mathrm{Spin}^c(d) = (\mathrm{Spin}(d) \times U(1))/\mathbb{Z}_2 \tag{17.29}$$

Exercise

Show that in general that for $\phi \in \Gamma(t,s) \subset C\ell_{s,-t}$, the norm $N(\phi)$ can be positive or negative.

Exercise

Show that e^{1234} is not connected to the identity in $\mathrm{Spin}(1,3)$.

Exercise

a.) Consider the quaternions realized as $C\ell(0,2)$. Show that $x\bar{x}$ is the norm of the quaternion.

b.) Show that we can identify $C\ell(0,2)^* \cong \mathbb{R}^4 - \{0\}$

17.2 The relation of Pin and Spin for definite signature

We consider the case of definite signature for simplicity. Then we define $\mathrm{Pin}^{\pm}(d)$ according to whether $e_i^2 = +1$ or $e_i^2 = -1$.

If $P \in O(n)$ is any reflection, then $\{1, P\} \cong \mathbb{Z}_2$ generates a $\mathbb{Z}_2$ subgroup of $O(n)$. In $Pin^+(n)$ this subgroup is covered by $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the double coverings $\pm\tilde{P}$ of the reflection squares to $1 \in Pin^+(n)$. On the other hand, in $Pin^-(n)$, this subgroup is covered by $\mathbb{Z}_4$ and the double coverings of the reflection $\pm\tilde{P}$ squares to $-1 \in Spin(n)$.

The difference between Pin^+ and Pin^- is that the double coverings of a reflection squares to 1 in Pin^+ and squares to -1 in Pin^-.

Now, from the definition it is clear that $\mathrm{Spin}(d) \subset \mathrm{Pin}(d)$ is a normal subgroup of index two. We can define an explicit homomorphism

$$\delta : \mathrm{Pin}(d) \to \{\pm 1\} \cong \mathbb{Z}_2 \tag{17.30}$$ `eq:DetRep`

by $\delta(\phi) := \det\widetilde{\mathrm{Ad}}(\phi)$, and $\mathrm{Spin}(d) = \ker(\delta)$. So

$$0 \to \mathrm{Spin}(d) \to \mathrm{Pin}^{\pm}(d) \quad \stackrel{\delta}{\to} \quad \mathbb{Z}_2 \to 0 \tag{17.31}$$

In the case of $\mathrm{Pin}^+(d)$ we can split this sequence by taking $s(-1) = e^1$ (or any other vector of unit norm). The associated automorphism of $\mathrm{Spin}(d)$ is:

$$\phi \to e^1 \phi e^1 \tag{17.32}$$

and in general is an outer automorphism. It follows that

$$\mathrm{Pin}^+(d) \cong \mathrm{Spin}(d) \rtimes \mathbb{Z}_2. \tag{17.33}$$

It turns out that $\mathrm{Pin}^-(d)$ is more complicated. To describe the group structure of $\mathrm{Pin}^-(d)$ we define an automorphism of $\mathbb{Z}_4$ on $\mathrm{Spin}(d)$. Choose any vector v with $v^2 = -1$ (e.g. it could be any of the generators). Then $\omega^j \in \mathbb{Z}_4$ acts by

$$\alpha_{\omega^j} : \mathcal{E} \to v^j \mathcal{E} (v^j)^{-1} \tag{17.34}$$

Using this automorphism construct the semidirect product $\mathrm{Spin}(d) \rtimes \mathbb{Z}_4$. Then we claim there is a well-defined surjective homomorphism

$$\mathrm{Spin}(d) \rtimes \mathbb{Z}_4 \to \mathrm{Pin}^-(d) \to 0 \tag{17.35}$$

given by $(\mathcal{E}, \omega^j) \to \mathcal{E} v^j$. Now, the kernel of this homomorphism is just $\mathbb{Z}_2$ with the nontrivial element being $(-1, -1)$.

To summarize:

$$\mathrm{Pin}^+(d) = \mathrm{Spin}(d) \rtimes \mathbb{Z}_2 \tag{17.36}$$ `eq:PinPlusSeq`

$$\mathrm{Pin}^-(d) = \left(\mathrm{Spin}(d) \rtimes \mathbb{Z}_4\right) / \mathbb{Z}_2 \tag{17.37}$$ `eq:PinMinSeq`

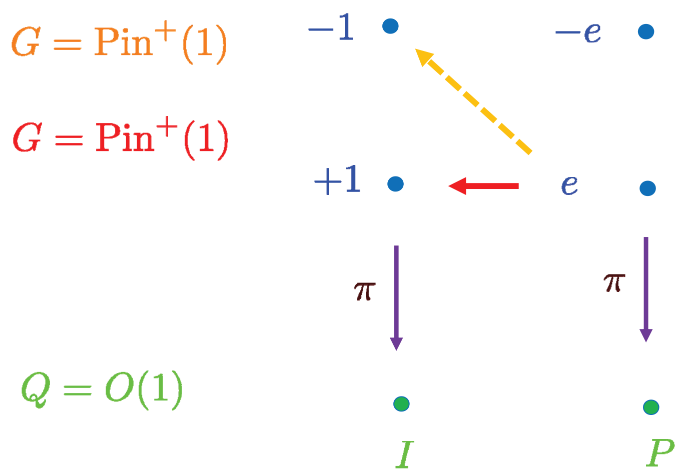

Figure 14: Illustrating $\mathrm{Pin}^{\pm}(1)$ double covering $O(1)$. The red arrow indicates $e^2 = +1$ in $\mathrm{Pin}^+(1)$ and the golden arrow indicates $e^2 = -1$ in $\mathrm{Pin}^-(1)$. Of course, $\mathrm{Spin}(1)$ is the $\mathbb{Z}_2$ subgroup double covering the identity I in $O(1)$. `fig:PinSpinOne`

17.3 Examples of low-dimensional Pin and Spin groups

In this section we give some explicit examples of Pin and Spin groups in low dimensions. These examples have the nice feature that one can easily parametrize the general group element in a way which makes the group multiplication simple. On the other hand, the reader should be warned that the topological properties in these cases are *not* representative of the general case.

17.3.1 $\mathrm{Pin}^{\pm}(1)$

$$\mathrm{Pin}^+(1) = \{\pm 1, \pm e\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad (17.38)$$ `eq:pinpl`

$$\mathrm{Pin}^-(1) = \{\pm 1, \pm e\} \cong \mathbb{Z}_4 \qquad (17.39)$$ `eq:pinpla`

Now, $O(1) = \{I, P\}$, where I is the identity and P is the nontrivial element, acts on the 1-dimensional vector space $\mathbb{R}e$ by $P : e \to -e$. We have

$$\widetilde{\mathrm{Ad}}(\pm 1) = I \qquad \widetilde{\mathrm{Ad}}(\pm e) = P \qquad (17.40)$$ `eq:homim`

Thus we have

$$0 \to \mathbb{Z}_2 \to Pin^{\pm}(1) \to O(1) \to 0 \tag{17.41}$$

eq:exts

There is a homomorphism $O(1) \to Pin^+(1)$ which splits the sequence, but there is no such homomorphism for $Pin^-(1)$.

Note that $\mathrm{Spin}(1) = \{\pm 1\}$ is the double cover of the identity element $I \in O(1)$.

See Figure fig:PinSpinOne 14.

17.3.2 $\mathrm{Pin}^+(2)$

Now consider $\mathrm{Pin}^+(2) \subset C\ell_{+2}^* \cong GL(2,\mathbb{R})$.

For $\alpha \sim \alpha + 2\pi$ we define:

$$\begin{aligned} \mathcal{O}(\alpha) &= \cos(\alpha)e^1 + \sin(\alpha)e^2 \\ \mathcal{E}(\alpha) &= \cos(\alpha) + \sin(\alpha)e^1e^2 \end{aligned} \tag{17.42}$$

eq:dfns

Then

$$Pin^+(2) = \{\mathcal{E}(\alpha)\} \amalg \{\mathcal{O}(\alpha)\} \tag{17.43}$$

eq:pinplt

has two components, both isomorphic to the circle. To compute the group structure note that $\mathcal{E}(\alpha)\mathcal{E}(\beta) = \mathcal{E}(\alpha+\beta)$, so these elements form a subgroup. This is the group Spin(2). The group multiplication for $\mathrm{Pin}^+(2)$ is easily worked out to be:

$$\begin{aligned} \mathcal{E}(\alpha)\mathcal{E}(\beta) &= \mathcal{E}(\alpha+\beta) \\ \mathcal{O}(\alpha)\mathcal{E}(\beta) &= \mathcal{O}(\alpha+\beta) \\ \mathcal{E}(\beta)\mathcal{O}(\alpha) &= \mathcal{O}(\alpha-\beta) \\ \mathcal{O}(\alpha)\mathcal{O}(\beta) &= \mathcal{E}(\beta-\alpha) \end{aligned} \tag{17.44}$$

eq:pinplgs

In particular Spin(2) is isomorphic to $U(1)$. Note that for any α, $\mathcal{O}(\alpha)\mathcal{O}(\alpha) = +1$.

If we consider the homomorphism $\widetilde{\mathrm{Ad}} : Pin^+(2) \to O(2)$ we have a matrix representation defined by

$$\widetilde{\mathrm{Ad}}(\phi)(e^a) = \sum_b (\widetilde{\mathrm{Ad}}(\phi))_{ba} e^b \tag{17.45}$$

eq:anron

Then in the ordered basis $\{e_1, e_2\}$:

$$\begin{aligned} \widetilde{\mathrm{Ad}}(\mathcal{E}(\beta)) &= \begin{pmatrix} \cos(2\beta) & \sin(2\beta) \\ -\sin(2\beta) & \cos(2\beta) \end{pmatrix} \\ \widetilde{\mathrm{Ad}}(\mathcal{O}(\beta)) &= \begin{pmatrix} -\cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & \cos(2\beta) \end{pmatrix} \end{aligned} \tag{17.46}$$

eq:mtrxrp

These relations nicely illustrate the exact sequence (eq:PinPlusSeq 17.36):

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}^+(2) \to O(2) \to 0 \tag{17.47}$$

eq:extsaa

Topologically, both Spin(2) and $SO(2)$ are copies of the circle. From the above we see that the double-covering Spin(2) $\to SO(2)$ is the *nontrivial* double-covering of a circle over a circle. The group $O(2)$ has two connected components, each component is a circle. The group Pin(2) also has two connected components, each is a circle nontrivially double-covering the circle in $O(2)$.

Note that we also can define a homomorphism $\phi : \mathrm{Pin}^+(2) \to \mathbb{Z}_2$ given by $\phi(x) = \det\widetilde{\mathrm{Ad}}(x)$. The kernel is $\mathrm{Spin}(2) \cong U(1)$ and the sequence

$$0 \to U(1) \to \mathrm{Pin}^+(2) \to \mathbb{Z}_2 \to 0 \tag{17.48}$$

`eq:PhiTwPlus`

splits by

$$s : \bar{T} \to T := \mathcal{O}(\alpha) \tag{17.49}$$

where we can choose any reflection $\mathcal{O}(\alpha)$ we please. Thus we recognize (17.48) as the ϕ-twisted extension we called M_2^+ in the example (6.12) above.

`eq:PhiTwPlus` `eq:Gtau`

17.3.3 $\mathrm{Pin}^-(2)$

Now consider $\mathrm{Pin}^-(2) \subset C\ell_{-2}^* \cong \mathbb{H}^* \cong \mathbb{R}_+ \times SU(2)$.

For $\alpha \sim \alpha + 2\pi$ we define:

$$\begin{aligned} \mathcal{O}(\alpha) &= \cos(\alpha)e^1 + \sin\alpha e^2 \\ \mathcal{E}(\alpha) &= \cos(\alpha) + \sin(\alpha)e^1 e^2 \end{aligned} \tag{17.50}$$

`eq:dfnsp`

Then

$$\mathrm{Pin}^-(2) = \{\mathcal{E}(\alpha)\} \amalg \{\mathcal{O}(\alpha)\} \tag{17.51}$$

`eq:pinplta`

has two components, both isomorphic to the circle. The group structure is then easily computed:

$$\begin{aligned} \mathcal{E}(\alpha)\mathcal{E}(\beta) &= \mathcal{E}(\alpha+\beta) \\ \mathcal{O}(\alpha)\mathcal{E}(\beta) &= \mathcal{O}(\alpha-\beta) \\ \mathcal{E}(\beta)\mathcal{O}(\alpha) &= \mathcal{O}(\alpha+\beta) \\ \mathcal{O}(\alpha)\mathcal{O}(\beta) &= \mathcal{E}(\alpha-\beta+\pi) \end{aligned} \tag{17.52}$$

`eq:pinplgsa`

If we consider the homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Pin}^-(2) \to O(2)$ we have a matrix representation

$$\begin{aligned} \widetilde{\mathrm{Ad}}(\mathcal{E}(\beta)) &= \begin{pmatrix} \cos(2\beta) & -\sin(2\beta) \\ \sin(2\beta) & \cos(2\beta) \end{pmatrix} \\ \widetilde{\mathrm{Ad}}(\mathcal{O}(\beta)) &= \begin{pmatrix} -\cos(2\beta) & -\sin(2\beta) \\ -\sin(2\beta) & \cos(2\beta) \end{pmatrix} \end{aligned} \tag{17.53}$$

`eq:mtrxrpa`

The first line shows the double-covering of Spin(2) over $SO(2)$. In the next line we have the set of all reflections in $O(2)$ double-covered by the elements $\mathcal{O}(\beta)$. In particular, we have (eq:PinMinSeq 17.37):

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}^-(2) \to O(2) \to 0 \tag{17.54}$$ eq:extsaa

Note that we also can define a homomorphism $\phi : \mathrm{Pin}^-(2) \to \mathbb{Z}_2$ given by $\phi(x) = \det\widetilde{\mathrm{Ad}}(x)$. The kernel is $\mathrm{Spin}(2) \cong U(1)$ and the sequence

$$0 \to U(1) \to \mathrm{Pin}^-(2) \to \mathbb{Z}_2 \to 0 \tag{17.55}$$ eq:PhiTwMin

does not split. The most general section we can choose is

$$s : \bar{T} \to T := \mathcal{O}(\alpha) \tag{17.56}$$

where we can choose any reflection $\mathcal{O}(\alpha)$ we please, but now reflections square to -1. Thus we recognize (eq:PhiTwPlus 17.48) as the ϕ-twisted extension we called M_2^- in Section §**** above.

17.3.4 Pin(1, 1)

Now consider $C\ell_{1,-1}$ and let the generators be $\{e_1, e_2\}$ with $e_1^2 = +1$ and $e_2^2 = -1$. It is useful to form the isotropic elements

$$e_+ := \frac{e_1 + e_2}{2} \qquad e_- := \frac{e_1 - e_2}{2} \tag{17.57}$$

so that

$$e_+^2 = e_-^2 = 0 \qquad \{e_+, e_-\} = 1 \tag{17.58}$$

Indeed $P_+ = e_-e_+ = \frac{1}{2}(1 + e_{12})$ and $P_- = e_+e_- = \frac{1}{2}(1 - e_{12})$ are orthogonal projection operators, and provide a basis for the even subalgebra.

We can define group elements

$$\begin{aligned} \mathcal{E}_{\chi_+,\chi_-}(\theta) &= \chi_+ e^{\theta} P_+ + \chi_- e^{-\theta} P_- \\ \mathcal{O}_{\chi_+,\chi_-}(\theta) &= \chi_+ e^{\theta} e_+ + \chi_- e^{-\theta} e_- \end{aligned} \tag{17.59}$$

where $\theta \in \mathbb{R}$ and $\chi_\pm \in \{\pm 1\}$. It is not difficult to show that these are the most general even and odd elements in Pin(1, 1). Thus, Spin(1, 1) has four connected components, each a copy of $\mathbb{R}$ as a manifold, while Pin(1, 1) has eight connected components, again each a copy of $\mathbb{R}$ as a manifold. This presentation of the group elements makes the computation of the group law especially transparent.

Exercise

Compute the action of Spin(1, 1) on $\mathbb{R}^{1,1}$ by twisted adjoint action:

$$\begin{aligned} \widetilde{\mathrm{Ad}}(\mathcal{E}_{\chi_+,\chi_-}(\theta))e_+ &= \chi_+\chi_- e^{-2\theta} e_+ \\ \widetilde{\mathrm{Ad}}(\mathcal{E}_{\chi_+,\chi_-}(\theta))e_- &= \chi_+\chi_- e^{2\theta} e_- \end{aligned} \tag{17.60}$$

For the components with $\chi_+\chi_- = 1$ the image is a boost of rapidity 2θ. For the components with $\chi_+\chi_- = -1$ the image is such a boost together with a PT transformation.

Exercise $\mathrm{Pin}^{\pm}(3)$

Using the fact that $C\ell^0_{\pm 3} \cong \mathbb{H}$ give an analogous presentation of the group structure of $\mathrm{Pin}^{\pm}(3)$ parametrizing even and odd elements in terms of unit quaternions.

17.4 Some useful facts about Pin ad Spin

17.4.1 The center

We focus on the case of definite signature.

For $d = 1$ the groups $\mathrm{Pin}^{\pm}(d)$ and $\mathrm{Spin}(d)$ are abelian as we saw above.

For $d = 2$ $\mathrm{Spin}(2)$ is abelian, while the center of $\mathrm{Pin}^{\pm}(2)$ is just $\{\pm 1\}$. This follows from the explicit discussion of the group law above.

To understand $d > 2$ note that any element of the center of $\mathrm{Pin}^{\pm}(d)$ or $\mathrm{Spin}(d)$ must map to the center of $O(d)$ or $SO(d)$ under $\widetilde{\mathrm{Ad}}$, respectively. But for $d > 2$ any element of the center of $O(d)$ or $SO(d)$ must be proportional to the identity matrix. The only orthogonal matrices of the form $\alpha 1_d$ are $\pm 1_d$. Viewing -1_d as a product of reflections in the planes orthogonal to $e_1, \dots, e_d$ it is clear that the inverse image of -1_d is $\pm\omega$ where $\omega = e_1 \cdots e_d$ is the volume form. Therefore, the center of $\mathrm{Pin}^{\pm}(d)$ and $\mathrm{Spin}(d)$ must be contained in the group $\{\pm 1, \pm\omega\}$.

To compute the structure of the group $\{\pm 1, \pm\omega\}$ recall that the square of the volume form is

$$\omega^2 = \begin{cases} \epsilon & d = 1\mathrm{mod}4 \\ -1 & d = 2\mathrm{mod}4 \\ -\epsilon & d = 3\mathrm{mod}4 \\ +1 & d = 4\mathrm{mod}4 \end{cases} \tag{17.61}$$

where $\epsilon = \pm$ for $\mathrm{Pin}^{\pm}(d)$. Thus $\{\pm 1, \pm\omega\}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ according to the above cases.

For d even the element ω is indeed in $\mathrm{Spin}(d)$ and clearly is in fact central. For d odd ω is not in $\mathrm{Spin}(d)$. Therefore:

$$Z(\mathrm{Spin}(d)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & d = 0\mathrm{mod}4 \\ \mathbb{Z}_4 & d = 2\mathrm{mod}4 \\ \mathbb{Z}_2 & d = 1\mathrm{mod}2 \end{cases} \tag{17.62}$$

eq:centerevn

For Pin the situation is reversed: If d is even then ω is not central and if d is odd then ω is central, and hence:

$$Z(\mathrm{Pin}^+(d)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & d = 1 \mathrm{mod} 4 \\ \mathbb{Z}_4 & d = 3 \mathrm{mod} 4 \\ \mathbb{Z}_2 & d = 0 \mathrm{mod} 2 \end{cases} \tag{17.63}$$

eq:centerevn

$$Z(\mathrm{Pin}^-(d)) = \begin{cases} \mathbb{Z}_4 & d = 1 \mathrm{mod} 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & d = 3 \mathrm{mod} 4 \\ \mathbb{Z}_2 & d = 0 \mathrm{mod} 2 \end{cases} \tag{17.64}$$

eq:centerevn

17.4.2 Connectivity

For $d = 1$, $\mathrm{Spin}(1) = \{\pm 1\}$ has two components, while $\mathrm{Pin}^\pm(1)$ has four components.

For $d > 1$, $\mathrm{Spin}(d)$ is a connected double-cover of the connected group $SO(d)$. The connectedness follows directly from the definition: Consider an element $v_1 \cdots v_{2n}$. Each vector v_s lies on the unit sphere $v^2 = +1$. But that sphere is connected, for $d > 1$. Therefore for each v_s we may choose a continuous path of vectors $v_s(x)$, $0 \le x \le 1$ of unit norm connecting v_s to some common vector, say e_1, at $x = 1$. Then $v_1(x) \cdots v_{2n}(x)$ is a continuous path of elements in $\mathrm{Spin}(d)$ connecting $v_1 \cdots v_{2n}$ to 1.

Although not strictly necessary, we can also exhibit an explicit path connecting -1 to $+1$ within $\mathrm{Spin}(d)$. Consider the path of elements:

$$r(t) := \cos t + \sin t e_1 e_2 \qquad\qquad 0 \le t \le \pi. \tag{17.65}$$

eq:BasicPath

This path is useful in discussing simple-connectivity below.

Applying the same argument to $\mathrm{Pin}^\pm(d)$ with $d > 1$ shows that it has exactly two components. For $v_1 \cdots v_n$ we choose paths $v_s(x)$ as before. At $x = 1$ we have $(e_1)^n$ which is $= 1$ for n even or e_1 for n odd. Since these map to disconnected components of $O(d)$ under the continuous map $\widetilde{\mathrm{Ad}}$ there cannot be any path connecting 1 and e_1. Each component double-covers the corresponding connected component of $O(n)$. These facts also follow from the group-theoretic discussion of the relation of Pin and Spin above, once we know Spin is connected.

For the general case of $\mathrm{Spin}(t,s)$ and $\mathrm{Pin}(t,s)$ with $t > 0$ and $s > 0$ and $d > 2$ a similar argument can be used to investigate the components: $\mathrm{Spin}(t,s)$ has 2 connected components and $\mathrm{Pin}(t,s)$ has 4 connected components. Each component is a nontrivial double cover of one of the 4 connected components of $O(t,s)$.

To prove the above claims note that the sphere in $\mathbb{R}^{t,s}$ given by

$$-x_1^2 - \cdots - x_t^2 + x_{t+1}^2 + \cdots + x_{t+s}^2 = 1 \tag{17.66}$$

eq:PsdSpr1

(where x_μ are the coordinates of v) is a bundle over $\mathbb{R}^t$ whose fiber is topologically S^{s-1}. The sphere S^0 has two components and S^n has one component for $n > 0$. Therefore the solution space of (eq:PsdSpr1 17.66) has one component for $s > 1$ and two for $s = 1$. By the same token

$$-x_1^2 - \cdots - x_t^2 + x_{t+1}^2 + \cdots + x_{t+s}^2 = -1 \tag{17.67}$$

eq:PsdSpr2

has one component for $t > 1$ and two for $t = 1$.

Now consider an arbitrary group element in $\mathrm{Pin}(t,s)$. Since $d > 2$ either $s > 1$ or $t > 1$ and this allows us to prove that -1 is connected to $+1$. Therefore the general group element is path connected to one of the form

$$v_1 \cdots v_n \tag{17.68}$$

If $s > 1$ and $t > 1$ then for each vector v_s with $v_s^2 = +1$ we choose a path of unit norm vectors connecting it to e_d. For each vector with $v_s^2 = -1$ we choose a path connecting it to e_1. Therefore, at the endpoint of our path we obtain a group element $\pm e_1^{\ell_1} e_d^{\ell_d}$ which is $\pm e_1^{\overline{\ell_1}} e_d^{\overline{\ell_d}}$, where $\overline{\ell_1}, \overline{\ell_d}$ are valued in $\{0,1\}$ and are congruent module two to ℓ_1, ℓ_d, respectively. Again, since -1 is connected to $+1$ we have shown that the arbitrary group element in $\mathrm{Pin}(t,s)$ is connected to one of $1, e_1, e_d, e_1 e_d$. But each of these projects under $\widetilde{\mathrm{Ad}}$ to each of the four components of $O(t,s)$, with e_1 projecting to a "time reflection," and e_d projecting to a "space reflection." If $t = 1$ or $s = 1$ the argument needs to be supplemented but the conclusion is unchanged. For example, if $t = 1$ then there are two components of the set of vectors with $v^2 = -1$. These vectors are pathwise connected to $\pm e_1$. But then, so long as $s > 1$, e_1 can be path connected to $-e_1$ in the group. $\diamondsuit$

17.4.3 Simple-Connectivity

Now consider the simple-connectivity. $\mathrm{Spin}(d)$ is a principal $\mathbb{Z}_2$ bundle over $SO(d)$. From the exact homotopy sequence for fibrations:

$$0 \to \pi_1(\mathrm{Spin}(d)) \to \pi_1(SO(d)) \to \pi_0(\mathbb{Z}_2) \to 0 \tag{17.69}$$

eq:FiberSeq

For sufficiently large d we can use the Bott song to see that $\pi_1(SO(d)) \cong \mathbb{Z}_2$ and therefore it follows that $\pi_1(\mathrm{Spin}(d)) = 0$. In fact this applies for $d > 2$. Of course $\pi_1(SO(2)) \cong \mathbb{Z}$ and hence $\pi_1(\mathrm{Spin}(2)) = \mathbb{Z}$.

Now, consider the path $r(t)$ in equation (17.65). Let us compute its image under $\widetilde{\mathrm{Ad}}$:

$$\begin{aligned} \widetilde{\mathrm{Ad}}(r(t))e_1 &= (\cos t + \sin t e_1 e_2) e_1 (\cos t - \sin t e_1 e_2) \\ &= \left(\cos^2 t - \sin^2 t\right) e_1 - 2 \cos t \sin t e_2 \\ &= \cos(2t) e_1 - \sin(2t) e_2 \\ \widetilde{\mathrm{Ad}}(r(t))e_2 &= \sin(2t) e_1 + \cos(2t) e_2 \end{aligned} \tag{17.70}$$

and of course $\widetilde{\mathrm{Ad}}(r(t))e_i = e_i$ for $i > 2$. The matrix representation is thus

$$\widetilde{\mathrm{Ad}}(r(t)) = \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} \oplus_{i>2} 1 \tag{17.71}$$

Thus the image is the path of rotations $R(2t)$ in the $e_1 e_2$ plane. Note in particular it is a closed path for $0 \le t \le \pi$. Thus this closed loop in $SO(d)$, which is homotopically nontrivial and in fact generates $\pi_1(SO(d), 1)$, lifts to an open loop in $\mathrm{Spin}(d)$ which connects $+1$ to -1, thus explicitly showing how the connecting homomorphism in (17.69) maps the basic loop to the nontrivial element of $\pi_0(\mathbb{Z}_2)$.

17.5 The Lie algebra of the spin group

:LieAlgSpinGrp

We consider the general situation of a quadratic form Q on a real vector space and compute the Lie algebra spin(Q). Since $\widetilde{\mathrm{Ad}}$ is a $2:1$ covering and a group homomorphism we are guaranteed that Lie algebra spin(Q) of Spin(Q) is isomorphic to that of $so(Q)$. However, since we are thinking of Spin(Q) as a subgroup of $C\ell(Q)^*$ we can also give a very nice description of this Lie algebra as a Lie subalgebra of $C\ell(Q)$. After all, the tangent space to Spin(Q) at the origin will be a linear subspace of $T_1 C\ell(Q) \cong C\ell(Q)$, as a vector space. Moreover, $C\ell(Q)$ can also be considered to be a a Lie algebra with the obvious Lie product $[a,b] := ab - ba$, and we will see that $L(\mathrm{Spin}(Q)) := \mathrm{spin}(Q)$ is a Lie subalgebra.

To motivate the construction we want to think of the identity element as the product $v^2 = 1$ for some vector $v \in \mathbb{R}^{t,s} \subset C\ell_{s,-t}$. Consider a path of elements $v_1(t)v_2(t)$ with $v_i(0) = v$ and $v_i^2(t) = 1$. The tangent vector is

$$\frac{d}{dt}|_0(v_1(t)v_2(t)) = \dot{v}_1 v + v\dot{v}_2 \tag{17.72}$$

where $\dot{v}_i = \frac{d}{dt}|_0 v_i(t)$. On the other hand, differentiating $v_i^2(t) = 1$ gives $v\dot{v}_i + \dot{v}_i v = 0$. It follows that we can equally well write the tangent vector as

$$\frac{d}{dt}|_0(v_1(t)v_2(t)) = \frac{1}{2}(\dot{v}_1 v - v\dot{v}_1) + \frac{1}{2}(v\dot{v}_2 - \dot{v}_2 v) = vw - wv \tag{17.73}$$

where $w = \frac{1}{2}(\dot{v}_2 - \dot{v}_1)$.

This suggests that

$$\mathrm{spin}(t,s) = \{v_1 v_2 - v_2 v_1 | v_1, v_2 \in \mathbb{R}^{t,s}\} \subset C\ell_{s,-t} \tag{17.74}$$

and indeed, by the above remark the RHS must be a subspace. On the other hand, it is easy to see that it is already the full dimension of spin(t,s), and hence the spaces are equal. Indeed, if e_μ is a basis for $\mathbb{R}^{t,s}$ with $Q(e_\mu, e_\nu) = \eta_{\mu,\nu}$ then a natural basis for spin(t,s) is the set of generators:

$$M_{\mu\nu} := \frac{1}{2}e_{\mu\nu} = \frac{1}{4}[e_\mu, e_\nu] \tag{17.75}$$

where $\mu \neq \nu$ and we need only take those for $\mu < \nu$ to get a basis since $M_{\mu\nu}$ is antisymmetric on μ, ν.

As a check of the isomorphism $\mathrm{spin}(t,s) \cong so(t,s)$ the reader should use the Clifford algebra relations to compute

$$[e_{\mu\nu}, e_{\lambda\rho}] = 2\Bigl(\eta_{\nu\lambda}e_{\mu\rho} - \eta_{\mu\lambda}e_{\nu\rho} - \eta_{\nu\rho}e_{\mu\lambda} + \eta_{\mu\rho}e_{\nu\lambda}\Bigr) \tag{17.76}$$

It is easy to remember this formula by specializing to the case $\nu = \lambda \neq \mu, \rho$, using the Clifford algebra property, and then imposing the antisymmetry on the pairs μ, ν and λ, ρ. Therefore we get the desired equation:

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda}M_{\mu\rho} \pm 3 \qquad \text{terms} \tag{17.77}$$

A natural choice of Cartan subalgebra is given by the span of

$$h_1 = \frac{1}{2}e_{12}, \quad h_2 = \frac{1}{2}e_{34}, \quad , \dots , \quad h_r = \frac{1}{2}e_{2r-1,2r} \tag{17.78}$$ eq:simple-coroots

where $r = [d/2]$ is the rank of the complex simple Lie algebra $\mathrm{so}(d) \otimes \mathbb{C}$. Indeed the h_i can be taken to be a set of simple coroots. ♣check ♣

Exercise
Show that $Ad(e_{\mu\nu})$ acts on V as a linear transformation

$$\begin{aligned}[e_{\mu\nu}, e_\lambda] &= [e_\mu e_\nu, e_\lambda] \\ &= e_\mu\{e_\nu, e_\lambda\} - \{e_\mu, e_\lambda\}e_\nu \\ &= 2e_\mu Q_{\nu\lambda} - 2e_\nu Q_{\mu\lambda}\end{aligned} \tag{17.79}$$

Exercise
Check that the linear transformation $Ad(e_{\mu\nu})$ preserves the quadratic form Q:

$$\begin{aligned}2Q([e_{\mu\nu}, e_\lambda], e_\rho) + 2Q(e_\lambda, [e_{\mu\nu}, e_\rho]) &= [e_{\mu\nu}, e_\lambda]e_\rho + e_\rho[e_{\mu\nu}, e_\lambda] \\ &\quad + e_\lambda[e_{\mu\nu}, e_\rho] + [e_{\mu\nu}, e_\rho]e_\lambda \\ &= [e_{\mu\nu}, e_\lambda e_\rho + e_\rho e_\lambda] = [e_{\mu\nu}, 2Q_{\lambda\rho}1] \\ &= 0\end{aligned} \tag{17.80}$$

Thus, $Ad(e_{\mu\nu})$ acts as linear transformations on V preserving the quadratic form Q on V and they generate the Lie algebra $so(Q)$.

Exercise
If $T_{\mu\nu}$ is the antisymmetric matrix with 1 (-1) in matrix element μ, ν (ν, μ) and zero elsewhere and $\exp(\frac{1}{2}\omega^{\mu\nu}T_{\mu\nu}) = g \in SO_0(r,s)$ then

$$\exp\left(\frac{1}{4}\omega^{\mu\nu}e_{\mu\nu}\right)e_\lambda \exp\left(-\frac{1}{4}\omega_{\mu\nu}e_{\mu\nu}\right) = g_{\rho\lambda}e_\rho \tag{17.81}$$

17.5.1 The exponential map

Let us consider the image of the exponential map. Note that for each pair $\mu < \nu$ we can write:

$$\exp\left(\frac{1}{2}\theta e_{\mu\nu}\right) = \begin{cases} \cos(\frac{1}{2}\theta) + \sin(\frac{1}{2}\theta)e_{\mu\nu} & \text{if} \quad \eta_{\mu\mu}\eta_{\nu\nu} = +1 \\ \cosh(\frac{1}{2}\theta) + \sinh(\frac{1}{2}\theta)e_{\mu\nu} & \text{if} \quad \eta_{\mu\mu}\eta_{\nu\nu} = -1 \end{cases} \tag{17.82}$$

Although it is guaranteed to be true, let us note that these expressions can be written as the product of two vectors of norm-square ± 1:

$$\exp\left(\frac{1}{2}\theta e_{\mu\nu}\right) = \begin{cases} e_\mu(\eta_{\mu\mu}\cos(\frac{1}{2}\theta)e_\mu + \sin(\frac{1}{2}\theta)e_\nu) & \text{if} \quad \eta_{\mu\mu}\eta_{\nu\nu} = +1 \\ e_\mu(\eta_{\mu\mu}\cosh(\frac{1}{2}\theta)e_\mu + \sinh(\frac{1}{2}\theta)e_\nu) & \text{if} \quad \eta_{\mu\mu}\eta_{\nu\nu} = -1 \end{cases} \tag{17.83}$$

Now, a small computation shows that the image under $\widetilde{\mathrm{Ad}}$ of $e_{\frac{1}{2}\theta e_{\mu\nu}}$ is a rotation (or boost) by angle θ in the $\mu\nu$ plane:

$$\widetilde{\mathrm{Ad}}(\exp\left(\frac{1}{2}\theta e_{\mu\nu}\right))(e_\lambda) = e_\lambda \qquad \lambda \neq \mu, \nu \tag{17.84}$$

$$\widetilde{\mathrm{Ad}}(\exp\left(\frac{1}{2}\theta e_{\mu\nu}\right))(e_\mu) = \mathrm{c}(\theta)e_\mu - \eta_{\mu\mu}\mathrm{s}(\theta)e_\nu \tag{17.85}$$

$$\widetilde{\mathrm{Ad}}(\exp\left(\frac{1}{2}\theta e_{\mu\nu}\right))(e_\nu) = \mathrm{c}(\theta)e_\nu + \eta_{\nu\nu}\mathrm{s}(\theta)e_\mu \tag{17.86}$$

where $\mathrm{c}(\theta)$ is $\cos\theta$ or $\cosh\theta$ according to the sign of $\eta_{\mu\mu}\eta_{\nu\nu}$, etc.

Now let us restrict to the case of definite signature, say all $+$, so $(e_\mu)^2 = 1$ (no sum on μ). Then the maximal torus is the subgroup of $\mathrm{Spin}(d)$ composed of elements of the form:

$$(\cos\theta_1 + e_{12}\sin\theta_1)(\cos\theta_2 + e_{34}\sin\theta_2)\cdots(\cos\theta_r + e_{(2r-1)(2r)}\sin\theta_r) \tag{17.87}$$

where $r = [d/2]$ and $\theta_i \sim \theta_i + 2\pi$. Recall that the reflection in a plane by two lines at angle θ is a rotation by 2θ. Since we can write $(\cos\theta_1 + e_{12}\sin\theta_1) = e_1(\cos\theta_1 e_1 + \sin\theta_1 e_2)$ the above element maps to

$$\begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ -\sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos 2\theta_r & \sin 2\theta_r \\ -\sin 2\theta_r & \cos 2\theta_r \end{pmatrix} \tag{17.88}$$

for $d = 2r$ and

$$\begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ -\sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos 2\theta_r & \sin 2\theta_r \\ -\sin 2\theta_r & \cos 2\theta_r \end{pmatrix} \oplus 1 \tag{17.89}$$

for $d = 2r+1$. In either case this gives a 2^r-fold covering of the maximal torus of $\mathrm{Spin}(d)$ over the maximal torus of $SO(d)$.

17.6 Pinors and Spinors

Let us now consider representations of $\mathrm{Spin}(t,s)$ and $\mathrm{Pin}(t,s)$. We first want to define *spinorial representations.*

Suppose $\rho : \mathrm{Spin}(t,s) \to \mathrm{Aut}(V)$ is a representation on a vector space V over a field κ. We automatically get a representation of the Lie algebra $\mathrm{spin}(t,s)$, and hence of the universal enveloping algebra $\mathcal{U}\mathrm{spin}(t,s)$:

$$\rho : \mathcal{U}\mathrm{spin}(t,s) \to \mathrm{End}(V) \tag{17.90}$$

subsec:LieAlgSpinGrp

On the other hand, the upshot of Section §17.5 is that there is an embedding of algebras

$$\iota : \mathcal{U}\mathrm{spin}(t,s) \to C\ell^0_{s,-t} \tag{17.91}$$

Moreover, $C\ell^0_{s,-t}$ is generated by products $v_1 v_2$ of vectors in $\mathbb{R}^{t,s}$ and since

$$v_1 v_2 = \frac{1}{2}(v_1 v_2 - v_2 v_1) + \frac{1}{2}(v_1 v_2 + v_2 v_1) = \frac{1}{2}(v_1 v_2 - v_2 v_1) + Q(v_1, v_2)1 \tag{17.92}$$

this embedding is surjective, and hence ι is in fact an isomorphism of algebras.

We say that the representation ρ "factors through" a representation of $C\ell^0_{s,-t}$ if we can write $\rho = \rho^{cliff} \circ \iota$ where ρ^{cliff} is a representation of $C\ell^0_{s,-t}$.

Definition

a.) A representation (ρ, V) of $\mathrm{spin}(t,s)$ is *spinorial* if it factors through a representation of $C\ell^0_{s,-t}$.

b.) A representation of $\mathrm{Spin}(t,s)$ is *spinorial* if its Lie algebra representation is spinorial.

c.) A *spinor representation* S is an irreducible representation of $C\ell^0_{s,-t}$ restricted to $\mathrm{Spin}(t,s)$. Typical vectors in S are called *spinors.*

d.) A *pinor representation* S is an irreducible representation of $C\ell_{s,-t}$, restricted to $\mathrm{Pin}(t,s)$. Typical vectors in S are called *pinors* (if we wish to emphasize that the representation extends to the other components).

Thanks to the isomorphism ι we know that irreducible representations of $C\ell^0_{s,-t}$ restrict to irreducible representations of $\mathrm{Spin}(t,s)$. Similarly, irreducible representations of $C\ell_{s,-t}$ restrict to irreducible representations of $\mathrm{Pin}(t,s)$.

In view of this relation of Spin to the even parts of the Clifford algebras, together with the relation

$$C\ell^0_d \cong C\ell_{1-d} \tag{17.93}$$

for $d > 0$, and using the tables above for the structure of the ungraded Clifford algebras we can immediately read off the spin representations of $\mathrm{Spin}(d)$:

d	$C\ell_d^0 \cong C\ell_{1-d}$	$\mathrm{Irreps}_{\mathbb{R}}(C\ell_d^0)$	$\mathrm{Irreps}_{\mathbb{C}}(C\ell_d^0) \cong \mathrm{Irreps}_{\mathbb{C}}(\mathbb{C}\ell_d^0)$	$\mathrm{Irreps}_{\mathbb{R}}(\mathrm{Pin}^-(d))$
2	$\mathbb{C}$	$S \cong \mathbb{C}$	$S_c^{\pm} \cong \mathbb{C}$	$\mathbb{H}$
3	$\mathbb{H}$	$S \cong \mathbb{H}$	$S_c \cong \mathbb{C}^2$	$\mathcal{S}^{\pm} \cong \mathbb{H}$
4	$\mathbb{H} \oplus \mathbb{H}$	$S^{\pm} \cong \mathbb{H}$, $\omega = \pm 1$	$S_c^{\pm} \cong \mathbb{C}^2$	$\mathcal{S} \cong \mathbb{H}^2$
5	$\mathbb{H}(2)$	$S \cong \mathbb{H}^2$	$S_c \cong \mathbb{C}^4$	$\mathcal{S} \cong \mathbb{C}^4$
6	$\mathbb{C}(4)$	$S \cong \mathbb{C}^4$	$S_c^{\pm} \cong \mathbb{C}^4$	$\mathcal{S} \cong \mathbb{R}^8$
7	$\mathbb{R}(8)$	$S \cong \mathbb{R}^8$	$S_c \cong \mathbb{C}^8$	$\mathcal{S}^{\pm} \cong \mathbb{R}^8$
8	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$S^{\pm} \cong \mathbb{R}^8$, $\omega = \pm 1$	$S_c^{\pm} \cong \mathbb{C}^8$	$\mathcal{S} \cong \mathbb{R}^{16}$
9	$\mathbb{R}(16)$	$S \cong \mathbb{R}^{16}$	$\mathcal{S}_c \cong \mathbb{C}^{16}$	$\mathcal{S} \cong \mathbb{C}^{16}$
10	$\mathbb{C}(16)$	$S \cong \mathbb{C}^{16}$	$S_c^{\pm} \cong \mathbb{C}^{16}$	$\mathcal{S} \cong \mathbb{H}^{16}$
11	$\mathbb{H}(16)$	$S \cong \mathbb{H}^{16}$	$S_c \cong \mathbb{C}^{32}$	$\mathcal{S}^{\pm} \cong \mathbb{H}^{16}$

We need to make a number of remarks about this table:

1. We left off the case $d = 1$ because $\mathrm{Spin}(1) = \{\pm 1\}$ is not a connected group, the argument based on the Lie algebra does not apply.

2. We could have stopped at $d = 9$ and invoked periodicity for higher d since the ungraded algebra just changes by multiplying by the appropriate number of factors of $\mathbb{R}(16)$. These merely shift the dimensions in the obvious way. We put in the last two rows because they are useful in physics.

3. The algebra $C\ell_d^0$ is ungraded and hence we are considering ungraded representations of $C\ell_{1-d}$ here.

4. Note we must be careful to distinguish representations over $\mathbb{R}$ from those over $\mathbb{C}$. As an example, consider $\mathrm{Spin}(2) \cong U(1)$. There is a unique irreducible representation over $\mathbb{R}$ which is the vector space $V \cong \mathbb{C}$ with representation matrix

$$\rho(\cos\theta + \sin\theta e_{12})z = e^{\mathrm{i}\theta} z \tag{17.94}$$

eq:Rep1

Of course, one might wonder about the representation

$$\rho(\cos\theta + \sin\theta e_{12})z = e^{-\mathrm{i}\theta} z \tag{17.95}$$

eq:Rep2

As a representation over $\mathbb{R}$ we can use the $\mathbb{R}$-linear intertwiner $z \to \bar{z}$ to prove that (17.94) is equivalent to (17.95). That is why there is a single spinor representation $S \cong \mathbb{R}$ in the table. On the other hand, as representations over $\mathbb{C}$ the representations on $V \cong \mathbb{C}$ given by (17.94) and (17.95) are inequivalent, because there is no complex-linear transformation on V which conjugates one into the other.

5. The presence of two inequivalent representations in the table can always be understood from the volume element ω, which in some contexts is called the chirality operator. For d even the volume element of $C\ell_d$ is in $C\ell_d^0$. It will be, up to a sign, the same as the volume element of $C\ell_{1-d}$. For $d = 0, 4\mathrm{mod}8$ it squares to $+1$ and hence we can define projection operators $P_\pm = \frac{1}{2}(1 \pm \omega)$ projecting onto the two simple summands in $C\ell_d^0$. For $d = 2, 6\mathrm{mod}8$ it squares to -1 and there are no such projection operators over $\mathbb{R}$. However, if we complexify the Clifford algebra, or the representation, then we can multiply ω by $\sqrt{-1}$ and then produce a projection operator, thus giving two inequivalent complex representations. In general the representations $S_c^\pm \cong \mathbb{C}^{2^{d/2}}$ for $d = 0\mathrm{mod}2$ with $\omega_c = \pm 1$ are known as *Weyl* or *chiral* or *semi-* spin representations. For the case of odd d the volume element is not in $C\ell_d^0$, and the algebra, and its complexification, are simple.

6. Of course, for representations of $\mathrm{Pin}^-(d)$ over $\mathbb{R}$ the situation is different and the Clifford algebra is not simple for $d = 3\mathrm{mod}4$, yielding two inequivalent pinor representations.

Now, for representations of $\mathrm{Spin}(t, s)$ more generally we can use Morita equivalence in the following sense:

If $C\ell_{s,-t}$ has an irreducible *graded* representation $V = V^0 \oplus V^1$ then V^0 and V^1 will be irreducible representations of the ungraded even subalgebra $C\ell_{s,-t}^0$. On the other hand, we can always write

$$C\ell_{s,-t} \cong C\ell_{\ell,-\ell} \widehat{\otimes} C\ell_\alpha \qquad (17.96)$$

eq:gencliff

along the lines of (13.171). Note the change of conventions so that $\alpha = -d_T \mathrm{mod}8$

$$d_T \mathrm{mod}8 = (t - s)\mathrm{mod}8 \qquad (17.97)$$

Now $C\ell_{\ell,-\ell}$ is a super-matrix algebra and has a unique graded irrep $U \cong \mathbb{R}^{s|s}$ where $s = 2^{\ell-1}$. Now it follows (Morita equivalence) that there is a one-one correspondence of graded irreps S of the definite signature Clifford algebra $C\ell_\alpha$ and those of $C\ell_{s,-t}$ given by $S \mapsto U \widehat{\otimes} S$. Therefore, there is also such a correspondence for the irreps of $\mathrm{Spin}(t, s)$ and $\mathrm{Spin}(d_T)$ with

♣check! ♣

$$S^0, S^1 \mapsto \mathbb{R}^s \otimes (S^0 \oplus S^1). \qquad (17.98)$$

♣Something not quite right here. This is not [illegible] ♣

$\boxed{\text{eq:MapReps}}$

Thus the properties of representations being real and or quaternionic is invariant under the shift $(s, t) \to (s + \ell, t + \ell)$. One example of this is physically very important: It relates Spinor representations for Lorentzian signature $\mathrm{Spin}(d + 1, 1)$ to those of the Spin representations of the space transverse to the lightcone $\mathrm{Spin}(d)$.

♣What about chirality? Even for $d = 2\mathrm{mod}4$ $S^0 \oplus S^1$ would have a real structure... ♣

$d_T \mathrm{mod} 8$	Real (Majorana)	Quaternionic (Pseudoreal)	Chiral (Weyl)	Majorana-Weyl
-4	-	yes	yes	-
-3	-	yes	-	-
-2	-	-	yes	-
-1	yes	-	-	-
0	yes	-	yes	yes
1	yes	-	-	-
2	-	-	yes	-
3	-	yes	-	-
4	-	yes	yes	-

Remark: In the physics literature the reality properties of spin representations are usually established by considering irreducible representations of the complex Clifford algebras and then showing the existence (or not) of intertwining matrices between Γ^μ and

$$\pm\Gamma^\mu, \pm(\Gamma^\mu)^{tr}, \pm(\Gamma^\mu)^*, \pm(\Gamma^\mu)^\dagger \tag{17.99}$$

Once one knows that the key properties of these intertwiners does not depend on a particular representation one can even use a particularly convenient one, such as that given by fermionic oscillators (see below) to compute explicit intertwiners.

♣SAY MORE. ♣

♣Really should discuss symplectic Majorana conditions as these are quite important in physics. ♣

17.7 Products of spin representations and antisymmetric tensors

For spinorial representations S of Spin and Pin some key constructions in physics involve the existence of covariant maps (morphisms of representations) of the following form:

1. $S \otimes S \to V$: Super-Poincaré and super-conformal algebras.
2. $S \otimes S \otimes V \to 1$: Kinetic terms in Lagrangians.
3. $S \otimes S \to 1$: Mass terms in Lagrangians.
4. $V \otimes S_1 \to S_2$: Dirac operators

The last morphism is given by Clifford mulitplication by $V \subset C\ell(V, Q)$ in the pinor representation. If the spin group admits chiral representations then it exchanges these chiral representations.

The existence and the properties of the first three kinds of morphisms depends on the signature and the dimension mod eight. This mod-eight dependence should be reminiscent of the graded Brauer group for $\kappa = \mathbb{R}$, and indeed one can find a conceptual discussion based on that in DelignSpinors [16]. We are going to take a much more concrete and down-to-earth approach to these matters, but the beautiful conceptual underpinnings are worth bearing in mind.

Our approach will be to decompose the products of spinors (bispinors) into sums of antisymmetric tensors. The existence and properties of the above morphisms will be a corollary of that discussion.

17.7.1 Statements

We begin with the Clifford algebra generated by

$$\{e_\mu, e_\nu\} = 2\eta_{\mu\nu} = 2\mathrm{Diag}\{+1^s, -1^t\}_{\mu\nu} \tag{17.100}$$

over $\mathbb{R}$. Choose an irreducible Clifford representation on a complex vector space S. Let $V = \mathbb{R}^{t,s} \otimes \mathbb{C} \cong \mathbb{C}^d$, with $d = s + t$. Then, thanks to $\widetilde{\mathrm{Ad}}$, V is an irreducible representation of $\mathrm{Spin}(t, s)$, and so are all the antisymmetric powers $\Lambda^k V$ for $0 \leq k \leq d$. ♣Notation S_c? ♣

We want to study $\mathrm{Spin}(t, s)$-equivariant maps

$$S \otimes S \to \oplus_{k=0}^{d} \Lambda^k V \tag{17.101}$$

eq:BiSpinMap

We immediately see that there is an important distinction between d even and odd. If d is even both the LHS and RHS are of complex dimension $2^{d/2} \times 2^{d/2} = 2^d$. Indeed, we will see that there is an isomorphism (eq:BiSpinMap 17.101). On the other hand, if d is odd then S is $2^{(d-1)/2}$ dimensional and the LHS is only $2^{(d-1)}$-dimensional so there can be no isomorphism.

For fixed k the space of intertwiners

$$\mathrm{Hom}^{\mathrm{Spin}(d)}(S \otimes S, \Lambda^k V) \tag{17.102}$$

is one-dimensional, when d is odd, and two-dimensional, when d is even. In Section §subsubsec:BiSpinorProofs 17.7.2, see equation (eq:antisymmtensa 17.131), we will show that - given a representation of the Clifford algebra by gamma matrices Γ^μ - there are canonical intertwiners Φ_k^ξ, where $\xi \in \{\pm 1\}$ is a sign which enters the construction. When d is even there are two distinct intertwiners labeled by ξ and when d is odd there is only one. Moreover ♣Need to justify this! ♣

- For $d = 1 \mathrm{mod} 4$ then $\xi = +1$.

- For $d = 3 \mathrm{mod} 4$ then $\xi = -1$.

Note that since the intetwiners are nonzero they are surjective by the reasoning of Schur's lemma, since the target representation is irreducible.

We want to understand how the maps (eq:BiSpinMap 17.101) behave when restricted to irreducible representations of $\mathrm{Spin}(t, s)$, what the symmetry properties are, and what the reality properties are. As we mentioned above, these properties depend on *both* s and t modulo eight. In order to express the answer we need to understand the roles of chirality, Hodge duality, and symmetry for these maps.

First, let us discuss the role of chirality:

Denote the representation of the volume form $\Gamma := \Gamma_{1\ldots d} = \rho(\omega)$. In representations over the complex numbers we can always diagonalize this operator. The eigenvalues of Γ are denoted by ζ and they obey

$$\zeta^2 = (-1)^{\frac{1}{2}d(d-1)+t} = (-1)^{\frac{1}{2}d_T(d_T+1)} \tag{17.103}$$

eq:zeta-sq

where we recall $d_T = t - s$ for $C\ell_{s,-t}$. For d even the trace of Γ in the Dirac representation S_c is zero. In order to state our results in a convention-independent way we let $\zeta_\pm$ denote the eigenvalue of Γ on the irreducible representation $S^\pm$. Then $\zeta_- = -\zeta_+$. When $\zeta^2 = 1$ it would be natural (but not necessary) to choose $\zeta_+ = +1$ and when $\zeta^2 = -1$ which root we assign to S^+ is a matter of convention.

Now, when d is even we have

$$\boxed{\Phi_k^\xi(\psi_1, \Gamma\psi_2) = (-1)^{k+d/2}\Phi_k^\xi(\Gamma\psi_1, \psi_2)} \tag{17.104}$$

eq:Phik-chiralId

The identity (eq:Phik-chiralId 17.104) implies that if ψ_1, ψ_2 have definite chiralities ζ_1, ζ_2, respectively then $\Phi_k(\psi_1, \psi_2)$ can only be nonzero if

$$\bar{\zeta}_1\zeta_2 = (-1)^{\frac{d}{2}+k} \tag{17.105}$$

We say that the spinors have opposite chirality if $\bar{\zeta}_1\zeta_2 = -1$ and the same chirality if $\bar{\zeta}_1\zeta_2 = 1$. This gives the following table summarizing when $\Phi_k(\psi_1, \psi_2)$ can be nonzero if ψ_1, ψ_2 have definite chirality:

	$k = 0\text{mod}2$	$k = 1\text{mod}2$
$d = 0\text{mod}4$	same	opposite
$d = 2\text{mod}4$	opposite	same

For d even the Φ_k can be assembled to give isomorphisms

$$\begin{aligned} S^+ \otimes S^+ \oplus S^- \otimes S^- &\cong \oplus_{k=\frac{d}{2}(2)}\Lambda^k\mathbb{C}^d \\ S^- \otimes S^+ \oplus S^+ \otimes S^- &\cong \oplus_{k=\left(\frac{d}{2}+1\right)(2)}\Lambda^k\mathbb{C}^d \end{aligned} \tag{17.106}$$

eq:ChiralPairing

Now let us consider the role of Hodge duality:

Given a metric on V and an orientation, expressed as a volume form vol of unit norm, the Hodge $*$ operator is the unique $\mathbb{C}$-linear operator $* : \Lambda^k V \to \Lambda^{d-k}V$ such that

$$f * f = \parallel f \parallel^2 \text{vol} \tag{17.107}$$

where $\parallel f \parallel^2$ is the norm-squared of the differential form f in the metric.

Using the metric $ds^2 = \eta_{\mu\nu}e^\mu \otimes e^\nu$ and the orientation

$$\text{vol} := e^1 \wedge \cdots \wedge e^d = \frac{1}{d!}(-1)^t\epsilon_{\mu_1\cdots\mu_d}e^{\mu_1} \wedge \cdots \wedge e^{\mu_d} \tag{17.108}$$

eq:orientation

the Hodge $*$ acts on the natural basis as

$$*\,(e^{\nu_1}\wedge\cdots\wedge e^{\nu_{d-k}}) = (-1)^t \frac{1}{k!}\epsilon^{\nu_1\cdots\nu_{d-k}}{}_{\mu_1\cdots\mu_k} e^{\mu_1}\wedge\cdots\wedge e^{\mu_k} \tag{17.109}$$

eq:Hodgestar

where $\epsilon^{\mu_1\cdots\mu_d}\in\{0,\pm1\}$ is the totally antisymmetric tensor normalized by $\epsilon^{1\cdots d}=+1$, and indices are raised and lowered with $\eta_{\mu\nu}$. Note especially that, restricted to $\Lambda^k V$ for any k we have the important sign:

$$*^2 = (-1)^t(-1)^{k(d-k)}. \tag{17.110}$$

When this is $+1$ we can diagonalize $*$ over the reals with eigenvalues ±1, and when it is -1 we can only diagonalize over the complex numbers.

The Hodge star commutes with the $\mathrm{Spin}(t,s)$ action (but not with the $\mathrm{Pin}(t,s)$ action! For orientation-reversing group elements it anti-commutes) and hence defines isomorphism of $\mathrm{Spin}(t,s)$ representations

$$\Lambda^k V \cong \Lambda^{d-k}V \tag{17.111}$$

eq:Star-Iso

Moreover, if d is odd then $*^2=(-1)^t$. Denote the two eigenvalues of $*$ by $\pm\varepsilon$, with $\varepsilon=+1$ for $t=0(2)$ and $\varepsilon=\mathrm{i}$ for $t=1(2)$. Then, as representations of $\mathrm{Spin}(t,s)$ we can decompose $\Lambda^* V$ into two equivalent (highly reducible) representations given by

$$\Lambda^* V \cong \left[\oplus_{k=0}^d \Lambda^k V\right]^{\varepsilon} \oplus \left[\oplus_{k=0}^d \Lambda^k V\right]^{-\varepsilon} \tag{17.112}$$

where the superscripts $\pm\varepsilon$ indicate the corresponding eigenspaces of $*$. Of course, thanks to (eq:Star-Iso 17.111) each of the summands is isomorphic to, say,

$$\oplus_{k<\frac{d}{2}}\Lambda^k V \tag{17.113}$$

as a $\mathrm{Spin}(t,s)$ representation.

When d is even then $*^2=(-1)^{t+k}$ and hence $\Lambda^* V$ decomposes into two subspaces. One is the subspace on which $*^2=+1$ and the other is the subspace on which $*^2=-1$. These two subspaces are distinguished by the parity of k. Each of these subspaces may be decomposed into $*$ eigenspaces. The "middle" space $\Lambda^{\frac{d}{2}}V$ splits into two representations given by the $*=\pm\varepsilon$ eigenspaces, where

$$\varepsilon = \begin{cases} +1 & (-1)^{t+d/2}=+1 \\ +\mathrm{i} & (-1)^{t+d/2}=-1 \end{cases} \tag{17.114}$$

Therefore, we can decompose $\Lambda^* V$ into eigenspaces of $*$ as

$$\begin{aligned}\Lambda^* V \cong & \left[\oplus_{k=\frac{d}{2}(2)}\Lambda^k V\right]^{\varepsilon} \oplus \left[\oplus_{k=\frac{d}{2}(2)}\Lambda^k V\right]^{-\varepsilon} \\ & \oplus \left[\oplus_{k=\left(\frac{d}{2}+1\right)(2)}\Lambda^k V\right]^{\mathrm{i}\varepsilon} \oplus \left[\oplus_{k=\left(\frac{d}{2}+1\right)(2)}\Lambda^k V\right]^{-\mathrm{i}\varepsilon}\end{aligned} \tag{17.115}$$

Now, returning to our equivariant maps $\Phi_k^\xi : S \otimes S \to \Lambda^k V$, the key identity which relates Hodge $*$ and chirality is

$$\boxed{*\Phi_k^\xi(\psi_1 \otimes \psi_2) = (-1)^{t+\frac{1}{2}d(d-1)}(-1)^{\frac{1}{2}k(k-1)}\Phi_{d-k}^\xi(\psi_1 \otimes \Gamma\psi_2)} \tag{17.116}$$

eq:HS-CHIR

This identity holds for d even or odd. If ψ_2 is an eigenstate of Γ of eigenvalue ζ_2 then by (eq:zeta-sq 17.103) we can simplify (eq:HS-CHIR 17.116) to

$$*\Phi_k^\xi(\psi_1 \otimes \psi_2) = \zeta_2^{-1}(-1)^{\frac{1}{2}k(k-1)}\Phi_{d-k}^\xi(\psi_1 \otimes \psi_2) \tag{17.117}$$

eq:HS-CHIR2

Again, this equation holds for d even or odd.

Now, for d odd, since each Φ_k is surjective, it follows that we have

♣True for $SO(d)$ reps but we should refine the statement to include $O(d)$ reps. ♣

$$S \otimes S \cong \oplus_{k<\frac{d}{2}}\Lambda^k V \cong \left[\oplus_{k=0}^d \Lambda^k V\right]^\varepsilon \cong \left[\oplus_{k=0}^d \Lambda^k V\right]^{-\varepsilon} \tag{17.118}$$

where we can form the self-dual or anti-self-dual linear combinations of Φ_k and Φ_{d-k} as we please, using (eq:HS-CHIR2 17.117).

For d even equation (eq:HS-CHIR2 17.117) implies that

$$\begin{aligned} S^+ \otimes S^+ &\cong \oplus_{k<\frac{d}{2},k=\frac{d}{2}(2)}\Lambda^k V \oplus \left[\Lambda^{\frac{d}{2}} V\right]^\varepsilon \\ &\cong \left[\oplus_{k=\frac{d}{2}(2)}\Lambda^{\frac{d}{2}} V\right]^\varepsilon \end{aligned} \tag{17.119}$$

eq:SplSq

$$\begin{aligned} S^- \otimes S^- &\cong \oplus_{k<\frac{d}{2},k=\frac{d}{2}(2)}\Lambda^k V \oplus \left[\Lambda^{\frac{d}{2}} V\right]^{-\varepsilon} \\ &\cong \left[\oplus_{k=\frac{d}{2}(2)}\Lambda^{\frac{d}{2}} V\right]^{-\varepsilon} \end{aligned} \tag{17.120}$$

eq:SmnSq

where

$$\varepsilon = \zeta_+^{-1}(-1)^{\frac{d(d-2)}{8}} \tag{17.121}$$

Meanwhile, because of the isomorphism (eq:Star-Iso 17.111) we also have

$$S^- \otimes S^+ \cong \oplus_{k<\frac{d}{2},k=\left(\frac{d}{2}+1\right)(2)}\Lambda^k V \tag{17.122}$$

eq:SplSmn

Finally, we can meaningfully ask how the symmetric and anti-symmetric decompositions of $S \otimes S$ (for d odd) and $S^+ \otimes S^+$, $S^- \otimes S^-$ (for d even) map to antisymmetric tensors. The key identity is now

$$\boxed{\Phi_k^\xi(\psi_1 \otimes \psi_2) = \xi^k \eta(\xi, d)(-1)^{\frac{1}{2}k(k-1)}\Phi_k^\xi(\psi_2 \otimes \psi_1)} \tag{17.123}$$

eq:Phik-SYMM

where $\eta(\xi, d)$ is a sign which depends on ξ and $d\text{mod}8$ and is given by

$d$mod8	$\xi = +1$	$\xi = -1$
0	+1	+1
1	+1	*
2	+1	−1
3	*	−1
4	−1	−1
5	−1	*
6	−1	+1
7	*	+1

It follows that for d odd we have

$$\mathrm{Sym}^2(S) \cong \oplus_{k=\frac{d\pm 1}{2}(4), k<\frac{d}{2}} \Lambda^k V \tag{17.124}$$

$$\Lambda^2(S) \cong \oplus_{k=\frac{d\pm 5}{2}(4), k<\frac{d}{2}} \Lambda^k V \tag{17.125}$$

On the other hand, if d is even then we have

$$\mathrm{Sym}^2(S^+) \cong \left[\oplus_{k=\frac{d}{2}\mathrm{mod}4} \Lambda^k V\right]^{\varepsilon} \tag{17.126}$$ eq:symmsp

$$\Lambda^2(S^+) \cong \oplus_{k=(\frac{d}{2}+2)\mathrm{mod}4, k<\frac{d}{2}} \Lambda^k V \tag{17.127}$$ eq:asymsp

$$\mathrm{Sym}^2(S^-) \cong \left[\oplus_{k=\frac{d}{2}\mathrm{mod}4} \Lambda^k V\right]^{-\varepsilon} \tag{17.128}$$ eq:symmsm

$$\Lambda^2(S^-) \cong \oplus_{k=\left(\frac{d}{2}+2\right)\mathrm{mod}4, k<\frac{d}{2}} \Lambda^k V \tag{17.129}$$ eq:asymsm

Returning to the questions at the beginning of this section the above identities easily reproduce Table 1.5.1 from DelignSpinors [16]. In the entries in the second column, "orthogonal" means there is a symmetric nondegenerate Spin-invariant form on the spin representation, while "symplectic" means there is an anti-symmetric nondegenerate Spin-invariant form.

$d$mod8	Forms on spinors	Symmetry of $S \otimes S \to V$
0	$S_c^{\pm}$ orthogonal	$S_c^+ \otimes S_c^- \to V$
1	orthogonal	symmetric
2	S_c^+ dual to S_c^-	symmetric (on $S_c^{\pm}$ separately)
3	symplectic	symmetric
4	$S_c^{\pm}$ symplectic	$S_c^+ \otimes S_c^- \to V$
5	symplectic	antisymmetric
6	S_c^+ dual to S_c^-	antisymmetric (on $S_c^{\pm}$ separately)
7	orthogonal	antisymmetric

Finally, we comment on the reality properties. For $d = 2\text{mod}4$ $S_c^{\pm}$ are complex conjugates of each other. In the other cases $S_c^{\pm}$ are self-conjugate.

The proofs of the key identities (17.104), (17.116), and (17.123) did not use complex conjugation. From our discussions of Clifford algebras we know that we can form a real representation of gamma matrices for $d_T = \pm 1, 0\text{mod}8$ and hence the same story holds with real representations. For quaternionic representations $d_T = \pm 3, 4\text{mod}8$ the story is a little more complicated. See [16] for one discussion. We will merely remark that the product of two quaternionic complex vector spaces carries a natural real structure, so that one can use the reality conditions to map to real antisymmetric tensors.

♣Should have a more extensive discussion about symplectic Majorana conditions. They are important in physics. ♣

17.7.2 Proofs

BiSpinorProofs

We begin with a definition of the intertwiners:

$$\Phi_k^{\xi} : S \otimes S \to \Lambda^k V \tag{17.130}$$

To define it choose an irreducible matrix representation Γ_μ of $C\ell_{r,-s}$. Think of spinors as column vectors and consider

$$\Phi_k^{\xi}(\psi_1, \psi_2) := \frac{1}{k!} \left(\psi_1^T C_\xi \Gamma^{\mu_1 \cdots \mu_k} \psi_2\right) e_{\mu_1} \wedge \cdots \wedge e_{\mu_k} \tag{17.131}$$

eq:antisymmtensa

The reason for the matrix C_ξ is the following: In the spin representation ψ transforms according to

$$\rho(e^{\frac{1}{2}\omega^{\mu\nu} e_{\mu\nu}}) \cdot \psi = \tilde{\psi} = e^{\frac{1}{2}\omega_{\mu\nu}\Gamma^{\mu\nu}} \psi \tag{17.132}$$

eq:spintrmn

so we need to know how the transpose transforms. (That is, we are looking at the dual representation space $S^\vee$.) This requires the introduction of an intertwiner C_ξ, which has the property

♣Explain more thoroughly about the dual representation. ♣

$$C_\xi \Gamma^\mu C_\xi^{-1} = \xi \Gamma^{\mu,tr} \tag{17.133}$$

eq:defcxi

where $\xi = \pm 1$. We will see below that such intertwiners always exist with the only restriction that $\xi = +1$ for $d = 1\text{mod}4$ and $\xi = -1$ for $d = 3\text{mod}4$.

Given a choice of C_ξ the representation on the dual space is

$$\tilde{\psi}^{tr} C_\xi = (\psi^{tr} C_\xi) e^{-\frac{1}{2}\omega_{\mu\nu}\Gamma^{\mu\nu}} \tag{17.134}$$

Consequently, the RHS of (17.131) is invariant if we replace $\psi_i \to \rho_S(g) \cdot \psi_i$ and $e_\mu \to \widetilde{\text{Ad}}(g)(e_\mu)$ for $g \in \text{Spin}(t,s)$.

Our first rule is about how k is correlated with the pairing of chirality. Note that:

$$(\Gamma^{1 \cdots d})^{tr} = \xi^d (-1)^{\frac{1}{2}d(d-1)} C_\xi \Gamma^{1 \cdots d} C_\xi^{-1} \tag{17.135}$$

eq:voltrnpse

This holds for any value of d, even or odd.

For d odd $\Gamma^{1 \cdots d}$ is represented by a scalar and therefore $\xi^d = (-1)^{\frac{1}{2}d(d-1)}$. Thus:

- For $d = 1\text{mod}4$ then C_ξ can only exist for $\xi = +1$. (And it turns out it does exist for $\xi = +1$.)

- For $d = 3\mathrm{mod}4$ then C_ξ can only exist for $\xi = -1$. (And it turns out it does exist for $\xi = -1$.)

For d even we may simplify (eq:voltrnpse 17.135) to

$$(\Gamma^{1\cdots d})^{tr} = (-1)^{d/2} C_\xi \Gamma^{1\cdots d} C_\xi^{-1} \tag{17.136}$$

Therefore we compute

$$\begin{aligned}\psi_1^T C_\xi \Gamma^{\mu_1\cdots\mu_k}\Gamma\psi_2 &= (-1)^k \psi_1^T (C_\xi \Gamma C_\xi^{-1}) C_\xi \Gamma^{\mu_1\cdots\mu_k}\psi_2 \\ &= (-1)^{d/2+k} (\Gamma\psi_1)^T C_\xi \Gamma^{\mu_1\cdots\mu_k}\psi_2\end{aligned} \tag{17.137}$$

eq:connectchiral

Thus proving (eq:Phik-chiralId 17.104).

Next we take into account Hodge duality and use the crucial identity:

$$\Gamma^{\mu_1\cdots\mu_k}\Gamma_{1\cdots d} = (-1)^{\frac{1}{2}k(k-1)} \frac{1}{(d-k)!} \epsilon^{\mu_1\cdots\mu_k\nu_1\cdots\nu_{d-k}} \Gamma_{\nu_1\cdots\nu_{d-k}} \tag{17.138}$$

eq:hodgegamma

where $\Gamma = \Gamma_{1\cdots d}$ and $\epsilon^{1\cdots d} = +1$.

We now compute

$$\begin{aligned} *\Phi_k^\xi(\psi_1\otimes\psi_2) &= \frac{1}{k!}\left(\psi_1^{tr} C_\xi \Gamma_{\mu_1\cdots\mu_k}\psi_2\right)(*e^{\mu_1\cdots\mu_k}) \\ &= (-1)^t \frac{1}{(d-k)!k!}\left(\psi_1^{tr} C_\xi \Gamma_{\mu_1\cdots\mu_k}\psi_2\right) \epsilon^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_{d-k}} e^{\nu_1\cdots\nu_{d-k}}) \end{aligned} \tag{17.139}$$

On the other hand,

$$\begin{aligned}\Phi_{d-k}^\xi(\psi_1\otimes\Gamma\psi_2) &= \frac{1}{(d-k)!}\left(\psi_1^{tr} C_\xi \Gamma^{\nu_1\cdots\nu_{d-k}}\Gamma\psi_2\right) e_{\nu_1\cdots\nu_{d-k}}) \\ &= (-1)^{\frac{1}{2}(d-k)(d-k-1)} (-1)^{k(d-k)} \frac{1}{(d-k)!k!}\left(\psi_1^{tr} C_\xi \Gamma_{\mu_1\cdots\mu_k}\psi_2\right) \epsilon^{\mu_1\cdots\mu_k\nu_1\cdots\nu_{d-k}} e_{\nu_1\cdots\nu_{d-k}}\end{aligned} \tag{17.140}$$

Comparing these equations and doing a little algebra leads to (eq:HS-CHIR 17.116).

Finally, for the symmetry properties note that from the definition of C_ξ we can compute:

$$(C_\xi \Gamma^{\mu_1\cdots\mu_k})^{tr} = \xi^k (-1)^{\frac{1}{2}k(k-1)} C_\xi \Gamma^{\mu_1\cdots\mu_k} C_\xi^{-1} C_\xi^{tr} \tag{17.141}$$

eq:symmetry

Now we can therefore say

$$\begin{aligned}\Phi_k^\xi(\psi_1\otimes\psi_2) &= \frac{1}{k!}\left(\psi_1^{tr} C_\xi \Gamma_{\mu_1\cdots\mu_k}\psi_2\right) e^{\mu_1\cdots\mu_k} \\ &= \frac{1}{k!}\left(\psi_2^{tr}\,(\Gamma_{\mu_1\cdots\mu_k})^{tr}\, C_\xi^{tr}\psi_1\right) e^{\mu_1\cdots\mu_k} \\ &= \xi^k (-1)^{\frac{1}{2}k(k-1)} \frac{1}{k!}\left(\psi_2^{tr} C_\xi \Gamma_{\mu_1\cdots\mu_k}(C_\xi^{-1}C_\xi^{tr})\psi_1\right) e^{\mu_1\cdots\mu_k}\end{aligned} \tag{17.142}$$

Now note that (eq:defcxi 17.133) implies

$$C_\xi^{tr}\Gamma^\mu C_\xi^{tr,-1} = \xi \Gamma^{\mu,tr} \tag{17.143}$$

eq:defcxia

and by Schur's lemma it follows that $C_\xi^{-1} C_\xi^{tr}$ is a scalar, and consistency of (eq:symmetry 17.141) implies that scalar is ± 1. The symmetry nature of the tensor product decompositions depends on that sign.

Again by Schur's lemma that sign cannot depend on the matrix representation. If we multiply matrices by $\sqrt{-1}$ we can change the signature but not the anti-symmetry properties, so we might as well choose signature $+1^d$ and compute in a specific representation. We will use the harmonic oscillator representation constructed in Section §subsubsec:ExplicitRepGamma 18.4.1 below For $d = 2n$, define $U = \Gamma^2 \Gamma^4 \cdots \Gamma^{2n}$, and then check that we can take:

♣Perhaps would be better to use the explicit representation (eq:evenodd 13.58) above. ♣

$$C_+ = \begin{cases} U & n \quad \text{even} \\ \Gamma_\omega U & n \quad \text{odd} \end{cases} \tag{17.144}$$

eq:plusinter

$$C_- = \begin{cases} \Gamma_\omega U & n \quad \text{even} \\ U & n \quad \text{odd} \end{cases} \tag{17.145}$$

eq:plusintera

and by explicit computation

$$U^{-1} U^{tr} = (-1)^{\frac{1}{2} n(n+1)} \qquad\qquad (\Gamma_\omega U)^{-1} (\Gamma_\omega U)^{tr} = (-1)^{\frac{1}{2} n(n-1)} \tag{17.146}$$

eq:stnnant

Finally, recall that for d odd, we may only use C_+ for n even, i.e. $d = 1 \mathrm{mod} 4$ and C_- for n odd, i.e. $d = 3 \mathrm{mod} 4$.

In this way we compute

$d\mathrm{mod}8$	$C_+^{-1} C_+^{tr}$	$C_-^{-1} C_-^{tr}$
0	+1	+1
1	+1	*
2	+1	−1
3	*	−1
4	−1	−1
5	−1	*
6	−1	+1
7	*	+1

This proves (eq:Phik-SYMM 17.123).

Note that from this table we deduce

	0	1	2	3	4	5	6	7
$C_+\Gamma^{0(4)}$	S	S	S	*	A	A	A	*
$C_-\Gamma^{0(4)}$	S	*	A	A	A	*	S	S
$C_+\Gamma^{1(4)}$	S	S	S	*	A	A	A	*
$C_-\Gamma^{1(4)}$	A	*	S	S	S	*	A	A
$C_+\Gamma^{2(4)}$	A	A	A	*	S	S	S	*
$C_-\Gamma^{2(4)}$	A	*	S	S	S	*	A	A
$C_+\Gamma^{3(4)}$	A	A	A	*	S	S	S	*
$C_-\Gamma^{3(4)}$	S	*	A	A	A	*	S	S

Across the top we have written the value of $d$mod8 and in the left-column $C_+\Gamma^{0(4)}$ means a matrix $C_+\Gamma^{\mu_1\cdots\mu_k}$ with $k = 0$mod4. The S,A in the table denotes symmetry or anti-symmetry, respectively. This leads to the final refinements (eq:symmsp 17.126), et. seq.

Remark: Note that for some columns, e.g. $d = 2$mod8 and $k = 0$ we can have both symmetric and antisymmetric matrices. This is not a contradiction because in such cases we are pairing spinors of opposite chirality.

Exercise *Checking dimensions*

a.) Show that: [40]

$$\begin{aligned}
\sum_{k=0(4)} \binom{d}{k} &= 2^{d-2} + 2^{\frac{1}{2}d-1}\cos(\frac{\pi d}{4}) \\
\sum_{k=1(4)} \binom{d}{k} &= 2^{d-2} + 2^{\frac{1}{2}d-1}\sin(\frac{\pi d}{4}) \\
\sum_{k=2(4)} \binom{d}{k} &= 2^{d-2} - 2^{\frac{1}{2}d-1}\cos(\frac{\pi d}{4}) \\
\sum_{k=3(4)} \binom{d}{k} &= 2^{d-2} - 2^{\frac{1}{2}d-1}\sin(\frac{\pi d}{4})
\end{aligned} \tag{17.147}$$

These identities hold for any positive integer d, even or odd.

b.) Using these identities check that the dimensions match in the various decompositions of products of spinors into antisymmetric tensors given above.

Exercise

Find the explicit linear combinations of Φ_k which project into the eigenspaces of $*$.

[40] *Answer*: Apply the binomial expansion to $(1+\kappa)^d$ for the four distinct fourth roots κ of 1.

Hints:

a.) It is useful to remark that

$$(-1)^{\frac{1}{2}(d-k)(d-k-1)} = \begin{cases} +1 & k = d, d+3\mathrm{mod}4 \\ -1 & k = d+1, d+2\mathrm{mod}4 \end{cases} \tag{17.148}$$

and hence

$$\begin{cases} (-1)^{\frac{1}{2}(d-k)(d-k-1)} = (-1)^{\frac{1}{2}k(k-1)} & k = \frac{d}{2}\mathrm{mod}2 \\ (-1)^{\frac{1}{2}(d-k)(d-k-1)} = -(-1)^{\frac{1}{2}k(k-1)} & k = \left(\frac{d}{2}+1\right)\mathrm{mod}2 \end{cases} \tag{17.149}$$

b.) In particular, if $k = \frac{d}{2}\mathrm{mod}2$ then

$$\begin{aligned} *\left(\Phi_k(\psi_1,\psi_2) + \Phi_{d-k}(\psi_1,\psi_2)\right) &= \zeta_2^{-1}\left((-1)^{\frac{1}{2}k(k-1)}\Phi_{d-k} + (-1)^{\frac{1}{2}(d-k)(d-k-1)}\Phi_k\right) \\ &= \zeta_2^{-1}(-1)^{\frac{1}{2}k(k-1)}\left(\Phi_k(\psi_1,\psi_2) + \Phi_{d-k}(\psi_1,\psi_2)\right) \end{aligned} \tag{17.150}$$

c.) On the other hand, if $k = \frac{d}{2}+1\mathrm{mod}2$ then $(-1)^{\frac{1}{2}k(k-1)} = -(-1)^{\frac{1}{2}(d-k)(d-k-1)}$ and hence

$$\begin{aligned} *\left(\Phi_k(\psi_1,\psi_2) - \mathrm{i}\Phi_{d-k}(\psi_1,\psi_2)\right) &= \zeta_2^{-1}\left((-1)^{\frac{1}{2}k(k-1)}\Phi_{d-k} - \mathrm{i}(-1)^{\frac{1}{2}(d-k)(d-k-1)}\Phi_k\right) \\ &= \mathrm{i}\zeta_2^{-1}(-1)^{\frac{1}{2}k(k-1)}\left(\Phi_k(\psi_1,\psi_2) - \mathrm{i}\Phi_{d-k}(\psi_1,\psi_2)\right) \end{aligned} \tag{17.151}$$

eq:PhiMap-2

d.) It follows that we can refine the decompositions to

$$S^+ \otimes S^+ \cong \left[\oplus_{k=\frac{d}{2}(4)}\Lambda^k V\right]^{\varepsilon} \oplus \left[\oplus_{k=\left(\frac{d}{2}+2\right)(4)}\Lambda^k V\right]^{-\varepsilon} \tag{17.152}$$

eq:SpSp

where the superscripts $\pm\varepsilon$ mean the spaces are eigenspaces of Hodge $*$ with eigenvalues

$$\varepsilon = \zeta_+^{-1}(-1)^{\frac{d(d-2)}{8}} \tag{17.153}$$

respectively. Here ζ_+ is the eigenvalue of Γ on S^+. For S^- we have the same story with $\zeta_- = -\zeta_+$ and so we get the complementary space

$$S^- \otimes S^- \cong \left[\oplus_{k=\frac{d}{2}(4)}\Lambda^k V\right]^{-\varepsilon} \oplus \left[\oplus_{k=\left(\frac{d}{2}+2\right)(4)}\Lambda^k V\right]^{+\varepsilon} \tag{17.154}$$

eq:SmSm

e.) Similarly, the map (eq:PhiMap-2 17.151) leads to

$$S^- \otimes S^+ \cong \left[\oplus_{k=\left(\frac{d}{2}+1\right)(4)}\Lambda^k V\right]^{+\varepsilon'} \oplus \left[\oplus_{k=\left(\frac{d}{2}+3\right)(4)}\Lambda^k V\right]^{-\varepsilon'} \tag{17.155}$$

eq:SpSp

where now the $\pm$ superscripts mean

$$\varepsilon' = \mathrm{i}\zeta_+^{-1}(-1)^{\frac{d(d+2)}{8}} \tag{17.156}$$

17.7.3 Fierz identities

For d even we can rewrite the main result of the previous section as

$$(\psi_1)_\alpha(\psi_2)_\beta = 2^{-d/2}\sum_{k=0}^{d}\frac{1}{k!}(\psi_1^{tr}C_\xi\Gamma_{\mu_1\cdots\mu_k}\psi_2)(\Gamma^{\mu_k\cdots\mu_1}C_\xi^{-1})_{\beta\alpha} \tag{17.157}$$

eq:Fierz

Proof: The Clifford algebra is simple as an ungraded algebra so $\Gamma^{\mu_1\cdots\mu_k})_{\alpha\beta}$ forms a linear basis for the full matrix algebra, and hence so does $(\Gamma^{\mu_k\cdots\mu_1}C_\xi^{-1})_{\beta\alpha}$. Therefore, we can certainly write

$$(\psi_1)_\alpha(\psi_2)_\beta = \sum_{k=0}^{d}\frac{1}{k!}N(\psi_1,\psi_2)_{\mu_1\ldots\mu_k}(\Gamma^{\mu_k\cdots\mu_1}C_\xi^{-1})_{\beta\alpha} \tag{17.158}$$

for some some totally antisymmetric tensors $N(\psi_1,\psi_2)_{\mu_1\ldots\mu_k}$ which are linear in ψ_1 and ψ_2.

Moreover, the trace in the Dirac representation has the property that

$$\mathrm{Tr}\Gamma^{\mu_1\cdots\mu_k} = 0 \tag{17.159}$$

for $0 < k \le d$. When k is odd this immediately follows by thinking of the Dirac representation as a $\mathbb{Z}_2$-graded representation. (Equivalently, we can insert $\Gamma_\chi^2 = 1$ and use cyclicity.) When k is even we can cycle, say, Γ^{μ_k}. Therefore it follows that

$$\mathrm{Tr}(\Gamma^{\mu_1\cdots\mu_k}\Gamma_{\nu_\ell\cdots\nu_1}) = \delta_{k,\ell}2^{d/2}\sum_{\sigma\in S_k}\mathrm{sgn}\,(\sigma)\delta^{\mu_1}_{\nu_{\sigma(1)}}\cdots\delta^{\mu_k}_{\nu_{\sigma(k)}} \tag{17.160}$$

Using this property of the trace we can determine $N(\psi_1,\psi_2)_{\mu_1\ldots\mu_k}$ as above.

Further contraction of (eq:Fierz 17.157) with spinors ψ_3,ψ_4 gives a way of rearranging products of spinor bilinears known as *Fierz rearrangement*.

Remark: Fierz rearrangements are frequently used in computations in perturbative quantum field theory and in computations involving supersymmetric field representations and invariant Lagrangians.

♣Give an exercise based on Fierz rearrangement in some physics computation. ♣

17.8 Digression: Spinor Magic

17.8.1 Isomorphisms with (special) unitary groups

The minimal dimensional irreps of the Spin group give insight into the special isomorphisms between the different classical Lie groups in low dimension.

We consider the definite signature Clifford algebra and study $\mathrm{Spin}(d)$. The irreducible representations on real vector spaces are of the form $\mathbb{K}^n$ where $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$ according to $d$mod8. After extension to complex scalars n is a power of 2 given $2^{[d-1]/2}$. If the signature is positive we can choose the representation matrices Γ^μ to be hermitian and if negative we can choose them to be anti-hermitian. In either case they are *unitary* matrices considered as complex matrices. In any case, the representation is given by a homomorphism ρ into the norm-preserving elements of $\mathbb{K}^n$ which we will denote $U(\mathbb{K}^n)$ with the understanding that

$$
\begin{aligned}
U(\mathbb{R}^n) &\cong SO(n;\mathbb{R}) \\
U(\mathbb{C}^n) &\cong U(n) \\
U(\mathbb{H}^n) &\cong USp(2n)
\end{aligned}
\tag{17.161}
$$

Now, using the irreps of $\mathrm{Spin}(d)$ over $\mathbb{R}$ we construct homomorphisms $\rho : \mathrm{Spin}(d) \to U(V)$ with

$$
\begin{aligned}
&\rho : \mathrm{Spin}(2) \to U(\mathbb{C}) \cong U(1) \\
&\rho : \mathrm{Spin}(3) \to U(\mathbb{H}) \cong USp(2) \cong SU(2) \\
&\rho : \mathrm{Spin}(4) \to U(\mathbb{H} \oplus \mathbb{H}) \cong USp(2) \times USp(2) \cong SU(2) \times SU(2) \\
&\rho : \mathrm{Spin}(5) \to U(\mathbb{H}^2) \cong USp(4) \\
&\rho : \mathrm{Spin}(6) \to U(\mathbb{C}^4) \cong U(4) \\
&\rho : \mathrm{Spin}(7) \to U(\mathbb{R}^8) \cong SO(8) \\
&\rho : \mathrm{Spin}(8) \to U(\mathbb{R}^8) \cong SO(8) \\
&\rho : \mathrm{Spin}(9) \to U(\mathbb{R}^{16}) \cong SO(16)
\end{aligned}
\tag{17.162}
$$

In the above the kernel of ρ is one. This follows since $\mathrm{Spin}(d)$ is simple, except for $d = 2$, where we can check the kernel explicitly, and for $d = 4$, which is why we mapped to the two chiral spin representations. The center of the spin group acts nontrivially on the spinor.

♣Need a better argument to rule out a kernel which is a finite group ♣

Now, we compare dimensions. In the cases where the dimensions match we obtain isomorphisms of Spin groups with (special) unitary groups. These isomorphisms are:

$$
\begin{aligned}
&\mathrm{Spin}(2) \cong U(\mathbb{C}) \cong U(1) \\
&\mathrm{Spin}(3) \cong U(\mathbb{H}) \cong USp(2) \cong SU(2) \\
&\mathrm{Spin}(4) \cong U(\mathbb{H}) \times U(\mathbb{H}) \cong USp(2) \times USp(2) \cong SU(2) \times SU(2) \\
&\mathrm{Spin}(5) \cong U(\mathbb{H}^2) \cong USp(4) \\
&\mathrm{Spin}(6) \cong SU(\mathbb{C}^4) \cong SU(4)
\end{aligned}
\tag{17.163}
$$

eq:SpecialIso

In general the image of ρ will only be a small subgroup of $U(V_d, \mathbb{K})$ since the dimension of V_d is growing like $\sim 2^{d/2}$ while the real dimension of the Spin group is $\frac{1}{2}d(d-1)$. The dimensions coincide for $d = 2, 3, 4, 5, 6$. By the time we get to $d = 9$ the Spin group is dimension 36 but the dimension of $SO(16)$ is 120.

****

Compatibility with $\widetilde{\mathrm{Ad}}$. Explain the relation to $Ax \cdot \Gamma A^{-1} = R(x) \cdot \Gamma$.

****

17.8.2 The spinor embedding of $\mathrm{Spin}(7) \to SO(8)$

Once we reach $d = 7$ we have an image of $\rho : \mathrm{Spin}(7) \to SO(8)$ and just counting dimensions we see that it cannot be onto. This hardly means the magic is over!

We should note that there is another natural homomorphism from Spin(7) to $SO(8)$ given by

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(7) \to SO(7) \to SO(8) \qquad (17.164)$$

where we embed $SO(7)$ into $SO(8)$ so that it is the stabilizer of a nonzero vector.

Now let us compare with the embedding by the spin representation. $C\ell(7_-)$ has two inequivalent representations on $\mathbb{R}^8$, giving two inequivalent representations of $\mathrm{Pin}(7_-)$ on $\mathbb{R}^8$. These become equivalent when restricted to Spin(7). A beautiful explicit matrix representation is obtained by considering the action of the seven imaginary units in the octonions on $\mathbb{O} \cong \mathbb{R}^8$ as described in (eq:octmult 13.188) et. seq. above.

Now, in this representation if we consider the stabilizer of a nonzero vector then we do not get $SO(7)$ but rather a completely different group, known as the exceptional group G_2. Thus, the spinor $\mathbf{8}$ of Spin(7) becomes reducible as a representation of G_2:

$$\mathbf{8} = \mathbf{1} \oplus \mathbf{7} \qquad (17.165)$$

where $\mathbf{7}$ is the smallest nontrivial representation of G_2.

♣Another definition of G_2 is that it is the subgroup of $SO(7)$ which stabilizes the 3-form defined by the structure constants of the octonions. Need to explain the relation between these definitions. ♣

From the octonionic description we obtain 7 explicit 8×8 real antisymmetric matrices. The two inequivalent representations of $C\ell_{-7}$ are related by $\gamma^i \to -\gamma^i$. The spinor representation is obtained by multiplying even numbers of vectors of norm-squared -1 or by exponentiating γ^{ij}. We will next use the matrices γ^i to construct the spinor representations of Spin(8).

17.8.3 Three inequivalent 8-dimensional representations of Spin(8)

Something very special happens at $d = 8$. Then there are three inequivalent 8-dimensional representations. Two of these are the spinor representations $S^\pm$. These are obtained by

$$\begin{aligned} e_{ij} &\to -\gamma_{ij} \qquad & 1 \le i < j \le 7 \\ e_{i8} &\to \pm\gamma_i \qquad & 1 \le i \le 7 \end{aligned} \qquad (17.166)$$

eq:twospinr

Put differently, we can form a representation of $C\ell(8+) \cong \mathbb{R}(16)$ by taking

$$\tilde{\gamma}_i = \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix} \qquad i = 1, \dots, 7 \qquad \tilde{\gamma}_8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (17.167)$$

eq:clfsxtn

so that if $\gamma_1 \cdots \gamma_7 = 1$ then the chirality matrix is

$$\tilde{\gamma}_\chi := \tilde{\gamma}_1 \cdots \tilde{\gamma}_8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (17.168)$$

eq:chrlty

Then the two representations of spin(8) are given by the two block diagonal components of $\tilde{\gamma}_{MN}$, $1 \le M < N \le 8$.

In addition, there is the vector, or defining representation of $SO(8)$ on $\mathbb{R}^8$, and thanks to the $\widetilde{\mathrm{Ad}}$ homomorphism, this is also a representation of Spin(8). Thus, we have constructed *three* irreducible representations of Spin(8) on $\mathbb{R}^8$. We claim that these representations are in fact inequivalent. One way to see this is to consider the representation of the *center* of Spin(8). This is $Z = \{\pm 1, \pm\omega\}$ where $\omega = e_1 \cdots e_8$ is the volume form. Note that $\omega^2 = 1$ so the center is $Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In the two spin representations $(\rho_\pm, S^\pm)$ the element ω is represented by the 11, and 22 blocks of $\tilde{\gamma}_\chi$ above. Thus, $\rho_+(\omega) = +1$ and $\rho_-(\omega) = -1$, and hence they are inequivalent. Of course, $-1 \in \mathrm{Spin}(8)$ is represented by -1 on both $S^\pm$ but is represented by $+1$ in the vector representation V: $\widetilde{\mathrm{Ad}}(-1) = +1$. Thus, there are three inequivalent 8-dimensional representations. Of course, the volume element $\omega = (e_{12})(e_{34})(e_{56})(e_{78})$ double covers (under $\widetilde{\mathrm{Ad}}$) a rotation by π in the 12, 34, 56, 78 planes and hence $\widetilde{\mathrm{Ad}}(\omega) = -1$. Thus, we have the following table (of course, the last column is the product of the first two):

$g \in \mathrm{Spin}(8)$	-1	ω	$-\omega$
ρ_+	-1	$+1$	-1
ρ_-	-1	-1	$+1$
$\widetilde{\mathrm{Ad}}$	$+1$	-1	-1

It turns out that the group of outer automorphisms of Spin(8) is isomorphic to the symmetric group S_3, and the automorphism can be detected by its action on the center:

$$\mathrm{Outer}(\mathrm{Spin}(8))/\mathrm{Inner}(\mathrm{Spin}(8)) \cong \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3 \qquad (17.169)$$

Moreover, S_3 permutes the three 8-dimensional representations amongst themselves. This very beautiful group of outer automorphisms of Spin(8) is known as the *triality group*, discovered by E. Cartan in 1925.

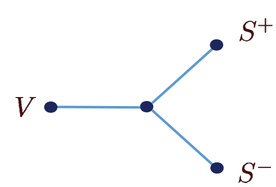

Figure 15: The Dynkin diagram of D_4 with nodes labeled by fundamental representations corresponding to the simple roots. fig:D4DYNKIN

Triality can be understood in several different ways. Here are a few of them: Label the three real eight-dimensional representations by $R_1 = S^+$, $R_2 = S^-$, $R_3 = V$.

1. The most direct way utilizes the relation between Lie groups and Lie algebras and the characterization of the Lie algebra by Dynkin diagrams. Since the Lie algebra can be reconstructed from its root system it suffices to give an automorphism of the

root system. The group of outer automorphisms of a simple Lie algebra is given by the automorphisms of its Dynkin diagram. The most symmetric case is that of $D_4 \cong so(8) \cong \mathrm{spin}(8)$ and shown in Figure fig:D4DYNKIN 15. The three legs can be permuted arbitrarily. In spin(8) the four simple coroots can be taken to be

$$\frac{1}{2}e_{12}, \quad \frac{1}{2}e_{34}, \quad \frac{1}{2}e_{56}, \quad \frac{1}{2}e_{78}. \tag{17.170}$$

Then the permutation $\sigma_{12} \in S_3$ can be lifted to the automorphism which acts on the Cartan subalgebra as [41]

$$\tilde{\sigma}_{12} \begin{pmatrix} e_{12} \\ e_{34} \\ e_{56} \\ e_{78} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_{12} \\ e_{34} \\ e_{56} \\ e_{78} \end{pmatrix} \tag{17.171}$$

A more nontrivial automorphism is

$$\tilde{\sigma}_{13} \begin{pmatrix} e_{12} \\ e_{34} \\ e_{56} \\ e_{78} \end{pmatrix} = H \begin{pmatrix} e_{12} \\ e_{34} \\ e_{56} \\ e_{78} \end{pmatrix} \tag{17.172}$$

eq:outeronethree

where

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{17.173}$$

is a matrix that squares to 1. Note that

$$(\tilde{\sigma}_{13}\tilde{\sigma}_{12})^3 = 1 \tag{17.174}$$

and hence $\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{13}$ generate a copy of the group S_3 in the group of automorphisms.

2. Using this we can write a group-theoretic version. [42] An outer automorphism σ_{12} which permutes $S^{\pm}$ holding V fixed is defined by its action on the generators

$$\begin{aligned} e_{ij} &\to e_{ij} \qquad 1 \le i,j \le 7 \\ e_{i8} &\to -e_{i8} \qquad i = 1, \dots, 7 \end{aligned} \tag{17.175}$$

To see this, note that one can write $\omega = e_{18}e_{28}\cdots e_{78}$. Thus σ_{12} exchanges ω for $-\omega$ holding -1 fixed. A glance at the table above showing the representation of the center shows that the represesentations R_1, R_2 i.e. $S^{\pm}$ are exchanged, holding V fixed.

---

[41] Rows 2 and 3 in this matrix are not misprints. They differ from the naive transformation by an inner automorphism.

[42] One should be careful not to interpret the transformation (eq:outeronethree 17.172) as an automorphism of the Clifford algebra. This would map ω to a projection operator.

To construct the permutation σ_{13} note that since $\omega = \exp\left(\pi\frac{1}{2}(e_{12}+e_{34}+e_{56}+e_{78})\right)$ it follows that the induced automorphism on the group, denoted σ_{13} is given by $\sigma_{13}^*(\omega) = -1$. We see that exchanging ω and -1 in the table above exchanges S^+ for V, leaving S^- fixed.

17.8.4 Trialities and division algebras

A very nice viewpoint on the triality automorphism of Spin(8) is provided by stepping back and thinking first about trialities more generally. We are here following the nice exposition in Baez [10] who is describing ideas of J.F. Adams.

If V_1, V_2 are two vector spaces over a field κ they are said to be in *duality* if there is a nondegenerate bilinear form

$$d : V_1 \times V_2 \to \kappa \tag{17.176}$$

This is also known as a perfect pairing. It establishes an isomorphism $V_1 \cong V_2^\vee$.

Thus, it is reasonable to say that three vector spaces V_1, V_2, V_3 are in *triality* if there exists a trilinear form

$$t : V_1 \times V_2 \times V_3 \to \kappa \tag{17.177}$$

which is nondegenerate in the sense that if we fix any two nonzero arguments we obtain a nonzero linear functional on the third vector space. This can be interpreted as defining maps

$$m_i : V_i \times V_{i+1} \to V_{i+2}^\vee \tag{17.178}$$

and nondegeneracy implies that if we choose any nonzero vector $v_i \in V_i$ then

$$m_i(v_i, \cdot) : V_{i+1} \cong V_{i+2}^\vee \tag{17.179}$$

$$m_i(\cdot, v_{i+1}) \cong V_i^\vee \tag{17.180}$$

Therefore, with a choice of nonzero vectors v_1, v_2 we have an isomorphism $V_2 \cong V_3^\vee \cong V_1$. Let us call the common vector space V. The triality defines a product

$$V \times V \to V \tag{17.181}$$

Since left and right multiplication by a nonzero vector is an isomorphism it follows that V is a division algebra! If we take the field $\kappa = \mathbb{R}$ then by a theorem of Kervaire-Bott-Milnor the dimension of V must be $1, 2, 4, 8$. ♣This is a little tricky. Explain it more carefully. ♣

Now, if we consider the representations over $\mathbb{R}$ of Spin(d) then we have irreps $S_d^\pm$ for $d = 0, 4 \bmod 8$ and unique irrep S_d otherwise. Taking $V \cong \mathbb{R}^d$ to be the vector representation we certainly have multiplication maps

$$\begin{aligned} m_d : V \times S_d^\pm \to S_d^\mp \qquad & d = 0, 4\bmod 8 \\ m_d : V \times S_d \to S_d \qquad & \text{else} \end{aligned} \tag{17.182}$$

Since the reps are self-dual we get trilinear maps

$$\begin{aligned} t_d : V \times S_d^+ \times S_d^- \to \mathbb{R} \qquad & d = 0, 4\bmod 8 \\ t_d : V \times S_d \times S_d \to \mathbb{R} \qquad & \text{else} \end{aligned} \tag{17.183}$$

Essentially, the coefficients of the map in a basis are the matrix elements $\Gamma^i_{\alpha\beta}$ of the gamma matrices.

In order for the gamma matrices to define a triality we must have an isomorphism $V \cong S_d$. One checks this only happens for $d = 1, 2, 4, 8$. Moreover, the form is nondegenerate. In this way we define the three division algebras

$$\begin{aligned} t_1 &: V_1 \times S_1 \times S_1 \to \mathbb{R} & \Rightarrow D &= \mathbb{R} \\ t_2 &: V_2 \times S_2 \times S_2 \to \mathbb{R} & \Rightarrow D &= \mathbb{C} \\ t_4 &: V_4 \times S_4^+ \times S_4^- \to \mathbb{R} & \Rightarrow D &= \mathbb{H} \\ t_8 &: V_8 \times S_8^+ \times S_8^- \to \mathbb{R} & \Rightarrow D &= \mathbb{O} \end{aligned} \tag{17.184}$$

Under the isomorphisms $V \cong \mathbb{O}$, and $S^\pm \cong \mathbb{O}$ the multiplication maps are

$$x \otimes y \to \bar{x} y \tag{17.185}$$

and the triality map is just

$$t(x_1, x_2, x_3) = \mathrm{Re}(\bar{x}_1 \bar{x}_2 \bar{x}_3) \tag{17.186}$$

(For more about this see DelignSpinors [16].)

The triality automorphism can be written very explicitly in terms of the unique (up to scale) nondegenerate trilinear coupling

$$t : S^+ \otimes S^- \otimes V \to \mathbb{R} \tag{17.187}$$

Given $g \in \mathrm{Spin}(8)$ there exist unique elements $g_\pm \in \mathrm{Spin}(8)$ such that, for all vectors $s_\pm \in S^\pm$ and $v \in V$,

$$t(\rho_+(g_+)s_+, \rho_-(g_-)s_-, \rho_V(g)v) = t(s_+, s_-, v) \tag{17.188}$$

Then the maps $\alpha_\pm : g \to g_\pm$ are outer automorphisms and descend (taking the quotient by inner automorphisms) to generators of the group of outer automorphisms. ♣Need to explain why! ♣

17.8.5 Lorentz groups and division algebras

Finally, we remark that there is a beautiful uniform prescription for the Lorentz groups in the special dimensions $2+1, 3+1, 5+1$ and $9+1$ where the transverse dimension $s-t$ is a power of 2. In terms of spinor representations $C\ell_1, C\ell_2, C\ell_4, C\ell_8$ have graded representations

$$S^+ \oplus S^- \tag{17.189}$$

where $S^\pm \cong \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ as a real vector space. Now

$$C\ell_{d+1,-1} \cong C\ell_{1,-1} \widehat{\otimes} C\ell_d \tag{17.190}$$

Let $\{v_0, v_1\}$ be a basis of even and odd vectors for the irreducible module $\mathbb{R}^{1|1}$ of $C\ell_{1,-1}$ in which the generators take the values

$$\rho(e_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho(e_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{17.191}$$

Then the even part of the irreducible $C\ell_{d+1,-1}$ module is

$$S^+ \otimes v_0 \oplus S^- \otimes v_1 \tag{17.192}$$

and hence we can think of spinors in these Minkowski spaces as pairs of elements of $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Moreover, bispinors can be related to 2×2 Hermitian matrices over the normed division algebra $\mathbb{K}$:

$$\mathbb{X} = \begin{pmatrix} x^0 + x^1 & x_t \\ x_t^* & x^0 - x^1 \end{pmatrix} \tag{17.193}$$

with $x^0 \pm x^1 \in \mathbb{R}$ light cone coordinates and $x_t \in \mathbb{K}$ a transverse coordinate. It makes sense to define the determinant of such an Hermitian matrix as

$$(x^0)^2 - (x^1)^2 - \parallel x_t \parallel^2 \tag{17.194}$$

The idea is that spin transformations should act on $\mathbb{X}$ as

$$\mathbb{X} \to S\mathbb{X}S^\dagger \tag{17.195}$$

The transformation S should have unit determinant and hence transformations of the spin group should act on spacetime as norm-preserving transformations. This works well and is quite useful for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. It is the basis of the spinor helicity formalism! It is problematic but suggestive for $\mathbb{K} = \mathbb{H}, \mathbb{O}$. There is, however, the suggestion of a beautiful and profound pattern:

$$\begin{aligned} \mathrm{Spin}(1,2) &\cong SL(2,\mathbb{R}) \\ \mathrm{Spin}(1,3) &\cong SL(2,\mathbb{C}) \\ \mathrm{Spin}(1,5) &\cong SL(2,\mathbb{H}) \\ \mathrm{Spin}(1,9) &\cong SL(2,\mathbb{O}) \end{aligned} \tag{17.196}$$

The last two lines require some nontrivial interpretation: It is nontrivial to say what these are as groups and to interpret the $\det(S) = 1$ condition. See Baez [10] and DelignSpinors [16] for the elegant details.

18. Fermions and the Spin Representation

ermionsSpinRep

We now return to quantum mechanics.

The central motivation for this chapter, in the context of these notes, is that important examples of the 10-fold way described above are provided by free fermions. They also appear in the Altland-Zirnbauer classification, and in applications to topological band structure.

Of course, the basic mathematics of free fermion quantization is very broadly applicable. In this chapter we give a summary of that quantization and comment on the relation to the Spin group and spin representations.

subsec:FDFS

18.1 Finite dimensional fermionic systems

A finite dimensional fermionic system (FDFS) is a quantum system based on a certain kind of operator algebra and its representation:

Definition: A finite dimensional fermionic system is the following data:

1. A finite-dimensional real vector space $\mathcal{M} \cong \mathbb{R}^N$, called the *mode space* with a positive symmetric bilinear form Q.

2. An extension of the complex Clifford algebra

$$\mathcal{A} = \mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C} \tag{18.1}$$

   to a $*$-algebra.

3. A choice of Hilbert space $\mathcal{H}_F$ together with a $*$ homomorphism of $\mathcal{A}$ into the algebra of $\mathbb{C}$-linear operators on $\mathcal{H}_F$. [43]

Here are a number of remarks about this definition:

1. As an algebra $\mathcal{A}$ is the complex Clifford algebra of $V := \mathcal{M} \otimes \mathbb{C}$ with Q extended $\mathbb{C}$-linearly.

2. From $(\mathcal{M}, Q)$ we can make the real Clifford algebra $\mathrm{Cliff}(\mathcal{M}, Q)$. In quantum mechanics we will want a $*$-algebra of operators and the observables will be the operators fixed by the $*$-action. For us the $*$-algebra structure on

$$\mathcal{A} := \mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C} \tag{18.2}$$

   is $\beta \otimes \mathcal{C}$, where β is the canonical anti-automorphism of $\mathrm{Cliff}(\mathcal{M}, Q)$ and $\mathcal{C}$ is complex conjugation on $\mathbb{C}$. Thus $*$ fixes $\mathcal{M}$ and is an anti-automorphism. (These conditions uniquely determine $*$.) Axioms of quantum mechanics would simply give us *some* $*$-algebra without extra structure. The fermionic system gives us the extra data $(\mathcal{M}, Q)$.

3. Since we have a $*$ structure on a $\mathbb{Z}_2$-graded algebra we must deal with a convention issue. Here we are taking the convention that $(ab)^* = b^* a^*$ for any a, b because this is the convention almost universally adopted in the physics literature. However, a systematic application of the Koszul sign rule in the definition of $*$ would require $(ab)^* = (-1)^{|a|\cdot|b|} b^* a^*$. One can freely pass between these two conventions and, if used consistently, the final results are the same. See Section §subsec:SuperHilbert 12.5 above for more discussion.

[43] The subscript "F" is for "Fock."

4. If Q is positive definite then we can diagonalize it to the unit matrix. If e_i is a choice of basis in which Q is δ_{ij} then the usual Clifford relations

$$e_i e_j + e_j e_i = 2\delta_{ij} \qquad\qquad i,j = 1,\dots,N \tag{18.3}$$ eq:FF-CCR

are known in this context as the *fermionic canonical commutation relations*. Because of our choice of $*$-structure we have $e_i^* = e_i$. Of course, the choice of basis is far from unique. Different choices are related by $O(N)$ transformations. Those transformations commute with the $*$ structure. The e_i are known in the literature as *real fermions* or *Majorana fermions*. In terms of the e_i the most general quantum observable is

$$\mathcal{O} = \mathcal{O}_0 + \sum_{k=1}^{d} \mathcal{O}_{i_1\dots i_k} e_{i_1\dots i_k} \tag{18.4}$$ eq:GenSelfAdj

where the coefficients are totally antisymmetric tensors such that $\mathcal{O}_0 \in \mathbb{R}$ and

$$\mathcal{O}^*_{i_1\dots i_k} = (-1)^{\frac{1}{2}k(k-1)} \mathcal{O}_{i_1\dots i_k}. \tag{18.5}$$

5. In quantum mechanics we must also have a Hilbert space representation of the $*$-algebra of operators so that $*$ corresponds to Hermitian conjugation in the Hilbert space representation. That is, we have an algebra homomorphism

$$\rho_F : \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}_F) \tag{18.6}$$

to the $\mathbb{C}$-linear operators on the Hilbert space $\mathcal{H}_F$. The is a $*$-homomorphism in the sense that

$$(\rho_F(a))^\dagger = \rho_F(a^*) \tag{18.7}$$

In the fermionic system we are assuming that $\mathcal{H}_F$ is a choice of an *irreducible* module for $\mathcal{A}$. We will describe explicit models for $\mathcal{H}_F$ in great detail below. (Of course, we have already discussed them at great length - up to isomorphism.)

6. The notation N is meant to suggest some large integer, since this is a typical case in the cond-matt applications. But we will not make specific use of that property.

18.2 Left regular representation of the Clifford algebra

The Clifford algebra acts on itself, say, from the left. On the other hand, it is a vector space. Thus, as with any algebra, it provides a representation of itself, called the left-regular representation (LRR).

Note that this representation is 2^N dimensional, and hence rather larger than the $\sim 2^{[N/2]}$ dimensional irreducible representations. Hence it is highly reducible. In order to find irreps we should "take a squareroot" of this representation.

We will now describe some ways in which one can take such a "squareroot." To motivate the construction we first step back to the general real Clifford algebra $C\ell_{r,-s}$ and interpret the LRR in terms of the exterior algebra. Recall that we identified

$$C\ell(r_+, s_-) \cong \Lambda^* \mathbb{R}^{r+s} \tag{18.8}$$ eq:vspi

as a vector spaces. Also, while the exterior algebra $\Lambda^*\mathbb{R}^{r,s}$ is an algebra we stressed that (18.8) is *not* an algebra isomorphism.

Nevertheless, since (18.8) is a vector space isomorphism this means that $\Lambda^*(\mathbb{R}^{r,s})$ must be a Clifford *module*, that is, a representation space of the Clifford algebra. We now describe explicitly its structure as a module.

If $v \in \mathbb{R}^{r,s}$ then we can define the contraction operator by

$$\mathfrak{i}(v)(v_{i_1} \wedge \cdots \wedge v_{i_k}) := \sum_{s=1}^{k} (-1)^{s-1} Q(v, v_{i_s}) v_{i_1} \wedge \cdots \wedge \widehat{v_{i_s}} \wedge \cdots \wedge v_{i_k} \tag{18.9}$$

eq:contract

where the hat superscript $\widehat{v}$ means we omit that factor. Similarly, we can define the wedge operator by

$$\mathfrak{w}(v)(v_{i_1} \wedge \cdots \wedge v_{i_k}) := v \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} \tag{18.10}$$

eq:wedgeoper

These operators are easily shown to satisfy the algebra:

$$\begin{aligned} \{\mathfrak{i}(v_1), \mathfrak{i}(v_2)\} &= 0 \\ \{\mathfrak{w}(v_1), \mathfrak{w}(v_2)\} &= 0 \\ \{\mathfrak{i}(v_1), \mathfrak{w}(v_2)\} &= Q(v_1, v_2) \end{aligned} \tag{18.11}$$

eq:rccrs

Using these relations we see that we can (using symmetry of Q) represent Clifford multiplication by v on $\Lambda^*\mathbb{R}^{r,s}$ by the operator:

$$\rho(v) = \mathfrak{i}(v) + \mathfrak{w}(v) \tag{18.12}$$

eq:CliffRep

Since the $v \in V$ generate the Clifford algebra we can then extend this to a representation of the entire Clifford algebra by taking $\rho(a_1 a_2) = \rho(a_1)\rho(a_2)$, and scalars in $\mathcal{A}$ act as scalars on $\Lambda^*\mathbb{R}^{r,s}$.

Of course, this representation is highly reducible! We have seen that it is isomorphic to the tensor product of spin representations. Thus, spin representations take a square root of this representation. In order to describe that square-root intrinsically in terms of the exterior complex we need to split the vector space V in half in an appropriate way. This is the topic of the next Section.

18.3 Spin representations from complex isotropic subspaces

pCplxIsotropic

Let us assume $\dim_{\mathbb{R}} \mathcal{M}$ is even so that $N = 2n$. The standard finite-dimensional fermionic Fock space construction begins by *choosing* a complex structure I on $\mathcal{M}$. As shown in (7.47) above we automatically have

♣Put remark below on the odd case. ♣

$$\mathcal{M} \otimes \mathbb{C} = V \cong W \oplus \overline{W} \tag{18.13}$$

given by the projection operators $P_\pm = \frac{1}{2}(1 \pm I \otimes i)$. Here we take

$$W := P_- \mathcal{M} \otimes \mathbb{C} = \mathrm{Span}_{\mathbb{C}}\{e - iIe | e \in \mathcal{M}\} \tag{18.14}$$

$$\overline{W} \cong P_+ \mathcal{M} \otimes \mathbb{C} = \mathrm{Span}_{\mathbb{C}}\{e + iIe | e \in \mathcal{M}\} \tag{18.15}$$

If a vector $P_{-}v = v$ then $Iv = iv$, so W is the $(1,0)$ space of V.

Now, we henceforth assume that I is compatible with the quadratic form Q so

$$Q(Iv_1, Iv_2) = Q(v_1, v_2) \tag{18.16}$$

eq:IQ-compat

We then extend Q to be a symmetric $\mathbb{C}$-*linear* form on V. Note that W is a maximal dimension isotropic complex subspace of V. For if $w_1, w_2 \in W$ then

$$Q(w_1, w_2) = Q(Iw_1, Iw_2) = Q(\mathrm{i}w_1, \mathrm{i}w_2) = \mathrm{i}^2 Q(w_1, w_2) = -Q(w_1, w_2) \tag{18.17}$$

and hence $Q(w_1, w_2) = 0$. Note we have crucially used the fact that the extension of Q is $\mathbb{C}$-linear.

Remark: Recall that the space of complex structures compatible with Q is a homogeneous space $\mathrm{CmptCplxStr}(\mathcal{M}, Q)$ isomorphic to $O(2n)/U(n)$. (See (7.23) above.) Once we have extended Q in this $\mathbb{C}$-linear fashion we can also understand the space of complex structures as the Grassmannian of maximal dimension complex isotropic subspaces in V. This interpretation is sometimes quite useful, especially in giving a geometrical interpretation of the spin representation in Section 18.4.7 below. We denote this Grassmannian by $\mathcal{G}(V, Q)$.

Now, given the decomposition $V \cong W \oplus \bar{W}$ it is fairly evident how to take a "square-root" of

$$\Lambda^* V \cong \Lambda^* W \otimes \Lambda^* \overline{W} \tag{18.18}$$

We could, for example, consider the vector space

$$\Lambda^* W = \oplus_{k=0}^{n} \Lambda^k W \tag{18.19}$$

We can make this vector space into an irreducible Clifford module for $\mathrm{Cliff}(V, Q)$ by similarly taking "half" of the representation (18.12):

1. For $w \in W$ we define $\rho_{F,W}(w) := \mathfrak{w}(w)$
2. For $\bar{w} \in \overline{W}$ we define $\rho_{F,W}(\bar{w}) = \mathfrak{i}(\bar{w})$.
3. Now define ρ_F on V by extending the above equations $\mathbb{C}$-linearly: $\rho_{F,W}(w_1 \oplus \overline{w_2}) := \rho_{F,W}(w_1) + \rho_{F,W}(\overline{w_2})$.

Now one checks that indeed

$$\{\rho_{F,W}(w_1 \oplus \overline{w_2}), \rho_{F,W}(u_1 \oplus \overline{u_2})\} = 2Q(w_1 \oplus \overline{w_2}, u_1 \oplus \overline{u_2})1 \tag{18.20}$$

so that the Clifford relations are satisfied and $\rho_{F,W}$ defines the structure of a Clifford module on $\Lambda^* W$. We will often denote this module as

$$\mathcal{H}_{F,W} := \Lambda^* W \tag{18.21}$$

and to lighten the notation we sometimes abbreviate $\rho_{F,W}$ by ρ_F if W is understood or drop it altogether if the context is clear.

In fact $(\rho_{F,W}, \mathcal{H}_{F,W})$ is naturally a graded representation with

$$\mathcal{H}^0_{F,W} \cong \Lambda^{\rm even} W := \oplus_{k=0(2)} \Lambda^k W \tag{18.22}$$

$$\mathcal{H}^1_{F,W} \cong \Lambda^{\rm odd} W := \oplus_{k=1(2)} \Lambda^k W \tag{18.23}$$

Now if we think of Spin$(2n)$ as a group of invertible elements in

$$\mathrm{Spin}(2n) \subset \mathrm{Cliff}(\mathcal{M}, Q) \subset \mathrm{Cliff}(V, Q) \tag{18.24}$$

then through ρ_F the group Spin$(2n)$ acts on $\mathcal{H}_{F,W}$, but not irreducibly. The operations of contraction and wedging with a vector change the parity of k, but Spin$(2n)$ involves the action of an even number of vectors so we see that, as a representation of Spin$(2n)$, $\mathcal{H}_{F,W} \cong S_c$ and this decomposes into:

$$S_c^+ \cong \Lambda^{\rm even} W \tag{18.25}$$

$$S_c^- \cong \Lambda^{\rm odd} W \tag{18.26}$$

♣Need to check conventions for chiral vs. antichiral! ♣

In the physical applications it is important to note that we can put an Hermitian structure on V by defining the sesquilinear form

$$h(v_1, v_2) := Q(\overline{v}_1, v_2) \tag{18.27}$$

where $\overline{v}$ is defined from the decomposition $W \oplus \overline{W}$. Note that $V = W \oplus \bar{W}$ is an orthogonal Hilbert space decomposition: W and $\bar{W}$ are separately Hilbert spaces and are orthogonal. To prove this note that orthogonality follows since W and $\bar{W}$ are isotropic with respect to Q. Then since W is maximal isotropic and Q is nondegenerate the sesquilinear form restricted to W must be nondegenerate. Moreover, since $Q > 0$, this defines a Hilbert space inner product on V.

The Fock space $\mathcal{H}_{F,W}$ now inherits a Hilbert space structure since we can define

$$h(w_1 \wedge \cdots \wedge w_k, w_1' \wedge \cdots \wedge w_\ell') := \delta_{k,\ell} {\rm det} h(w_i, w_j') \tag{18.28}$$ `eq:HilbStruct`

for $k, \ell > 0$. We extend this to $\Lambda^0 W$ by declaring it orthogonal to the subspaces $\Lambda^k W$ with $k > 0$ and normalizing:

$$h(1,1) := 1 \tag{18.29}$$ `eq:Normalize-Vac`

Note that $\rho_{F,W}(e_i)$ are self-adjoint operators so that $\rho_{F,W}$ is indeed a $*$ homomorphism, as desired. Moreover, $\rho_{F,W}(e_{ij})$ are anti-self-adjoint. Therefore with this Hilbert space structure $\mathcal{H}_{F,W}$ is a *unitary* representation of Spin$(2n)$. Indeed, the operators representing the group Spin$(2n)$ are of the form $\exp[\frac{1}{2}\omega^{ij}\rho_{F,W}(e_{ij})]$ with real ω^{ij}.

The upshot of this discussion is the theorem:

Theorem: There is a bundle of $\mathbb{Z}_2$-graded finite dimensional Hilbert spaces over

$$\mathrm{CmptCplxStr}(\mathcal{M}, Q) \cong \mathcal{G}(V, Q) \cong O(2n)/U(n) \tag{18.30}$$

whose fiber at a complex maximal isotropic subspace $W \subset V$ is the fermionic Fock space

$$\mathcal{H}_{F,W} \cong \Lambda^{\text{even}} W \oplus \Lambda^{\text{odd}} W \tag{18.31}$$

The homogeneous subspaces in the fibers are naturally unitary chiral representations of the spin group $\mathrm{Spin}(2n)$.

Remark: The reader might well be wondering: "Why not choose $\Lambda^* \overline{W}$?" Indeed that works too. Exchanging I and $-I$ is equivalent to exchanging W and $\overline{W}$. So, with our construction, $\rho_{F,\overline{W}}$ is simply the module we would get from complex structure $-I$. The space of complex structures $O(2n)/U(n)$ has two connected components. These can be distinguished by the extra data of a choice of orientation of $\mathcal{M}$. For example, we could associate to any basis in which I is of the form

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{18.32}$$

the orientation $e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}$. Then, thanks to (eq:Stab-IO-p 7.19) above, this association is well-defined.

♣Maybe this discussion should be in Section subsec:CplxS 7.1. ♣

subsec:FermOsc

18.4 Fermionic Oscillators

Now let us connnect the construction of the spin representation in subsec:SpinRepCplxIsotropic 18.3 to the usual discussion in the physics literature using fermionic harmonic oscillators. In particular, we would like to justify the terminology "fermionic Fock space" for $\mathcal{H}_{F,W}$.

Given a complex structure I on $\mathcal{M}$ compatible with Q we can find an ON basis e_i for $\mathcal{M}$ such that

$$\begin{aligned} Ie_{2j-1} &= -e_{2j} \\ Ie_{2j} &= e_{2j-1}, \qquad\qquad j = 1, \dots, n \end{aligned} \tag{18.33}$$

eq:FermOsc-1

Put differently, the ordered basis:

$$\{\mathfrak{e}\}_{\alpha=1}^{2n} := \{e_1, e_3, \dots, e_{2n-1}, e_2, e_4, \dots, e_{2n}\}, \tag{18.34}$$

eq:e-ord-alt

is a basis in which

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{18.35}$$

Once again: The choice of such a basis is far from unique. Different choices are related by a subgroup of $O(2n)$ isomorphic to $U(n)$, as described in Section §subsec:CplxStrRealVS 7.1. We will explore this in detail below.

♣Opposite sign from I_0 in Section subsec:CplxStrRealVS 7.1. ♣

Then applying projection operators gives us a basis for W and $\bar{W}$, respectively:

$$\begin{aligned} \bar{a}_j &= P_- e_{2j-1} = \frac{1}{2}(e_{2j-1} + \mathrm{i} e_{2j}) \\ a_j &= P_+ e_{2j-1} = \frac{1}{2}(e_{2j-1} - \mathrm{i} e_{2j}) \end{aligned} \tag{18.36}$$

eq:FermOsc-2

$$\begin{aligned} e_{2j-1} &= a_j + \bar{a}_j \\ e_{2j} &= \mathrm{i}(a_j - \bar{a}_j) \end{aligned} \tag{18.37}$$

eq:FermOsc-3

We easily compute the fermionic CCR's in this basis to get the usual fermionic harmonic oscillator algebra:

$$\begin{aligned} \{a_j, a_k\} &= \{\bar{a}_j, \bar{a}_k\} = 0 \\ \{a_j, \bar{a}_k\} &= \delta_{j,k} \end{aligned} \tag{18.38}$$

The space $\Lambda^* W$ has a natural basis $1, \bar{a}_j, \dots$ where the general basis element is given by $\bar{a}_{j_1} \cdots \bar{a}_{j_\ell}$ for $j_1 < \cdots < j_\ell$. In particular, note that [44]

$$\rho_{F,W}(a_i) \cdot 1 = 0 \tag{18.39}$$

where $1 \in \Lambda^0 W \cong \mathbb{C}$. We build up the other basis vectors by acting with $\rho_{F,W}(\bar{a}_j)$ on 1.

The transcription to physics notation should now be clear. The *vacuum line* is the complex vector space $\Lambda^0 W \cong \mathbb{C}$. Physicists usually choose an element of that line and denote it $|0\rangle$. Moreover, they drop the heavy notation $\rho_{F,W}$, so, in an irreducible module we have just

$$a_i|0\rangle = 0. \tag{18.40}$$

The state $|0\rangle$ is variously called the *Dirac vacuum*, the *Fermi sea*, or the *Clifford vacuum*. However, irrespective of whose name you wish to name the state after, it must be stressed that these equations only determine a line, not an actual vector, and, when considering families of representations this can be important. Indeed, some families of quantum field theories are inconsistent because there is no way to assign an unambiguous vacuum vector to every element in the family which varies with sufficient regularity.

In our case we have a canonical choice $1 \in \Lambda^0 W \leftrightarrow |0\rangle \in \mathcal{H}_F$, where $\mathcal{H}_F$ is our notation for the fermionic Fock space. Then, $\Lambda^k W$ is the same as the subspace spanned by $\bar{a}_{j_1} \cdots \bar{a}_{j_k}|0\rangle$.

In physical interpretations $\Lambda^k W$ is a subspace of a Fock space describing states with k-particle excitations above the vacuum $|0\rangle$. It is very convenient to introduce the *fermion number operator*

$$\mathcal{F} := \sum_{i=1}^{n} \bar{a}_i a_i = \frac{n}{2} - \frac{\mathrm{i}}{4} \sum_{\alpha,\beta} e_\alpha I_{\alpha\beta} e_\beta \tag{18.41}$$

so that $\Lambda^k W$ is the subspace of "fermion number k."

The operator $(-1)^{\mathcal{F}}$ commutes with the spin group and decomposes the Fock space into even and odd subspaces. That is, the eigenspaces $(-1)^{\mathcal{F}} = \pm 1$ are isomorphic to the chiral spin representations.

Finally, consider the Hilbert space structure. With respect to the Hilbert space structure (eq:HilbStruct 18.28) we find that indeed

$$\rho_{F,W}(\bar{a}_i) = \rho_{F,w}(a_i)^\dagger, \tag{18.42}$$

[44] Note that, if we drop the $\rho_{F,W}$ then the equation would be wrong!

so in physics we would just write $\bar{a}_i \to a_i^\dagger$. The normalization condition (eq:Normalize-Vac 18.29) is written in physics notation as

$$\langle 0|0\rangle = 1. \tag{18.43}$$

Remarks:

1. In the physics literature the decomposition of $V = W \oplus \bar{W}$ into orthogonal Hilbert spaces given by a bilinear form and compatible complex structure is sometimes referred to as a *Nambu structure*. Note that we therefore have *two* Hilbert spaces associated to the system of free fermions. This is important in the K-theory classification.

2.

Exercise *Change of basis between fermionic harmonic oscillators and Majorana operators*

a.) Compute the matrix for the change of ordered basis $\{\mathfrak{e}_\alpha\}$ for $V = \mathcal{M} \otimes \mathbb{C}$ to the ordered basis

$$\{\mathfrak{a}_\alpha\} := \{\bar{a}_1, \dots, \bar{a}_n, a_1, \dots, a_n\} \tag{18.44}$$

eq:full-osc-bas

♣S is bad notation since it is already used for a spinor rep. ♣

Answer: $\mathfrak{a}_\alpha = u_{\beta\alpha}\mathfrak{e}_\beta$ with

$$u = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{18.45}$$

eq:e-to-aabar

b.) Check that

$$u^{-1} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \qquad u^{tr}u = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad uu^{tr} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18.46}$$

These identities are useful in Section §subsubsec:BogTmn 18.4.3.

c.) Show that

$$\begin{aligned} Q(a_i, a_j) &= Q(\bar{a}_i, \bar{a}_j) = 0 \\ Q(a_i, \bar{a}_j) &= Q(\bar{a}_j, a_i) = \frac{1}{2}\delta_{i,j} \end{aligned} \tag{18.47}$$

eq:Q-in-aabar

Exercise

Show that

$$e_{2j-1}e_{2j} = \mathrm{i}(2\bar{a}_j a_j - 1) \tag{18.48}$$

so this has eigenvalues $\pm\mathrm{i}$ and the representation of the volume element for the orientation $\omega = e_1 \cdots e_{2n}$ is

$$\rho_{F,W}(\omega) = \mathrm{i}^n \prod_{j=1}^n (2a_j^\dagger a_j - 1) \tag{18.49}$$

---

plicitRepGamma 18.4.1 An explicit representation of gamma matrices

The Fock space $\mathcal{H}_F$ gives a nice representation of the full complex Clifford algebra $\mathbb{C}\ell_{2n}$.

Consider first the case $n = 1$. It is useful to make a change of notation:

$$|0\rangle := |-\frac{1}{2}\rangle \qquad a^\dagger|0\rangle := |+\frac{1}{2}\rangle \tag{18.50}$$

We will write $|\pm\rangle = |\pm\frac{1}{2}\rangle$ for brevity. This labelling will be useful later for representations of the spin group. It follows that

$$a^\dagger|-\rangle = |+\rangle \qquad\qquad a|+\rangle = |-\rangle \tag{18.51}$$

Now taking

$$x_1|+\rangle + x_2|-\rangle \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{18.52}$$

we have the representation

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \rho(e_2) = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \tag{18.53}$$

We recognize one of our standard graded irreducible representations of $\mathbb{C}\ell_2$. According to eq:M2-min (13.44), with the choice of orientation $\omega_c = ie_{12}$ and taking the upper component as the even subspace it is M_2^-.

Now, with n oscillator pairs we have a natural basis for a 2^n dimensional Fock space:

$$(a_n^\dagger)^{s_n+\frac{1}{2}}(a_{n-1}^\dagger)^{s_{n-1}+\frac{1}{2}}\cdots(a_1^\dagger)^{s_1+\frac{1}{2}}|0\rangle \tag{18.54}$$ eq:SpinWtBs

where $s_i = \pm\frac{1}{2}$. We identify these states with the basis for the tensor product of representations

$$|s_n, s_{n-1}, \dots, s_1\rangle = |s_n\rangle\widehat{\otimes}|s_{n-1}\rangle\widehat{\otimes}\cdots\widehat{\otimes}|s_1\rangle \tag{18.55}$$

Note that because the $a_j^\dagger$ anticommute we are really taking a graded tensor product.

Let $\Gamma^j_{(n-1)}$ be the $2^{n-1}\times 2^{n-1}$ representation matrices of e_j for a collection of $(n-1)$ oscillators. Then when we add the n^{th} oscillator pair we get

$$\begin{aligned}
\rho_n(e_j) = \Gamma^j_{(n)} &= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \otimes \Gamma^j_{(n-1)} \qquad\qquad j = 1,\dots,2n-2 \\
\rho_n(e_{2n-1}) = \Gamma^{2n-1}_{(n)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1_{2^{n-1}} \\
\rho_n(e_{2n}) = \Gamma^{2n}_{(n)} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes 1_{2^{n-1}}
\end{aligned} \tag{18.56}$$ eq:ExplctRep

Note the first factor in the first line of eq:ExplctRep (18.56). It is a direct manifestation of the fact that we are taking a graded tensor product.

For example, for $n=2$ we have

$$\begin{aligned}
\Gamma^1_{(2)} &= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\Gamma^2_{(2)} &= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \\
\Gamma^3_{(2)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix} \\
\Gamma^4_{(2)} &= \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix}
\end{aligned} \tag{18.57}$$

By induction we see that the volume form ω_c is represented by ♣Conventions! ♣

$$\Gamma_\omega = (-i)^n \Gamma^1 \cdots \Gamma^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18.58}$$

eq:cplxvol

where there are n factors. Note that this can be expressed in terms of the fermion number operator as

$$\Gamma_\omega = (-1)^n (-1)^{\mathcal{F}} \tag{18.59}$$

♣Better to redefine Γ_ω but then must be careful about intertwiners. ♣

For $d = 2n+1$ we still take n pairs of oscillators and set $\Gamma^{2n+1} = \Gamma_\omega$.

When we consider the Fock space as a representation of Spin(2n) the vectors $(s_n, \dots, s_1)$ become the *spinor weights* of the spinor representations of $so(2n;\mathbb{C})$. That is, we choose a basis for a Cartan subalgebra, in this case - $(M_{2n-1,2n}, \dots, M_{12})$. Then these operators are simultaneously diagonalizable, and the basis (eq:SpinWtBs 18.54) is a simultaneous eigenbasis for these operators with vector of eigenvalues given by $(s_n, \dots, s_1)$. Note that the weights of S^+ and S^- are distinguished by the parity of $\sum_i (s_i - \frac{1}{2})$.

♣Again, check conventions! ♣

Remark *Explicit intertwiners.* We can now fill a gap in our discussion above. In our explicit basis Γ^i are real and symmetric for i odd, and imaginary ($= i\times$ real) and antisymmetric for i even. Our explicit intertwiners are

$$\begin{aligned}
B_\pm \Gamma^i B_\pm^{-1} &= \pm (\Gamma^i)^* \\
C_\pm \Gamma^i C_\pm^{-1} &= \pm (\Gamma^i)^{tr}
\end{aligned} \tag{18.60}$$

Note that in this basis we can take $B_\pm = C_\pm$. Because of the simple reality and symmetry properties we can easily construct the intertwiners using the operator $U := \Gamma^2\Gamma^4 \cdots \Gamma^{2n}$. In particular, we have

$$C_+ = B_+ = \begin{cases} U & neven \\ \Gamma_\omega U & nodd \end{cases} \tag{18.61}$$

eq:plusinter

$$C_- = B_- = \begin{cases} \Gamma_\omega U & neven \\ U & nodd \end{cases} \tag{18.62}$$

eq:plusintera

It is now a matter of straightforward computation to compute the scalars $C_\xi^{-1} C_\xi^{tr}$ and $B_\xi^* B_\xi$.

18.4.2 Characters of the spin group

The character of a representation is the function on the group given by the trace. The character is a class function and therefore determined by its restriction to the maximal torus.

Parametrize the Cartan subalgebra by $x \cdot M := x_1 M_{12} + x_2 M_{34} + \cdots + x_n M_{2n-1,2n}$ where $n = [N/2]$. Then the character functions for $\mathrm{Spin}(N)$ are given by

$$\mathrm{ch}_{S_c}(x) := \mathrm{Tr}_{S_c}(e^{x\cdot M}) = \prod_{i=1}^{n}(e^{\frac{1}{2}x_i} + e^{-\frac{1}{2}x_i}) = \prod_{i=1}^{n}(2\cosh x_i/2) \tag{18.63}$$

This follows immediately from the above oscillator construction of the spin representation since $M_{2j-1,2j}$ has eigenvalue s_j. The sum over the representation is a sum over the spinor weights $(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})$.

When $N = 2n$ is even the representation is reducible. If we look back at the oscillator construction we see that the volume form acts on a basis state $|s_n, s_{n-1}, \dots, s_1\rangle$ as

$$\Gamma_\omega |s_n, s_{n-1}, \dots, s_1\rangle = (-1)^{\sum_{i=1}^{n}(s_i - \frac{1}{2})}|s_n, s_{n-1}, \dots, s_1\rangle = (-1)^{\mathcal{F}}|s_n, s_{n-1}, \dots, s_1\rangle \tag{18.64}$$

Exercise

a.) Show that:

$$\mathrm{Tr}_{S_c}\left[(-1)^{\mathcal{F}} e^{x\cdot M}\right] = \prod_{i=1}^{n}(e^{\frac{1}{2}x_i} - e^{-\frac{1}{2}x_i}) = \prod_{i=1}^{n}(2\sinh x_i/2) \tag{18.65}$$

This formula is important in index theory.

b.) Deduce that

$$\mathrm{ch}_{S_c^{\pm}}(x) = \frac{1}{2}\left[\prod(2\cosh x_i/2) \pm \prod(2\sinh x_i/2)\right] \tag{18.66}$$

c.) Check the special isomorphisms with unitary groups with these formulae.

d.) Check the identity on characters implied by the decomposition of $S \otimes S$ and its variants.

---

subsec:BogTmn

18.4.3 Bogoliubov transformations

We now return to the fact that we had to choose a complex structure to construct an irreducible spin representation in Section §18.3 (subsec:SpinRepCplxIsotropic). However, as we saw in equation (7.23) (eq:Cplx-Compat) above there is a whole family of complex structures I which we can use to effect the construction. On the other hand, the irreducible spin representations $S_c^{\pm}$ are unique up to isomorphism. Therefore there must be an isomorphism between the constructions: this isomorphism is known in physics as a "Bogoliubov transformation." It can have nontrivial physical consequences.

To a mathematician, there is just one isomorphism class of chiral spin representation S_c^+ or S_c^- (distinguished, invariantly, by the volume element). However, in physics, the fermionic oscillators represent physical degrees of freedom: Nature chooses a vacuum, and if, as a function of some control parameters a new vacuum becomes preferred when those parameters are varied then the Bogoliubov transformation has very important physical implications. A good example of this is superconductivity.

♣Explain somewhere what a polarization is. ♣

Returning to mathematics, suppose we choose one complex structure I_1 with a compatible basis $\{e_\alpha\}$ satisfying (18.33) for I_1. Next, we consider a different complex structure I_2 with corresponding basis $\{f_\alpha\}$ satisfying (18.33) for I_2.

♣We need to explain that although $\mathrm{Spin}(2n)$ acts on $\mathcal{H}_{F,W}$ for fixed W we can also make it (through the map to $O(2n)$) act on the space of complex structures. So we could ask how the actions on different complex structures is defined. Thus, we could also ask about whether the bundle $\mathcal{H}_F \to \mathcal{G}(V,Q)$ can be interpreted as an equivariant bundle. ♣

With the different basis $\{f_\alpha\}$ we can form fermionic oscillators according to (18.36):

$$\begin{aligned} \bar{b}_j &= \frac{1}{2}(f_{2j-1} + \mathrm{i} f_{2j}) \\ b_j &= \frac{1}{2}(f_{2j-1} - \mathrm{i} f_{2j}) \end{aligned} \tag{18.67}$$

Now we must have a transformation of the form

$$\begin{aligned} \bar{b}_i &= A_{ji}\bar{a}_j + C_{ji}a_j \\ b_i &= B_{ji}\bar{a}_j + D_{ji}a_j \qquad\qquad 1 \le i,j \le n \end{aligned} \tag{18.68}$$

Observation: For a general transformation of the form (18.68), with complex $n \times n$ matrices A, B, C, D the fermionic CCR's are preserved iff the matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{18.69}$$

satisfies:

$$g^{tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18.70}$$

That is

$$\begin{aligned} A^{tr}D + C^{tr}B &= 1 \\ A^{tr}C &= -(A^{tr}C)^{tr} \\ D^{tr}B &= -(D^{tr}B)^{tr} \end{aligned} \tag{18.71}$$

Proof: The proof is a straightforward computation. ♢

The proposition characterizes the general matrices which preserve the CCR's. We recognize (18.70) as the definition of the complex orthogonal group for the quadratic form

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18.72}$$

So, Bogoliubov transformations can be effected by complex orthogonal transformations

$$O(q;\mathbb{C}) := \{g \in GL(2n;\mathbb{C}) | g^{tr} q g = q\} \tag{18.73}$$

The form q has signature (n,n) over the real numbers but is, of course, equivalent to the standard Euclidean form over the complex numbers. Indeed, the transformation $\sqrt{2}u$ transforms between them thanks to equations (18.45) et. seq. above.

The change of complex structure does not induce the most general Bogoliubov transformation. From (7.23) we know that there must exist an orthogonal matrix $R \in O(2n)$ in the compact orthogonal group such that

$$f_\alpha = R_{\beta\alpha} e_\beta \tag{18.74}$$

and

$$I_2 = R I_1 R^{tr} \tag{18.75}$$

Using the change of basis (18.45) we check that (with $\mathfrak{b}_\alpha$ defined analogously to (18.44)):

$$\begin{aligned} \mathfrak{b}_\alpha &= u_{\beta\alpha} f_\beta \\ &= u_{\beta\alpha} R_{\gamma\beta} e_\gamma \\ &= (u^{-1} R u)_{\delta\alpha} \mathfrak{a}_\delta \end{aligned} \tag{18.76}$$

and hence

$$R = u g u^{-1} \tag{18.77}$$ eq:R-to-g

Remarks:

1. There are some elementary matrix multiplications it would be well to record here. Note that the transformation of the matrix g related to R via (18.77) indeed produces a complex orthogonal matrix:

   $$R \in O(2n;\mathbb{C}) := \{R \in GL(2n;\mathbb{C}) | R^{tr} R = 1\}. \tag{18.78}$$

   so

   $$O(q;\mathbb{C}) = u^{-1} O(2n;\mathbb{C}) u \tag{18.79}$$

   If $R \in O(2n;\mathbb{C})$ is written in block form as

   $$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{18.80}$$

   then

   $$g = u^{-1} R u \tag{18.81}$$

has

$$\begin{aligned} A &= \frac{1}{2}(\alpha+\delta+\mathrm{i}(\beta-\gamma)) \\ D &= \frac{1}{2}(\alpha+\delta-\mathrm{i}(\beta-\gamma)) \\ B &= \frac{1}{2}(\alpha-\delta-\mathrm{i}(\beta+\gamma)) \\ C &= \frac{1}{2}(\alpha-\delta+\mathrm{i}(\beta+\gamma)) \end{aligned} \tag{18.82}$$

eq:R-to-g-p

It follows that the image of $O(2n) := O(2n;\mathbb{R})$ in $O(q;\mathbb{C})$ consists precisely of those matrices $g \in O(q;\mathbb{C})$ such that $A = D^*$ and $B = C^*$. We claim that this is precisely the intersection $U(2n) \cap O(q;\mathbb{C})$. To prove this observe that if $g \in U(2n)$ has block form (eq:OnnBlockForm 18.69) then

$$g^{-1} = \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \tag{18.83}$$

eq:OqOnn-1

On the other hand if $g \in O(q;\mathbb{C})$ then

$$g^{-1} = q g^{tr} q = \begin{pmatrix} D^{tr} & B^{tr} \\ C^{tr} & A^{tr} \end{pmatrix} \tag{18.84}$$

eq:OqOnn-2

The inverse of g is unique. Therefore, if $g \in U(2n) \cap O(q;\mathbb{C})$, then (eq:OqOnn-1 18.83) equals (eq:OqOnn-2 18.84), but this is true iff $A = D^*$ and $B = C^*$, which is precisely the defining relation of $u^{-1}O(2n;\mathbb{R})u$.

2. In particular, note that a Bogoliubov transformation preserves the $*$-automorphism $(a_i)^* = \overline{a_i}$ iff $g \in U(2n) \cap O(q;\mathbb{C})$. A general Bogoliubov transformation would change the hermitian structure on the Fock space.

Exercise

Show that if $g \in O(q;\mathbb{C})$ is written in block form (eq:OnnBlockForm 18.69) then it is also true that

$$\begin{aligned} AD^{tr} + BC^{tr} &= 1 \\ BA^{tr} &= -(BA^{tr})^{tr} \\ CD^{tr} &= -(CD^{tr})^{tr} \end{aligned} \tag{18.85}$$

eq:Onn-Block-Rels

and that (eq:Onn-Block-Rels 18.71) is true iff (eq:Onn-Block-Rels-alt 18.85) is true.

Exercise

Consider the case of $n = 1$.

a.) Show that $O(q;\mathbb{C})$ has two components, given by matrices of the form

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \lambda \in \mathbb{C}^* \tag{18.86}$$ eq:cco2-1

or

$$g = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} \qquad \lambda \in \mathbb{C}^* \tag{18.87}$$ eq:cco2-2

b.) Show that under the transformation $R = SgS^{-1}$ with $R \in O(2)$, those Bogoliubov transformations arising from orthogonal transformations in $O(2)$ connected to the identity correspond to (18.86) while those in the nonidentity component correspond to (18.86) with $\lambda = e^{\mathrm{i}\theta}$.

Exercise

Given a complex structure I there is always a second complex structure $-I$. Of course, since I has a stabilizer, there are many transformations $R \in O(2n)$ which conjugate I to $-I$, but since exchanging I and $-I$ exchanges the projection operators $P_\pm = \frac{1}{2}(1 \pm I \otimes \mathrm{i})$ the most natural transformation is

$$\begin{aligned} \bar{b}_j &= a_j \\ b_j &= \bar{a}_j \end{aligned} \tag{18.88}$$ eq:I-flip

If we exchange I for $-I$ then we exchange W for $\bar{W}$ so the corresponding Fock space is

$$\mathcal{H}_{F,\bar{W}} \cong \Lambda^* \bar{W} \tag{18.89}$$

Interpret the unitary transformation from $\mathcal{H}_{F,W}$ to $\mathcal{H}_{F,\bar{W}}$ in terms of the Hodge $*$ operation.

♣Important. Spell this out more. ♣

18.4.4 The spin representation and $U(n)$ representations

As we stressed at the end of Section §18.3, the Fock space constructure provides us with a Hilbert bundle of fermionic Fock spaces over the base manifold $O(2n)/U(n)$. Now $O(2n)$ acts on the base manifold and hence a natural question is whether this action "lifts" to the bundle $\mathcal{H}_F$ so that it is an "equivariant bundle." Let us define this term.

Suppose $\pi : E \to X$ is a general fiber bundle over a topological space X and that a group G acts on X, say, as a left-action. So we write $\psi_g(x) = g \cdot x$ for the action of $g \in G$ on $x \in X$. We say that E is an *equivariant bundle* if there exists a lift of this action to E. That is, for each $g \in G$ there should be a bundle map so that $\tilde{\psi}_g$ fits into the commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\tilde{\psi}_g"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\ X \arrow[r, "\psi_g"] & X \end{tikzcd} \tag{18.90}$$

In more concrete terms, for each x there should be a map $\tilde{\psi}_{g,x} : E_x \to E_{g\cdot x}$ so that the maps compose according to the group law of G:

$$\tilde{\psi}_{g_1,g_2\cdot x}\tilde{\psi}_{g_2,x} = \tilde{\psi}_{g_1g_2,x} \tag{18.91}$$

The statement "$\tilde{\psi}_g$ is a bundle map" means that it must preserve the structures on the fiber. Thus, if E is a vector bundle then $\tilde{\psi}_g$ must be linear on the fibers. Note that this means that the isotropy group $H_x \subset G$ at a point x must act on the fiber at x.

In our case we will now show that for an $O(2n)$ action on CmptCplxStr$(\mathcal{M}, Q)$ the stabilizer group is isomorphic to $U(n)$ but this $U(n)$ group does *not* act compatibly on $\mathcal{H}_{F,W}$. Rather a double cover acts.

Let us choose a complex structure I and a compatible basis (eq:FermOsc-1 18.33). Order the basis $\mathfrak{e}_\alpha$ given in (eq:e-ord-alt 18.34) above. Now, as we discussed in Section §subsec:CplxStrRealVS 7.1 (see equations (eq:Stab-IO 7.18), (eq:Stab-IO-p 7.19)) the stabilizer of a complex structure is a subgroup of $O(2n)$ isomorphic to $U(n)$. In particular, we have the embedding $\iota : U(n) \to SO(2n)$ defined on $u \in U(n)$ by writing it in terms of its real and imaginary parts $u = \alpha + \mathrm{i}\beta$ and defining

$$\iota(u) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SO(2n) \tag{18.92}$$

The unitary subgroup can be thought of as follows. The Bogoliubov group $O(q;\mathbb{C})$ acts on the vector space of oscillators V. The most general Bogoliubov transformation which stabilizes the decomposition $V = W \oplus \bar{W}$ expresses the $\bar{b}_i$ as linear combinations of $\bar{a}_i$ and b_i as linear combinations of a_i. In other words $C_{ji} = 0$ and $B_{ji} = 0$, and therefore $A = D^{tr,-1} \in GL(n,\mathbb{C})$. Those transformations which preserve the $*$ structure have $A = D^*$ and therefore we are lead to the unitary group $A \in U(n)$.

The $U(n)$ group of Bogoliubov transformations that stabilizes $\mathcal{H}_{F,W}$ in fact commutes with the fermion number operator $\mathcal{F}$. We might therefore expect that there is a well-defined action of $U(n)$ on the k-particle subspaces $\Lambda^k W$. Indeed, acting on the oscillators, W is just the defining representation of $U(n)$ and hence there is a very natural action of $U(n)$ on $\Lambda^k W$. But is this action actually compatible with the Spin-representation?

We have to be a little careful here. The stabilizer group $U(n)$ discussed here is a subgroup

$$U(n) \to SO(2n) \tag{18.93}$$

acting on the space of oscillators - the vector representation of Spin$(2n)$. But $SO(2n)$ only acts *projectively* on $\mathcal{H}_{F,W}$. Indeed the spin double-cover is nontrivial, the sequence given by $\widetilde{\mathrm{Ad}}$ does not split, and $-1 \in \mathrm{Spin}(2n)$ acts nontrivially on the spin representation. It is therefore not *a priori* obvious that we can make $U(n)$ act on the spin representation in a way compatible with its action on V in the vector representation. That is, we want an action $\rho(u)$ of $U(n)$ on $\mathcal{H}_{F,W}$ so that such that

$$\rho(u)\left(\rho_{F,W}(v)\psi\right) \stackrel{?}{=} \rho_{F,W}(\iota(u)\cdot v)\rho(u)\psi \tag{18.94}$$ eq:compat-condit

for $u \in U(n)$, $v \in V \subset \mathrm{Cliff}(V,Q)$ and $\psi \in \mathcal{H}_{F,W}$. Here $\iota(u)\cdot v = \widetilde{\mathrm{Ad}}(\iota(u))v$. (Although there are two lifts of $\iota(u)$ to Spin$(2n)$ they differ by sign so $\widetilde{\mathrm{Ad}}(\iota(u))$ is well-defined.) A less-formal way of stating (eq:compat-condit 18.94) is that we want to say how $g \in U(n)$ acts on $a^\dagger_{i_1} \cdots a^\dagger_{i_k}|0\rangle$ and we should surely have

$$g \cdot a^\dagger_{i_1} \cdots a^\dagger_{i_k}|0\rangle = (g a^\dagger_{i_1} g^{-1}) \cdots (g a^\dagger_{i_k} g^{-1}) g|0\rangle \tag{18.95}$$

where $g a^\dagger g^{-1}$ is determined by the vector representation of $O(2n)$.

Since $U(n)$ is a "small" subgroup of $SO(2n)$ one might think that we do not need to worry, but in fact we do! The subtlety is typical of some of the tricky points one encounters when working with spin groups, so we will describe the problem and its resolution in some detail.

Since $\mathrm{Spin}(2n) \to SO(2n)$ is a double-covering the problem is going to be a sign problem in defining $\rho(u)$ on $\mathcal{H}_{F,W}$. It therefore suffices to examine the Cartan subgroups. Every element can be conjugated into the Cartan subgroup, and a sign ambiguity in the conjugating operators will cancel out.

Let us start with the $U(1)$ subgroup of matrices in $U(n)$ proportional to the identity . From the embedding (eq:Stab-IO 7.18) we see that, in the ordered basis $\{e_1, e_2, \dots, e_{2n}\}$ this embeds $e^{\mathrm{i}\theta} 1_n \in U(n)$ as the $SO(2n)$ matrix:

$$\iota(e^{\mathrm{i}\theta}) = R(\theta) \oplus \cdots \oplus R(\theta) \tag{18.96}$$

eq:u1-embed

The action on the oscillators is just

$$\begin{aligned} a_i &\to e^{\mathrm{i}\theta} a_i \\ \bar{a}_i &\to e^{-\mathrm{i}\theta} \bar{a}_i. \end{aligned} \tag{18.97}$$

What should be the compatible action on the Fock space $\mathcal{H}_{F,W}$? Under the twisted adjoint map $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(2n) \to SO(2n)$ (eq:u1-embed 18.96) lifts to a nonclosed loop in $\mathrm{Spin}(2n)$:

$$\widetilde{\mathrm{Ad}} : \exp[\frac{\theta}{2}(e_{12} + e_{34} + \cdots + e_{2n-1,2n})] \to R(\theta) \oplus \cdots \oplus R(\theta) \tag{18.98}$$

eq:u1-embed-lft

When n is odd this will have a sign problem: The $U(1)$ is only projectively represented on $\mathcal{H}_{F,W}$!

One way of discussing the problem is this. The Fock space is a representation of $\mathrm{Spin}(2n)$. Now we have the diagram:

$$\begin{tikzcd} & \mathrm{Spin}(2n) \arrow[d, "\widetilde{\mathrm{Ad}}"] \\ U(1) \arrow[r, "\iota"] & SO(2n) \end{tikzcd} \tag{18.99}$$

eq:lifting-prob-0

and we ask whether we can find a homomorphism f such that it can be completed to a commutative diagram

$$\begin{tikzcd} & \mathrm{Spin}(2n) \arrow[d, "\widetilde{\mathrm{Ad}}"] & ??? \\ U(1) \arrow[ur, dotted, "f"] \arrow[r, "\iota"] & SO(2n) \end{tikzcd} \tag{18.100}$$

eq:lifting-prob

Since the Spin representation is given by a homomorphism ρ from $\mathrm{Spin}(2n)$, in order to represent the $U(1)$ subgroup of $U(n)$ in a way compatible with $\widetilde{\mathrm{Ad}}$ we need to lift the homomorphism ι to f as in equation (18.100). However, what we have just observed is that no such lift exists! On the other hand, we "almost" managed to lift it: We "just" missed by a minus sign. Mathematically what we can do about this is consider the double-cover of $\pi : U(1) \to U(1)$ given by $\pi(e^{\mathrm{i}\theta}) = e^{\mathrm{i}2\theta}$ and then there is indeed a homomorphism $\tilde{\iota}$ enabling a commutative diagram:

$$\begin{tikzcd} U(1) \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & \mathrm{Spin}(2n) \arrow[d, "\widetilde{\mathrm{Ad}}"] \\ U(1) \arrow[r, "\iota"] & SO(2n) \end{tikzcd} \tag{18.101}$$

In order to extend this idea to the full group $U(n)$ we define double coverings:

$$U(n)^{\pm} := \{(u,\lambda) \in U(n) \times U(1) | \lambda^{\pm 2} = \det(u)\} \tag{18.102}$$

Then, (18.101) generalizes to

$$\begin{tikzcd} U(n)^{\pm} \arrow[r, "\iota^{\pm}"] \arrow[d, "\pi"] & \mathrm{Spin}(2n) \arrow[d, "\widetilde{\mathrm{Ad}}"] \\ U(n) \arrow[r, "\iota"] & SO(2n) \end{tikzcd} \tag{18.103}$$

and under $\rho \circ \iota^{\pm}$, where ρ is the spin representation $U(n)^{\pm}$ takes the k-particle space $\Lambda^k W$ to itself and acts as

$$\rho \circ \iota^{\pm}(u,\lambda) = \lambda^{\mp 1} \Lambda^k(u) \tag{18.104}$$

In order to prove (18.103), (18.104) we use the group $\mathrm{Spin}^c(2n)$ (see equation (17.27) above). Note that $\mathrm{Spin}(2n) \times U(1)$ acts on $\mathcal{H}_F$ with the $U(1)$ acting by scalars. Therefore $(-1,-1)$ acts trivially and we have an action of $\mathrm{Spin}^c(2n) = \mathrm{Spin}(2n) \times U(1)/\mathbb{Z}_2$ on $\mathcal{H}_F$. Now we *can* solve the lifting problem

$$\begin{tikzcd} & \mathrm{Spin}^c(2n) \arrow[d, "p"] \\ U(n) \arrow[ur, dotted, "F_{\pm}"] \arrow[r, "f_{\pm}"] & SO(2n) \times U(1) \end{tikzcd} \tag{18.105}$$

where $p(g,z) := (\widetilde{\mathrm{Ad}}(g), z^2)$ is a two-fold cover of $\mathrm{Spin}^c(2n) \to SO(2n) \times U(1)$, and $f_{\pm}(u) = (\iota(u), (\det u)^{\pm 1})$. This can be proven using abstract covering theory from algebraic topology, but in this example we can in fact give an explicit description:

As we mentioned above, we need only define the lifting on a Cartan torus in $U(n)$. Thus we take

$$u = \mathrm{Diag}\{e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}\} \tag{18.106}$$

and therefore

$$\iota(u) = R(\theta_1) \oplus R(\theta_2) \oplus \cdots \oplus R(\theta_n) \tag{18.107}$$

Then we take

$$F_{\pm}(u) := [\prod_{i=1}^{n} e^{\frac{1}{2}\theta_i e_{2i-1}e_{2i}}, e^{\pm\frac{\mathrm{i}}{2}\sum_i \theta_i}]. \tag{18.108}$$

Note that each of the angles θ_i is only defined modulo 2π, that is, we identify $\theta_i \sim \theta_i + 2\pi$. Therefore neither $e^{\frac{1}{2}\theta_i e_{2i-1}e_{2i}}$ nor $e^{\pm\frac{\mathrm{i}}{2}\sum_i \theta_i}$ is well-defined: They both change by a minus sign if we shift $\theta_i \to \theta_i + 2\pi$. However, the *pair* is well-defined in $\mathrm{Spin}^c(2n) = (\mathrm{Spin}(2n) \times U(1))/\mathbb{Z}_2$.

The main conclusion from the above discussion is that the Fock space bundle $\mathcal{H}_F \to \mathcal{G}(V,Q) \cong O(2n)/U(n)$ constructed in Section §subsec:SpinRepCplxIsotropic 18.3 is not an equivariant bundle for $O(2n)$. However, we could also consider the homogeneous space $\mathrm{CmptCplxStr}(\mathcal{M}, Q)$ to be a homogeneous space for $\mathrm{Pin}^{-}(2n)$ and we will show below that there is a lift the $\mathrm{Pin}^{-}(2n)$ action.

Lie algebra level:

It is quite interesting to see how the decomposition of the spin representation works as a representation of Lie algebras. Recall that $\mathrm{spin}(2n) \cong so(2n)$ with $\frac{1}{2}e_{\mu\nu} \cong M_{\mu\nu}$.

If we choose a complex structure I with $+i$ eigenspace of V given by W together with a compatible set of harmonic oscillators $a_i, \bar{a}_i$ then, in terms of the oscillator basis we have generators of $su(n)$ given by

$$T^i{}_j = \left(\bar{a}_i a_j - \frac{1}{n}\delta^i{}_j \mathcal{F}\right) \qquad i,j = 1,\dots,n \tag{18.109}$$

Note that $\sum_i T^i{}_i = 0$. One easily computes

$$[T^i{}_j, T^k{}_\ell] = \delta^k{}_j T^i{}_\ell - \delta^i{}_\ell T^k{}_j \tag{18.110}$$

which is a standard presentation of the Lie algebra $su(n)$ in terms of generators and structure constants. Note that this is not a real basis, rather, the general element of the $su(n)$ Lie algebra (which is a Lie algebra over $\kappa = \mathbb{R}$) is $t = \sum x_i{}^j T^i{}_j$ such that $t^* = -t$. Thus $x_i{}^i$ are pure imaginary and $(x_i{}^j)^* = -x_j{}^i$.

To complete the Lie algebra of $U(n)$, namely, $u(n) = su(n) \oplus u(1)$ we must again be careful. The generator $t \in u(1)$ of the $U(1)$ subgroup of $U(n) \subset SO(2n)$ of diagonal matrices lifts to

$$\tilde{t} = \frac{1}{2}(e_{12} + \cdots + e_{2n-1,2n}) \tag{18.111}$$

In terms of harmonic oscillators we write $e_{2j-1} = a_j + \bar{a}_j$ and $e_{2j} = \mathrm{i}(a_j - \bar{a}_j)$ so that

$$\tilde{t} = \mathrm{i}\mathcal{F} - \mathrm{i}\frac{n}{2} \tag{18.112}$$

Thus:

1. The vacuum is not invariant under the $U(1)$

2. When n is odd $\tilde{t}$ only exponentiates to give a projective representation of $U(1)$, in accord with our discussion above.

The point of this exercise is that if one is sufficiently careful with normalizations of generators one can detect topological subtleties.

Remarks:

1. One interesting implication of the above formulae is an important formula in Kähler geometry. Let K denote the determinant representation of $U(n)$. Then we consider the projective representation $K^{1/2}$. This has precisely the same cocycle as the projective $U(n)$ representation on the Fock space. These two $\mathbb{Z}_2$-valued cocycles cancel if we consider $S \otimes K^{1/2}$, which becomes a true representation of $U(n)$. Thus we have the identities of true $U(n) \subset SO(2n)$ representations:

$$S^+ \otimes K^{1/2} \cong \oplus_{k=0(2)} \Lambda^k W \cong \oplus_{k=0(2)} \Lambda^{k,0} V \tag{18.113}$$ eq:Splus-Khalf

$$S^- \otimes K^{1/2} \cong \oplus_{k=1(2)} \Lambda^k W \cong \oplus_{k=0(2)} \Lambda^{k,0} V \tag{18.114}$$ eq:Sminus-Khalf

where $W \cong \mathbb{C}^n$ is the defining representation of $U(n)$. If we exchange the complex structure I for $-I$ then we exchange W and $\bar{W}$. Then we have ♣CHECK! ♣

$$S^+ \otimes K^{-1/2} \cong \oplus_{k=0(2)} \Lambda^k \bar{W} \cong \oplus_{k=0(2)} \Lambda^{0,k} V \tag{18.115}$$ eq:Splus-Khalf

$$S^- \otimes K^{-1/2} \cong \oplus_{k=1(2)} \Lambda^k \bar{W} \cong \oplus_{k=0(2)} \Lambda^{0,k} V \tag{18.116}$$ eq:Sminus-Khalf

♣This is confusing unless you explain the role of orientations in these isomorphisms. ♣

2. The identities (eq:Splus-Khalf 18.115) and (eq:Sminus-Khalf 18.116) are very important in Kahler geometry where we can exchange Dirac operators for Dolbeault operators HitchinHarmonicSpinors [26]:

$$\not{D} \leftrightarrow \partial + \partial^\dagger \tag{18.117}$$

We will explain this a little bit by considering $M = \mathbb{R}^{2n}$ with the Euclidean metric. To define the Dirac operator we consider Cliff(T^*M). Choosing standard coordinates we can use an ON basis $e_\alpha = dx^\alpha$ and represent $\rho(e_\alpha)$ on a Dirac representation and form the Spinor bundle $\mathcal{S} = M \times S_c$. Spinor fields will be functions on M valued in S_c. We denote the space of spinor fields as the sections of the spin bundle $\Gamma(\mathcal{S})$. Then the Dirac operator $\not{D}$ is defined by the exterior derivative $d : \Gamma(\mathcal{S}) \to \Omega^1(\mathcal{S})$ followed by Clifford contraction back to $\Gamma(\mathcal{S})$. In explicit equations

$$\not{D} = \rho(e_\alpha)\frac{\partial}{\partial x^\alpha} = \Gamma^\alpha \frac{\partial}{\partial x^\alpha} \tag{18.118}$$ eq:DiracOp

Of course $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is $\mathbb{Z}_2$-graded and $\not{D}$ is odd:

$$\not{D}^\pm : \Gamma(\mathcal{S}^\pm) \to \Gamma(\mathcal{S}^\mp) \tag{18.119}$$

Now choose a complex structure I so that we can split $T^*M \otimes \mathbb{C} \cong T^{*(1,0)}M \oplus T^{*(0,1)}M$. Let us choose the complex structure **** above. Then

$$\begin{aligned} \bar{a}_j &= \frac{1}{2}(dx^{2j-1} + \mathrm{i} dx^{2j}) := \frac{1}{2} dz^j \\ a_j &= \frac{1}{2}(dx^{2j-1} - \mathrm{i} dx^{2j}) := \frac{1}{2} d\bar{z}^j \end{aligned} \tag{18.120}$$

We have introduced standard complex coordinates so

$$\begin{aligned}\frac{\partial}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} - \mathrm{i}\frac{\partial}{\partial x^{2j}}\right)\\ \frac{\partial}{\partial \bar{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + \mathrm{i}\frac{\partial}{\partial x^{2j}}\right)\end{aligned} \tag{18.121}$$

In terms of complex coordinates the Dirac operator becomes

$$D\!\!\!/ = \sum_{j=1}^{n}\left(\rho(2\bar{a}_j)\frac{\partial}{\partial z^j} + \rho(2a_j)\frac{\partial}{\partial \bar{z}^j}\right) \tag{18.122}$$

Now under the isomorphism (eq:Splus-Khalf 18.115), etc. $\rho(\bar{a}_j)$ becomes wedging with $\frac{1}{2}dz^j$ and $\rho(a_j)$ becomes contraction with $2(\frac{\partial}{\partial z^j})$, so we can identify $D\!\!\!/$ with $\partial + \partial^\dagger$.

♣Need to explain hermitian structure and † better. ♣

18.4.5 Bogoliubov transformations and the spin Lie algebra

Above we identified an action of $O(q;\mathbb{C})$ on V preserving the harmonic oscillator algebra. By considering one-parameter subgroups of matrices satisfying (eq:Onn-Block-Rels 18.71) we see that the Lie algebra of this group consists of matrices of the form

$$m = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^{tr}\end{pmatrix} \in Mat_{2n}(\mathbb{C}) \tag{18.123}$$ eq:liegeni

with β, γ antisymmetric. The α, β, γ are otherwise arbitrary complex $n \times n$ matrices. Note that m is antihermitian iff $\alpha^\dagger = -\alpha$ and $\beta^\dagger = -\gamma$. Such antihermitian matrices exponentiate to elements of $U(2n)$ and $U(2n) \cap O(q;\mathbb{C}) \cong O(2n;\mathbb{R})$.

For matrices (eq:liegeni 18.123) with $m \in o(q;\mathbb{C})$ define a corresponding element of the Clifford algebra:

$$\begin{aligned}\widetilde{m} &:= \sum_{i,j=1}^{n}\left(\alpha_{ji}\bar{a}_j a_i + \frac{1}{2}\gamma_{ij}a_i a_j + \frac{1}{2}\beta_{ij}\bar{a}_i\bar{a}_j\right)\\ &= \frac{1}{2}\sum_{i,j=1}^{n}\left(\alpha_{ji}(\bar{a}_j a_i - a_i\bar{a}_j) + \gamma_{ij}a_i a_j + \beta_{ij}\bar{a}_i\bar{a}_j\right) + \frac{1}{2}\mathrm{Tr}(\alpha)1\end{aligned} \tag{18.124}$$ eq:legn

Note that m is antihermitian, so that $\alpha^\dagger = -\alpha$ and $\beta^\dagger = -\gamma$ if and only if $\tilde{m}$ is a pure imaginary element of the $*$-algebra $\mathcal{A}$: $\tilde{m}^* = -\tilde{m}$.

We claim that the element $\widetilde{g} := \exp[\widetilde{m}]$ in the complex Clifford algebra conjugates the column vector $\mathfrak{a}_\alpha$ defined in (eq:full-osc-bas 18.44) according to the matrix $g := e^m \in O(q;\mathbb{C})$:

$$\widetilde{g}\mathfrak{a}_\alpha\widetilde{g}^{-1} = g_{\beta\alpha}\mathfrak{a}_\beta \tag{18.125}$$ eq:SNTNE

with

$$g = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \tag{18.126}$$

satisfying (eq:Onn-Block-Rels 18.71).

To prove this, use $[AB,C]=A\{B,C\}-\{A,C\}B$ to check that

$$\begin{aligned}
[\widetilde{m},\bar{a}_i] &= \alpha_{ji}\bar{a}_j+\gamma_{ji}a_j \\
[\widetilde{m},a_i] &= \beta_{ji}\bar{a}_j-\alpha_{ij}a_j
\end{aligned} \tag{18.127}$$

In other words, if we define a vector $\mathfrak{a}_\alpha$ from the ordered basis (18.44) then

$$[\widetilde{m},\mathfrak{a}_\alpha]=m_{\beta\alpha}\mathfrak{a}_\beta \tag{18.128}$$

`eq:mtilde-action`

This formula exponentiates to give (18.125).

The matrices defined in (18.123) span the Lie algebra $o(q;\mathbb{C})$. This does not imply that the corresponding elements $\widetilde{m}$ generate an isomorphic Lie algebra! That is, the map $o(q;\mathbb{C})\to \mathrm{Cliff}(\mathcal{M};Q)\otimes\mathbb{C}$, need not be a Lie algebra homomorphism. The origin of the problem is that the relations (18.128) would also be satisfied if we shifted $\widetilde{m}$ by any scalar. Indeed, given m the general element of the Clifford algebra satisfying (18.128) is of the form $\tilde{m}$ plus a scalar.

One can compute the commutator $[\tilde{m}_1,\tilde{m}_2]$ using the relation

$$[AB,CD]=A\{B,C\}D-\{A,C\}BD+CA\{B,D\}-C\{A,D\}B \tag{18.129}$$

and a small computation shows that in fact

$$[\widetilde{m_1},\widetilde{m_2}]=\widetilde{[m_1,m_2]}-\frac{1}{2}\mathrm{Tr}(\beta_1\gamma_2-\beta_2\gamma_1)1 \tag{18.130}$$

`eq:centrl`

The term proportional to 1 is easily computed from the VEV $\langle 0|\cdots|0\rangle$ of the LHS and the RHS. The expression $\omega(m_1,m_2):=\mathrm{Tr}(\beta_1\gamma_2-\beta_2\gamma_1)$ is a two-cocycle on the Lie algebra $so(2n)$. It follows that the elements $\tilde{m}$ of $\mathcal{A}$ do not close to form a Lie sub-algebra of $\mathcal{A}$, but rather they generate a Lie algebra $\mathfrak{g}$ which fits in a central extension of Lie algebras:

$$0\to\mathbb{C}\to\mathfrak{g}\to o(q;\mathbb{C})\to 0 \tag{18.131}$$

`eq:centrls`

See Appendix B below for a very brief précis of the relation of Lie algebra cohomology to central extensions. As explained there, the extension is only nontrivial if the cocycle is nontrivial. In fact, in the present case the cocycle can be trivialized! To see this note that in the block decomposition (18.123) we have

$$([m_1,m_2])_{11}=[\alpha_1,\alpha_2]+\beta_1\gamma_2-\beta_2\gamma_1 \tag{18.132}$$

and therefore, the linear functional $f(m):=\frac{1}{2}\mathrm{Tr}(\alpha)$ trivializes ω, i.e. $\omega=df$, where d is the Chevalley-Eilenberg differential. In particular, if we define

$$\widehat{m}:=\tilde{m}-\frac{1}{2}\mathrm{Tr}(\alpha)\cdot 1=\frac{1}{2}\sum_{i,j=1}^{n}\left(\alpha_{ji}(\bar{a}_ja_i-a_i\bar{a}_j)+\gamma_{ij}a_ia_j+\beta_{ij}\bar{a}_i\bar{a}_j\right) \tag{18.133}$$

then we can compute

$$\begin{aligned}
[\widehat{m_1},\widehat{m_2}] &= [\widetilde{m_1}-\frac{1}{2}\mathrm{Tr}(\alpha_1)\cdot 1,\widetilde{m_2}-\frac{1}{2}\mathrm{Tr}(\alpha_2)\cdot 1] \\
&= \widetilde{[m_1,m_2]}-\frac{1}{2}\mathrm{Tr}(\beta_1\gamma_2-\beta_2\gamma_1)1 \\
&= \widehat{[m_1,m_2]}
\end{aligned} \tag{18.134}$$

and therefore $m \mapsto \widehat{m}$ is a homomorphism of Lie algebras.

Note that if we express $\widehat{m}$ in terms of e_i using (eq:FermOsc-2 18.36) then we obtain an element of $\mathrm{spin}(2n) \otimes \mathbb{C}$, which becomes an element of $\mathrm{spin}(2n)$ when we impose the condition $m^* = -m$. It follows that

$$\mathfrak{g} \cong \mathrm{spin}(2n) \otimes \mathbb{C} \oplus \mathbb{C} \tag{18.135}$$

At the group level we know from (eq:pinone 17.22) that $\Gamma(t,s) \cong \mathrm{Pin}(t,s) \times \mathbb{R}_+$ and hence we can identify $\mathfrak{g}$ with the Lie algebra of the complexified Clifford group $\Gamma_c(d)$.

Corresponding to the central extension of Lie algebras there is a central extension of groups:

$$1 \to \mathbb{C}^* \to \Gamma_c(2n) \to O(q;\mathbb{C}) \to 1 \tag{18.136}$$

eq:csntne

In particular, we have the group multiplication:

$$\widetilde{g_1}\widetilde{g_2} = c(g_1,g_2)\widetilde{g_1 g_2} \tag{18.137}$$

where $c(g_1,g_2)$ is a group cocycle related to ω. Locally, the extension splits, thanks to the splitting of the Lie algebras, but the extension does not split at the group level, ultimately because the cover $\mathrm{Spin}(2n) \to O(2n)$ is nontrivial.

:FockSpinEquiv

18.4.6 The Fock space bundle as a $\mathrm{Spin}(2n)$-equivariant bundle

Let us now return to the question of lifting the $\mathrm{Spin}(2n)$ action on $\mathrm{CmptCplxStr}(\mathcal{M},Q) \cong \mathcal{G}(V,Q)$ to the bundle $\mathcal{H}_F$. We summarize the situation so far. We have described a bundle of Fock spaces

$$\mathcal{H}_F \to \mathrm{CmptCplxStr}(\mathcal{M},Q) \cong O(2n)/U(n) \tag{18.138}$$

where the isomorphism is obtained by choosing a complex structure I on $\mathcal{M}$, or, equivalently

$$\mathcal{H}_F \to \mathcal{G}(V,Q) \cong O(q;\mathbb{C})/\mathcal{LD} \tag{18.139}$$

♣$\mathcal{LD}$ introduced below. Description of $\mathcal{G}(V,Q)$ as a homogeneous space should be in a previous section. ♣

where the isomorphism is obtained by choosing a maximal isotropic subspace $\bar{W}$ in V. The two fibrations are related by identifying $\bar{W}$ with the $I = -\mathrm{i}$ eigenspace in V. We have seen that neither $O(2n)$ nor $O(q;\mathbb{C})$ lifts to define an equivariant structure on $\mathcal{H}_F$. We will show that rather, the spin double covers do lift.

The key will be to understand how Bogoliubov transformations change the vacuum line. Suppose we make one choice of harmonic oscillators $\{\bar{a}_i, a_i\}$ with a corresponding vacuum line generated, say, by the state $|0\rangle_W$. Thus $a_i|0\rangle_W = 0$. Now consider a Bogoliubov transformation generated by $g = \exp(m) \in O(q;\mathbb{C})$. If the Bogoliubov transformation is implemented by the matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{18.140}$$

then

$$e^{\widetilde{m}}|0\rangle_W \tag{18.141}$$

will generate the vacuum line relative to the new oscillators $\{\mathfrak{b}_\alpha\} = \{\bar{b}_i, b_i\}$ defined in (eq:BogT-3 18.68).

Let us now try to write the new vacuum state more explicitly in terms of the operators $\mathfrak{a}_\alpha$ and $|0\rangle_W$. *If D is invertible* then we can write

$$\begin{aligned} g &= \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix} \\ &= \exp\left[\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\right] \exp[\begin{pmatrix} x & 0 \\ 0 & -x^{tr} \end{pmatrix} \exp\left[\begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}\right] \end{aligned} \tag{18.142}$$

eq:GaussDecomp-1

where

$$\begin{aligned} S &= BD^{-1} \\ R &= D^{-1}C \\ A' &= A - BD^{-1}C \\ e^x &= A' \\ e^{-x^{tr}} &= D \end{aligned} \tag{18.143}$$

eq:GaussDecomp-2

Note that by the defining relations of $O(q;\mathbb{C})$ the matrices R, S are antisymmetric. Therefore we have

$$e^{\widetilde{m}} = \kappa e^{\frac{1}{2}\sum_{i,j} S_{ij}\bar{a}_i\bar{a}_j} e^{\sum_{i,j=1}^n x_{ji}\bar{a}_j a_i} e^{\frac{1}{2}\sum_{i,j} R_{ij}a_i a_j} \tag{18.144}$$

where κ is a nonzero scalar and therefore the new vacuum line is spanned by

$$e^{\widetilde{m}}|0\rangle = \kappa e^{\frac{1}{2}\sum_{i,j} S_{ij}\bar{a}_i\bar{a}_j}|0\rangle \tag{18.145}$$

This is the fermionic analog of a *squeezed state.*

More precisely, for a complex anti-symmetric $n \times n$ matrix S define the squeezed state

$$|S\rangle_W := \rho_{F,W}\left(e^{\frac{1}{2}\sum_{i,j} S_{ij}\bar{a}_i\bar{a}_j}\right)|0\rangle_W \tag{18.146}$$

eq:F-SqzdSt

Now consider $\tilde{g}$ which is a lift of

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{18.147}$$

eq:BlockForm-iii

We want to compute $\rho_{F,W}(\tilde{g})|S\rangle$. Now using (eq:GaussDecomp-1 18.142) and (eq:GaussDecomp-2 18.143) we know that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & AS + B \\ C & CS + D \end{pmatrix} = \begin{pmatrix} 1 & g \cdot S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \tag{18.148}$$

eq:g-on-ODL

where

$$g \cdot S := (AS + B)(CS + D)^{-1} \tag{18.149}$$

eq:Def-gdotS

follows immediately from the general formulae of (eq:GaussDecomp-2 18.143). Let $\mathcal{LD} \subset O(q;\mathbb{C})$ be the group of block-lower-diagonal matrices. That is, those with $B = 0$. Then S is a coordinate in a dense open set of the homogeneous space $O(q;\mathbb{C})/\mathcal{LD}$. Because there is a group action from the left we know that

$$g_1 \cdot (g_2 \cdot S) = (g_1 g_2) \cdot S \tag{18.150}$$

provided the relevant matrices are invertible so that the formula makes sense.

Therefore, thanks to (eq:Def-gdotS 18.149) we know that

$$\rho_{F,W}(\tilde{g})|S\rangle = \kappa(\tilde{g},S)|g\cdot S\rangle_W \tag{18.151}$$

eq:ActOnVac

where $\kappa(\tilde{g},S)$ is a scalar, at least for those transformations such that $\det(CS+D)\neq 0$.

Once we know how the vacuum transforms, that is, once we have computed $\kappa(\tilde{g},S)$ in (eq:ActOnVac 18.151) we can lift the transformation $\tilde{g}$ on $\mathcal{G}(V,Q)$ to the entire Fock bundle since the other states are obtained by acting with oscillators, and these just transform in the vector representation. (See (eq:compat-condit 18.94) above.)

In equations (eq:compt-kappa-1 18.189)- (eq:compt-kappa-4 18.193) below we will see how to compute $\kappa(\tilde{g},S)$ once we have a good formula for the overlaps $\langle S_1|S_2\rangle$. Therefore, in order to compute $\kappa(\tilde{g},S)$ we need a key identity for the the overlaps of squeezed states. One way of stating the identity is: [45]

$$\boxed{\langle 0|\exp\left(-\frac{1}{2}\sum_{ij}S_{ij}a_ia_j\right)\exp\left(\frac{1}{2}\sum_{ij}T_{ij}\bar{a}_i\bar{a}_j\right)|0\rangle = \sqrt{\det(1-ST)} := \mathrm{pf}(1-ST)} \tag{18.152}$$

eq:Overlap-1

♣ would be good to give a proof based on on fermionic quantum mechanics path integrals, reducing this to a finite dimensional fermionic integral. ♣

Remarks

1. Note that we could replace the operator in the vev with the group commutator, so that this is an exponentiated version of the identity (eq:centrl 18.130).

2. The notation $\mathrm{pf}(1-ST)$ is something of an abuse because neither 1 nor ST nor the difference $1-ST$ is an antisymmetric matrix. The notation is just meant to denote the canonical squareroot of $\det(1-ST)$ when S,T are antisymmetric.

3. Note that the bra- dual to the ket $|S\rangle$ is

$$\langle S| = {}_W\langle 0|\rho_{F,W}(e^{-\frac{1}{2}\sum_{i,j}\bar{S}_{ij}a_ia_j}) \tag{18.153}$$

So in math notation we would write instead:

$$h(e^{\frac{1}{2}S_{ij}\bar{a}_i\bar{a}_j}, e^{\frac{1}{2}T_{ij}\bar{a}_i\bar{a}_j}) = \mathrm{pf}(1-\bar{S}T) \tag{18.154}$$

eq:Overlap-2

We need to define $\mathrm{pf}(1-ST)$ and prove these formulae. Here we will follow the very elegant discussion of Pressley and Segal (PS-LoopGroups) [35] In order to do this we need the notion of the Pfaffian of an antisymmetric matrix.

Reminder: *Properties of Pfaffians*

[45] There is an easier proof based on fermionic Gaussian integrals and the fermionic coherent state representation. See Section (subsec:BosonicSqueezed) 21.6 below.

1. If S_{ij} is an $N \times N$ antisymmetric matrix then $\mathrm{pf}(S) := 0$ for N odd, and if $N = 2n$ is even the Pfaffian of S, denoted $\mathrm{pf}(S)$ is defined by

$$\begin{aligned} \mathrm{pf}(S) &:= \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) S_{\sigma(1)\sigma(2)} \cdots S_{\sigma(2n-1),\sigma(2n)} \\ &= S_{12} S_{34} \cdot S_{2n-1,2n} \pm \cdots \end{aligned} \tag{18.155}$$

   In particular, if S is skew-diagonal

$$S = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \tag{18.156}$$

   then

$$\mathrm{pf}(S) = \prod_{i=1}^{n} \lambda_i \tag{18.157}$$

2. Some important properties are the following (See UsersManual [34] Sections 23.12 and 24.4 for more details):

3. $\mathrm{pf}(S)^2 = \mathrm{det} S$

4. $\mathrm{pf}(RSR^{tr}) = \det(R)\mathrm{pf}(S)$ ♣Does R have to be invertible? ♣

5. If we define the 2-form $\omega_S := \frac{1}{2} S_{ij} \bar{a}_i \bar{a}_j \in \Lambda^2 W$ then

$$\frac{\omega_S^n}{n!} = \mathrm{pf}(S) \bar{a}_1 \cdots \bar{a}_n \tag{18.158}$$

Now consider expanding the exponential to get:

$$\exp\left(\frac{1}{2} \sum_{ij} S_{ij} \bar{a}_i \bar{a}_j \right) |0\rangle = |0\rangle + \sum_I \mathrm{pf}(S_I) |I\rangle \tag{18.159}$$

Here we denote an ordered multi-index by $I = \{i_1 < i_2 < \cdots i_{2\ell}\}$. We need only consider multi-indices of even length $|I| = 2\ell$. We denote

$$|I\rangle := \bar{a}_{i_1} \cdots \bar{a}_{i_{2\ell}} |0\rangle \tag{18.160}$$

Also, given an multi-index we let S_I be the $(2\ell) \times (2\ell)$ antisymmetric matrix obtained by retaining the rows and columns enumerated by the elements of I. For example, if $I = \{1, 3\}$ and S and $n > 1$ then

$$S_I = \begin{pmatrix} 0 & a_{13} \\ -a_{13} & 0 \end{pmatrix} \tag{18.161}$$

Now let us consider the inner product of such fermionic squeezed states. The Dirac conjugate of $|I\rangle$ is the linear functional:

$$\langle I| = \langle 0| a_{i_{2\ell}} \cdots a_{i_1} \tag{18.162}$$

Note that

$$\langle I|J\rangle = \delta_{I,J} \tag{18.163}$$

It thus follows that

$$\langle 0|\exp\left(-\frac{1}{2}\sum_{ij} S_{ij}a_i a_j\right)\exp\left(\frac{1}{2}\sum_{ij} T_{ij}\bar{a}_i\bar{a}_j\right)|0\rangle = 1+\sum_I \mathrm{pf}(S_I)\mathrm{pf}(T_I) \tag{18.164}$$

Note well that this is a polynomial in the matrix elements of S and T.

Now, suppose that S is any invertible $N\times N$ matrix and T is any $N\times N$ matrix. Then, we certainly have:

$$\det(1-ST) = \det S\det(S^{-1}-T) \tag{18.165}$$

If S,T are anti-symmetric and S is invertible then it is therefore sensible to *define*

$$\mathrm{pf}(1-ST) := \mathrm{pf}(S)\mathrm{pf}(S^{-1}-T) \tag{18.166}$$

eq:pf1ST-def

The RHS is a rational expression in the matrix elements S_{ij} but since its square is a polynomial it must in fact be a polynomial.

Now, for a multi-index I let I' be the ordered complementary multi-index. So, for example, if $2n=6$ and $I=\{1<4\}$ then $I'=\{2<3<5<6\}$. Let ε_I be the sign of the permutation

$$\{1,2,\dots,2n\}\to\{I,I'\} = \{i_1,\dots,i_{2k},i'_1,\dots,i'_{2k'}\} \tag{18.167}$$

where $k+k'=n$. Then we claim that if R and T are antisymmetric matrices then

$$\mathrm{pf}(R+T) = \sum_I \varepsilon_I \mathrm{pf}(R_I)\mathrm{pf}(T_{I'}) \tag{18.168}$$

In this sum we have included $I=\emptyset$, with the definition $\mathrm{pf}(R_\emptyset):=1$, and similarly for I'. The same convention will be used in similar sums below.

The proof is to consider the associated 2-forms ω_R and ω_T and then expand

$$\begin{aligned}\frac{1}{n!}(\omega_R+\omega_T)^n &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\omega_R^k\omega_T^{n-k}\\ &= \sum_k \frac{1}{k!}\omega_R^k\frac{1}{(n-k)!}\omega_T^{n-k}\\ &= \sum_k\sum_{|I|=k}\mathrm{pf}(R_I)\mathrm{pf}(T_{I'})\bar{a}_I\bar{a}_{I'}\\ &= \left(\sum_I \varepsilon_I\mathrm{pf}(R_I)\mathrm{pf}(T_{I'})\right)\bar{a}_1\cdots\bar{a}_{2n}\end{aligned} \tag{18.169}$$

Applying this to (18.166) we find

$$\mathrm{pf}(1-ST) = \sum_I \varepsilon_I(-1)^{|I'|/2}\mathrm{pf}(S)\mathrm{pf}((S^{-1})_I)\mathrm{pf}(T_{I'}) \tag{18.170}$$

eq:Pf-Almt

so now we must simplify $\mathrm{pf}(S)\mathrm{pf}((S^{-1})_I)$.

Now we use a curious identity from linear algebra

$$\mathrm{pf}(S)\mathrm{pf}((S^{-1})_{I'}) = (-1)^{|I|/2}\varepsilon_I \mathrm{pf}(S_I) \tag{18.171}$$ eq:JDI-PF

The proof is discussed in the remark below.

Now, finally, we apply (18.171) to (18.170) to get

$$\mathrm{pf}(1-ST) = \sum_I \mathrm{pf}(S_I)\mathrm{pf}(T_I) \tag{18.172}$$

This completes the proof of (18.152). $\diamondsuit$

Remarks: Here we give an extended commentary on the identity (18.171).

1. An identity from linear algebra known as *Jacobi's determinantal identity* (see, e.g. eq. 11 of [13]) states that, for *any* matrix S

$$\det S \det((S^{-1})_{I'}) = \det(S_I) \tag{18.173}$$ eq:JacobiDetIdent

   with the sign as above, provided I and I' inherit their ordering from $\{1, \dots, n\}$.

2. One proof of (18.173) makes use of Grassmann integrals. Consider the integral

$$\int \prod_{i=1}^n d\theta_i d\chi_i \prod_{a\in I'}^n d\eta_a d\nu_a \exp[\chi_i S_{ij}\theta_j + \eta_a\chi_a + \nu_a\theta_a] \tag{18.174}$$

   Integrating out first (θ,χ) and then (η,ν) gives $\det S \det(S^{-1})_{I'}$. On the other hand, integrating out first (η,ν) and then (θ,χ) gives $\det S_I$. This elegant proof was pointed out to me by N. Arkani-Hamed.

♣Fix signs. Then do it for real fermions to get the result for Pfaffians. ♣

3. Now, if in addition S is antisymmetric it follows from (18.173) that

$$\mathrm{pf}(S)\mathrm{pf}((S^{-1})_{I'}) = \pm \mathrm{pf}(S_I) \tag{18.175}$$

   The sign cannot depend on the matrix elements s_{ij} since the LHS and RHS are rational expressions in the matrix elements, but it can depend on I and I'. To check the sign, evaluate left and right hand sides for the case that S is skew diagonal.

4. There is a nice conceptual interpretation of the identity (18.173). Suppose V is a vector space over κ of dimension n. Then we claim that there is a *canonical* isomorphism

$$c : \Lambda^n V \otimes \left(\Lambda^{n-k}V^\vee\right) \to \Lambda^k V \tag{18.176}$$

   it is defined by

$$(v_1 \wedge \cdots \wedge v_n) \otimes (\alpha_1 \wedge \cdots \wedge \alpha_{n-k}) \mapsto \sum_I \varepsilon_I \langle v_{I'}, \alpha\rangle v_I \tag{18.177}$$

where

$$\langle v_{I'}, \alpha \rangle := \det\langle v_{i_s}, \alpha_j \rangle \tag{18.178}$$

Now, let $\{u_i\}$ be an ordered basis for V and $\{\check{u}_i\}$ the dual basis. Choose another basis $w_i = S_{ji}u_j$ so that $\check{w}_i = (S^{tr,-1})_{ji}\check{w}_j$. Then, by naturalness we must have

$$c : (w_1 \wedge \cdots \wedge w_n) \otimes \check{w}_{I'} \mapsto w_I \tag{18.179}$$

as well as

$$c : (u_1 \wedge \cdots \wedge u_n) \otimes \check{u}_{I'} \mapsto u_I \tag{18.180}$$

That is the meaning of naturalness: We have the same formula in any basis. On the other hand, expressing the w's in terms of the u's we are lead to the identity (eq:JacobiDetIdent 18.173).

The next preliminary we need is the observation that if $g \in O(q;\mathbb{C})$ is in block form (eq:OnnBlockForm 18.69) and S is antisymmetric then

$$\det(CS + D) \tag{18.181}$$

is a *square* of a polynomial in the matrix elements of S. This follows from our key identity (eq:Overlap-1 18.152) above, for if D is invertible then by the defining relations (eq:Onn-Block-Rels-alt 18.85) of $O(q;\mathbb{C})$ we know that $D^{-1}C$ is an antisymmetric matrix but then we can write

$$\det(CS + D) = \det(D)\det\left(1 + (D^{-1}C)S\right) \tag{18.182}$$

and now the RHS has a squareroot as a polynomial in the matrix elements in S. This formula can be extended to D noninvertible. ♣Need to extend to D noninvertible. ♣

Now the expression $\det(CS + D)$ has a very nice group multiplication property. If $g_1 g_2 = g_3$ and so

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \tag{18.183}$$

then we claim that

$$C_3 S + D_3 = (C_1(g_2 \cdot S) + D_1)(C_2 S + D_2) \tag{18.184}$$ eq:Oq-grp-22

and hence

$$\det(C_3 S + D_3) = \det(C_1(g_2 \cdot S) + D_1)\det(C_2 S + D_2) \tag{18.185}$$ eq:Oq-grp-cocyc

The proof of (eq:Oq-grp-22 18.184) is sketched in an exercise below.

The fact that $\det(CS+D)$ admits a square root as a polynomial in S allows us to give a convenient formulation of the Spin group: [46]

Proposition: Consider the group of pairs (g, f) where $g \in O(q;\mathbb{C})$ and f is a polynomial function on $n \times n$ antisymmetric matrices S such that

$$(f(S))^2 = \det(CS + D) \tag{18.186}$$ eq:alt-def-spin

[46] This generalizes nicely to the metaplectic double cover of the symplectic group in the case of bosons.

The group multiplication is

$$(g_1, f_1)(g_2, f_2) = (g_1 g_2, f_3) \tag{18.187}$$

where (see (eq:Oq-grp-cocyc 18.185)):

$$f_3(S) = f_1(g_2 \cdot S) f_2(S) \tag{18.188}$$

Claim: This group is isomorphic to $\mathrm{Pin}^-(2n)_c$.

Proof: The group is clearly a $2:1$ covering of the group $O(q;\mathbb{C})$ since $f(S)$ is determined by g up to sign. We need only show it is nontrivial on the connected component of the identity. The proof is that the group cocycle in defining the sign of f is precisely that used to define the Pin double cover. We can demonstrate this by considering an elementary loop in the group $O(q;\mathbb{C})$. Consider, for example the loop which rotates $\{e_1, e_2\}$ by $R(\theta)$ and leaves the remaining e_i fixed. Using the transformations (eq:R-to-g-p 18.82) these transformations correspond to a loop of elements $g(\theta) \in O(q;\mathbb{C})$ with $B = C = 0$ and $D = A^{tr,-1}$ is a diagonal matrix with all diagonal entries 1 except for the 11 element which is $e^{-\mathrm{i}\theta}$. For this loop of matrices $f(S)^2 = e^{-\mathrm{i}\theta}$ and taking the squareroot of f produces the necessary cocycle. $\diamondsuit$

We are now finally ready to compute $\kappa(\tilde g, S)$ in (eq:ActOnVac 18.151). We restrict to $\tilde g$ such that $\rho_{F,W}(\tilde g)$ are unitary, so $\tilde g \in \mathrm{Spin}(2n)$ and $g \in U(2n) \cap O(q;\mathbb{C}) \cong O(2n)$. Then we have

$$\begin{aligned} {}_W\langle S_1 | S_2 \rangle_W &= {}_W\langle S_1 | \rho_{F,W}(\tilde g)^\dagger \rho_{F,W}(\tilde g) | S_2 \rangle_W \\ &= \kappa(\tilde g, S_1)^* \kappa(\tilde g, S_2) {}_W\langle g \cdot S_1 | g \cdot S_2 \rangle_W \end{aligned} \tag{18.189}$$

eq:compt-kappa-1

and hence

$$\kappa(\tilde g, S_1)^* \kappa(\tilde g, S_2) \mathrm{pf}(1 - \overline{g \cdot S_1} g \cdot S_2) = \mathrm{pf}(1 - \overline{S_1} S_2) \tag{18.190}$$

eq:compt-kappa-2

Now, using the property that $g \in U(2n)$ it is straightforward to compute:

$$\begin{aligned} \det(1 - \overline{g \cdot S_1} g \cdot S_2) &= \frac{\det\left[(\bar C \bar S_1 + \bar D)^{tr}(C S_2 + D) + (\bar A \bar S_1 + \bar B)^{tr}(A S_2 + B) \right]}{\det(\bar C \bar S_1 + \bar D)^{tr} \det(C S_2 + D)} \\ &= \det(C S_1 + D)^{*,-1} \det(C S_2 + D)^{-1} \det(1 - \overline{S_1} S_2) \end{aligned} \tag{18.191}$$

where we have just used antisymmetry of S and unitarity of g. It follows from (eq:compt-kappa-2 18.190) that

$$(\kappa(\tilde g, S_1)^* \kappa(\tilde g, S_2))^2 = (\det(C S_1 + D))^* \det(C S_2 + D) \tag{18.192}$$

eq:compt-kappa-3

and therefore

$$\kappa(\tilde g, S)^2 = \det(C S + D) \tag{18.193}$$

eq:compt-kappa-4

so $\kappa(\tilde g, S)$ is one of the two squareroots. From this, and the above characterization of the Spin group we have finally constructed $\mathcal{H}_F$ as an equivariant bundle over $\mathcal{G}(V, Q)$, at least on a dense open set.

♣Need to explain more that it is the "right" square root for the spin action. ♣

We summarize this long computation in the following beautiful statement:

Theorem: *The action of* $\tilde g \in \mathrm{Spin}(2n)$ *on squeezed states is given by*

$$\boxed{\rho_{F,W}(\tilde g) | S \rangle_W = \sqrt{\det(C S + D)} | g \cdot S \rangle_W} \tag{18.194}$$

where $g = \widetilde{\mathrm{Ad}}(\tilde{g}) \in SO(2n)$ *has block decomposition* (eq:BlockForm-iii 18.147) *when considered as an element of* $O(q;\mathbb{C})$ *and* $g\cdot S = (AS+B)(CS+D)^{-1}$. *The inverse image* $\tilde{g}$ *of* g *under* $\widetilde{\mathrm{Ad}}$ *determines the choice of the square root. That is, the fermionic vacuum transforms as an automorphic form of weight* $1/2$.

Remarks:

1. With a little algebraic geometry we can extend it to the entire isotropic Grassmannian $\mathcal{G}(V,Q)$. For details see Pressley and Segal.

2. A very similar story holds for representations of the metaplectic group by systems of free bosons. See sec:Bosons 21 below.

3. It is worth giving a more geometrical interpretation to some of these expressions which will be useful in Section §subsubsec:GeometricSpinRep 18.4.7 below. The Grassmannian $\mathcal{G}(V,Q)$ is a homogeneous space for $O(q;\mathbb{C})$. If we think of W as the span of the oscillators $\{\bar{a}_i\}$ and $\bar{W}$ as the span of $\{a_i\}$ then it acts according to (eq:BogT-3 18.68). More invariantly, given $\bar{W}$, a maximal isotropic subspace of V, we identify $\bar{W}$ with the $I = -\mathrm{i}$ subspace of a complex structure I on $\mathcal{M}$ and then define W to be the $I = +\mathrm{i}$ subspace. Then $O(q;\mathbb{C})$ transforms $W \oplus \bar{W}$ to $W' \oplus \bar{W}'$ with $W' = \mathrm{Span}\{\bar{b}_i\}$ and $\bar{W}' = \mathrm{Span}\{b_i\}$. (Warning: Since we are using general Bogoliubov transformations we are changing the $*$-structure.) Since we are focusing on the vacuum line through $|0\rangle_W$ which is, by definition, the annihilator of $\bar{W}$ it is more natural to think of $\mathcal{G}(V,Q)$ as the space of the $\bar{W}$'s. The stabilizer of the span of $\{a_i\}$ under the action (eq:BogT-3 18.68) is the subgroup of $O(q;\mathbb{C})$ with $B_{ij} = 0$:

$$\mathcal{LD} = \{g = \begin{pmatrix} A & 0 \\ C & A^{tr,-1} \end{pmatrix}\} \subset O(q;\mathbb{C}) \tag{18.195}$$

Then, on the subset of $O(q;\mathbb{C})/\mathcal{LD}$ where D is invertible (eq:GaussDecomp-1 18.142) shows that we can regard the antisymmetric matrix S as a set of coordinates. More invariantly, we can interpret S_{ij} as the matrix of an operator $S : \bar{W} \to W$. Given such an operator its graph is the linear subspace of V

$$\mathrm{Graph}(S) := \{S(\bar{w}) \oplus \bar{w} | \bar{w} \in \bar{W}\} \subset V \tag{18.196}$$

Note that $\mathrm{Graph}(S)$ is isotropic iff S is skew symmetric. This follows from the identity:

$$Q(S(\bar{w}_1) \oplus \bar{w}_1, S(\bar{w}_2) \oplus \bar{w}_2) = Q(S(\bar{w}_1), \bar{w}_2) + Q(\bar{w}_1, S(\bar{w}_2)). \tag{18.197}$$

Therefore $\mathrm{Graph}(S)$ is isotropic iff in oscillator basis the matrix S_{ij} is antisymmetric. the new oscillators $b_i = S_{ji}\bar{a}_j + a_i$ span $\bar{W}'$ and from $\bar{W}'$ we construct W'. The squeezed state $|S\rangle$ spans a line which is the line annihilated by $\bar{W}'$. The set of isotropic subspaces which can be written as $\mathrm{Graph}(S)$ for some antisymmetric matrix S forms a dense open set $\mathcal{U}_{\bar{W}}$ of one component of $\mathcal{G}(V,Q)$. The complement of $\mathcal{U}_{\bar{W}}$ in that component is the set of $\bar{W}'$ so that $\bar{W}' \cap W \neq \{0\}$ and is of complex codimension one. The open sets $\mathcal{U}_{\bar{W}}$ for different choices of maximal isotropic subspaces form an atlas for $\mathcal{G}(V,Q)$. The map from isotropic spaces Graph(S) to the vacuum line through the squeezed state $|S\rangle_W$ will play an important role in the next section.

Exercise
Let S be a $(2n) \times (2n)$ matrix and let I and I' be complementary multi-indices. Prove or give a counterexample to the hypothetical equation

$$\mathrm{pf}(S_I)\mathrm{pf}(S_{I'}) \stackrel{?}{=} \mathrm{pf}(S) \tag{18.198}$$

Exercise
Compute the generating function

$$\langle 0|\exp\left(-\frac{1}{2}\sum_{ij} S_{ij}a_i a_j\right) e^{x_i \bar{a}_i} e^{y_i a_i} \exp\left(\frac{1}{2}\sum_{ij} T_{ij}\bar{a}_i\bar{a}_j\right)|0\rangle \tag{18.199}$$

Hint: Consider the x_i, y_i to be generators of a Grassmann algebra.

Exercise *Proof of equation* (eq:0g-grp-22 18.184)
Prove (eq:0g-grp-22 18.184).

One answer: Compute

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \tag{18.200}$$

in two ways using the refinement of (eq:g-on-ODL 18.148)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & g\cdot S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (CS+D)^{tr,-1} & 0 \\ 0 & (CS+D) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \tag{18.201}$$

(corresponding to a coordinate system on a homogeneous space). Therefore

$$
\begin{aligned}
&\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} 1 & g_2 \cdot S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (C_2 S + D_2)^{tr,-1} & 0 \\ 0 & (C_2 S + D_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & g_1 \cdot (g_2 \cdot S) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (C_1 S' + D_1)^{tr,-1} & 0 \\ 0 & (C_1 S' + D_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} (C_2 S + D_2)^{tr,-1} & 0 \\ 0 & (C_2 S + D_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & g_1 \cdot (g_2 \cdot S) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (C_1 S' + D_1)^{tr,-1} & 0 \\ 0 & (C_1 S' + D_1) \end{pmatrix} \begin{pmatrix} (C_2 S + D_2)^{tr,-1} & 0 \\ 0 & (C_2 S + D_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}
\end{aligned}
\tag{18.202}
$$

where $S' = g_2 \cdot S$.

Exercise *Alternative proof of* (18.152)

a.) Check equation (18.152) for the case that S and T are *simultaneously* skew-diagonalizable. (This is easy.)

b.) Now consider a family of fermionic states:

$$|\Omega(t)\rangle := \exp[-\frac{t}{2} S_{ij} a_i a_j] \exp[\frac{t}{2} T_{ij} \bar{a}_i \bar{a}_j] |0\rangle \tag{18.203}$$

Note that $c(t) = \langle 0 | \Omega(t) \rangle$ is equal to 1 for $t = 0$ and we wish to compute it for $t = 1$. Show that $|\Omega(t)\rangle$ is annihilated by the set of operators

$$(1 - t^2 ST)_{ji} a_j + t T_{ji} \bar{a}_j \qquad i = 1, \dots, n \tag{18.204}$$

♣check ♣

c.) Derive a simple differential equation for $c(t)$ and use it to reduce the proof of (18.152) to the easy case where S, T are simultaneously skew-diagonalizable. [47]

♣Add more hints ♣

18.4.7 Digression: A geometric construction of the spin representation

In this section we are again following closely the beautiful presentation of Pressley and Segal in [35]. One reason for this digression is that the geometrical interpretation of representation theory is very beautiful, unifying as it does different mathematical perspectives on a single object. Another reason is that it provides an elegant and rigorous approach to the quantization of fermionic quantum field theories, and this was in fact one of the central points of [35].

Let us begin with a broader view of a geometric interpretation of representations of Lie groups more generally. To do this we must take another step back and describe an important general idea in complex geometry. That idea is the correspondence between

[47] I thank Y. Nidaiev for suggesting this line of proof.

holomorphic line bundles over a complex manifold X and maps of X into a projective space based on the following two constructions:

Construction 1: Let X be a complex manifold. Then given a holomorphic map:

$$f : X \to \mathbb{P}(V) \tag{18.205}$$

eq:HoloMap-1

(where $\mathbb{P}(V)$ is the projective space of a complex vector space V) we can produce a holomorphic line bundle L_f over X.

The map $f \to L_f$ is easy: We view the projective space $\mathbb{P}(V)$ as the space of complex lines in V. Therefore to give a map (eq:HoloMap-1 18.205) is to give, for each $x \in X$, a complex line $f(x) \subset V$. But this is just the data needed to construct a line bundle $\pi_1 : L_f \to X$. It is a subbundle of the trivial bundle $X \times V$ whose fiber at x is just the line $(L_f)|_x = f(x)$, i.e.

$$L_f := \{(x, v)|v \in f(x)\} \subset X \times V. \tag{18.206}$$

eq:Lf-def

We take $\pi_1(x, v) = x$, so the fiber is a complex line. Everything varies holomorphically, so $\pi_1 : L_f \to X$ is a holomorphic line bundle.

Note that the projection $\pi_2 : X \times V \to V$ is holomorphic so that if $\alpha \in V^\vee$ then $s_\alpha := \alpha \circ \pi_2$ defines a map

$$s_\alpha : L_f \to \mathbb{C} \tag{18.207}$$

which is, moreover, linear on the fibers of L_f. Because it is defined for every fiber and is linear on each fiber we can view s_α as a holomorphic section of $L^\vee$ and hence we have an injective map

$$\Psi_f : V^\vee \to \Gamma(X; L_f^\vee). \tag{18.208}$$

eq:HoloSecV-vee

♣Need to explain when the vector spaces are isomorphic. ♣

Construction 2: Suppose X is a complex manifold and $Q \to X$ is a holomorphic line bundle which admits holomorphic sections such that for every x there exists $s \in \Gamma(X; Q)$ such that $s(x) \neq 0$. Then we can construct a map

$$f_Q : X \to \mathbb{P}(V_Q) \tag{18.209}$$

where

$$V_Q := \Gamma(X; Q)^\vee \tag{18.210}$$

The most conceptual way to construct the map f_Q is to recall that for any vector space there is a natural 1-1 correspondence between lines in V and hyperplanes in $V^\vee$: Given a line $\ell \subset V$ the annihilator in $V^\vee$ is a hyperplane. Apply that to $V = V_Q$. Thus, it suffices to show how, for each $x \in X$, to construct a hyperplane in $\Gamma(X; Q)$. This is simply done by considering the set of sections such that $s(x) = 0$.

Let us give a slightly more concrete description of f_Q. Define a map $f_Q : X \to \mathbb{P}(V_Q)$ by evaluation at x as follows: For any x choose a local trivialization $\psi_x : Q_x \cong \mathbb{C}$. Then define a linear functional on $\Gamma(X; Q)$ by

$$\ell_{x,\psi} : s \mapsto \psi_x(s(x)). \tag{18.211}$$

This is clearly a nonzero linear functional on $\Gamma(X;Q)$ and hence a vector in V_Q. This vector depends on x as well as on the choice of trivialization ψ_x. However, any two trivializations of Q_x differ by an element of $\mathrm{Aut}(\mathbb{C})$ which is canonically a nonzero complex number. Therefore the line $\mathcal{L}_{x,Q} \subset V_Q$ through $\ell_{x,\psi}$ does not depend on the choice of local trivialization ψ_x. The line $\mathcal{L}_{x,Q}$ is, by definition, $f_Q(x)$.

Given these constructions it is natural to ask how L_{f_L} compares with $L^\vee$ and how $f_{L_f^\vee}$ compares with f, and whether Ψ_f above is an isomorphism. Under "good" conditions, for example if X is a smooth compact Kähler manifold and L has a suitable first Chern class we will indeed have

$$L_{f_L} \cong L^\vee \tag{18.212}$$

$$f_{L_f^\vee} \cong f \tag{18.213}$$

and Ψ_f will be an isomorphism. This is the Kodaira embedding theorem. See GriffithsHarris [23]. ♣Improvement needed here. ♣

Example: We illustrate these constructions with the important example of holomorphic line bundles over $\mathbb{C}P^N = \mathrm{Gr}_1(\mathbb{C}^{N+1})$. Denote a point in $\mathbb{C}P^N$ by $[X_0 : X_1 : \cdots : X_N]$ where the square brackets indicate the usual equivalence relation

$$[X_0 : X_1 : \cdots : X_N] = [\lambda X_0 : \lambda X_1 : \cdots : \lambda X_N] \tag{18.214}$$

and of course at least one X_i is nonzero. Choose a positive integer d and consider the vector space $V_d = V_d(\mathbb{C}^{N+1})$ of homogeneous degree d polynomials on $\mathbb{C}^{N+1}$. Think of such polynomials as expressions [48]

$$\sum_{|I|=d} c_I u_0^{i_0} \cdots u_N^{i_N} \tag{18.215}$$

Now, we claim that there is a canonical map $f_d : \mathbb{C}P^N \to \mathbb{P}(V_d^\vee)$, for if we have $[X_0 : \cdots : X_N] \in \mathbb{C}P^N$ then for each choice $(X_0, \dots, X_N)$ in the equivalence class we have a nonzero linear functional on V_d taking

$$\sum_{|I|=d} c_I u_0^{i_0} \cdots u_N^{i_N} \mapsto \sum_{|I|=d} c_I X_0^{i_0} \cdots X_N^{i_N} \tag{18.216}$$

Therefore, associated to the equivalence class $[X_0 : \cdots : X_N] \in \mathbb{C}P^N$ is a well-defined line in $V_d^\vee$. Since we have assigned a line in $V_d^\vee$ to each point in $\mathbb{C}P^N$ we have defined the map f_d.

Another more concrete way of describing the maps f_d might be helpful. First choose $N = 1$. Then a map

$$f_d : \mathbb{C}P^1 \to \mathbb{C}P^d \tag{18.217}$$

can be defined by

$$f_d : [X_0 : X_1] \mapsto [X_0^d : X_0^{d-1} X_1 : X_0^{d-2} X_1^2 : \cdots : X_0 X_1^{d-1} : X_1^d] \tag{18.218}$$

[48] Unlike the multi-indices we have been using until now, here repeated entries are allowed.

On the other hand, if $d = 1$ and we take any N then we could of course take f_1 to be the identity map:

$$f_1 : [X_0 : \cdots : X_N] \mapsto [X_0 : \cdots : X_N] \tag{18.219}$$

The general case with $d > 1$ and $N > 1$ is a simple generalization of these. It is helpful to state it more invariantly. Let U be a complex vector space of dimension $N + 1$. Then the general case is a map

$$f_d : \mathbb{P}(U) \to \mathbb{P}(\mathrm{Sym}^d(U)) \tag{18.220}$$

We can write it explicitly by choosing a basis for U and then using that to construct a corresponding basis of

$$\left(\binom{N+1}{d}\right) = \frac{(N+1)(N+2)\cdots(N+d)}{d!} \tag{18.221}$$

homogeneous expressions. [49]

What holomorphic line bundles do we get from these functions f_d? The line L_{f_1} is particularly easy to describe: It is called the *tautological line bundle*. If we take $f_1 : \mathbb{P}(U) \to \mathbb{P}(U)$ to be the identity map the line bundle L_{f_1} is just the subbundle of $\mathbb{P}(U) \times U$ whose fiber above a *point* $\ell \subset U$ in $\mathbb{P}(U)$ is the *one-dimensional subspace* $\ell \subset U$. The bundle L_{f_1} is commonly denoted $\mathcal{O}(-1)$.

The tautological line bundle is very important. Its first Chern class generates the integral cohomology of $\mathbb{C}P^N$ and all holomorphic line bundles are powers of $\mathcal{O}(-1)$. In particular L_{f_d} turns out to be $(L_{f_1})^{\otimes d}$. It is usually denoted $\mathcal{O}(-d)$.

Following through the above definitions (and using the fact that $\mathbb{C}P^N$ is smooth, compact, and Kähler) one can check that the space of holomorphic sections of the line bundle $L_{f_d}^\vee$ (also denoted $\mathcal{O}(d)$) is naturally isomorphic to the vector space of homogeneous degree d polynomials in $N + 1$ variables. We can think - informally - of the holomorphic sections as homogeneous degree d polynomials $\sum_{|I|=d} c_I X_0^{i_0} \cdots X_N^{i_N}$. Although trying to assign a value to such a polynomial at a point of $\mathbb{C}P^N$ does not make sense, the zero set of the polynomial in $\mathbb{C}P^N$ is a well-defined subvariety of $\mathbb{C}P^N$. For example for $N = 1$ there will be precisely d zeroes and hence d points in $\mathbb{C}P^1$, counted with multiplicity. This turns out to be quite significant because the *theory of divisors* shows how zero-sets of sections of holomorphic line bundles can be used to characterize them uniquely. See [23].

If we return to the line bundle $L_{f_d} \cong \mathcal{O}(-d)$ then we see it has no holomorphic sections. After all, any putative holomorphic section $t^\vee$ of $\mathcal{O}(-d)$ would have to pair with a holomorphic section s of $\mathcal{O}(d)$ to produce a holomorphic function $\langle s, t^\vee \rangle$, and by Liouville's theorem this function would have to be constant. But then, if s has zeroes, $t^\vee$ would have to have poles, so it wouldn't be holomorphic. Note that there is a big difference between $\Gamma(X; L)^\vee$ and $\Gamma(X; L^\vee)$!

Finally, note that if $f : X \to \mathbb{P}(V)$ then the line bundle $L_f \to X$ defined in (18.206) is just the pullback $f^*\mathcal{O}_{\mathbb{P}(V)}(-1)$.

[49] In the literature on algebraic geometry the map f_d is known as a *Veronese map*. Veronese considered the case $N = 2$ and $d = 2$.

Now, let us apply these general constructions to the representation theory of compact Lie groups G. Here is a lightning summary of some basic definitions.

1. Up to conjugation, G has a unique maximal abelian subgroup, the *Cartan torus* $T \subset G$. We have $T \cong U(1)^r$ for some integer r known as the *rank* of G.

2. If $\rho : G \to U(V)$ is a unitary representation of G on a complex vector space V then, restricted to T the representation must decompose as a sum of one-dimensional representations of T, $V \cong \oplus L_\mu$ where $\mu \in \mathrm{Hom}(T, U(1))$ are characters on T. (Unitary irreps of T are in one-one correspondence with such characters.)

3. The set of homomorphisms $\mathrm{Hom}(T, U(1))$ is a lattice because if μ_1 and μ_2 are characters then so is $\mu_1^{n_1}\mu_2^{n_2}$ for any integers $n_1, n_2 \in \mathbb{Z}$. This lattice is known as the *weight lattice of* G and characters in this lattice are referred to as *weights* in this context. The characters μ which appear in the decomposition $V \cong \oplus L_\mu$ of a unitary representation V of G are known as the *weights of the representation.*

4. It is often useful to use the exponential map to view the weight lattice as a subspace of $\mathrm{Hom}(\mathfrak{t}, \mathbb{R})$, where $\mathfrak{t}$ is the Lie algebra of T. One can choose a basis for $\mathfrak{t}$ of *simple roots* (see below) with corresponding simple coroots H_i so that T is the set of group elements $t = \exp[\sum_{s=1}^r \theta_s H_s]$ where $\theta_s \sim \theta_s + 2\pi$. Then the most general weight is of the form
$$\mu(t) = \prod_s e^{\mathrm{i} n_s \theta_s} \tag{18.222}$$
for integers n_s and the corresponding element of $\mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ maps
$$\sum_s \tilde{\theta}_s H_s \mapsto \sum_s \tilde{\theta}_s n_s \tag{18.223}$$
where $\tilde{\theta}_s \in \mathbb{R}$. We freely will pass between the multiplicative and additive interpretation of weights below. ♣DON'T DO THAT! ♣

5. The nonzero weights of the adjoint representation $\mathfrak{g}_c = \mathfrak{g} \otimes \mathbb{C}$ play a special role and are called *roots*. Now henceforth assume that G is simple. A key step in representation theory is to show that for each root there is a canonically associated subalgebra of $\mathfrak{g}_c$ which is isomorphic to $sl(2, \mathbb{C})$. We denote it $sl(2, \mathbb{C})_\alpha \subset \mathfrak{g}_c$. Its intersection with $\mathfrak{t}$ is generated by a canonically normalized generator H_α known as a *coroot.* The normalization conditions is (viewing weights additively) $\alpha(H_\alpha) = 2$, and indeed $\beta(H_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ where $(\cdot, \cdot)$ is the Killing form. [50] Then $sl(2, \mathbb{C})_\alpha$ has canonical generators $E_{\pm\alpha}$ and H_α with
$$[E_\alpha, E_{-\alpha}] = H_\alpha \qquad [H_\alpha, E_{\pm\alpha}] = \pm 2 E_{\pm\alpha} \tag{18.224}$$

6. One can prove that if α is a root then so is $-\alpha$. A choice of *positive roots* is a maximal set of roots not containing the pair $\{\alpha, -\alpha\}$. If α, β are roots $\alpha + \beta$ might

[50]Note that the expression is linear in β but not in α. It is true that $H_{-\alpha} = -H_\alpha$.

(or might not) be a root but $\alpha - \beta$, if nonzero, is never a root. Therefore, given a set of positive roots there is a canonically defined set of *simple roots* α_i which cannot be decomposed as sums of other positive roots. The H_i above are the corresponding simple coroots.

7. Given a choice of positive roots a *dominant weight* λ is a weight such that (considered additively) $\lambda(H_\alpha) \geq 0$ for all $\alpha > 0$ and an anti-dominant weight λ is one such that $-\lambda$ is dominant. Given a choice of positive roots there is a 1-1 correspondence between irreducible representations V of G and dominant (or anti-dominant) weights. Roughly speaking, the representation V_λ corresponding to a dominant weight λ has a unique highest weight vector which is annihilated by E_α for all $\alpha > 0$. One can then build the representation by acting on this vector with lowering operators $E_{-\alpha}$ for $\alpha > 0$. If $\lambda \in \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ is suitably quantized (which is guaranteed if it exponentiates to a character in $\mathrm{Hom}(T, U(1))$) and we mod out by null vectors the resulting representation V_λ is finite dimensional.

8. If λ is dominant then $V_\lambda^\vee$ has lowest weight vector with weight $-\lambda$. ♣check! ♣

Now, to bring in holomorphic geometry we summarize a few more facts. The complexification G_c of G is a holomorphic manifold. Roughly speaking, we exponentiate the generators of the Lie algebra $\mathfrak{g}$ of G with complex coefficients. Put differently, we exponentiate the complex Lie algebra $\mathfrak{g}_c$ and complete to form a group. Now, given a choice of Cartan subgroup T together with a choice of positive roots there is a canonically determined "upper triangular" subgroup $B^+ \subset G_c$ given by exponentiating H_α and E_α for $\alpha > 0$. A good example to bear in mind is the group $U(n)$ with T the subgroup of diagonal matrices. Then with respect to a standard choice of positive roots B^+ is just the subgroup of $GL(n, \mathbb{C})$ of upper triangular matrices.

Now, if $\chi : T \to U(1)$ is a unitary character then it has a holomorphic extension to a multiplicative character $\chi : B^+ \to \mathbb{C}^*$. Therefore, we can define an associated holomorphic line bundle over G_c/B^+

$$\mathcal{L}_\chi = G_c \times_{B^+} \mathbb{C} = \{[g, z]| \qquad [gb, z] = [g, \chi(b)z] \qquad \forall b \in B^+, g \in G, z \in \mathbb{C}\} \tag{18.225}$$

Notice that there is a well-defined projection $\pi : \mathcal{L}_\chi \to G_c/B^+$ given by $\pi : [g, z] \mapsto gB^+$. Now, a crucial point is that

The vector space of ($\mathcal{C}^\infty$, holomorphic) sections of the bundle $\pi : \mathcal{L}_\chi \to G_c/B^+$ is naturally isomorphic to the vector space of ($\mathcal{C}^\infty$, holomorphic) B^+-equivariant functions $f : G_c \to \mathbb{C}$.

The phrase "B^+-equivariant" just means that $f(gb) = \chi(b^{-1})f(g)$. The equivalence is established as follows: Suppose first we have a section s of $\pi : \mathcal{L}_\chi \to G_c/B^+$. Now, choose $g \in G_c$. Form the coset gB^+. Then the section $s(gB^+)$ gives us an equivalence class $[h, z]$. Since it is in the fiber above gB^+ we have $hB^+ = gB^+$. In particular, that equivalence class must have a representative (g, z_g) where the first entry is exactly g. We use that representative to define $f(g) := z_g$. The resulting function is clearly equivariant. Proof: We have $s(gB^+) = s(gbB^+)$ and so $[g, z_g] = [gb, z_{gb}]$ but, by definition, $[gb, z_{gb}] = [g, \chi(b) z_{gb}]$. Putting these two equations together we see that $f(g) = \chi(b) f(gb)$. Conversely, given such an equivariant function we can define a section: $s : gB^+ \mapsto [g, f(g)]$. (Because f is equivariant this formula for s is well-defined.)

Moreover, the vector space of ($\mathcal{C}^\infty$, holomorphic) B^+-equivariant functions $f : G_c \to \mathbb{C}$. is naturally a representation of G_c. Indeed, given an equivariant function f and $g_0 \in G_c$ we can define a new equivariant function $L(g_0) \cdot f$ whose values are

$$(L(g_0) \cdot f)(g) := f(g_0^{-1} g) \tag{18.226}$$

The reason for the annoying inverse in g_0^{-1} on the RHS is that this way we get a representation $L(g_0)L(g_0') = L(g_0 g_0')$. Note that since the multiplication by g_0^{-1} is on the left the equivariance property is not spoiled, even for G nonabelian. The representation we produce this way depends on the character χ and is known as an *induced representation*. If we take $\mathcal{C}^\infty$ sections then it is infinite dimensional and has no reason to be irreducible. However, if we take *holomorphic sections* then, it can be shown, $\Gamma(\mathcal{L}_\chi, G_c/B^+)$ is finite dimensional and irreducible. This representation is the *holomorphically induced representation*.

The finite-dimensionality follows once one realizes that $G_c/B^+ \cong G/T$ is compact and we are essentially solving a Cauchy-Riemann like equation $\bar{\partial} s = 0$. The irreducibility follows from a basic decomposition theorem of matrices known as the *Bruhat decomposition*. Let N^- be the group generated by exponentiating $E_{-\alpha}$ for $\alpha > 0$. For $G_c = GL(n, \mathbb{C})$ this would be the lower triangular matrices with 1 on the diagonal. The orbits of N^- on G_c/B^+ are cells of dimensions related to properties of the Weyl group. There is one open dense orbit of maximal dimension. Now, if $\Gamma(G_c/B^+; \mathcal{L}_\chi)$ were reducible there would be two linearly independent lowest weight vectors s_1 and s_2. But these are invariant under N^-. Therefore, therefore s_1/s_2 is constant on the N^- orbit of $1 \cdot B^+$. But this is a function, which if constant off of a codimension one subspace must be constant everywhere, contradicting linear independence of s_1 and s_2.

Now, conversely suppose G is a compact simple Lie group, and suppose it has an irreducible representation on a complex finite-dimensional vector space V and we choose positive roots so we can identify $V = V_\lambda$ where λ is a dominant weight. Then the representation extends to a holomorphic representation of the complexification $\rho_c : G_c \to GL(V)$, and there is a multiplicative holomorphic character $\chi_\lambda : B^+ \to \mathbb{C}^*$. The dual representation $V_\lambda^\vee$ has a lowest weight vector v and the action $\rho^\vee(B^+)$ on v is via the character χ_λ^{-1}. The lowest weight vector generates a line $v\mathbb{C} \subset V_\lambda^\vee$. Now $\rho_c^\vee(g)$ acts on the projective space $\mathbb{P}(V_\lambda^\vee)$ since a linear transformation takes lines to lines. It is a transitive action and the stabilizer of $v\mathbb{C}$ is just B^+. Therefore, we get a map

$$f_\lambda : G_c/B^+ \to \mathbb{P}(V_\lambda^\vee) \tag{18.227}$$

From our Construction 1 above we automatically get a holomorphic line bundle $L_{f_\lambda} \to G_c/B^+$. Tracing through the definitions one can show that this line is exactly the associated line bundle discussed above: $L_{f_\lambda} = \mathcal{L}_{\chi_\lambda}$. Moreover, thanks to (eq:HoloSecV-vee 18.208) we get an injective map

$$\Psi_{f_\lambda} : V_\lambda \to \Gamma(\mathcal{L}^\vee_{\chi_\lambda}) \tag{18.228}$$

Again, following through definitions one can check that this map is G_c-equivariant. Therefore, this is an isomorphism of representations, thus giving a beautiful geometrical interpretation to the irreducible representations of G. ♣more detail. ♣

The result of all the above is the very beautiful Borel-Weil-Bott theorem:

Theorem : Let G be a simple Lie group. Choose a system of positive roots, thus determining a Borel subgroup $B^+ \subset G_c$. For any weight λ let $\mathcal{L}_{\chi_\lambda} \to G_c/B^+$ be the induced holomorphic line bundle from the character on T.

a.) $\mathcal{L}_{\chi_\lambda}$ has no holomorphic sections unless λ is anti-dominant.

b.) If λ is dominant then $\Gamma(G_c/B^+, \mathcal{L}^\vee_{\chi_\lambda})$ is a representation of G_c which is isomorphic to the representation V_λ.

♣We didn't explain (a). ♣

Example 1: *Representations of* $SU(2)$. We take T to be the subgroup of $SU(2)$ of diagonal matrices. It is isomorphic to $U(1)$ and hence the characters are labeled by $\lambda \in \mathbb{Z}$:

$$\chi_\lambda : \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mapsto e^{i\lambda\theta} \tag{18.229}$$

We choose positive roots so that B^+ is the group of upper triangular matrices

$$b = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \tag{18.230}$$

with $b_{11}b_{22} = 1$. With this choice of positive roots we have

$$E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18.231}$$

The holomorphic extension of χ_λ is $\chi_\lambda(b) = b_{11}^\lambda$. A holomorphic section of $\mathcal{L}_{\chi_\lambda}$ is equivalent to an equivariant holomorphic function

$$f : SL(2,\mathbb{C}) \to \mathbb{C} \tag{18.232}$$

such that

$$f(gb) = \chi_\lambda(b^{-1}) f(g) \tag{18.233}$$

Let us unpack what this means:

Equivariance with respect to matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in B^+ \qquad x \in \mathbb{C} \tag{18.234}$$

implies

$$f(\begin{pmatrix} r & s \\ t & u \end{pmatrix}) = f(\begin{pmatrix} r & rx+s \\ t & tx+u \end{pmatrix}) \qquad \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in SL(2,\mathbb{C}) \tag{18.235}$$

which implies that f is only a function of (r,t). [51] Next, invariance with respect to diagonal matrices

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in B^+ \qquad x \in \mathbb{C}^* \tag{18.236}$$

implies

$$f(\begin{pmatrix} rx & sx^{-1} \\ tx & ux^{-1} \end{pmatrix}) = x^{-\lambda} f(\begin{pmatrix} r & s \\ t & u \end{pmatrix}) \qquad\qquad \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in SL(2,\mathbb{C}) \tag{18.237}$$

Therefore, $f(r,t)$ is homogeneous of degree $-\lambda$. If $\lambda > 0$ there are no holomorphic functions, as promised by the above theorem. If $\lambda \leq 0$ then $f(r,t)$ is a homogeneous polynomial of degree $d := |\lambda|$. The space $V_d(\mathbb{C}^2)$ is well-known to be a standard presentation of the irreducible spin $j = d/2$ representation of $SU(2)$ of dimension $d+1$. Indeed, if

$$g_0^{-1} = \begin{pmatrix} r_0 & s_0 \\ t_0 & u_0 \end{pmatrix} \in SU(2) \tag{18.238}$$

and if we choose a basis for $V_d(\mathbb{C}^2)$ of the form $f_n(r,t) := r^n t^{d-n}$, $0 \leq n \leq d$, then we can compute the matrix elements of the representation. By definition:

$$(L(g_0) \cdot f_n)(r,t) = f_n(r_0 r + s_0 t, t_0 r + u_0 t) \tag{18.239}$$

and hence

$$L(g_0) \cdot f_n = \sum_{n'} \mathcal{D}^{(d)}_{n'n}(g_0) f_{n'} \tag{18.240}$$

$$\mathcal{D}^{(d)}_{n'n}(g_0) = \sum_{p+q=n'} \binom{n}{p} \binom{d-n}{q} r_0^p t_0^q s_0^{n-p} t_0^{d-n-q} \tag{18.241}$$

The functions $\mathcal{D}^{(d)}_{n'n}(g_0)$ on $SL(2,\mathbb{C})$ are - up to normalization - known as *Wigner functions* ♣CHECK! ♣ and special cases include standard functions such as Legendre, associated Legendre, and spherical harmonics.

Example 2: *Antisymmetric tensors of* $U(n)$. We now consider the geometrical interpretation of the k^{th} antisymmetric representation of $U(n)$. Consider the Grassmannian of k-planes in an n-dimensional complex vector space V:

$$\mathrm{Gr}_k(V) := \{W \subset V | \dim_{\mathbb{C}} W = k\} \tag{18.242}$$

This is a complex manifold. Then there is a natural holomorphic line bundle DET $\to \mathrm{Gr}_k(V)$. The fiber above a subspace $W \in \mathrm{Gr}_k(V)$ is $\Lambda^k(W)$. It corresponds to a holomorphic map $f_{\mathrm{DET}} : \mathrm{Gr}_k(V) \to \mathbb{P}(\Lambda^k V)$ defined by mapping the subspace W to the complex line $\Lambda^k W$ which is a line in the $\binom{n}{k}$-dimensional vector space $\Lambda^k V$.

[51] If $rt \neq 0$ then we can always choose one x to set $rx + s = 0$ and another to set $tx + u = 0$, so f cannot be a function of s or u. If $r = 0$ then $st = -1$. In particular s is not independent of t and we can choose an x to set $tx + u = 0$. If $t = 0$ then $ru = 1$ and a similar argument applies.

For $\dim_{\mathbb{C}} V = n$ we claim that, as representations of $U(n)$, $\Lambda^k(V)$ is isomoprhic to $\Gamma(\mathrm{Gr}_k(V), \mathrm{DET}^\vee)$. According to our general principle (18.208) above we have a map

$$\Lambda^k(V)^\vee \to \Gamma(\mathrm{Gr}_k(V), \mathrm{DET}^\vee) \tag{18.243}$$

eq:BWB-map-lk

In our case it can be defined directly as follows: First $\Lambda^k(V)^\vee \cong \Lambda^k(V^\vee)$. It suffices to define the map for elements $\alpha \in \Lambda^k V^\vee$ of the form $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ with $\alpha_i \in V^\vee$ and then extend by linearity. The corresponding section s_α is a holomorphic map $\mathrm{DET} \to \mathbb{C}$ which is linear on the fibers. An element of DET in the fiber above W is of the form $w_1 \wedge \cdots \wedge w_k$ for some vectors $w_i \in W$. We then define

$$s_\alpha(w_1 \wedge \cdots \wedge w_k) := \det(\alpha_i(w_j)) \tag{18.244}$$

The map (18.243) is clearly injective. Some algebraic geometry allows one to show that it is surjective, so we get an isomorphism. ($\mathrm{Gr}_k(V)$ is smooth compact and Kähler.) It is clearly equivariant. Thus $\mathrm{DET}^\vee$ has holomorphic sections and therefore DET has no holomorphic sections. This is in accord with our discussion of holomorphic line bundles over $\mathbb{C}P^N = \mathrm{Gr}_1(\mathbb{C}^{N+1})$.

Remark: The map f_{DET} is very important in algebraic geometry. It is known as the *Plücker embedding*. Let us describe it a bit more explicitly. If we choose a basis for V then, given a basis for W we associate a $k \times n$ complex matrix whose rows are the components of the basis elements of W. Therefore, the space of k-dimensional subspaces together with ordered basis can be identified with the subspace of the matrices $M_{k\times n}(\mathbb{C})$ which have rank k. Call this subspace $M^0_{k\times n}(\mathbb{C})$. Left action by $GL(k,\mathbb{C})$ corresponds to a change of basis for W and hence we can identify

$$\mathrm{Gr}_k(V) \cong GL(k,\mathbb{C})\backslash M^0_{k\times n}(\mathbb{C}). \tag{18.245}$$

To give the Plücker coordinates of a point in the Grassmannian we start with W, choose a basis for W and therefore a matrix $\Lambda \in M^0_{k\times n}(\mathbb{C})$ and associate to it the vector of $k \times k$ minors of Λ. The map descends to a map from the quotient $GL(k,\mathbb{C})\backslash M^0_{k\times n}(\mathbb{C})$ to the projective space $\mathbb{P}(\Lambda^k V) \cong \mathbb{C}P^{\binom{n}{k}}$. To see that the map is an embedding note that for $[\omega] \in \mathbb{P}(\Lambda^k V)$ we can define a subspace $V_\omega \subset V$ as the set of vectors such that $v \wedge \omega = 0$. If $[\omega]$ is in the image of the Plücker map applied to W then clearly $W \subset V_\omega$. On the other hand, if $w_1, \dots, w_k$ is a basis for W then we can extend it to a basis for V to show that in fact $V_\omega = W$. (Indeed, simple considerations of linear algebra show that for any $[\omega] \in \mathbb{P}(\Lambda^k V)$ the map $v \mapsto v \wedge \omega$ has kernel of dimension $\le k$.) Therefore, we can reconstruct W from the equation $v \wedge \omega = 0$ and hence the Plücker map is an embedding.

♣Should have discussion of decomposable and indecomposable elements. ♣

One can show that these Plücker coordinates satisfy a set of quadratic relations which in fact define the image of the Grassmannian under the Plücker embedding. This exhibits the Grassmannian as an explicit algebraic variety, indeed as an intersection of quadrics. See [24].

♣Should give an example and/or a simple explanation. ♣

♣Plucker relations have a nice interpretation in terms of bosonization of fermions. Explain? ♣

Let us now apply these ideas to the Spin group to get a nice geometric insight into one sense in which the Spin representation is a "squareroot." (We are again following Pressley and Segal, chapter 12.)

We apply the above correspondence between maps to projective space and holomorphic line bundles. In our context of fermions note that given a point in the Grassmannian $\mathcal{G}(V,Q)$ of maximal complex isotropic subspaces of V we automatically have a Fock space and in particular a *vacuum line*. That is, the quantum vacuum defines a map

$$f_{\rm vac} : \mathcal{G}(V,Q) \to \mathbb{P}(\Lambda^* W) = \mathbb{P}(S_c) \tag{18.246}$$

To define this more precisely, choose a decomposition $V = W \oplus \bar{W}$. Then $f_{\rm vac}$ maps $\bar{W}' \in \mathcal{G}(V,Q)$ to the line in $\Lambda^* W$ annihilated by $\bar{W}'$. The corresponding line bundle is called the vacuum line bundle $\mathfrak{Vac} \to \mathcal{G}(V,Q)$. (Pressley and Segal call this the Pfaffian bundle $\mathrm{PF} \to \mathcal{G}(V,Q)$ for reasons explained below.) We then have a BWB-type interpretation of the spin representation:

Theorem The pin representation of $\mathrm{Pin}^-(2n)$ can be identified with the holomorphic sections $\Gamma(\mathfrak{Vac}^\vee)$.

This is the geometical interpretation of the spin representation we wanted to find. Now we have two geometrical results which beautifully reflect representation-theory facts.

Let

$$\mathrm{Gr}(\bar{W}) = \amalg_{k=0}^n \mathrm{Gr}_k(\bar{W}) \tag{18.247}$$

be the complete Grassmann variety of $\bar{W}$. There is a natural embedding of $\mathrm{Gr}(\bar{W})$ into $\mathcal{G}(V,Q)$. If $\bar{W}_1 \subset \bar{W}$ then we can define $\bar{W}' := W_1^\perp \oplus \bar{W}_1 \in \mathcal{G}(V,Q)$. Here $W_1^\perp \subset W$ is the orthogonal complement in the Hilbert space inner product h. The space $\bar{W}'$ is maximal isotropic in V. The map ι_1 which takes $\bar{W}_1 \mapsto \bar{W}'$ embeds the Grassmannian of the n-dimensional complex vector space $\bar{W}$ into the isotropic Grassmannian of (V,Q).

We thus have the diagram

$$\begin{tikzcd}
\mathrm{Gr}(W) \arrow[r, "\iota_1"] \arrow[d, "f_{\rm DET}"] & \mathcal{G}(V,Q) \arrow[r, "\iota_2"] \arrow[d, "f_{\rm vac}"] & \mathrm{Gr}_n(V) \arrow[d, "f_{\rm DET}"] \\
\mathbb{P}(\Lambda^* W) \arrow[r, equal] & \mathbb{P}(\Lambda^* W) & \mathbb{P}(\Lambda^* V)
\end{tikzcd} \tag{18.248}$$

where $f_{\rm DET}$ is the Plücker embedding.

To check that the square on the left is commutative note that we can interpret $W_1^\perp \oplus \bar{W}_1$ as a space of annihilation operators of the form:

$$\begin{aligned} b_i &= \bar{a}_i + \sum_{j=k+1}^n R_{ij} a_j \qquad 1 \le i \le k \\ b_i &= a_i \qquad i = k+1, \dots, n \end{aligned} \tag{18.249}$$

by choosing $\bar{a}_i$, $i = 1, \dots, k$ to be a basis for $W_1^\perp$ and a_i, $i = k+1, \dots, n$ to be a basis for $\bar{W}_1$. Then the line annihilated by the $\{b_i\}_{i=1}^n$ is generated by $\bar{a}_1 \cdots \bar{a}_k |0\rangle$. Since the ♣Check. This is not quite right. ♣

square commutes we have

$$\iota_1^*(\mathfrak{Vac}) \cong \mathrm{DET}(W) \tag{18.250}$$ eq:geom-emb-1

This is related to the fact that $U(n)$ (or rather, a double cover) acts on the spaces $\Lambda^k W$ as the k^{th} anti-symmetric power of the fundamental.

Secondly

$$\iota_2^*(\mathrm{DET}(V)) \cong (\mathfrak{Vac})^2 \tag{18.251}$$ eq:geom-emb-2

reflecting the fact that the spin representation is a squareroot of the left regular representation $\Lambda^* V$ of the Clifford algebra.

Remarks

1. The beautiful story of the Borel-Weil-Bott theorem goes further. One can show that $G/T \cong G_c/B^+$ as manifolds, and indeed with a choice of positive roots G/T can be given a complex structure so that these are isomorphic as complex manifolds. G/T is obviously compact and G_c/B^+ is obviously holomorphic.

   One can also define natural symplectic forms on G/T so that, if G is compact, it has finite symplectic volume. These forms are compatible with the complex structures and make G/T into a Kähler manifold with left-invariant metric.

   The Lie algebra $\mathfrak{g}$ has a natural adjoint action of the group. For matrix groups $\mathrm{Ad}(g) : X \mapsto gXg^{-1}$. The dual representation will be represented by the transpose inverse. To be precise we define the *coadjoint* action on $\mathfrak{g}^*$ as follows: If $v \in \mathfrak{g}^*$ and $g \in G$ then

   $$\langle \mathrm{Ad}^*(g)v, x\rangle := \langle v, \mathrm{Ad}(g^{-1})x\rangle \qquad \forall x \in \mathfrak{g} \tag{18.252}$$

   We can therefore study the *orbits* of G acting on $\mathfrak{g}^*$. By definition, the orbit $\mathcal{O}(v_0)$ through $v_0 \in \mathfrak{g}^*$ is isomorphic as a manifold to G/K where K is the stabilizer of v_0 under Ad^*. The Kirillov-Kostant-Souriau theorem states that these orbits are in fact naturally symplectic manifolds: To see this define an antisymmetric form on $\mathfrak{g}$ by:

   $$\omega_{v_0}(X, Y) := v_0([X, Y]) \tag{18.253}$$

   The annihilator of this form is, almost by definition, the Lie algebra of K. Now, antisymmetric forms on $\mathfrak{g}$ are two-forms on $\mathfrak{g}^*$ (since cotangent vectors on $\mathfrak{g}^*$ can be identified with elements of $\mathfrak{g}$). The 2-form on $\mathfrak{g}^*$ can be pulled back to $\mathcal{O}(v_0)$. Since the annihilator is $Lie(K)$ the 2-form is nondegenerate. Moreover, it is easily seen to be left-invariant, and hence it defines a symplectic form on $\mathcal{O}(v_0)$.

   It we introduce a Killing form $B(X, Y) = \mathrm{Tr}(XY)$ on $\mathfrak{g}$ then we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and define a symplectic form on the G-orbits in $\mathfrak{g}$ of the form $\omega_{v_0}(X, Y) = \mathrm{Tr}(v_0[X, Y])$

   ♣Also explain by using the structure constants to define $\mathfrak{g}$ as a Poisson manifold. ♣

   If we choose a Cartan subalgebra in $\mathfrak{g}$ then without loss of generality we can take v_0 to be in $\mathfrak{t}^*$. It turns out that if $v_0 = \lambda \in \Lambda_{wt}$ then the orbit $\mathcal{O}(\lambda)$ has integral symplectic volume. We can therefore expect to quantize this symplectic manifold.

The resulting Hilbert space will be a representation of G and its dimension will be finite: Up to quantum corrections it will be the symplectic volume.

**** DO EXAMPLE OF $SU(2)$ ****

Viewed as a quantum system the action $\int(pdq - Hdt)$ is

$$\int(\mathrm{Tr}[\Lambda_0(g^{-1}\dot{g})] - H)dt \tag{18.254}$$

If $H(t) = \mathrm{Tr}[\Lambda_0 h(t)]$ for a Lie-algebra valued function $h(t)$ then the partition function on the circle will just be

$$\mathrm{Tr}_R P\exp - \int h(t)dt \tag{18.255}$$

♣This all needs a lot more explanation to be comprehensible... ♣

If we introduce a choice of positive roots then we can also take a holomorphic viewpoint. The metric $g(X,Y) = \omega_\lambda(X, IY)$ is a homogeneous Kähler metric for λ a dominant weight. When it is integral ω is properly normalized for quantization of the phase space. Now, in the Kähler quantization - also known as the coherent state formalism - the wavefunctions are holomorphic sections of the holomorphic bundle $L_\chi \to G/T$. The important property mentioned above that $\Gamma(\mathcal{L}_{\chi_\lambda})$ is finite dimensional is now easily understood: On a compact phase space there should be a finite dimensional space of quantum states.

Some references:

1. Kirillov, Elements of the theory of representations.

2. Perelomov book on coherent states.

3. Raoul Bott, "On induced representations," in *Mathematical Heritage of Hermann Weyl*, or Collected Papers 48 (1994): 402.

4. In the physics literature there are several papers interpreting these facts in terms of quantum mechanical path integrals [2][5][6]. The holomorphic interpretation we stressed above can be naturally incorporated by thinking about the supersymmetric quantum mechanics on G/T using the Kähler structure. [ref to cite??]

2. Comment on infinite dimensions....

Exercise

Describe the line bundles $\mathcal{O}(\pm d)$ over $\mathbb{C}P^N$ in terms of patches and transition functions. Use the natural patches $\mathcal{U}_i$ defined by the points with $X_i \neq 0$.

18.4.8 The real story: spin representation of $\mathrm{Spin}(n,n)$

Finally, we note a purely real analog of the above construction which is useful in geometry and supersymmetric quantum mechanics.

We begin with an example:

Let W be a real vector space and consider of dimension n and consider $V = W \oplus W^\vee$. Note that V admits a natural nondegenerate quadratic form of signature $(+1^n, -1^n)$ where we take $W, W^\vee$ to be isotropic and use the pairing $W \times W^\vee \to \mathbb{R}$. That is, if we choose a basis w_i for W and a dual basis $\hat{w}^i$ for $W^\vee$ then with respect to this basis

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18.256}$$

The resulting Clifford algebra is $C\ell_{n,-n} \cong \mathrm{End}(\mathbb{R}^{2^{n-1}|2^{n-1}})$.

We know there is a unique irrep up to isomorphism. One way to construct it is by taking the representation space to be $\Lambda^* W^\vee$. In close analogy to the complex case we let $\rho(\check{w})$ for $\check{w} \in W^\vee$ be defined by wedge product, $\check{w}\wedge$ and we let $\rho(w)$ for $w \in W$ be defined by $\rho(w) = \iota(w)$ where $\iota(w)$ is the contraction operator:

$$\iota(w)(\hat{w}^{i_1} \wedge \cdots \wedge \hat{w}^{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} \langle w, \hat{w}^{i_j} \rangle \hat{w}^{i_1} \wedge \cdots \wedge \hat{w}^{i_{j-1}} \wedge \hat{w}^{i_{j+1}} \wedge \cdots \wedge \hat{w}^{i_n} \tag{18.257}$$ eq:contracta

We then extend to V by linearity. A simple computation shows that

$$\begin{aligned} \{\rho(w), \rho(w')\} &= 0 \\ \{\rho(\hat{w}), \rho(\hat{w}')\} &= 0 \\ \{\rho(w), \rho(\hat{w}')\} &= \langle w, \hat{w}' \rangle \end{aligned} \tag{18.258}$$ eq:sensf

and thus the Clifford relations are satisfied.

An important example where this appears is in the quantization of fermions in supersymmetric quantum mechanics. If M is a manifold we can consider $TM \oplus T^*M$ which has a natural quadratic form of signature (n,n) since TM and T^*M are dual spaces. Note that $W = TM$ a maximal isotropic subspace, and a natural choice of complementary isotropic subspace is $U = T^*M$. Then the Clifford algebra acts on the DeRham complex $\Lambda^* T^* M$. Now $\psi^\mu = \rho(dx^\mu)$ is the action by wedge product, and $\chi_\mu = \rho(w_\mu) = \iota(\frac{\partial}{\partial x^\mu})$ acts by contraction. Thus we realize the fermionic CCR's

$$\begin{aligned} \{\psi^\mu, \psi^\nu\} &= 0 \\ \{\chi_\mu, \chi_\nu\} &= 0 \\ \{\psi^\mu, \chi_\nu\} &= \delta^\mu{}_\nu \end{aligned} \tag{18.259}$$

on a Hilbert space - the DeRham complex at a point $\phi \in M$ given by the bosonic coordinate.

The above construction can be generalized as follows:

Suppose V is $2n$-dimensional with a nondegenerate metric of signature (n,n). Thus $C\ell(n_+, n_-) \cong \mathbb{R}(2^n)$ and we wish to construct the 2^n-dimensional irrep. Suppose we have a decomposition of V into two maximal isotropic subspaces $V = W \oplus U$ where W, U are maximal isotropic. That is, with respect to this decomposition we have

$$Q = \begin{pmatrix} 0 & q \\ q^{\dagger} & 0 \end{pmatrix} \tag{18.260}$$

where $q : U \to W$ is an isomorphism.

Then, we claim, the exterior algebra $\Lambda^*(V/W)$ is naturally a 2^n dimensional representation of the Clifford algebra on V.

$u \in U$ acts on $\Lambda^*(V/W)$ by wedge product: Note that V/W acts via wedge product. Since U is a subspace of V it descends to a subspace of V/W and hence it acts by wedge product. On the other hand, $w \in W$ acts by contraction

$$\iota(w)([v_{i_1}] \wedge \cdots \wedge [v_{i_n}]) = \sum_{j=1}^{n} (-1)^{j-1} Q(w, v_{i_j}) [v_{i_1}] \wedge \cdots \wedge [v_{i_{j-1}}] \wedge [v_{i_{j+1}}] \wedge \cdots \wedge [v_{i_n}] \tag{18.261}$$

eq:contractaa

Note that the expression $Q(w, v_{i_j})$ is unambiguous because W is isotropic.

There is an alternative description of the same representation since one can show that $V/W \cong W^*$. To see this note that given v, $\ell_v : w \mapsto (v, w)$ is an element of W^* and $\ell_v = \ell_{v+w}$ for $w \in W$ (since W is isotropic). Thus we could also have represented the Clifford algebra on $\Lambda^* W^*$. Elements of W act by contraction and elements of U act by wedge product (where one needs to use the isomorphism $V/W \cong W^*$.)

As in the complex case there is a family of such decompositions, parametrized by a Grassmannian of isotropic subspaces.

19. Free fermion dynamics and their symmetries

ec:FF-Dynamics

19.1 FDFS with symmetry

ec:FDFS-WthSym

Finally, let us define formally what it means for a FDFS to have a symmetry.

Definition: Let (G, ϕ) be a $\mathbb{Z}_2$-graded group with $\phi : G \to \mathbb{Z}_2$. We will say that (G, ϕ) *acts as a group of symmetries of the FDFS* if

1. There is a homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}_F)$ of $\mathbb{Z}_2$-graded groups. That is, $\mathcal{H}_F$ is a ϕ-rep of G. (See Section §8.1.) subsec:PhiRepBasics

2. There is a compatible automorphism α of the $*$-algebra $\mathcal{A}$ so that $\mathcal{A}$ is a ϕ-representation of G. That is, $\alpha(g)$ is $\mathbb{C}$-linear or anti-linear according to ϕ and ρ and α are compatible in the sense that:
$$\rho(g)\rho_F(a)\rho(g)^{-1} = \rho_F(\alpha(g) \cdot a) \tag{19.1}$$
eq:compatible

3. The automorphism preserves the real subspace $\mathcal{M} \subset \mathcal{A}$, and hence we have a group homomorphism: $\alpha : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{M}, Q) = O(\mathcal{M}, Q) \cong O(N)$.

Remarks:

1. Assuming ρ_F is faithful and surjective (as happens for example if N is even and we choose an irreducible Clifford module for $\mathcal{H}_F$) the map $a \mapsto a'$ defined by

$$\rho(g)\rho_F(a)\rho(g)^{-1} = \rho_F(a') \tag{19.2}$$

   defines the automorphism of $\mathcal{A}$. When $\mathcal{A}$ is a central simple algebra it must be inner. The condition (3) above puts a further restriction on what elements we can conjugate by.

2. We put condition (3) because we want the symmetry to preserve the notion of a fermionic field. The mode space $\mathcal{M}$ is the space of real fermionic fields. It should then preserve Q because we want it to preserve the canonical commutation relations. In terms of operators on $\mathcal{H}_F$:

$$\rho(g)\rho_F(e_j)\rho(g)^{-1} = \sum_m S_{mj}\rho_F(e_m) \tag{19.3}$$

   where $g \mapsto S(g) \in O(N)$ is a representation of G by orthogonal matrices.

3. When constructing examples it is natural to start with a homomorphism $\alpha : G \to O(N)$. We then automatically have an extension to an automorphism of $\mathrm{Cliff}(\mathcal{M}, Q)$. There is no *a priori* extension to an automorphism of $\mathcal{A}$. The data of the ϕ-representation determines that extension because $a \mapsto \rho_F(a)$ is $\mathbb{C}$-linear. It follows that $\rho(g)$ is conjugate linear iff $\alpha(g)$ is conjugate linear. This tells us how to extend α to $\mathrm{Aut}_{\mathbb{R}}(\mathcal{A})$.

Examples

1. By its very construction, the group $G = \mathrm{Pin}^+(N)$ with $\phi = 1$ is a symmetry group of the FDFS generated by $(\mathcal{M}, Q)$ for $\mathcal{M}$ of dimension N. We can simply take $\rho = \rho_F$. This forces us to take $\alpha = \mathrm{Ad}$. [52]

2. What about $G = \mathrm{Pin}^-(N)$? In fact we can make $G = \mathrm{Pin}^c(N)$ (which contains both $\mathrm{Pin}^{\pm}(N)$ as subgroups) act. We think of generators of $\mathrm{Pin}^c(N)$ as ζe_i where $|\zeta| = 1$ is in $U(1)$. Then $\rho(\zeta e_i) = \zeta\rho_F(e_i)$ and $\alpha(\zeta e_i) = \mathrm{Ad}(e_i)$. Again we take $\phi = 1$ in this example.

3. Now we can ask what $\mathbb{Z}_2$-gradings we can give, say, $G = \mathrm{Pin}^+(N)$. Since we take ϕ to be continuous $\phi = 1$ on the connected component of the identity. Then if we take $\phi(v) = -1$ for some norm-one vector then if v' is any other norm-one vector $vv' \in \mathrm{Spin}(N)$ and hence $\phi(vv') = 1$ so $\phi(v') = -1$. Therefore the only nontrivial $\mathbb{Z}_2$-grading is given by the determinant representation described in (eq:DetRep 17.30) above. If we use this then in general there is no consistent action of (G, ϕ) on the N-dimensional FDFS.

♣Maybe when $C\ell_N$ has real reps it is ok? ♣

[52] Note that it is Ad and not $\widetilde{\mathrm{Ad}}$. This leads to some important signs below.

4. To give a very simple example with $\phi \neq 1$ consider $N = 2$, hence a single oscillator $a, \bar{a}$ and let $G = \mathbb{Z}_4 = \langle T | T^4 = 1 \rangle$. Then, in the explicit representation of subsubsec:ExplicitRepGamma 18.4.1 take

$$\begin{aligned} \rho(T)|+\rangle &= -|-\rangle \\ \rho(T)|-\rangle &= |+\rangle \end{aligned} \tag{19.4}$$

and extend by linearity for $\phi(T) = +1$, and by anti-linearity for $\phi(T) = -1$, to define $\rho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}_F)$. In either case $\alpha(T) \cdot e_1 = -e_1$, but a small computation shows that

$$\alpha(T) \cdot e_2 = \begin{cases} e_2 & \phi(T) = +1 \\ -e_2 & \phi(T) = -1 \end{cases} \tag{19.5}$$

Note that

$$\begin{aligned} \alpha(T) \cdot a &= -\bar{a} \\ \alpha(T) \cdot \bar{a} &= -a \end{aligned} \tag{19.6}$$

in *both cases* $\phi(T) = \pm 1$.

5. Inside the real Clifford algebra generated by e_i is a *group* $\mathcal{E}_N$ generated by e_i. This group is discrete, has 2^{N+1} elements and is a nonabelian extension of $\mathbb{Z}_2^N$ with cocycle determined from $e_i e_j e_i^{-1} e_j^{-1} = -1$ for $i \neq j$. $\mathcal{E}_N$ is known as an *extraspecial group*. Suppose T_i, $i = 1, \dots, k$ with $2k \leq N$ generate an extraspecial group $\mathcal{E}_k$ of order 2^{k+1}. Thus, $T_i^2 = 1$ and $T_i T_j T_i^{-1} T_j^{-1} = -1$ for $i \neq j$. Then there are many $\mathbb{Z}_2$ gradings ϕ of $\mathcal{E}_k$ because we can choose the sign of $\phi(T_i)$ independently for each generator. For each such choice $(\mathcal{E}_k, \phi)$ acts as a symmetry group of the N-dimensional FDFS. Using the basis for the explicit representation of subsubsec:ExplicitRepGamma 18.4.1 we can take $\rho(T_i) = \rho_F(e_{2i-1})$. Since the latter matrix is real the operators $\rho(T_i)$ can be consistently anti-linearly extended in the basis of subsubsec:ExplicitRepGamma 18.4.1. A small computation shows that

$$\alpha(T_i) \cdot e_{2j} = \begin{cases} -e_{2j} & \phi(T_i) = +1 \\ e_{2j} & \phi(T_i) = -1 \end{cases} \tag{19.7}$$

but

$$\alpha(T_i) \cdot \bar{a}_j = \begin{cases} a_i & j = i \\ -a_j & j \neq i \end{cases} \tag{19.8}$$

$$\alpha(T_i) \cdot a_j = \begin{cases} \bar{a}_i & j = i \\ -\bar{a}_j & j \neq i \end{cases} \tag{19.9}$$

independent of the choice of ϕ.

♣Now comment on possibilities for ϕ and what the Dyson classes would be. ♣

♣Put general comments on dynamics here? ♣

19.2 Free fermion dynamics

In general, the Hamiltonian is a self-adjoint element of the operator $*$-algebra and thus has the form (eq:GenSelfAdj 18.4). We will distinguish a $*$-invariant element $h \in \mathcal{A}$ from the Fock space Hamiltonian $H := \rho_F(h)$.

Usually, for reasons of rotational invariance, physicists restrict attention to Hamiltonians in the even part of the Clifford algebra, so then

$$h = h_0 + \sum_{k=0(2)} h_{i_1 \dots i_k} e_{i_1 \dots i_k} \tag{19.10}$$

with $h_0 \in \mathbb{R}$ and $h^*_{i_1 \dots i_k} = (-1)^{k/2} h_{i_1 \dots i_k}$. These elements generate a one-parameter group of automorphisms $\mathrm{Ad}(u(t))$ on $\mathcal{A}$ where $u(t) = e^{-\mathrm{i}th}$. Related to this is a one-parameter group of unitary operators

$$U(t) = \rho_F(u(t)) = e^{-\mathrm{i}tH} \tag{19.11}$$

on $\mathcal{H}_F$ representing time evolution in the Schrödinger picture.

In the Heisenberg picture $\mathrm{Ad}(u(t))$ induces a one-parameter group of automorphisms of the algebra of operators and in particular the fermions themselves evolve according to

$$u(t)^{-1} e_i u(t) = e_i + \sqrt{-1} t \sum_I h_I [e_I, e_i] + \mathcal{O}(t^2) \tag{19.12}$$

where we have denoted a multi-index $I = \{i_1 < \cdots < i_k\}$. Terms with $k > 2$ will clearly not preserve the subspace $\mathcal{M}$ in $\mathcal{A}$.

By definition, a *free fermion dynamics* is generated by a Hamiltonian h such that $\mathrm{Ad}(u(t))$ preserves the subspace $\mathcal{M}$. (Note well, when expressed in terms of harmonic oscillators relative to some complex structure it might or might not commute with $\mathcal{F}$.) The most general Hamiltonian defining free fermion dynamics is a self-adjoint element of $\mathcal{A} = \mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C}$ which can be written with at most two generators. Therefore, the general free fermion Hamiltonian is

$$h = h_0 + \frac{\sqrt{-1}}{4} \sum_{i,j} A_{jk} e_j e_k \tag{19.13}$$

eq:FF-Hamiltonian

where $A_{ij} = -A_{ji}$ is a *real antisymmetric matrix.*

Remarks

1. Note well that A_{ij} is an element of the real Lie algebra $so(N)$ and indeed

$$\frac{1}{4} \sum_{j,k} A_{jk} e_j e_k \tag{19.14}$$

   is the corresponding element of $\mathrm{spin}(N) \cong so(N)$.

2. As we remarked, there are two Hilbert spaces associated to the fermionic system. In the Fock space $\mathcal{H}_F$ we have Hamiltonian

$$H = h_0 + \frac{\sqrt{-1}}{4} \sum_{i,j} A_{jk} \rho_F(e_j e_k) \tag{19.15}$$

eq:FF-Hamiltonian

and, up to a trivial evolution by $e^{-\mathrm{i}h_0 t}$, the free fermion dynamics is the action of a one-parameter subgroup $U(t)$ of $\mathrm{Spin}(2N)$ acting on the spin representation, in the Schrödinger picture. In the Heisenberg picture the corresponding dynamical evolution preserves the real subspace $\mathcal{M} \subset \mathcal{A}$ is given by the real vector representation: $\widetilde{\mathrm{Ad}}(u(t))$.

3. Upon choosing a complex structure we have a second Hilbert space, the Dirac-Nambu Hilbert space $\mathcal{H}_{DN} := V \cong W \oplus \bar{W}$ and, (only in the case of free fermion dynamics) $U(t)$ induces an action on V. This is simply $\widetilde{\mathrm{Ad}}(u(t))$ on $\mathcal{M}$ extended $\mathbb{C}$-linearly to $V = \mathcal{M} \otimes \mathbb{C}$. The "Dirac-Nambu Hamiltonian" is therefore just $\rho_{DN}(h) := \mathrm{Ad}(h)$ acting on V, thought of as a subspace of $\mathrm{Cliff}(V, Q)$.

4. Any real antisymmetric matrix can be skew-diagonalized by an orthogonal transformation. That is, given A_{ij} there is an orthogonal transformation R so that

$$RAR^{tr} = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \tag{19.16}$$

   The Bogoliubov transformation corresponding to R can be implemented unitarily and hence if h_0 is zero then the spectrum of $\widehat{H}$ must be symmetric about zero. Therefore this is a system in which it is possible to have symmetries with $\chi \neq 0$. In this basis we simply have (with $h_0 = 0$)

$$h = \sum \lambda_j \bar{a}_j a_j - \frac{1}{2}(\lambda_1 + \cdots + \lambda_n) \tag{19.17}$$

   The spectrum of the Hamiltonian on $\mathcal{H}_{DN}$ is $\{\pm\lambda_j\}$ and on $\mathcal{H}_F$ is $\{\frac{1}{2}\sum_i \epsilon_i \lambda_i\}$ where $\epsilon_i \in \{\pm 1\}$.

Exercise

Compute the time evolution on $\mathcal{M}$ of the one-parameter subgroup generated by the self-adjoint operator e_i. [53]

19.3 Symmetries of free fermion systems

Now suppose we have a $\mathbb{Z}_2$-graded group (G, ϕ) acting as a group of symmetries of a finite dimensional fermion system. We therefore have the following data: $(\mathcal{M}, Q)$ together with a $*$-representation of $\mathcal{A} = \mathrm{Cliff}(V; Q)$ on the Hilbert space $\mathcal{H}_F$ together with the homomorphisms α and ρ satisfying (eq:compatible 19.1).

Suppose furthermore that we have a free fermionic system, hence a Hamiltonian of the form (eq:FF-Hamiltonian 19.13).

[53] *Answer:* $e_j(t) = \cos(2t)e_j + \mathrm{i}\sin(2t)e_i e_j$ for $j \neq i$ and $e_i(t) = e_i$.

Definition: We say that G is acting as a group of symmetries of the dynamics of the free fermionic system if

$$\rho(g)U(s)\rho(g) = U(s)^{\tau(g)} \tag{19.18}$$

eq:symm-dynamics

for *some* homomorphism $\tau : G \to \mathbb{Z}_2$. Here $U(s) = \exp[-isH/\hbar]$ is the one-parameter time evolution operator. If (19.18) holds then we declare g with $\tau(g) = -1$ to be time-reversing symmetries.

1. The above definition looks like a repeat of our previous definition of a symmetry of the dynamics from Section §9. The data $(\mathcal{M}, Q, \mathcal{H}_F, G, \phi, \alpha, \rho, H)$ determine $\rho(g)H\rho(g)^{-1}$. With our logical setup here, a symmetry of the fermionic system is a symmetry of the dynamics if there is *some* homomorphism $\chi : G \to \mathbb{Z}_2$ so that

$$\rho(g)H\rho(g)^{-1} = \chi(g)H \tag{19.19}$$

eq:c-homom

Then because general quantum mechanics requires $\phi\tau\chi = 1$, we will *declare* g to be time-orientation preserving or reversing according to $\tau(g) := \phi(g)\chi(g)$. This logic is reversed from our standard approach where we consider ϕ determined by an *a priori* given homomorphism $G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ together with an *a priori* given homomorphism τ determined by an *a priori* action on spacetime.

2. There will be physical situations, e.g. a single electron moving in a crystal where there is an *a priori* notion of what time-reversing symmetries should be and how they should act on fermion fields.

3. Let us see what the above definition implies for the transformation of the oscillators under $\widetilde{\mathrm{Ad}}$. Choose an ON basis for $(\mathcal{M}, Q)$ satisfying (18.3). Then, in terms of operators on $\mathcal{H}_F$:

$$\rho(g)\rho_F(e_j)\rho(g)^{-1} = \sum_m S_{mj}\rho_F(e_m) \tag{19.20}$$

♣Maybe S_{mj} should be $\alpha(g)_{mj}$.♣ eq:Rot-Ferm-Osc

Or, equivalently:

$$\alpha(g) \cdot e_j = \sum_m S_{mj} e_m \tag{19.21}$$

so

$$\begin{aligned} \rho(g)H\rho(g)^{-1} &= h_0 + \phi(g)\frac{\mathrm{i}}{4}\sum_{m,n}(SAS^{tr})_{mn}\rho_F(e_m e_n) \\ &= \chi(g)H \end{aligned} \tag{19.22}$$

This shows that

1. If $\chi(g) = -1$ for any $g \in G$ then $h_0 = 0$.
2. The matrix A must satisfy

$$\boxed{S(g)AS(g)^{tr} = \tau(g)A} \tag{19.23}$$

eq:gA

for all $g \in G$, where $\tau(g)$ is either prescribed, or deduced from $\tau = \phi \cdot \chi$, depending on what logical viewpoint we are taking.

The condition (19.23) can be expressed more invariantly: Given $\alpha : G \to O(\mathcal{M}, Q)$ there is an induced action $\mathrm{Ad}_{\alpha(g)}$ on $o(\mathcal{M}, Q)$, and we are requiring that

$$\mathrm{Ad}_{\alpha(g)} A = \tau(g) A. \tag{19.24}$$

eq:gA-p

19.4 The free fermion Dyson problem and the Altland-Zirnbauer classification

There is a natural analog of the Dyson problem suggested by the symmetries of free fermionic systems:

Given a finite dimensional fermionic system $(\mathcal{M}, Q, \mathcal{H}_F, \rho_F)$ and a $\mathbb{Z}_2$-graded group (G, ϕ) acting as a symmetry on the FDFS via (α, ρ), what is the ensemble of free Hamiltonians for the FDFS such that (G, ϕ) is a symmetry of the dynamics?

Note well! We have changed the Dyson problem for the ϕ-rep $\mathcal{H}_F$ of G in a crucial way by restricting the ensemble to free fermion Hamiltonians.

Our analysis above which led to (19.23) above shows that the answer, at one level, is immediate from (19.23): We have the subspace in $o(Q; \mathbb{R})$ satisfying (19.23). Somewhat surprisingly, this answer depends only on α and τ as is evident from (19.24). For a given τ there can be more than one choice for ϕ and χ.

However, the answer can be organized in a very nice way as noticed by Altland and Zirnbauer [4]: Such free fermion ensembles can be identified with the tangent space at the origin of classical Cartan symmetric spaces. This result was proved more formally in a subsequent paper of Heinzner, Huckleberry, and Zirnbauer [25]. In the next section we explain the main idea.

19.4.1 Classification by compact classical symmetric spaces

subsec:AZ-Cpt-SS

Let us consider two subspaces of $o(2n; \mathbb{R})$:

$$\mathfrak{k} := \{A | Ad_{\alpha(g)}(A) = A\} \tag{19.25}$$

eq:Def-k-fermion

$$\mathfrak{p} := \{A | Ad_{\alpha(g)}(A) = \tau(g) A\} \tag{19.26}$$

eq:Def-p-fermion

$\mathfrak{p}$ is of course the ensemble we want to understand. If $\tau = 1$ it is identical to $\mathfrak{k}$ but in general, when $\tau \neq 1$ it is *not* a Lie subalgebra of $o(2n; \mathbb{R})$ because the Lie bracket of two elements in $\mathfrak{p}$ is in $\mathfrak{k}$. This motivates us to define a Lie algebra structure on

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{19.27}$$

by

$$[k_1 \oplus p_1, k_2 \oplus p_2] := ([k_1, k_2] + [p_1, p_2]) \oplus ([p_1, k_2] + [k_1, p_2]) \tag{19.28}$$

One can check this satisfies the Jacobi relation.

Note that we have an automorphism of the Lie algebra which is $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$, so this is a Cartan decomposition.

Both $\mathfrak{k}$ and $\mathfrak{g}$ are classical Lie algebras: This means that they are Lie subalgebras of matrix Lie algebras over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ preserving a bilinear or sesquilinear form.

To prove this for $\mathfrak{k}$: Note that we have a representation of G on $o(\mathcal{M}; Q) \cong o(2n; \mathbb{R})$. If G is compact this representation must decompose into irreducible representations. The group algebra is therefore a direct sum of algebras of the form $n\mathbb{K}(m)$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. By the Weyl duality theorem (8.45),(8.46) the commutant is $m\mathbb{K}(n)$. Since $\mathfrak{k}$ is, by definition, the commutant, when restricted to each irreducible representation $\exp[\mathfrak{k}]$ must generate a matrix algebra over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Therefore, $\mathfrak{k}$ is a classical Lie algebra.

A similar argument works to show that $\mathfrak{g}$ is a classical Lie algebra. There is a Lie algebra homomorphism

$$\mathfrak{g} \to o(\mathcal{M}, Q) \oplus o(\mathcal{M}, Q) \tag{19.29}$$

given by

$$k \oplus p \to (k + p) \oplus (k - p) \tag{19.30}$$

Now, we can characterize $\mathfrak{g}$ as the commutant of a representation of G on $\mathcal{M} \oplus \mathcal{M}$ given by

♣Should $\alpha(g)$ be denoted $S(g)$? ♣

$$g \mapsto \begin{pmatrix} \alpha(g) & 0 \\ 0 & \alpha(g) \end{pmatrix} \qquad \tau(g) = 1 \tag{19.31}$$

$$g \mapsto \begin{pmatrix} 0 & \alpha(g) \\ \alpha(g) & 0 \end{pmatrix} \qquad \tau(g) = -1 \tag{19.32}$$

We embed $\mathfrak{g}$ into $o(\mathcal{M}) \oplus o(\mathcal{M})$. The matrices in the commutant of the form $x \otimes 1_2$ is isomorphic to $\mathfrak{k}$ and the matrices in the commutant of the image of G which are of the form $x \otimes \sigma^3$ is isomorphic to $\mathfrak{p}$. Hence $\mathfrak{k}$ and $\mathfrak{g}$ are both classical real Lie algebras.

♣explain why we don't need to worry about other kinds of matrices in the commutant. ♣

Next, note that the Killing form of $o(\mathcal{M}; Q)$ restricts to a Killing form on $\mathfrak{k}$ and on $\mathfrak{g}$. It is therefore negative definite. Hence the real Lie algebras $\mathfrak{k}$ and $\mathfrak{g}$ are of compact type.

This proves the theorem of [25]:

Theorem: The ensemble $\mathfrak{p}$ of free fermion Hamiltonians in $\mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C}$ compatible with (α, τ) is the tangent space at the identity of a classical compact symmetric space G/K.

We have collected a few definitions and facts about symmetric spaces in Appendix C.

19.4.2 Examples of AZ classes

1. Let $G = \mathrm{Spin}(2)$. Choose an ON basis $\{e_i\}$ for $\mathcal{M}$ and consider G to be the subgroup generated by $\frac{1}{2}e_{12} + \cdots + \frac{1}{2}e_{2n-1,2n}$. Then take $\rho = \rho_F$ and $\alpha = \widetilde{\mathrm{Ad}}$. If we choose the complex structure (18.33) then the group commutes with the Fermion number operator $\mathcal{F}$ and the action of

$$\mathrm{Ad}\left(\exp[\theta(\frac{1}{2}e_{12} + \cdots + \frac{1}{2}e_{2n-1,2n})]\right) \tag{19.33}$$

takes

$$a_i \to e^{2i\theta} a_i \qquad\qquad \bar{a}_i \to e^{-2i\theta} \bar{a}_i \tag{19.34}$$

Since $G = \mathrm{Spin}(2)$ is connected we must take $\phi = \chi = \tau = 1$. Therefore the free fermion Hamiltonians which respect this symmetry have the form

$$h = h_0 + \sum_{i,j=1}^{n} h_{ij}\bar{a}_i a_j \tag{19.35}$$

where h_{ij} is an Hermitian matrix. We easily compute that in this example $\mathfrak{k} \cong u(n)$ and $\mathfrak{p} \cong u(n)$ (as a vector space) so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \cong u(n) \oplus u(n)$ (as a Lie algebra), where $\mathfrak{k}$ is the diagonal and $\mathfrak{p}$ is the antidiagonal. In this case

$$G/K = (U(n) \times U(n))/U(n) \tag{19.36}$$

2. Let $G = \mathbb{Z}_2 \cong \{1, \bar{T}\}$ and choose $\tau(\bar{T}) = -1$ and let $\alpha(\bar{T})$ be

$$\begin{pmatrix} 1_\ell & 0 \\ 0 & -1_{N-\ell} \end{pmatrix} \tag{19.37}$$ eq:alphbarT

in an ON basis for $(\mathcal{M}, Q)$. Then $\mathfrak{k} \cong o(\ell) \oplus o(N-\ell)$ is the Lie subalgebra of $N \times N$ of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{19.38}$$

and $\mathfrak{p}$ is the subspace of matrices of the form

$$\begin{pmatrix} 0 & B \\ -B^{tr} & 0 \end{pmatrix} \tag{19.39}$$

so the symmetric space is

$$G/K = O(N)/O(\ell) \times O(N-\ell). \tag{19.40}$$

Explicitly, this class of Hamiltonians is:

$$h = \frac{\mathrm{i}}{2} \sum_{j=1}^{\ell} \sum_{\ell+1}^{N} B_{jk} e_j e_k \tag{19.41}$$

where B_{jk} is a real $\ell \times (N-\ell)$ matrix.

♣Note we said nothing about ρ, ϕ, χ. ♣

3. Let $G = \mathrm{Pin}^c(1)$. This has two components, consisting of $\zeta \in U(1)$ and ζT with $T^2 = 1$ and $\tau(\zeta T) = -1$. We suppose $N = 2n$ and for $\zeta = e^{\mathrm{i}\theta}$ we let

$$\rho(\zeta) = \rho_F \left(\exp[\theta(\frac{1}{2}e_{12} + \cdots + \frac{1}{2}e_{2n-1,2n})] \right) \tag{19.42}$$

and we take $\rho(T)$ so that $\alpha(T)$ has the form (eq:alphbarT 19.37) with $\ell = 2k$. Then $\mathfrak{k} \cong u(k) \oplus u(n-k)$ and $\mathfrak{p}$ is the tangent space to

$$G/K = U(n)/(U(k) \times U(n-k)). \tag{19.43}$$

eq:CanonCS

4. Returning to $G = \mathbb{Z}_2$ suppose $\alpha(\bar{T}) = I_0$ where I_0 is given in (7.9) and $\tau(\bar{T}) = -1$. Then $\mathfrak{k} \cong u(n)$. Writing $I_0 = 1_n \otimes \epsilon$ we see that $\mathfrak{p}$ consists of matrices of the form $b \otimes (x_1\sigma^1 + x_2\sigma^3)$ where b is real antisymmetric and x_1, x_2 are real. Thus, using the oscillators suited to I_0 the Hamiltonian is of the form

$$h = \frac{1}{2}\sum_{i,j=1}^{n}\left(\beta_{ij}\bar{a}_i\bar{a}_j + \beta_{ij}^* a_j a_i\right) \tag{19.44}$$

where β_{ij} is complex antisymmetric. In this case $G/K = SO(2n)/U(n)$.

It is interesting to compare the AZ ensembles with the ensembles of Hamiltonians one would meet in Dyson's 3-fold way or in the 10-fold way in the above examples. In each example there are two relevant Hilbert spaces to consider, namely $\mathcal{H}_{DN}$ and $\mathcal{H}_F$.

In example 1 above, for example, $\mathcal{H}_{DN}$ has isotypical decomposition:

$$\mathcal{H}_{DN} \cong \mathbb{C}^n \otimes V_2 \oplus \mathbb{C}^n \otimes V_{-2} \tag{19.45}$$

where V_q denotes the one-dimensional irrep of Spin(2) of charge q (normalized to be integral). The commutant for these irreps is $D = \mathbb{C}$. The ensemble of commuting Hamiltonians is therefore $\mathrm{Herm}_n(\mathbb{C}) \times \mathrm{Herm}_n(\mathbb{C})$. Applied to the Fock space the isotypical decomposition is

$$\mathcal{H}_F \cong \oplus_{k=0}^{n}\mathbb{C}^{\binom{n}{k}} \otimes V_{2k-n} \tag{19.46}$$

and so Dyson's ensemble is $\prod_k \mathrm{Herm}_{\binom{n}{k}}(\mathbb{C})$.

In example 2 above we have a group with more than one component and hence, in order even to begin discussing the 3-fold or the 10-fold way classification of ensembles on $\mathcal{H}_{DN}$ or $\mathcal{H}_F$ we need to choose ϕ and χ. There are two possibilities: $(\phi(\bar{T}) = +1, \chi(\bar{T}) = -1)$ and $(\phi(\bar{T}) = -1, \chi(\bar{T}) = +1)$. We discuss each of these in turn. ♣Answers here are a bit strange and need to be rechecked! ♣

If $(\phi(\bar{T}) = +1, \chi(\bar{T}) = -1)$ then a (ϕ,χ)-rep must be a *graded* rep of $\mathbb{Z}_2$ and there is one irrep, which is up to isomorphism $V \cong \mathbb{C}^{1|1}$ with $\rho(\bar{T}) = \sigma^1$. Now, in order to have a "gapped Hamiltonian" with 0 not in the spectrum we must have $2\ell = N$. Then the isotypical decomposition of the Dirac-Nambu space is

$$\mathcal{H}_{DN} \cong \mathbb{R}^\ell \otimes V \tag{19.47}$$

The supercommutant of V is generated over $\mathbb{C}$ by 1 and ϵ and is isomorphic to $\mathbb{C}\ell_1$. Therefore, the supercommutant in $\mathcal{H}_{DN}$ is $\mathrm{Mat}_\ell(\mathbb{C}\ell_1)$. Typical elements can be written as $A + \mathrm{i}B\epsilon$ where A, B are $\ell \times \ell$ complex matrices (and the factor of i in front of B is chosen for convenience). When we impose the Hermiticity condition we see that A and B are Hermitian and the ensemble is therefore $\mathrm{Herm}_\ell(\mathbb{C}) \times \mathrm{Herm}_\ell(\mathbb{C})$. ♣Discuss $\mathcal{H}_F$. ♣

Now let us consider the possibility $(\phi(\bar{T}) = -1, \chi(\bar{T}) = +1)$. In this case $\mathbb{Z}_2$-graded group M_2 has two irreducible ϕ-representations, namely $V_\pm \cong \mathbb{C}$ with $\rho(\bar{T})$ acting by $z \to \pm\bar{z}$. Here the commutant is $D_{V_\pm} = \mathbb{R}$ for both irreps.

Now, for simplicity take $\ell = 2k$ and $N = 2n$. Given the action $\alpha(\bar{T})$ on the e_j we extend it to $\mathcal{A}$ using ϕ and get:

$$\begin{aligned} \alpha(\bar{T}) : \bar{a}_j &\leftrightarrow a_j \qquad\qquad j = 1, \dots, k \\ \bar{a}_j &\leftrightarrow -a_j \qquad\qquad j = k+1, \dots, n \end{aligned} \tag{19.48}$$

Now the Dirac-Nambu Hilbert space has isotypical decomposition:

$$\mathcal{H}_{DN} \cong \mathbb{R}^k \otimes (V_+ \oplus V_-) \oplus \mathbb{C}^{n-k} \otimes (V_+ \oplus V_-) \tag{19.49}$$

and hence the Dyson ensemble is $\mathcal{E} = \mathrm{Herm}_n(\mathbb{R}) \times \mathrm{Herm}_n(\mathbb{R})$. Now consider the ensemble for $\mathcal{H}_F$. When we make $\alpha(T) : \bar{a} \leftrightarrow -a$ compatible with ρ_F we find a surprise: There is no consistent action! Rather, in harmony with the general principles described in Section sec:PhiTwistedExts 6 (see especially (eq:Gtau 6.12)) only the ϕ-twisted extension M_2^- acts on $\mathcal{H}_F$, which now must carry a quaternionic structure, and the ensemble of commuting Hamiltonians is again different. ♣FINISH THIS! ♣

One lesson we learn is that the different choices of (ϕ, χ) for fixed τ lead to different ensembles, so when discussing a "10-fold way" one must be very careful about the precise physical question under consideration!

Exercise

Analyze the Dyson ensembles for both $\mathcal{H}_{DN}$ and $\mathcal{H}_F$ for the remaining examples above.

19.4.3 Another 10-fold way

Remarks

1. Cartan classified the compact symmetric spaces. They are of the form G/K where G and K are Lie groups. There are some exceptional cases and then there are several infinite series analogous to the infinite series A, B, C, D of simple Lie algebras. These can be naturally organized into a series of 10 distinct classical symmetric spaces. Thus, the Altland-Zirnbauer argument provides a 10-fold classification of ensembles of free fermionic Hamiltonians. This gives yet another 10-fold way! We will relate it to the 10 Morita equivalence classes of Clifford algebras (and thereby implicitly to the 10 real super-division algebras) below. That relation will involve K-theory.

2. Using the description of the 10 classes given in (eq:ClassifSpaceCart C.7) - (eq:ClassSpace-10 C.17) one can give a description of the 10 AZ classes along the following lines. Recalling (eq:legn 18.124) we can, with a suitable choice of complex structure as basepoint write the free fermion hamiltonian as

$$h = \sum_{i,j} W_{ij} \bar{a}_i a_j + \frac{1}{2} \sum_{i,j} \left(Z_{ij} \bar{a}_i \bar{a}_j + \bar{Z}_{ij} a_j a_i \right) \tag{19.50}$$

where W_{ij} is hermitian and Z_{ij} is a complex antisymmetric matrix. Then the 10 cases correspond to various restrictions on W_{ij} and Z_{ij}. See Table 1 of Zirnbauer2 [44]. ♣Should probably reproduce that table here and explain the entries in detail. ♣

19.5 Realizations in Nature and in Number Theory

1. For realizations of the various AZ classes in physical systems see the descriptions in Zirnbauer1,Zirnbauer2 [43, 44].

2. For realizations of the various classes in Number Theory see the review by Conrey Conrey [14]. ♣Find a more up to date review. ♣

20. Symmetric Spaces and Classifying Spaces

ricClassifying

20.1 The Bott song and the 10 classical Cartan symmetric spaces

Now we will give an elegant description of how the 10 classical symmetric spaces arise directly from the representations of Clifford algebras. This follows a treatment by Milnor MilnorMorse [31]. Then, thanks to a paper of Atiyah and Singer AtiyahSingerSkew [9] we get a connection to the classifying spaces of K-theory. Milnor's construction was discussed in the context of topological insulators by Stone et. al. in Stone [38].

We begin by considering the complex Clifford algebra $\mathbb{C}\ell_{2d}$ and an irreducible representation, which, as a graded representation is $\mathcal{S}_c = \mathbb{C}^{2^{d-1}|2^{d-1}}$. However, we will here consider the Clifford algebra as an ungraded algebra and hence we forget the grading on the representations. Give it the standard Hermitian structure. We can then take the representation of the generators $J_i = \rho(e_i)$ so that $J_i^2 = 1$, $J_i^\dagger = J_i$ and hence J_i are unitary. Then we define a sequence of groups

♣Surely it would be better to keep the grading... ♣

$$G_0 \supset G_1 \supset G_2 \supset \cdots \tag{20.1}$$

We take $G_0 = U(2r)$ where we have denoted $2^d = 2r$ and we define

$$G_k = \{g \in G_0 | g J_s = J_s g \qquad s = 1, \dots, k\} \tag{20.2}$$

We claim that $G_1 \cong U(r) \times U(r)$. One way to see this is to note that G_k is the commutant of the image of $\mathbb{C}\ell_k$ in $\mathrm{End}(\mathcal{S}_c)$. As an ungraded algebra $\mathbb{C}\ell_1$ has two irreps and so we can write $\mathcal{S}_c$ as a sum of ungraded irreps of $\mathbb{C}\ell_1$ and it is easy to show (see below) that they occur as:

$$\mathcal{S}_c \cong r N_1^+ \oplus r N_1^- \tag{20.3}$$

and therefore the algebra $\rho(\mathbb{C}\ell_1)$ has Wedderburn type

$$r\mathbb{C} \oplus r\mathbb{C} \tag{20.4}$$

so the commutant must have Wedderburn type

$$\mathbb{C}(r) \oplus \mathbb{C}(r) \tag{20.5}$$

and the intersection with $\mathrm{Aut}(\mathcal{S}_c)$, which gives precisely G_1, must be

$$G_1 \cong U(r) \times U(r) \tag{20.6}$$

As a check on this reasoning note that we could represent

$$\rho(e_1) = J_1 = \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix} \tag{20.7}$$

and hence the matrices which commute with it are of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{20.8}$$

But such matrices are unitary iff $(A \pm B)$ are unitary. So the group of such unitary matrices is isomorphic to $U(r) \times U(r)$, as claimed. The more abstract argument will be useful in the real case below.

Next for G_2, $\mathbb{C}\ell_2 \cong M_2(\mathbb{C})$, so $\rho(\mathbb{C}\ell_2)$ has Wedderburn type $r\mathbb{C}(2)$ and hence the commutant is $2\mathbb{C}(r)$ so the group G_2 is isomorphic to $U(r)$. As r is a power of 2 we clearly have periodicity so that our sequence of groups is isomorphic to

$$U(2r) \supset U(r) \times U(r) \supset U(r) \supset \cdots \tag{20.9}$$

The successive quotients give the two kinds of symmetric spaces $U(2r)/(U(r) \times U(r))$ and $(U(r) \times U(r))/U(r)$.

♣Don't get Grassmannians $\mathrm{Gr}_{n,m}$ with $n \neq m$. ♣

Now let us move on to the real Clifford algebra $C\ell_{-8d}$. We choose a real graded irreducible representation, $\mathrm{End}(\mathbb{R}^{N|N})$, with $2N = 2^{4d}$. It is convenient to define an integer r by $2N = 16r$. Again, we will regard the Clifford algebras as ungraded and the representation $\mathcal{S} \cong \mathbb{R}^{2N}$. Denote the representations of the generators $J_i = \rho(e_i)$, so of course

$$J_s J_t + J_t J_s = -2\delta_{s,t}. \tag{20.10}$$

We can give $\mathcal{S}$ a Euclidean metric so that the representation of $\mathrm{Pin}^{-}(8d)$ is orthogonal. Therefore, $J_i^{\dagger} = -J_i$, so $J_i^{tr} = -J_i$, and hence $J_i \in o(2N)$. However, since $J_i^2 = -1$ we have $J_i^{tr} = J_i^{-1}$ and hence we also have $J_i \in O(2N)$.

Now we define a sequence of groups

$$G_0 \supset G_1 \supset G_2 \supset \cdots \tag{20.11}$$

These are defined by taking $G_0 := O(2N)$ and for $k > 0$ defining

$$G_k = \{g \in G_0 | g J_s = J_s g \qquad s = 1, \dots, k\} \tag{20.12}$$

Now, we claim that the series of groups is isomorphic to

$$\begin{aligned} O(16r) \supset U(8r) \supset Sp(4r) \supset Sp(2r) \times Sp(2r) \supset Sp(2r) \supset \\ \supset U(2r) \supset O(2r) \supset O(r) \times O(r) \supset O(r) \supset \cdots \end{aligned} \tag{20.13}$$

We will show that this follows easily from the basic Bott genetic code:

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{R}, \cdots \tag{20.14}$$

The argument proceeds as follows. Note that G_k is in the commutant of the image of the Clifford algebra $C\ell_{-k} \subset C\ell_{-8d}$. Now, we decompose $\mathcal{S}$ in terms of ungraded irreps of $C\ell_{-k}$. For $k \neq 3\mathrm{mod}4$ there is a unique irrep N_k up to isomorphism, and for $k = 3\mathrm{mod}4$ there are two $N_k^{\pm}$. Therefore, $\mathcal{S} \cong N_k^{\oplus s_k}$ for $k \neq 3\mathrm{mod}4$ and $\mathcal{S} \cong (N_k^{+})^{\oplus s_k} \oplus (N_k^{-})^{\oplus s_k}$ for $k = 3\mathrm{mod}4$. The number of summands is the same $N_k^{\pm}$ for $k = 3\mathrm{mod}4$ because the decomposition is effected by the projection operator using the volume form $P_{\pm} = \frac{1}{2}(1 \pm \omega_k)$ and $\mathrm{Tr}_{\mathcal{S}}(\omega_k) = 0$ for all k. Now, the image of the Clifford algebra in $\mathrm{End}(\mathcal{S})$ (as an ungraded algebra) will have be isomorphic to $s_k\mathbb{K}(t_k)$ for $k \neq 3\mathrm{mod}4$ and $s_k\mathbb{K}(t_k) \oplus s_k\mathbb{K}(t_k)$

for $k = 3 \bmod 4$. Therefore, by the Weyl theorem the commutant $Z(\rho(C\ell_{-k}))$ will be isomorphic to $t_k\mathbb{K}(s_k)$ for $k \neq 3 \bmod 4$ and $t_k\mathbb{K}(s_k) \oplus t_k\mathbb{K}(s_k)$ for $k = 3 \bmod 4$. When we intersect $Z(\rho(C\ell_{-k}))$ with $\mathrm{Aut}(\mathcal{S}) \cong O(16r)$ we get the group G_k. In this way we determine the following table

k	Bott clock	$\rho(C\ell_{-k})$	$Z(\rho(C\ell_{-k}))$	G_k
0	$\mathbb{R}$	$16r\mathbb{R}$	$\mathbb{R}(16r)$	$O(16r)$
1	$\mathbb{C}$	$8r\mathbb{C}$	$\mathbb{C}(8r)$	$U(8r)$
2	$\mathbb{H}$	$4r\mathbb{H}$	$\mathbb{H}^{\mathrm{opp}}(4r)$	$Sp(4r)$
3	$\mathbb{H}\oplus\mathbb{H}$	$2r\mathbb{H}\oplus 2r\mathbb{H}$	$\mathbb{H}^{\mathrm{opp}}(2r)\oplus\mathbb{H}^{\mathrm{opp}}(2r)$	$Sp(2r)\times Sp(2r)$
4	$\mathbb{H}$	$2r\mathbb{H}(2)$	$2\mathbb{H}^{\mathrm{opp}}(2r)$	$Sp(2r)$
5	$\mathbb{C}$	$2r\mathbb{C}(4)$	$4\mathbb{C}(2r)$	$U(2r)$
6	$\mathbb{R}$	$2r\mathbb{R}(8)$	$8\mathbb{R}(2r)$	$O(2r)$
7	$\mathbb{R}\oplus\mathbb{R}$	$r\mathbb{R}(8)\oplus r\mathbb{R}(8)$	$8\mathbb{R}(r)\oplus 8\mathbb{R}(r)$	$O(r)\times O(r)$
8	$\mathbb{R}$	$r\mathbb{R}(16)$	$16\mathbb{R}(r)$	$O(r)$

We should stress that the entries for $\rho(C\ell_{-k})$, $Z(\rho(C\ell_{-k}))$, and G_k just give the isomorphism type. Of course, r is some power of 2 and for large d we can repeat the periodic sequence down many steps.

♣NEED TO DO COMPLEX CASE OF CARTAN EMBEDDING ♣

The series of homogeneous spaces G_k/G_{k+1} for $k \geq 0$ provide examples of the Cartan symmetric spaces (for ranks which are a power of two!). Note that the tangent space at $1 \cdot G_{k+1}$ has an elegant description. First define $\mathfrak{g}_0 := T_1 G_0 = o(2N)$. Now for $k > 0$ define:

$$\mathfrak{g}_k := T_1 G_k = \{a \in o(2N) | a J_s = J_s a \qquad s = 1, \dots, k\}. \tag{20.15}$$

Observe that, for $k \geq 0$, the map $\theta_k(a) = J_{k+1} a J_{k+1}^{-1}$ acts as an involution on $\mathfrak{g}_k$ and that the eigenspace with $\theta_k = +1$ is just $\mathfrak{g}_{k+1}$. Therefore we can identify

$$\mathfrak{p}_k := T_{G_{k+1}} G_k/G_{k+1} = \{a \in o(2N) | a J_s = J_s a \qquad s = 1, \dots, k \quad \& \quad a J_{k+1} = -J_{k+1} a\} \tag{20.16}$$

so that

$$\mathfrak{g}_k = \mathfrak{g}_{k+1} \oplus \mathfrak{p}_k \tag{20.17}$$

20.2 Cartan embedding of the symmetric spaces

The involution θ_k described above extends to a global involution $\tau_k : G_k \to G_k$ defined by conjugation with J_{k+1}:

$$\tau_k(g) = J_{k+1} g J_{k+1}^{-1} \tag{20.18}$$

Of course, the fixed subgroup of τ_k in G_k is G_{k+1} so the Cartan symmetric space is G_k/G_{k+1}. Moreover, the Cartan embedding of this symmetric space is just

$$\mathcal{O}_k = \{g \in G_k | \tau_k(g) = g^{-1}\} \subset G_k \subset O(2N) \qquad k \geq 0. \tag{20.19}$$ eq:Ok-ORB-1

Let us unpack this definition: The condition $\tau_k(g) = g^{-1}$ is equivalent to the condition $(J_{k+1}g)^2 = -1$. Therefore, writing $\tilde{g} = J_{k+1}g$ we can also write

$$\tilde{\mathcal{O}}_k = \{\tilde{g} \in O(2N) | \tilde{g}^2 = -1 \qquad \{\tilde{g}, J_s\} = 0 \quad s = 1, \dots, k\} \tag{20.20}$$ eq:Ok-ORB-2

The map $g \mapsto \tilde{g} = J_{k+1}g$ is a simple diffeomorphism so $\tilde{\mathcal{O}}_k \cong G_k/G_{k+1}$, and $\tilde{\mathcal{O}}_k$ is also embedded in $O(2N)$. This manifestation of the homogeneous space will be more convenient to work with. Note that:

$$\cdots \subset \tilde{\mathcal{O}}_{k+1} \subset \tilde{\mathcal{O}}_k \subset \cdots \tag{20.21}$$

When we wish to emphasize the dependence on N we will write $\tilde{\mathcal{O}}_k(N)$.

Since $g = \pm 1$ is in $\mathcal{O}_k$ we have $\pm J_{k+1} \in \tilde{\mathcal{O}}_k$, as is immediately verified from the definition. (Note that ± 1 are *not* elements of $\tilde{\mathcal{O}}_k$.) Let us compute the tangent space to $\tilde{\mathcal{O}}_k$ at J_{k+1}. A path through J_{k+1} must be of the form $J_{k+1}e^{ta}$ where $a \in T_1\mathcal{O}_k = \mathfrak{p}_k$. Therefore there is an isomorphism $T_1\mathcal{O}_k \leftrightarrow T_{J_{k+1}}\tilde{\mathcal{O}}_k$ given simply by left-multiplication by J_{k+1}. Now $a \in T_1\mathcal{O}_k$ iff $a^{tr} = -a$, $[a, J_s] = 0$ for $s = 1, \dots, k$ and $\{a, J_{k+1}\} = 0$ and therefore

$$\tilde{\mathfrak{p}}_k := T_{J_{k+1}}\tilde{\mathcal{O}}_k = \{\tilde{a} \in o(2N) | \{\tilde{a}, J_s\} = 0, \qquad s = 1, \dots, k+1\} \tag{20.22}$$ eq:Ok-ORB-3

20.3 Application: Uniform realization of the Altland-Zirnbauer classes

The characterization (20.20) of $\tilde{\mathfrak{p}}_k$ is nicely suited to a realization of 8 of the 10 AZ classes of free fermion Hamiltonians. We take a FDFS based on $\mathcal{M} = \mathbb{R}^{2N}$ with Q the Euclidean metric. We take as our symmetry group $G = \mathrm{Pin}^-(k+1)$ with Clifford generators T_i. We choose the nontrivial option for τ on G, thus $\tau(T_i) = -1$ for $i = 1, \dots, k+1$. For α we choose the embedding of G into $O(2N)$ using $\alpha(T_i) = \mathrm{Ad}(e_i)$ (not $\widetilde{\mathrm{Ad}}$) acting on $\mathcal{M} \subset C\ell_{-8d}$. Comparing the definitions (19.25) and (19.26) we find that we have precisely

$$\begin{aligned} \mathfrak{k} &= \mathfrak{g}_k \\ \mathfrak{p} &= \tilde{\mathfrak{p}}_k \end{aligned} \tag{20.23}$$

thus neatly exhibiting examples of 8 of the AZ 10 classes.

The remaining two AZ classes follow from completely analogous manipulations for the series $U(2r) \supset U(r) \times U(r) \supset U(r) \supset \cdots$.

Remarks:

1. Note that our fermionic oscillators are a basis for the spin representation of $\mathrm{Spin}(8d)$. So *their* Hilbert space will be a representation of the much larger group $\mathrm{Spin}(2N)$ of dimension $2^N = 2^{2^{8r}} = 2^{2^{4d-1}}$.

2. This example can be extended to compute the 3- and 10-fold classes on $\mathcal{H}_{DN}$ and $\mathcal{H}_F$. Again there are two options $(\phi(T_i) = +1, \chi(T_i) = -1)$ and $(\phi(T_i) = -1, \chi(T_i) = +1)$. Representing the J_i by real matrices on $\mathcal{H}_F$ we can take $\rho = \rho_F$ restricted to $C\ell_{-k-1}$.

♣Finish by analyzing the classes with this choice. ♣

20.4 Relation to Morse theory and loop spaces

The homogeneous spaces $\tilde{\mathcal{O}}_k$ have a further beautiful significance when we bring in some ideas from Morse theory.

We consider the quantum mechanics of a particle moving on these manifolds using the action

$$S[q] = -\int dt \mathrm{Tr}(q^{-1}\frac{dq}{dt})^2 \tag{20.24}$$

eq:ParticleAction

where $q(t)$ is a path in the orthogonal group or one of the $\tilde{\mathcal{O}}_k$.

We begin with quantum mechanics on $SO(2N)$. We choose boundary conditions and define $\mathcal{P}_0$ to be the space of (continuously differentiable) paths $q : [0,1] \to SO(2N)$ such that $q(0) = +1_{2N}$ and $q(1) = -1_{2N}$. We are particularly interested in the *minimal action paths*. Such paths will be geodesics in the left-right-invariant metric. The geodesics are well known to be of the form $q(t) = \exp[\pi t A]$ with $A \in so(2N)$. We can always conjugate A to the form

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & a_N \\ -a_N & 0 \end{pmatrix} \tag{20.25}$$

where $a_i \in \mathbb{R}$. This has action $2\pi^2 \sum a_i^2$ and the boundary conditions imply that a_i are odd integers. Therefore the minimal action paths have $a_i = \pm 1$ and hence the space of minimal action paths is precisely given by the conjugacy class of $A \in o(2N)$ with $A^2 = -1$. Moreover, such paths have a very simple form:

$$q(t) = \cos \pi t + A \sin \pi t \tag{20.26}$$

Now, notice a trivial but significant fact:

1. If $g \in O(2N)$ is an orthogonal matrix and $g^2 = -1$ then $g^{tr} = -g$ and hence $g \in o(2N)$ is also in the Lie algebra.

2. If $A \in o(2N)$ is in the Lie algebra and $A^2 = -1$ then $A^{tr} = A^{-1}$ and hence $A \in O(2N)$ is also in the Lie group.

Therefore, the space of minimal action paths in $\mathcal{P}_0$ is naturally identified with

$$\tilde{\mathcal{O}}_0 := \{g \in O(2N) | g^2 = -1\} \subset O(2N). \tag{20.27}$$

Of course $\tilde{\mathcal{O}}_0$ is $J_1 \mathcal{O}_0$ where $\mathcal{O}_0$ is the Cartan embedding of $G_0/G_1 = O(2N)/U(N)$.

Now let us consider the quantum mechanics of a particle on the orbit $\tilde{\mathcal{O}}_0$, again with the action (20.24). We choose boundary conditions so that $\mathcal{P}_1$ consists of maps $q : [0,1] \to \tilde{\mathcal{O}}_0$ such that $q(0) = J_1$ and $q(1) = -J_1$. The solutions to the equations of motion are of the form [54] $g(t) = J_1 \exp[\pi t A]$ where now $A \in \mathfrak{p}_0$ implies $\{A, J_1\} = 0$, which guarantees that the path indeed remains in $\tilde{\mathcal{O}}_0$. Again, the boundary conditions together with the minimal action criterion implies that $A^2 = -1$, so we can write:

$$g(t) = J_1 \exp[\pi t A] = J_1 \cos \pi t + (J_1 A) \sin \pi t = J_1 \cos \pi t + \tilde{A} \sin \pi t \tag{20.28}$$

[54] We use the fact that $\tilde{\mathcal{O}}_0$ is totally geodesic.

Because $A \in \mathfrak{p}_0$ both A and $\tilde{A} = J_1 A$ are both antisymmetric and square to -1: $A^2 = -1 = \tilde{A}^2$. We can therefore consider A and $\tilde{A}$ to be in $O(2N)$ and hence the minimal action paths on $\tilde{\mathcal{O}}_0$ are parametrized by A, or better by $\tilde{A}$, and hence the space of minimal actions paths is naturally identified, via the mapping $g(t) \mapsto \tilde{A}$ with $\tilde{\mathcal{O}}_1 \subset O(2N)$.

This stunningly beautiful pattern continues: We take

$$\mathcal{P}_k := \{q : [0,1] \to \tilde{\mathcal{O}}_k | q(0) = J_{k+1} \quad q(1) = -J_{k+1}\} \tag{20.29}$$

Since $\tilde{\mathcal{O}}_k$ is totally geodesic the solutions to the equations of motion are of the form $J_{k+1}\exp[\pi A t]$ with $A \in \mathfrak{p}_k$. The minimal action paths have $A^2 = -1$ and hence they are of the form

$$g(t) = J_{k+1}\exp[\pi t A] = J_{k+1}\cos\pi t + (J_{k+1}A)\sin\pi t = J_{k+1}\cos\pi t + \tilde{A}\sin\pi t \tag{20.30}$$

eq:MinActPath

But now $\tilde{A}^2 = -1$ and $\tilde{A} \in \tilde{\mathcal{O}}_{k+1}$, so we can identify the space of minimal action paths in $\mathcal{P}_k$ with $\tilde{\mathcal{O}}_{k+1}$.

The space of minimal action paths in the set $\mathcal{P}_k$ of all smooth paths $[0,1] \to \tilde{\mathcal{O}}_k$ from J_{k+1} to $-J_{k+1}$ is naturally identified with $\tilde{\mathcal{O}}_{k+1}$ by equation (eq:MinActPath 20.30).

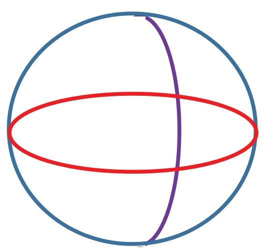

Figure 16: The minimal length geodesics on S^N from the north pole to the south pole are parametrized by S^{N-1}. Similarly, the geodesics in $\tilde{\mathcal{O}}_k$ from J_{k+1} to $-J_{k+1}$ are parametrized by $\tilde{\mathcal{O}}_{k+1}$.

fig:DegenerateMor

Remark: A good analogy to keep in mind is the length of a path on the N-dimensional sphere. If we consider the paths on S^N from the north pole $\mathfrak{N}$ to the south pole $\mathfrak{S}$ then the minimal length paths are great circles and are hence parametrized by their intersection with the equator S^{N-1}. See Figure fig:DegenerateMorse 16.

The great significance of this comes about through Morse theory. The action eq:ParticleAction (20.24) for the paths is a (degenerate) Morse function on $\mathcal{P}_k$ and the critical manifolds allow us to describe the homotopy type of $\mathcal{P}_k$. One considers a series of "approximations" to $\mathcal{P}_k$ by looking at paths with bounded action:

$$\mathcal{P}_k^s := \{q \in \mathcal{P}_k | S[q] \leq s\} \tag{20.31}$$

As we have seen, the minimal action space is $\mathcal{P}_k^{s_{min}} \cong \tilde{\mathcal{O}}_k(N) \subset O(2N)$. Now – it turns out – that the solutions of the equations of motion which are non-minimal have many unstable modes. The number of unstable modes is the "Morse index." The number of unstable modes is linear in N. The reason this is important is that in homotopy theory the way $\mathcal{P}_k^s$ changes as s crosses a critical value is

$$\mathcal{P}_k^{s_*+\epsilon} \sim (\mathcal{P}_k^{s_*-\epsilon} \times D^\lambda)/\sim \tag{20.32}$$

where λ is the number of unstable modes at the critical value s_* and D^λ is a ball of dimension λ. This operation does not change the homotopy groups π_j for $j < \lambda$. Therefore, in this topological sense, $\tilde{\mathcal{O}}_k(N)$ gives a "good approximation" to $\mathcal{P}_k(N)$.

On the other hand, the spaces $\mathcal{P}_k$ have the same homotopy type as the *based loop spaces* $\Omega_*\tilde{\mathcal{O}}_k$. Indeed, choosing any standard path from $-J_{k+1}$ to J_{k+1} we can use it to convert any path in $\mathcal{P}_k$ to a loop $\Omega_*\tilde{\mathcal{O}}_k$ based, say, at J_{k+1} by composition. Conversely, composing the (inverse of) the standard path with any loop gives a path in $\mathcal{P}_k$. The importance of relating these spaces to loop spaces is that

1. We get a nice proof of Bott periodicity MilnorMorse [31].

2. We thereby make a connection to generalized cohomology theory through the notion of a *spectrum*.

♣Both of these points require a lot more explanation. ♣

20.5 Relation to classifying spaces of K-theory

The fact that the Morse index for the space of paths $\mathcal{P}_k(N)$ (where the N-dependence comes from the fact that the paths are in $\tilde{\mathcal{O}}_k(N) \subset O(2N)$) grows linearly in N suggests that it will be interesting to take the $N \to \infty$ limit. We can do this as follows:

We make a real Hilbert space by taking a countable direct sum of copies of simple modules of the real Clifford algebra $C\ell_{-(k+1)}$. Specifically we define, for $k \geq 0$, [55]

$$\mathcal{H}_R^k := \begin{cases} N_{k+1} \otimes \ell^2(\mathbb{R}) & k \neq 2(4) \\ (N_{k+1}^+ \oplus N_{k+1}^-) \otimes \ell^2(\mathbb{R}) & k = 2(4) \end{cases} \tag{20.33}$$

and for an integer n let $\mathcal{H}_R^k(n)$ be the sum of the first n representations N_{k+1} or $(N_{k+1}^+ \oplus N_{k+1}^-)$. Now define a subspace of the space of orthogonal operators $\Omega_k(n) \subset O(\mathcal{H}_R^k)$. These are operators which satisfy the following three conditions:

[55] Recall that N_k denote irreducible ungraded Clifford modules.

1. They preserve separately $\mathcal{H}^k_R(n)$ and $\mathcal{H}^k_R(n)^\perp$.

2. They are just given by $A = J_k$ on $\mathcal{H}^k_R(n)^\perp$

3. On $\mathcal{H}^k_R(n)$ they satisfy:

$$\begin{aligned} A^2 &= -1 \\ \{A, J_i\} &= 0 \qquad i = 1, \dots, k-1 \end{aligned} \tag{20.34}$$

We recognize that $\Omega_k(n) \cong \tilde{\mathcal{O}}_{k-1}(N)$ where N and n are linearly related. (We define $\tilde{\mathcal{O}}_{-1}(N) := O(2N)$ so this holds for $k \geq 0$.) Now, from this description it is easy to see that there are embeddings

$$\Omega_k(n) \hookrightarrow \Omega_k(n+1) \tag{20.35}$$

and we can take a suitable "$n \to \infty$ limit" and norm closure to produce a set of operators $\Omega_k(\infty)$ on $\mathcal{H}^k_R$. In AtiyahSingerSkew [9] Atiyah and Singer show that this set of operators is closely related to a set of Fredholm operators on $\mathcal{H}^k_R$.

Define $\mathfrak{F}^0$ to be the set of all Fredholm operators on $\mathcal{H}^k_R$, and let $\mathfrak{F}^1 \subset \mathfrak{F}^0$ denote the subspace of skew-adjoint Fredholm operators: $A^{tr} = -A$. (Formally, this is the Lie algebra of $O(\mathcal{H}^k_R)$.) Now for $k \geq 2$ define $\mathfrak{F}^k \subset \mathfrak{F}^1$ to be the subspace such that [56]

$$TJ_i = -J_i T \qquad\qquad i = 1, \dots, k-1 \tag{20.36}$$ eq:Fk-def

Now, the space of Fredholm operators has a standard topology using the operator norm topology. Using this topology Atiyah and Singer prove

♣You are changing k's here. Need to clarify. ♣

1. $\mathfrak{F}^k \sim \Omega_{k-1}(\infty) \cong \tilde{\mathcal{O}}_{k-2}(\infty)$, $k \geq 1$, where $\sim$ denotes homotopy equivalence.

2. $\mathfrak{F}^{k+1} \sim \Omega \mathfrak{F}^k$, and in fact, the homotopy equivalence is given by

$$A \mapsto J_{k+1} \cos \pi t + A \sin \pi t \qquad 0 \leq t \leq 1 \tag{20.37}$$

   which should of course be compared with (eq:MinActPath 20.30).

The relation to Fredholm operators implies a relation to K-theory because one way of defining the real KO-theory groups of a topological space X is via the set of homotopy classes:

$$KO^{-k}(X) := [X, \mathfrak{F}^k] \qquad k \geq 0 \tag{20.38}$$

We summarize with a table

[56] For $k = 3 \text{mod} 4$ the subspace of $\mathfrak{F}^1$ satisfying (eq:Fk-def 20.36) in fact has three connected components in the norm topology. Two of these are contractible but one is topologically nontrivial and we take $\mathfrak{F}^k$ to be that component. In fact for T satisfying (eq:Fk-def 20.36) one can show that $\omega_{k-1} T$ is self-adjoint, where $\omega_{k-1} = J_1 \cdots J_{k-1}$ is the volume form. The contractible components are those for which $\omega_{k-1} T$ is positive or negative - up to a compact operator.

k	$\mathfrak{F}^k \sim G_{k-2}/G_{k-1}$	Cartan's Label
0	$(O/(O\times O))\times\mathbb{Z}$	BDI
1	O	D
2	O/U	DIII
3	U/Sp	AII
4	$(Sp/(Sp\times Sp))\times\mathbb{Z}$	CII
5	Sp	C
6	Sp/U	CI
7	U/O	AI

and the complex case is

k	$\mathfrak{F}_c^k \sim G_{k-2}/G_{k-1}$	Cartan's Label
0	$(U/(U\times U))\times\mathbb{Z}$	AIII
1	U	A

where $\mathfrak{F}_c$ is the space of Fredholm operators on a complex separable Hilbert space, and is the classifying space for $K^0(X)$ and $\mathfrak{F}_c^1$ is the subspace of skew-adjoint Fredholm operators and is the classifying space for $K^{-1}(X)$.

Remark: We indicate how this discussion of $KO(X)$ is related to what we discussed in Section §13.5 above. We take $X = pt$. Then, $KO^0(pt) \cong \mathbb{Z}$. In terms of Fredholm operators T the isomorphism is given by $T \mapsto \mathrm{Index}(T) := \mathrm{dimker}T - \mathrm{dimcok}T$. Thus, "invertible part of T cancels out." The idea that if T is invertible then it defines a trivial class was the essential idea in the definition in Section §13.5. It is also worth noting the Fredholm interpretation of $KO^{-1}(pt) \cong \mathbb{Z}_2$ in this context. For a skew-adjoint Fredholm operator $\ker(T) = \ker(T^\dagger)$ so the usual notion of index is just zero. However we can form the "mod-two index," which is defined to be $\mathrm{dimker}T\mathrm{mod}2$. This is indeed continuous in the norm topology and provides the required isomorphism.

♣Improve this discussion by rephrasing the AS results in terms of $\mathbb{Z}_2$-graded Hilbert spaces. ♣

****

END OF COURSE. FALL 2013

****

21. Analog for free bosons

sec:Bosons

Much of the material of Sections 18 and 19 have direct analogs for bosons. There are some interesting and significant sign changes. Roughly speaking, orthogonal and symplectic groups are exchanged. The physics is of course radically different.

21.1 Symplectic vector spaces and the Heisenberg algebra

We begin with a mode space $\mathcal{M} \cong \mathbb{R}^{2n}$ now equipped with a nondegenerate *anti-symmetric* form ω, i.e. a symplectic form. The automorphism group $\mathrm{Aut}(\mathcal{M}, \omega)$ will be isomorphic to a real symplectic group. By definition a *Darboux basis* is an ordered basis $\{v_i\}$ for $\mathcal{M}$ in which the matrix $\omega(v_i, v_j)$ is given by

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{21.1}$$

It is convention to write a basis of this form as Q_i, P^i such that

♣Important convention here! ♣

$$\begin{aligned} \omega(Q_i, Q_j) &= 0 \\ \omega(P^i, P^j) &= 0 \\ \omega(Q_i, P^j) &= -\omega(P^j, Q_i) = \delta_{i,j} \end{aligned} \tag{21.2}$$

Then

$$\mathrm{Aut}(\mathcal{M}, \omega) \cong Sp(2n; \mathbb{R}) := \{g \in GL(2n, \mathbb{R}) | gJg^{tr} = J\} \tag{21.3}$$

eq:def-SP

The conditions on the block-diagonal form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{21.4}$$

are now

$$\begin{aligned} A^{tr}D - C^{tr}B = 1 \\ A^{tr}C = (A^{tr}C)^{tr} \\ B^{tr}D = (B^{tr}D)^{tr} \end{aligned} \tag{21.5}$$

eq:Sympl-Block-Co

or equivalently

$$\begin{aligned} AD^{tr} - CD^{tr} = 1 \\ AB^{tr} = (AB^{tr})^{tr} \\ CD^{tr} = (CD^{tr})^{tr} \end{aligned} \tag{21.6}$$

eq:Sympl-Block-Co

Note the sign changes from ***** above.

The analog of the real Clifford algebra is the Poisson algebra $\mathrm{Poiss}(\mathcal{M}, \omega)$ of real-algebraic functions on $\mathcal{M}$. It is infinite-dimensional and generated by functions p_i, q^i which can be thought of as a dual basis: $p_i(P^j) = \delta_i^j$, etc. If we regard ω as a 2-form on $\mathcal{M}$, i.e. $\omega \in \Lambda^2\mathcal{M}^*$ then we have $\omega = \sum_{i=1}^n dp^i \wedge dq_i$

♣check sign! ♣

Quantization of the symplectic manifold $(\mathcal{M}, \omega)$ means producing a complex Hilbert space $\mathcal{H}_F$ [57] and a $*$-representation ρ_F of a complex $*$-algebra $\mathcal{A}$. In this case $\mathcal{A}$ which

[57] The subscript F is again for Fock

is a deformation of the Poisson algebra Poiss$(\mathcal{M},\omega)\otimes\mathbb{C}$. We can identify $\mathcal{A}$ with the Heisenberg algebra

$$\mathcal{A} = \mathrm{Heis}(\mathcal{M},\omega) := (T(\mathcal{M}^*)\otimes\mathbb{C})/\mathcal{I} \tag{21.7}$$

where $\mathcal{I}$ is the ideal generated by $vv' - v'v - \sqrt{-1}\omega(v,v')\cdot 1$. In particular (dropping the ρ_F):

$$\begin{aligned}
[q^i, q^j] &= 0 \\
[p_i, p_j] &= 0 \\
[q^i, p_j] &= \sqrt{-1}\delta_{i,j}
\end{aligned} \tag{21.8}$$

♣Need several comments on $\hbar$. ♣

There are two standard ways to produce irreducible $*$-representations of $\mathcal{A}$.

21.2 Bargmann representation

The first way, which is most directly analogous to the method we used for fermions: We complexify $V := \mathcal{M}\otimes\mathbb{C}$ and extend ω $\mathbb{C}$-linearly. Then we *choose* a compatible complex structure I on V:

$$\omega(Iv_1, Iv_2) = \omega(v_1, v_2) \tag{21.9}$$

♣Need to explain positivity property $\omega(Iv,v)>0$. ♣

Now we decompose $V = W\oplus\bar{W}$ into $I = \mathrm{i}$ and $I = -\mathrm{i}$ eigenspaces and define a representation of the Heisenberg algebra on

$$\mathcal{H}_F \cong \mathrm{Sym}(W) \tag{21.10}$$

which we can interpret as algebraic holomorphic functions on $\bar{W}$. Of course, unlike the fermionic case, this is an infinite-dimensional vector space. Issues of functional analysis now enter. For example, $\rho_F(q^i)$ and $\rho_F(p_i)$ will be unbounded self-adjoint operators and can only have a dense domain of definition. These kinds of subtleties are in general not important for many standard physical considerations.

♣Need to define Hilbert space structure. ♣

Let us choose a Darboux basis as above and take $I = J$ itself, so that $I: q^i \to p_i$ and $I: p_i \to -q^i$. Then if we define

$$\begin{aligned}
a_i &= \frac{1}{\sqrt{2}}(p_i - \mathrm{i}q^i) \\
\bar{a}^i &= \frac{1}{\sqrt{2}}(p_i + \mathrm{i}q^i)
\end{aligned} \tag{21.11}$$

$$\begin{aligned}
q^i &= \frac{\mathrm{i}}{\sqrt{2}}(a_i - \bar{a}^i) \\
p_i &= \frac{1}{\sqrt{2}}(a_i + \bar{a}^i)
\end{aligned} \tag{21.12}$$

we have $I: a_i \to -\sqrt{-1}a_i$ and $I: \bar{a}^i \to +\sqrt{-1}\bar{a}^i$. A small computation gives the standard CCR's for bosonic oscillators:

$$\begin{aligned}
[a_i, a_j] &= [\bar{a}^i, \bar{a}^j] = 0 \\
[a_i, \bar{a}^j] &= \delta_i^{\ j}
\end{aligned} \tag{21.13}$$

eq:Bos-Osc-1

A nice way to represent this is to consider $\mathcal{H}_F = \mathrm{Hol}(\mathbb{C}^n)$ holomorphic functions $\psi(\bar{z}^1, \dots, \bar{z}^n)$ which are L^2 with respect to the inner product

$$\langle\psi_1|\psi_2\rangle = \int \prod_i \frac{dz_i \wedge d\bar{z}^i}{-2\pi \mathrm{i}} e^{-\sum z_i \bar{z}^i} \overline{\psi_1(\bar{z}^i)} \psi_2(\bar{z}^i) \tag{21.14}$$

Then $\bar{a}^i$ is represented by multiplication by $\bar{z}^i$ and a_i is represented by $\frac{\partial}{\partial \bar{z}^i}$. The normalized vacuum is $\psi = 1$ and with this Hilbert space product $\bar{a}^i = (a_i)^\dagger$.

****

1. discuss operator kernels etc.

2. relation to Kähler quantization. Interpret ψ as holomorphic sections of a trivialized hermitian line bundle.

****

21.3 Real polarization

The second way is to form the finite-dimensional *Heisenberg group*. This is a central extension of the additive group $\mathcal{M}$ (considered as an abelian group under vector addition)

$$1 \to U(1) \to \mathrm{Heis}(\mathcal{M}, \omega) \to \mathcal{M} \to 0 \tag{21.15}$$

The cocycle is

$$c(v_1, v_2) = e^{-\frac{\mathrm{i}}{2}\omega(v_1, v_2)} \tag{21.16}$$

and hence the group law could be written as:

$$(z_1, v_1) \cdot (z_2, v_2) := (z_1 z_2 e^{-\frac{\mathrm{i}}{2}\omega(v_1, v_2)}, v_1 + v_2) \tag{21.17}$$

This formula will strike some readers as strange. Perhaps a more congenial way to write it is to represent group elements at ze^v, with group multiplication

$$(z_1 e^{v_1}) \cdot (z_2 e^{v_2}) := z_3 e^{v_1 + v_2} \tag{21.18}$$

$$z_3 = z_1 z_2 e^{-\frac{\mathrm{i}}{2}\omega(v_1, v_2)}. \tag{21.19}$$

The Heisenberg group is a finite-dimensional group. For example if $\mathcal{M} = \mathbb{R}^2$ it is isomorphic to the group of 3×3 real upper-triangular matrices.

By the Stone-von Neumann theorem $\mathrm{Heis}(\mathcal{M}, \omega)$ has a unique irreducible unitary representation - *up to isomorphism* - where $U(1)$ acts as scalars.

One way to exhibit the representation is to choose a Lagrangian decomposition of $\mathcal{M} = \mathcal{Q} \oplus \mathcal{P}$, where $\mathcal{Q}, \mathcal{P}$ are maximal Lagrangian subspaces and take $\mathcal{H}_F = L^2(\mathcal{Q}, d\mu)$ where $d\mu$ is the Euclidean measure $\mathcal{Q}$. Now $\mathcal{Q}$ and $\mathcal{P}$ are represented by multiplication and translation operators, respectively: ♣extra data? ♣

$$\begin{aligned} \left[\rho_F\left(e^{\mathrm{i}\alpha_j q^j}\right)\psi\right](q) &= e^{\mathrm{i}\alpha_j q^j}\psi(q) \\ \left[\rho_F\left(e^{\mathrm{i}\beta^j p_j}\right)\psi\right](q) &= \psi(q + \beta) \end{aligned} \tag{21.20}$$

Dropping the ρ_F's one can check that the Heisenberg group relations are indeed satisfied: ♣dual space muddled up here ♣

$$\operatorname{expi}\left(\alpha_j q^j + \beta^j p_j\right) \operatorname{expi}\left(\gamma_j q^j + \delta^j p_j\right) = e^{-\frac{\mathrm{i}}{2}\omega(v_1,v_2)} \operatorname{expi}\left((\alpha_j+\gamma_j)q^j + (\beta^j+\delta^j)p_j\right) \tag{21.21}$$

In this representation p_i acts as a differential operator

$$\rho_F(p_i) = -\sqrt{-1}\frac{\partial}{\partial q^i} \tag{21.22}$$

while $\rho_F(q^i)$ is a multiplication operator. (We will henceforth drop the tedious ρ_F.) These are unbounded operators with dense domains. The unitary groups they generate are defined on the entire Hilbert space.

In this representation a_i is the differential operator

$$a_i = -\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{\partial}{\partial q^i} + q^i\right) \tag{21.23}$$

so that the unique vacuum vector is proportional to $\Psi_0 = e^{-\frac{1}{2}\sum q^i q^i}$. This leads immediately to the isomorphism with the Bargmann representation. ♣Explain more. ♣

21.4 Metaplectic group as the analog of the Spin group

From (eq:def-SP 21.3) we get that the Lie algebra $sp(2n;\kappa)$ of the symplectic group is

$$sp(2n;\kappa) = \{m \in M_{2n}(\kappa) | m^{tr}J + Jm = 0\} \tag{21.24}$$ eq:Sympl-LA

Note well that $m \in sp(2n;\kappa)$ iff mJ is a *symmetric* matrix.

As in the fermionic case we can write Lie algebra elements in the form

$$m = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^{tr} \end{pmatrix} \in Mat_{2n}(\mathbb{C}) \tag{21.25}$$ eq:liegeni-bos

where now β,γ are symmetric matrices over κ. Note that m is antihermitian iff $\alpha^\dagger = -\alpha$ and $\beta^\dagger = -\gamma$. Such antihermitian matrices exponentiate to elements of $USp(2n) = U(2n) \cap Sp(2n;\mathbb{C}) \cong O(2n;\mathbb{R})$. ♣check ♣

For matrices (eq:liegeni-bos 21.25) with $m \in sp(2n;\mathbb{C})$ define a corresponding element of the Heisenberg algebra:

$$\begin{aligned} \widetilde{m} &:= \sum_{i,j=1}^{n}\left(\alpha_{ji}\bar{a}_j a_i + \frac{1}{2}\gamma_{ij}a_i a_j + \frac{1}{2}\beta_{ij}\bar{a}_i\bar{a}_j\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{n}\left(\alpha_{ji}(\bar{a}_j a_i + a_i\bar{a}_j) + \gamma_{ij}a_i a_j + \beta_{ij}\bar{a}_i\bar{a}_j\right) - \frac{1}{2}\mathrm{Tr}(\alpha)1 \end{aligned} \tag{21.26}$$ eq:legn-bos

Now by an argument completely parallel to the fermionic case we use the identity

$$[AB,CD] = A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B \tag{21.27}$$

to compute

$$[\widetilde{m_1},\widetilde{m_2}] = \widetilde{[m_1,m_2]} - \frac{1}{2}\mathrm{Tr}(\beta_1\gamma_2 - \beta_2\gamma_1)1 \tag{21.28}$$

eq:centrl-bos

and again the cocycle is trivializable.

An important special case is $sp(2;\mathbb{R}) \cong sl(2;\mathbb{R})$. The isomorphism is explicitly seen by taking

$$e := \frac{\mathrm{i}}{2}p^2 \qquad h := \frac{\mathrm{i}}{2}(qp+pq) \qquad f := \frac{\mathrm{i}}{2}q^2 \tag{21.29}$$

eq:Heis-HO-1

and computing

$$[h,e] = -2e \qquad [e,f] = h \qquad [h,f] = +2f \tag{21.30}$$

A basis of $sl(2;\mathbb{R})$ satisfying these relations is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{21.31}$$

(Note the signs carefully!)

Standard facts about the harmomic oscillator Hamiltonian now show that in the real polarization with $p = -\mathrm{i}\frac{d}{dq}$ acting on $L^2(\mathbb{R})$ the Lie algebra (21.29) exponentiates to a double cover of $Sp(2;\mathbb{R})$. Indeed, consider

$$e + f = \frac{\mathrm{i}}{2}(p^2+q^2) = \mathrm{i}(\bar{a}a + \frac{1}{2}) \tag{21.32}$$

This is well-known to have spectrum $\mathrm{i}(n+\frac{1}{2})$, $n = 0,1,2,\dots$. Therefore, the one-parameter subgroup $\exp[\theta(e+f)]$ has period $\theta \sim \theta + 4\pi$. Now compare with the representation above generating $SL(2;\mathbb{R})$. For this representation the one parameter group

$$\exp[\theta(e+f)] = \cos\theta + \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{21.33}$$

has period $\theta \sim \theta + 2\pi$.

Remark: For a very beautiful discussion of why the metaplectric group cannot be a matrix group, and of the relation of the one parameter subgroup $\exp[\theta(e+f)]$ to the Fourier transform see Section 17 of [36].

Exercise
Write $SU(1,1)$ generators in terms of quadratic expressions in a and $a^\dagger$.

21.5 Bogoliubov transformations

We again consider the Bogoliubov transformations for bosonic oscillators, which have exactly the same form as in the fermionic case:

$$\begin{aligned} \bar{b}_i &= A_{ji}\bar{a}_j + C_{ji}a_j \\ b_i &= B_{ji}\bar{a}_j + D_{ji}a_j \qquad\qquad 1 \le i,j \le n \end{aligned} \tag{21.34}$$

eq:BogT-osc-1

but now this gives an automorphism of the CCR's iff A, B, C, D satisfy (21.5) (or equivalently) (21.6) and hence $g \in Sp(2n; \mathbb{C})$.

****

1. Bundle of Fock spaces over $Sp(2n; \mathbb{R})/Gl(n; \mathbb{R})$ with $A \in GL(n; \mathbb{R})$ embedded as

$$\begin{pmatrix} A & 0 \\ 0 & A^{tr,-1} \end{pmatrix} \tag{21.35}$$

Note that in contrast to the fermionic case this space is *noncompact*. Below we will relate this to the fact that the bosonic Fock space is infinite-dimensional, in contrast with the fermionic Fock space.

2. Holomorphic presentation $Sp(2n; \mathbb{C})/\mathcal{LD}$.

****

osonicSqueezed

21.6 Squeezed states and the action of the metaplectic group

Define the squeezed state $|S\rangle$ to be the state which in the Bargmann representation is $\psi_S(\bar{z}) = \exp[-\frac{1}{2} S_{ij} \bar{z}^i \bar{z}^j]$. At least formally we can take S_{ij} to be any complex symmetric matrix.

Then we can compute the Gaussian integral (again formally) to be

$$\langle S|T\rangle = \frac{1}{\sqrt{\det(1 - \bar{S}T)}} \tag{21.36}$$

A quick and dirty way to get the answer is to do the Gaussian integral

$$\int \prod \frac{dz_i \wedge d\bar{z}_i}{-2\pi \mathrm{i}} e^{-\frac{1}{2}\bar{S}_{ij} z_i z_j - z_i \bar{z}_i - \frac{1}{2} T_{ij} \bar{z}_i \bar{z}_j} \tag{21.37}$$

by pretending that z_i and $\bar{z}_i$ are independent variables. One first does the Gaussian integral on the z_i giving a determinant $(\det \bar{S})^{-1/2}$ and we evaluate the action at the stationary point $z_i = -\bar{S}^{-1}_{ij} \bar{z}_j$. Then one does the Gaussian integral over $\bar{z}_i$ to get $(\det(\bar{S}^{-1} - T))^{-1/2}$. ♣Explain why it works and discuss convergence. ♣

Give action of metaplectic group on $|S\rangle$:

$$\tilde{g}|S\rangle = \frac{1}{\sqrt{\det(CS + D)}} |g \cdot S\rangle \tag{21.38}$$

where precisely the same reasoning as in (18.142) (now with R, S symmetric matrices) leads to

$$g \cdot S = (AS - B)(CS - D)^{-1} \tag{21.39}$$

♣check signs! ♣

****

1. Again choice of square-root leads to action of the metaplectic group. Give a definition of that group analogous to the definition (18.186) of the spin group above.

1. Infinite dimensions and Shale's theorem.

2. Coherent state (Bargmann) representation in fermionic case gives an easy derivation of (18.152) above.

3. Compute "particle number creation"

4. Infinite dimensions and Shale's theorem.

****

21.7 Induced representations

21.8 Free Hamiltonians

Up to a constant the general free boson Hamiltonian is an element of $\mathcal{A}$ of the form

$$h = h^{ij} v_i v_j \tag{21.40}$$

This should be $*$ invariant and hence h^{ij} must be a real symmetric matrix. Now, notice that from (eq:Sympl-LA 21.24) that we can therefore identify hJ with an element of the symplectic Lie algebra. Thus,

The space of free boson Hamiltonians is naturally identified with $sp(2n;\mathbb{R})$.

21.9 Analog of the AZ classification of free bosonic Hamiltonians

Now we define a symmetry of the bosonic dynamics to be a group G with $\rho : G \to \mathrm{End}(\mathcal{H}_F)$ such that *****

An argument completely analogous to that for (eq:gA 19.23) applies. The symmetry operators act by

$$\rho(g)\rho_F(v_j)\rho(g)^{-1} = \sum_m S_{mj}(g)\rho_F(v_m) \tag{21.41}$$

eq:Rot-Bos-Osc

where now $S(g) \in Sp(2n;\mathbb{R})$. The result is that the symmetry condition is just that $A = hJ \in sp(2n;\mathbb{R})$ is in the space

$$\mathfrak{p} := \{A \in sp(2n;\mathbb{R}) | S(g)AS(g)^{-1} = \chi(g)A\} \tag{21.42}$$

eq:Bos-Sym-Cond

For bosons the Hamiltonian will have an infinite spectrum. It is natural to assume that the Hamiltonian is bounded below, in which case $\chi = 1$. From a purely mathematical viewpoint one could certainly consider quadratic forms with Hamiltonian unbounded from above or below. Consider, e.g., the upside down harmonic oscillator. Thus, one could still contemplate systems with $\chi \neq 1$, although they are a bit unphysical.

****

1. Same argument applies and $\mathfrak{p}$ is now tangent to a noncompact symmetric space.
2. Most interesting case is where $\mathfrak{p}$ can be considered as subalgebra of a symplectic Lie algebra.
3. Again use involutions to classify etc. etc.
4. Bosons + fermions: Use osp(***) etc.

****

21.10 Physical Examples

21.10.1 Weakly interacting Bose gas

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p + \frac{g}{V} \sum_{k,p} a_k^\dagger a_p^\dagger a_p a_k \tag{21.43}$$

In the groundstate all particles have $p = 0$. To get low-lying excitations $a_p \to \delta_{p,0}N + a_p$ in a BEC. In the low density approximation

$$H = H_0 + \sum_{p\neq 0} \frac{p^2}{2m} a_p^\dagger a_p + \frac{\kappa g N}{V} \sum_p (a_p^\dagger a_{-p}^\dagger + a_p a_{-p} + 2 a_p a_p^\dagger) \qquad (21.44)$$

ETC.

Reference: R.K. Pathria and P.D. Beale, *Statistical Mechanics*

21.10.2 Particle creation by gravitational fields

Ref: Birrell and Davies

21.10.3 Free bosonic fields on Riemann surfaces

Operator formalism. State associated to Riemann surface, point, and local coordinate. Etc.

22. Reduced topological phases of a FDFS and twisted equivariant K-theory of a point

sec:FDFS-Point

22.1 Definition of G-equivariant K-theory of a point

22.2 Definition of twisted G-equivariant K-theory of a point

There is a general notion of a "twisting" of a generalized cohomology theory. This can be defined in terms of some sophisticated topology (like using nontrivial bundles of spectra) but in practice it often amounts to introducing some extra signs of phases. This is not always the case: Degree shift in K-theory can be viewed as an example of twisting.

A simple example of a twisting of ordinary cohomology theory arises when one has a double cover $\pi : \tilde{X} \to X$. Then the "twisted cohomology" of X refers to using cocycles, coboundaries, etc. on $\tilde{X}$ that are odd under the deck transformation.

In the case of equivariant K-theory of a point, a "twisting of $K_G(pt)$" is an isomorphism class of a central extension of G. These are classified by $H^2(G, U(1))$ and in general twistings of K-theory are classified by certain cohomology groups.

Let τ denote such a class of central extensions. A "twisted G-bundle over a point" is a representation of a corresponding central extension G^τ. The τ-twisted G-equiviariant K-theory of a point is then just the G^τ-equivariant K-theory of a point:

$$K_G^\tau(pt) = K_{G^\tau}(pt) \qquad (22.1)$$

22.3 Appliction to FDFS: Reduced topological phases

23. Groupoids

sec:Groupoids

Category.

Group as category.

Definition of groupoid.

Examples:

Equivalence of groupoids.

Vector bundles on groupoids.

24. Twisted equivariant K-theory of groupoids

iv-K-Groupoids

Central extensions as line bundles over a groupoid

ϕ-twisted extensions

Twisting for the K-theory of a groupoid $\mathcal{G}$.

(ϕ, χ)-twisted bundle over a groupoid

twisted equivariant K-theory: Definition

25. Applications to topological band structure

po-Band-Struct

Recall magnetic crystallographic group $G(C)$. SEE SECTION 4.1 ABOVE.

Bloch theory: Comment on Berry connection.

Insulators: $\mathcal{E}^{\pm}$.

*** The canonical twisting of $T//P$.

Comparison with literature.

Localization: A means of exhibiting some new invariants.

A. Simple, Semisimple, and Central Algebras

:CentralSimple

A.1 Ungraded case

We review here some standard material from algebra which is not often covered in physics courses. Some references include [12][17][1] Nevertheless, the results are very powerful and worth knowing. They are used at several points in the main text.

We consider associative algebras over a field κ.

Definition An algebra A is *central* if its center $Z(A)$ is precisely κ.

In general the center of an algebra can be larger than κ. For $A = M_n(\kappa)$ the algebra is indeed central. For the algebra $B = M_n(\kappa) \oplus M_m(\kappa)$ the center is the set of matrices

$$Z(B) = \{x 1_n \oplus y 1_m | x, y \in \kappa\} \tag{A.1}$$

and is isomorphic to $\kappa \oplus \kappa$, and hence B is not a central algebra.

In the literature one finds at least three different definitions of the notion of a *simple algebra*:

1. A simple algebra is an algebra isomorphic to a matrix algebra over a division ring D which contains κ in its center.

2. A simple algebra is an algebra where the product is nonzero and there are no non-trivial two-sided ideals.

3. A simple algebra is an algebra where the operator $L(a)$ in the left regular representation are simple - i.e. diagonalizable.

♣Explain more about this. ♣

Definition 2 is usually adopted for mathematical purity and then the equivalence with Definition 1 is regarded as a theorem, where it is known as the *Wedderburn theorem.* In a proper mathematical exposition we would stop here and prove that these three definitions are in fact equivalent.

An algebra is *semi simple* if it is isomorphic to a direct sum of simple algebras. If there is more than one nontrivial summand then it is not simple because the simple summands define nontrivial two-sided ideals.

Examples

1. Division algebras themselves are simple algebras. This is trivial by definition one. In terms of definition two, suppose that $I \subset D$ is a nonzero ideal in a division algebra. If $a \in I$ is nonzero then on the one hand a has an inverse b (since D is a division algebra) but then $ab = 1 \in I$, since I is an ideal. If an ideal contains 1 then then $I = D$.

2. An example of algebras which are not semisimple are the Grassmann algebras $\kappa[\theta_1, \dots, \theta_n]$. We refer to general elements as "superfields." The Grassmann algebra is filtered by the minimal number of θ's in the expansion of the "superfield." Let $\mathcal{F}^k$ be the subspace of linear combinations of elements with at least k θ's, so $\mathcal{F}^k \supset \mathcal{F}^{k+1} \supset \cdots$. All of the $\mathcal{F}^k$ are two-sided graded ideals.

3. The group algebra $L^2(G)$ of a finite group is a semisimple algebra. This follows by decomposing it as a direct sum of matrix algebras according to the Peter-Weyl theorem.

A semisimple algebra has the important property that, if (ρ, V) is a representation and $W \subset V$ is a subrepresentation, then there is a complementary representation U so that $V \cong W \oplus U$ as a representation. For example, if there is an inner product on V which is compatible with the algebra then $U = W^{\perp}$. This is what happens with group algebras.

Some important facts about simple algebras are:

Proposition : The center of A is a field which contains κ.

Proof: It is obvious that the center of A is a commutative ring which contains κ. The nontrivial fact is that if $a \in Z(A)$ is nonzero then it is invertible. To see why, consider $\ker L(a)$. This is an ideal in A, for if $L(a)b = 0$ then $ab = 0$ and then if c is any element of A we have $a(bc) = (ab)c = 0$ and $a(cb) = c(ab) = 0$, because a is central. But if a is nonzero then $L(a)1 = a \neq 0$, so $\ker L(a) \neq A$ and therefore $\ker L(a) = 0$. But then the linear transformation $L(a)$ must also be surjective. So $L(a)b = 1$ has a solution for some b and therefore a is invertible. $\diamondsuit$

In a large number of places in these notes we use the following basic property of simple algebras:

Theorem: A simple algebra over a field κ has a unique nonzero irreducible representation, up to isomorphism, and all other representations are completely reducible and isomorphic to direct sums of this unique irrep.

Proof: Let (ρ, V) be any representation of $M_n(D)$ for a division algebra D over a field κ. Then V is a vector space over κ and

$$\rho : M_n(D) \to \mathrm{End}_\kappa(V) \tag{A.2}$$

is a homomorphism of algebras. Consider

$$\ker(\rho) := \{M | \rho(M) = 0\} \tag{A.3}$$

Then one checks that $\ker(\rho)$ is a two-sided ideal in $M_n(D)$. Therefore, since $M_n(D)$ is simple, either $\ker(\rho) = M_n(D)$, in which case $\rho = 0$ or $\ker(\rho) = \{0\}$. Since we assume $\rho \neq 0$ it is has no kernel as a linear transformation of κ vector spaces. Therefore $P_i = \rho(e_{ii})$ is nonzero for all i. Consider $\rho(1) = \sum_i P_i$. Clearly, $\rho(1)$ is a central projection operator in the image of ρ. Let $W = \rho(1)V$, and $W_i = P_i V$. Then we claim that

$$W = \oplus W_i \tag{A.4}$$

clearly, if $w \in W$ then $w = \sum_i P_i w$ so the W_i span, but also $P_i P_j = \rho(e_{ii}e_{jj}) = 0$ for $i \neq j$ and hence the spaces W_i are all linearly independent. Moreover, note that there are canonical isomorphisms

$$\begin{aligned} \rho(e_{ij}) &: W_j \to W_i \\ \rho(e_{ji}) &: W_i \to W_j \end{aligned} \tag{A.5}$$

since $\rho(e_{ij})\rho(e_{ji}) = P_i$ and $\rho(e_{ji})\rho(e_{ij}) = P_j$.

Now suppose $D = \kappa$ and choose an ordered basis $w^{(\alpha)}$, $\alpha = 1, \dots, k$ for V_1 and define $w_j^{(\alpha)} := \rho(e_{j1})w^{(\alpha)}$. Then $\{w_j^{(\alpha)}\}_{\alpha=1,\dots,k; j=1,\dots,n}$ is a basis for W. (For a nice block-diagonal matrix realization of the representation use lexicographic ordering: First order by j then by α.) Let $W_j^{(\alpha)}$ denote the span of $w_j^{(\alpha)}$. Note that we have

$$\rho(e_{ij})w_k^{(\alpha)} = \rho(e_{ij})\rho(e_{k1})w^{(\alpha)} = \delta_{j,k} w_i^{(\alpha)} \tag{A.6}$$

Therefore, for any fixed α, $W^{(\alpha)} := \oplus_{j=1}^n W_j^{(\alpha)}$ is clearly isomorphic to the defining representation $\kappa^{\oplus n}$ of $M_n(\kappa)$ and

$$W \cong \oplus_{\alpha=1}^k W^{(\alpha)} \tag{A.7}$$

is then a direct sum of copies of the defining representation. Then $V = W \oplus (1 - \rho(1))V$ is a sum of these defining representations and the zero representation.

For the general division algebra D over κ we use a similar argument to show first that the general representation is of the form $D^{\oplus k}$ and then note that each V_i must be an isomorphic representation of D. $\diamondsuit$

♣Prove this. Then let $w^{(\alpha)}$ generate a copy of D in V_1. ♣

An algebra A over a field κ is said to be *central simple* if it is simple and moreover $Z(A) \cong \kappa$, that is, it is also central. The matrix agebras $M_n(\kappa)$ are central simple algebras over κ. The complex numbers $\mathbb{C}$ can be regarded as a two-dimensional simple algebra over $\mathbb{R}$. However, $\mathbb{C}$ is not a central simple algebra over $\mathbb{R}$ because its center is $\mathbb{C}$, and contains the ground field $\mathbb{R}$ as a proper subfield. Of course, $\mathbb{C}$ is a central simple algebra over $\mathbb{C}$!

When A is central simple there are some special nice properties:

1. If B is simple and A is central simple then any two embeddings of B into A are conjugate. In particular, an automorphism of A is an embedding of A into itself and therefore must be inner. This is known as the Skolem-Noether theorem.

2. If B is simple and A is central simple then $A \otimes_\kappa B$ is simple, and $Z(A \otimes_\kappa B) \cong Z(B)$.

3. If B is a simple subalgebra of a central simple algebra A then $C = Z(B)$, the centralizer of B in A is itself simple, and $Z(C) \cong B$. If B is central simple then $A \cong B \otimes_\kappa C$.

Example: ILLUSTRATE THESE CLAIMS WITH MATRIX ALGEBRAS. $M_n(\mathbb{C})$.

There is always a map

$$LR : A \otimes_\kappa A^{\mathrm{opp}} \to \mathrm{End}_\kappa(A) \tag{A.8}$$

given by $LR(a \otimes b) : x \mapsto axb$. One can show (Drozd [17], Theorem 4.3.1) that this map is an isomorphism iff A is central simple over κ.

Example 1: To see how this can fail when the algebra is not simple consider the Grassmann algebra $\kappa[\theta_1, \dots, \theta_n]$. In terms of the filtration $\mathcal{F}^k$ described above note that any map of the form $x \mapsto axb$ with a, x, b in the Grassmann algebra must be nondecreasing on the filtration. For example, we cannot produce linear transformations that take $\theta_{i_1} \cdots \theta_{i_k}$ to superfields involving fewer than k θ's.

Example 2: Consider the algebra $\mathcal{A} = M_n(\kappa)$ of $n \times n$ matrices over the field κ. The general linear transformation in $\mathrm{End}_\kappa(\mathcal{A})$ can be expressed relative to a basis of matrix units e_{ij} as

$$T : e_{ij} \to \sum_{k,l} T_{kl,ij} e_{kl} \tag{A.9}$$

Then

$$T = \sum_{i,j,k,l} T_{kl,ij} LR(e_{ki} \otimes e_{jl}) \tag{A.10}$$

Exercise

Consider the map $\rho : M_n(\mathbb{C}) \to M_1(\mathbb{C})$ given by the determinant. Why can't we use this to define $\mathbb{C}$ as a left $M_n(\mathbb{C})$ module?

A.2 Generalization to superalgebras

Of course, a superalgebra over κ is graded-central, or super-central if $Z_s(\mathcal{A}) \cong \kappa$.

In [?] Wall defines a graded ideal $J \subset \mathcal{A}$ to be an ideal such that $J = J^0 \oplus J^1$. Thus, the even and odd parts of elements of the ideal are "independent." Not all ideals will be of this form. For example, in $\mathbb{C}\ell_1$ the subalgebra $x(1+e)$, $x \in \mathbb{C}$ is an ungraded ideal, but not a graded ideal. Then Wall defines a super-algebra to be (super-) simple if there are no nontrivial two-sided graded ideals.

Deligne in [16] instead takes the definition:

Definition A super-algebra over κ is *central simple* if, after extension of scalars to an algebraic closure $\bar{\kappa}$ it is isomorphic to a matrix super algebra $\mathrm{End}(V)$ or to $\mathrm{End}(V)\widehat{\otimes}D$ where D is a superdivision algebra.

This is the definition one finds in Section 3.3 of Deligne's *Notes on Spinors*. The super-analog of the Wedderburn theorem shows the equivalence of these two definitions. It is essentially proved in Wall's paper [40].

Example: The Clifford algebras over $\kappa = \mathbb{R}, \mathbb{C}$ are not always central simple in the ungraded sense but are always central simple in the graded sense.

A.3 Morita equivalence

There is a very useful equivalence relation on (super)-algebras known as *Morita equivalence*.

The basic idea of Morita equivalence is that, to algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent if their "representation theory is the same." More technically, if we consider the categories $\mathrm{Mod}^L(\mathcal{A}_i)$ of left $\mathcal{A}_i$-modules then the categories are equivalent.

Example 1: $\mathcal{A}_1 = \mathbb{C}$ and $\mathcal{A}_2 = M_n(\mathbb{C}) = \mathbb{C}(n)$ are Morita equivalent ungraded algebras. The general representation of $\mathcal{A}_1$ is a sum of n copies of its irrep $\mathbb{C}$. So the general left $\mathcal{A}_1$-module M is isomorphic to

$$M \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m \quad \text{times}} \tag{A.11}$$

for some positive integer m. On the other hand, the general representation N of $\mathcal{A}_2$ can similarly be written

$$N \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{m \quad \text{times}} \tag{A.12}$$

again, for some positive integer m. Now, $\mathbb{C}^n$ is a left $\mathcal{A}_2$-module, but is also a right $\mathcal{A}_1$-module. Then, if M is a general left $\mathcal{A}_1$-module we can form $\mathbb{C}^n \otimes_{\mathcal{A}_1} M$ which is now a left $\mathcal{A}_2$-module. Conversely, given a left $\mathcal{A}_2$-module N we can recover a left $\mathcal{A}_1$-module from

$$M = \mathrm{Hom}_{\mathcal{A}_2}(\mathbb{C}^n, N) \tag{A.13}$$

Example 2: More generally, if $\mathcal{A}$ is a unital algebra then $\mathcal{A}$ and $M_n(\mathcal{A})$ are Morita equivalent, by considerations similar to those above.

In more general terms, a criterion for Morita equivalence is based on the notion of *bimodules*. An $\mathcal{A}_1 - \mathcal{A}_2$ bimodule $\mathcal{E}$ is a vector space which is simultaneously a left $\mathcal{A}_1$ module and a right $\mathcal{A}_2$ module (so that the actions of $\mathcal{A}_1$ and $\mathcal{A}_2$ therefore commute).

Now, given an $\mathcal{A}_1 - \mathcal{A}_2$ bimodule $\mathcal{E}$ we can define two functors:

$$F : \mathrm{Mod}^L(\mathcal{A}_1) \to \mathrm{Mod}^L(\mathcal{A}_2) \tag{A.14}$$

$$G : \mathrm{Mod}^L(\mathcal{A}_2) \to \mathrm{Mod}^L(\mathcal{A}_1) \tag{A.15}$$

as follows: For $M \in \mathrm{Mod}^L(\mathcal{A}_1)$ we define

$$F(M) := \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, M) \tag{A.16}$$

Note that this is in fact a *left* $\mathcal{A}_2$ module. To see that suppose that $a \in \mathcal{A}_2$ and $T : \mathcal{E} \to M$ commutes with the left $\mathcal{A}_1$-action. Then we define $(a \cdot T)(p) := T(pa)$ for $p \in \mathcal{E}$. Then we compute

$$\begin{aligned}(a_1 \cdot (a_2 \cdot T))(p) &= (a_2 \cdot T)(pa_1) \\ &= T(pa_1a_2) \\ &= ((a_1a_2) \cdot T)(p)\end{aligned} \tag{A.17}$$

On the other hand, given a left $\mathcal{A}_2$-module N we can define a left $\mathcal{A}_1$-module by

$$G(N) = \mathcal{E} \otimes_{\mathcal{A}_2} N \tag{A.18}$$

For Morita equivalence we would like F, G to define equivalences of categories so there must be natural identifications of

$$M \cong \mathcal{E} \otimes_{\mathcal{A}_2} \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, M) \cong \left(\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee\right) \otimes_{\mathcal{A}_1} M \tag{A.19}$$

$$N \cong \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}_2} N) \cong \left(\mathcal{E}^\vee \otimes_{\mathcal{A}_1} \mathcal{E}\right) \otimes_{\mathcal{A}_2} N \tag{A.20}$$

Therefore, for Morita equivalence $\mathcal{E}$ must be invertible in the sense that there is an $\mathcal{A}_2 - \mathcal{A}_1$ bimodule $\mathcal{E}^\vee$ with

$$\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee \cong \mathcal{A}_1 \tag{A.21}$$

as $\mathcal{A}_1 - \mathcal{A}_1$ *bimodules* together with

$$\mathcal{E}^\vee \otimes_{\mathcal{A}_1} \mathcal{E} \cong \mathcal{A}_2 \tag{A.22}$$

as $\mathcal{A}_2 - \mathcal{A}_2$ *bimodules.* In fact we can recover one algebra from the other

$$\mathcal{A}_2 \cong \mathrm{End}_{\mathcal{A}_1}(\mathcal{E}) \tag{A.23}$$

$$\mathcal{A}_1 \cong \mathrm{End}_{\mathcal{A}_2}(\mathcal{E}) \tag{A.24}$$

and within the algebra of κ-linear transformations $\mathrm{End}(\mathcal{E})$ we have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are each others commutant: $\mathcal{A}_1' = \mathcal{A}_2$.

Moreover, $\mathcal{E}$ determines $\mathcal{E}^\vee$ by saying

$$\mathcal{E}^\vee \cong \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E},\mathcal{A}_1) \qquad \text{as} \quad \text{left} \quad \mathcal{A}_2 \quad \text{module,} \tag{A.25}$$

$$\mathcal{E}^\vee \cong \mathrm{Hom}_{\mathcal{A}_2}(\mathcal{E},\mathcal{A}_2) \qquad \text{as} \quad \text{right} \quad \mathcal{A}_1 \quad \text{module.} \tag{A.26}$$

Another useful characterization of Morita equivalent algebras is that there exists a full idempotent [58] $e \in \mathcal{A}_1$ and a positive integer n so that

$$\mathcal{A}_2 \cong e M_n(\mathcal{A}_1) e. \tag{A.27}$$

Example $\mathcal{A}_1 = M_n(\kappa)$ is Morita equivalent to $\mathcal{A}_2 = M_m(\kappa)$ by the bimodule $\mathcal{E}$ of all $n \times m$ matrices over κ. Indeed, one easily checks that

$$\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee \cong \mathcal{A}_1 \tag{A.28}$$

(Exercise: Explain why the dimensions match.) and

$$\mathcal{E}^\vee \otimes_{\mathcal{A}_2} \mathcal{E} \cong \mathcal{A}_2 \tag{A.29}$$

Similarly, we can check the other identities above.

Remark: One reason Morita equivalence is important is that many aspects of representation theory are "the same." In particular, one approach to K-theory emphasizes algebras. Roughly speaking, $K_0(\mathcal{A})$ is defined to be the Grothendieck group or group completion of the monoid of finite-dimensional projective left $\mathcal{A}$-modules. The K-theories of two Morita equivalent algebras are isomorphic abelian groups.

The above discussion generalizes straightforwardly to superalgebras: Two superalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be Morita equivalent if there is a matrix superalgebra $\mathrm{End}(V)$ such that

$$\mathcal{A}_1 \cong \mathcal{A}_2 \widehat{\otimes} \mathrm{End}(V) \tag{A.30}$$

or the other way around. This is useful because $\mathrm{End}(V)$ has essentially a unique representation (actually V and ΠV) and hence the representation theory of $\mathcal{A}_1$ and $\mathcal{A}_2$ are essentially the same.

Tensor product induces a multiplication structure on Morita equivalence classes of (super) algebras.

$$[A] \cdot [B] := [A \otimes_\kappa B] \tag{A.31}$$

If we take the algebra consisting of the ground field κ itself then we have an identity element $[\kappa] \cdot [A] = [A]$ for all algebras over κ. If A is central simple then there is an isomorphism

$$A \widehat{\otimes} A^{\mathrm{opp}} \cong \mathrm{End}_\kappa(A) \tag{A.32}$$

where on the RHS we mean the algebra of linear transformations of A as a κ vector space. Since A is assumed finite dimensional this is isomorphic to a matrix algebra over κ and hence Morita equivalent to κ itself. Therefore the above product defines a group operation and not just a monoid. If we speak of ordinary algebras then this group is known as the *Brauer group* of κ, and if we speak of superalgebras we get the *graded Brauer group* of κ.

[58] If R is a ring then an idempotent $e \in R$ satisfies $e^2 = e$. It is a *full idempotent* if $ReR = R$.

ec:WallTheorem

A.4 Wall's theorem

The classification of real super-division algebras is based on Wall's theorem Wall [40], which we quote here:

Theorem Is $\mathcal{A}$ is a central simple superalgbra over a field κ then:

1. As ungraded algebras, either $\mathcal{A}$ or $\mathcal{A}^0$ is central simple over κ, but not both. We call these cases $\epsilon = +1$ and $\epsilon = -1$, respectively.

2. By Wedderurn's theorem we can associate a division algebra over κ, denoted D by $\mathcal{A} \cong M_n(D)$ in case $\epsilon = +1$ or $\mathcal{A}^0 = M_n(D)$ in case $\epsilon = -1$.

3. In case $\epsilon = +1$, there exists an element $\omega \in \mathcal{A}^0$, unique up to multiplication by elements of κ^*, characterized by the condition that $\omega^2 = a \neq 0$ and the centralizer of $\mathcal{A}^0$ in $\mathcal{A}$, as an ungraded algebra is $\kappa \oplus \kappa\omega$ and $y\omega = -\omega y$ for all $y \in \mathcal{A}^1$.

4. In case $\epsilon = -1$, there exists an element $\omega \in \mathcal{A}^1$, unique up to multiplication by elements of κ^*, characterized by the condition that $\omega^2 = a \neq 0$, the center of $\mathcal{A}$ as an ungraded algebra is $\kappa + \kappa\omega$ and $\mathcal{A}^1 = \omega\mathcal{A}^0$.

5. The triple of invariants $\epsilon \in \{\pm 1\}$, D, and $a \in \kappa^*/(\kappa^*)^2$ characterize the central simple superalgebra $\mathcal{A}$ up to Morita equivalence.

B. Summary of Lie algebra cohomology and central extensions

LieAlgebraCoho

A central extension of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{z}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ such that we have an exact sequence of Lie algebras:

$$0 \to \mathfrak{z} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

with $\mathfrak{z}$ mapping into the center of $\widetilde{\mathfrak{g}}$. As a *vector space* (but not necessarily as a Lie algebra) $\widetilde{\mathfrak{g}} = \mathfrak{z} \oplus \mathfrak{g}$ so we can denote elements by (z, X) and the Lie bracket has the form

$$[(z_1, X_1), (z_2, X_2)] = (c(X_1, X_2), [X_1, X_2])$$

where $c : \Lambda^2\mathfrak{g} \to \mathfrak{z}$ is known as a *two-cocycle* on the Lie algebra. That is $c(X, Y)$ is bilinear, it satisfies

$$c(X, Y) = -c(Y, X) \tag{B.1}$$ twococyci

and the Jacobi relation requires

$$c([X_1, X_2], X_3) + c([X_3, X_1], X_2) + c([X_2, X_3], X_1) = 0. \tag{B.2}$$ twococyci

Two different cocycles can define isomorphic Lie algebras. If there is a *linear* function $f : \mathfrak{g} \to \mathfrak{z}$ such that

$$c(X, Y) = df(X, Y) := f([X, Y]) \tag{B.3}$$ cobodn

then the cocycle is said to be trivial, and the central extension is isomorphic to $\mathfrak{z} \oplus \mathfrak{g}$ as a Lie algebra. Indeed,

$$\psi : X \mapsto (f(X), X)$$

defines an explicit Lie algebra homomorphism $\psi : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ splitting the sequence. (Exercise!) More generally, if two cocycles differ by a cocycle of the form df then they define isomorphic Lie algebras. Thus, again, classifying isomorphism classes of central extensions is a cohomology problem, in this case, Lie algebra cohomology of degree two.

Finally, Suppose that $\widetilde{G}$ is a Lie group central extension of a Lie group G. A central extension of the Lie group, defined by the group cocycle $c_G(g_1, g_2)$, also defines a corresponding 2-cocycle on the Lie algebra by

$$c_{\mathfrak{g}}(X_1, X_2) = \frac{1}{2\pi i}\frac{d}{dt_1}|_0 \frac{d}{dt_2}|_0 \log\left[\frac{c_G(e^{t_1 X_1}, e^{t_2 X_2})}{c_G(e^{t_2 X_2}, e^{t_1 X_1})}\right] \tag{B.4}$$

twoconscy

If $c_{\mathfrak{g}}$ is nontrivial then c_G will be. However, the converse statement is not correct. Indeed, the Spin representation discussed above provides a counterexample.

B.1 Lie algebra cohomology more generally

To put this into a broader context consider the the vector spaces $\Lambda^k \mathfrak{g}^*$ of k-forms on the Lie algebra $\mathfrak{g}$. We can assemble them into a complex by introducing a differential

$$d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^* \tag{B.5}$$

eq:lacoh

defined by the equation

$$d\omega(X_1, \dots, X_{k+1}) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots \hat{X}_i, \dots, \hat{X}_j \dots). \tag{B.6}$$

eq:extderi

The resulting differential (B.5) may also be usefully expressed in terms of a Grassmann algebra. To do this introduce a basis T_a for $\mathfrak{g}$ (so $a = 1, \dots, \dim\mathfrak{g}$) and corresponding structure constants

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{B.7}$$

and let θ^a be the dual basis so $\theta^a(T_b) = \delta^a{}_b$. We can then identify $\Lambda^* \mathfrak{g}^*$ with the Grassmann algebra $\Lambda^*[\theta^a]$ where θ^a are of degree 1. We then *define* the differential to be:

$$d\theta^a := -\frac{1}{2} f_{bc}{}^a \theta^b \theta^c \tag{B.8}$$

eq:lacohii

Exercise
Check that this is a differential, that is, that $d^2 = 0$.

The cohomology of the complex $(\Lambda^* \mathfrak{g}^*, d)$ is known as *Lie algebra cohomology* and denoted $H^*(\mathfrak{g})$. Note that it can be formulated purely algebraically. The differential defined by (B.6) or, equivalently, (B.8) is sometimes called the *Chevalley-Eilenberg differential.*

Remark: In the theory of the topology of Lie groups there is a theorem, the Hopf-Samelson theorem, which states that if G is compact and connected then

$$H^*_{DR}(G) \cong H^*(\mathfrak{g}) \tag{B.9}$$

The proof used both the connectedness and the compactness of G. To see that compactness is essential consider the abelian Lie group $\mathbb{R}^n$ under addition. We have

$$H^j_{DR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & j = 0 \\ 0 & else \end{cases} \tag{B.10}$$

eq:arrenn

but the Lie algebra cohomology is:

$$H^*_{Lie\ Algebra}(\mathbb{R}^n) = \Lambda^*[\theta^1, \dots, \theta^n] \tag{B.11}$$

eq:liealrn

since $d\theta^a = 0$, since the structure constants vanish. The two are very different! For a nice discussion of a theory that replaces this one in the noncompact case see Bott, "On the continuous cohomology..."

B.2 The physicist's approach to Lie algera cohomology

Suppose we have a Lie algebra $\mathfrak{g}$ with basis T_a, a is an index running over the generators. Let us introduce the Clifford algebra:

$$\{c^a, b_{a'}\} = \delta^a{}_{a'} \tag{B.12}$$

eq:ghosti

where $c^a, b_{a'}$ are referred to as ghosts and antighosts, respectively.

We can quantize the Clifford algebra by choosing a Clifford vacuum

$$b_{a'}|0\rangle = 0 \tag{B.13}$$

eq:ghostii

and the resulting Hilbert space is spanned by $|0\rangle, c^a|0\rangle, ...$

The Hilbert space is graded by the "ghost number operator" $N = \sum_a c^a b_a$, and we have an isomorphism of the vector space of states of ghost number k with $\Lambda^k \mathfrak{g}^*$:

$$\omega \leftrightarrow \frac{1}{k!}\omega_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}|0\rangle \tag{B.14}$$

eq:ghostiii

Under the isomorphism (eq:ghostiii B.14) the Chevalley-Eilenberg differential becomes what is known as the *BRST operator*:

$$Q := -\frac{1}{2} f_{a_2 a_3}{}^{a_1} c^{a_2} c^{a_3} b_{a_1} \tag{B.15}$$

eq:ghostiv

Exercise

a.) Prove that the differential d of (eq:lacohii B.8) maps to Q of (eq:ghostiv B.15) under the isomorphism (eq:ghostiii B.14).

b.) Show directly that $Q^2 = 0$.

The BRST cohomology is the cohomology of Q, and is graded by ghost number.

BRST cohomology enters physics in the quantization of theories with local gauge symmetry. In this context it is important to use a very natural generalization. Suppose we have a representation ρ of the Lie algebra $\mathfrak{g}$. We can then consider the complex

$$\Lambda^*\mathfrak{g}^* \otimes V \tag{B.16}$$

and introduce a differential

$$Q = c^a t_a - \frac{1}{2} f_{a_2 a_3}{}^{a_1} c^{a_2} c^{a_3} b_{a_1} \tag{B.17}$$ eq:ghostv

where $t_a = \rho(T_a)$ are the representation matrices of the rep. V.

Geometrically, the cohomology $H_Q^*(\Lambda^*\mathfrak{g}^* \otimes V)$ can be identified, for G compact and connected, with the cohomology $H_{DR}^*(G;\mathcal{V})$ of a homogeneous vector bundle over the group G.

Exercise
Check that $Q^2 = 0$

C. Background material: Cartan's symmetric spaces

ymmetricSpaces

Definition: A *symmetric space* is a (pseudo) Riemannian manifold (M, g) such that every point p is an isolated fixed point of an involutive isometry τ_p.

Near any point p, the involutive isometry τ_p can be expressed as the inversion of the geodesics through p. That is, if $(x^1, \dots, x^n)$ are normal coordinates in a neighborhood of p with $\vec{x} = 0$ the coordinate of p then $\tau_p(\vec{x}) = -\vec{x}$. Importantly, τ_p extends to an involutive isometry of the full Riemannian space (M, g)

One can show that the Riemannian curvature is covariantly constant, and hence there are three families of examples where the scalar curvature (which is must be constant) is positive, zero, or negative.

Cartan classified the symmetric spaces and found that they are always homogeneous spaces of Lie groups. The positive curvature examples are of the form G/K where G is a compact Lie group and K is a Lie subgroup.

Let us first examine G/K at the Lie algebra level. The tangent space of G at 1 is the Lie algebra $\mathfrak{g}$ and the tangent space of K at 1 is the Lie subalgebra $\mathfrak{k}$. If we write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{C.1}$$ eq:CartanDecomp

then there is a natural identification of $\mathfrak{p}$ with $T_K(G/K)$. The involutive isometry τ_p where $p = 1 \cdot K$ has a differential $\theta = d\tau_p : \mathfrak{p} \to \mathfrak{p}$ which in fact can be shown to be the restriction of an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$. That is, θ is a Lie algebra homomorphism $\theta([X, Y]) = [\theta(X), \theta(Y)]$ which is an isomorphism of vector spaces and $\theta^2 = Id$. The $+1$ eigenspace is $\mathfrak{k}$ and the -1 eigenspace is $\mathfrak{p}$. The property that it is a Lie algebra automorphism implies that

♣subsets not proper? ♣

$$
\begin{aligned}
[\mathfrak{k},\mathfrak{k}] &\subset \mathfrak{k} \\
[\mathfrak{k},\mathfrak{p}] &\subset \mathfrak{p} \\
[\mathfrak{p},\mathfrak{p}] &\subset \mathfrak{k}
\end{aligned}
\tag{C.2}
$$

The decomposition (C.1) associated with θ is called a *Cartan decomposition.* [59]

Now let us consider G/K at the global level. The reduction of the τ_p to a single involutive automorphism θ of $\mathfrak{g}$ has a global analog: There is an involutive automorphism τ of the group. (That is, τ is a group automorphism and $\tau^2 = Id$) such that $d\tau = \theta$ at the identity. Given such an involutive automorphism τ we can define a subgroup K to be the fixed points of τ:

$$
K = \{g \in G | \tau(g) = g\} \tag{C.3}
$$

Given such an involution we have a *Cartan embedding* by the "anti-fixed points":

$$
G/K \hookrightarrow \mathcal{O} := \{g \in G | \tau(g) = g^{-1}\} \tag{C.4}
$$

$$
gK \mapsto \tau(g)g^{-1}. \tag{C.5}
$$

Note that this is well-defined and indeed $\tau(\tau(g)g^{-1}) = (\tau(g)g^{-1})^{-1}$ because τ is an involution. One checks it is an embedding by looking at the neighborhood of $g = 1$. Then we identify $d\tau_1 = \theta$. To see it is surjective note that $\mathcal{O}$ admits a left G-action by twisted adjoint action: If $g_0 \in G$ and $g \in \mathcal{O}$ then $\tau(g_0)gg_0^{-1} \in \mathcal{O}$, and the isotropy group of this action at $g = 1$ is precisely K. The metric tensor is just the pullback of the usual left-right-invariant metric $-\mathrm{Tr}(g^{-1}dg) \otimes (g^{-1}dg)$. The inversion τ_{g_*} through $g_* \in \mathcal{O}$ required by the definition is $\tau_{g_*} : g \mapsto g_* g^{-1} g_*$. One easily checks that this takes $\mathcal{O} \to \mathcal{O}$ and is an isometry of the metric. To see that g_* is an isolated fixed point of τ_{g_*} use the left G-action to translate to $g_* = 1$ and use the involution θ on $\mathfrak{g}$ above. We see that infinitesimally it is the exponential of elements of $\mathfrak{p}$ which lie in $\mathcal{O}$ in the neighborhood of 1.

It is also worth noting that the Cartan embedding $\mathcal{O}$ of G/K is a totally geodesic submanifold, as follows from the same reasoning used at the end of 2.3

Now that we have these definitions we give the 10 classes of compact classical symmetric spaces:

Whenever G is a compact simple Lie group the homogeneous space $(G \times G)/G_{\mathrm{diag}}$ is a symmetric space. Suppose the action of the diagonal subgroup is on the right, then we have an isomorphism of manifolds:

$$
(G \times G)/G_{\mathrm{diag}} \cong G \tag{C.6}
$$

where we take $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Warning! This is *not* a group homomorphism. The involution τ is just $\tau : (g_1, g_2) \mapsto (g_2, g_1)$. In particular, if we take $G = U(n, \mathbb{R}) = O(n)$, $G = U(n, \mathbb{C}) = U(n)$, or $G = U(n, \mathbb{H}) = Sp(n)$ then we get a series of 3 classical symmetric spaces:

$$
(O(n) \times O(n))/O(n) \tag{C.7}
$$

[59] A *Cartan involution* of a Lie algebra is an involutive Lie algebra automorphism s such that $B(X, sY)$ is positive definite. θ is related to a Cartan involution. ♣ Clarify some confusing terminology. See Helgason III.7 for the straight story.♣

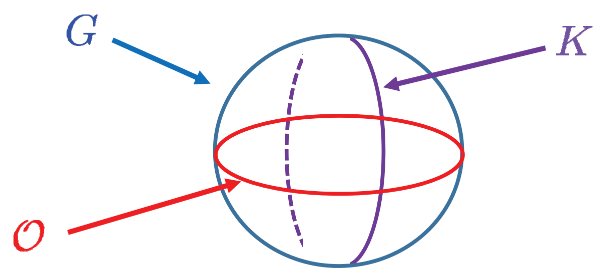

Figure 17: K and G/K locally divide the group into a product. fig:CartanEmbeddi

$$(U(n)\times U(n))/U(n) \tag{C.8}$$ eq:ClassCartSpace

$$(Sp(n)\times Sp(n))/Sp(n) \tag{C.9}$$ eq:ClassCartSpace

Another natural series of classical symmetric spaces are the Grassmannians. These arise from the involutive automorphism coming from conjugation

$$\tau(g) = g_0 g g_0^{-1} \qquad g_0 = \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix} \tag{C.10}$$

We can consider Grassmannians in real, complex, and quaternionic vector spaces to get

$$\mathrm{Gr}_k(\mathbb{R}^n) \cong O(n)/(O(k)\times O(n-k)) \tag{C.11}$$ eq:ClassCartSpace

$$\mathrm{Gr}_k(\mathbb{C}^n) \cong U(n)/(U(k)\times U(n-k)) \tag{C.12}$$ eq:ClassCartSpace

$$\mathrm{Gr}_k(\mathbb{H}^n) \cong Sp(n)/(Sp(k)\times Sp(n-k)) \tag{C.13}$$ eq:ClassCartSpace

With a little charity (regarding cases with $k\neq n-k$ as nonzero index analogs of the cases with $k=n-k$) we can consider this to be three more series of classical symmetric spaces.

Finally, as discussed in Section §7 (sec:RCH-VS), we can put real, complex, or quaternionic structures on real, complex, or quaternionic spaces (when this makes sense). When these structures are made compatible with standard Euclidean metrics we obtain moduli spaces of structures. This gives us:

Real structures on complex vector spaces: $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$. Moduli space

$$U(n)/O(n) \tag{C.14}$$ eq:ClassCartSpace

This comes from $\tau(u) = u^*$.

Complex structures on real vector spaces: $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Moduli space:

$$O(2n)/U(n) \tag{C.15}$$

eq:ClassCartSpace

This comes from $\tau(g) = I_0 g I_0^{-1}$ where I_0 is (7.9).

Complex structures on quaternionic vector spaces: $\mathbb{C}^n \hookrightarrow \mathbb{H}^n$. Moduli space:

$$Sp(n)/U(n) \tag{C.16}$$

eq:ClassCartSpace

Viewing $Sp(n)$ as unitary $n \times n$ matrices over the quaternions the involution is $\tau(g) = -\mathfrak{i} g \mathfrak{i}$, i.e. conjugation by the unit matrix times $\mathfrak{i}$.

Quaternionic structures on complex vector spaces: $\mathbb{C}^{2n} \cong \mathbb{H}^n$. Moduli space:

$$U(2n)/Sp(n) \tag{C.17}$$

eq:ClassCartSpace

Viewing $Sp(n)$ as $USp(2n) := U(2n) \cap Sp(2n; \mathbb{C})$ we can use the involutive automorphism $\tau(g) = I_0^{-1} g^* I_0$ on $U(2n)$. The fixed points in $U(2n)$ are the group elements with $g I_0 g^{tr} = I_0$, but this is the defining equation of $Sp(2n, \mathbb{C})$.

When Cartan classified compact symmetric spaces he found the 10 series above (C.7) - (C.17) together with a finite set of exceptional cases. [60]

References

Albert [1] A.A. Albert, *Structure of Algebras*, AMS monograph

lekseev:1988vx [2] A. Alekseev, L. D. Faddeev and S. L. Shatashvili, "Quantization of symplectic orbits of compact Lie groups by means of the functional integral," J. Geom. Phys. **5**, 391 (1988).

tlandZirnbauer [3] A. Altland and M. Zirnbauer, "Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures," Phys. Rev. B55 (1997) pp. 1142-1161.

Altland:1997zz [4] A. Altland and M. R. Zirnbauer, "Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures," arXiv:cond-mat/9602137 ; Phys. Rev. B **55**, 1142 (1997).

Alvarez:1989zv [5] O. Alvarez, I. M. Singer and P. Windey, "Quantum Mechanics and the Geometry of the Weyl Character Formula," Nucl. Phys. B **337**, 467 (1990).

Alvarez:1991xn [6] O. Alvarez, I. M. Singer and P. Windey, "The Supersymmetric sigma model and the geometry of the Weyl-Kac character formula," Nucl. Phys. B **373**, 647 (1992) [hep-th/9109047].

ABS [7] M.F. Atiyah, R. Bott, and A. Shapiro, "Clifford Modules", Topology 3 (1964), ((Suppl. 1)): 338

iyahHirzebruch [8] M.F. Atiyah and F. Hirzebruch, "Vector Bundles and Homogeneous Spaces," Proc. Symp. Pure Math. **3** (1961)53

iyahSingerSkew [9] M.F. Atiyah and I.M. Singer, "Index theory for skew-adjoint Fredholm operators," Publ. Math. IHES, **37** (1969) pp. 5-26

[60] The exceptional cases are presumably all related to the nonassociative real division algebra $\mathbb{O}$, but explaining how this comes about appears to be nontrivial [19].

Baez [10] J.C. Baez, "The Octonions." arXiv:math/0105155 [math.RA]; Bull.Am.Math.Soc.39:145-205,2002

Bargmann [11] V. Bargmann, "Note on Wigner's Theorem on Symmetry Operations," J. Math. Phys. **5** (1964) 862.

Bourbaki [12] Bourbaki, *Algebra*, ch. 8

Brualdi [13] R.A. Brualdi and H. Schneider, *Determinantal Identities: Gauss, Schur, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley*, Linear Algebra and its applications, Vol. 52-53, (1983), pp. 769-791

Conrey [14] B. Conrey, http://arxiv.org/pdf/math/0005300v1.pdf

IAS-VOL1 [15] P. Deligne, et. al. , *Quantum Fields and Strings: A Course for Mathematcians*, vol. 1, AMS, 1999

DelignSpinors [16] P. Deligne, Notes on Spinors [hep-th/9710198].

Drozd [17] Yu. A. Drozd and V.V. Kirichenko, *Finite Dimensional Algebras*, Springer 1994

Dyson3fold [18] F.J. Dyson, "The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics,' J. Math. Phys. **3**, 1199 (1962)

Eschenburg [19] J.-H. Eschenburg, "Symmetric spaces and division algebras."

fz1 [20] Lukasz Fidkowski, Alexei Kitaev, " Topological phases of fermions in one dimension," arXiv:1008.4138

freedwigner [21] Freed, arXiv:1211.2133.

Freed:2012uu [22] D. S. Freed and G. W. Moore, "Twisted equivariant matter," arXiv:1208.5055 [hep-th].

riffithsHarris [23] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*,

Harris [24] J. Harris, *Algebraic Geometry*, Springer GTM 133

HHZ [25] P. Heinzner, A. Huckleberry, and M. Zirnbauer, "Symmetry classes of disordered fermions," arXiv:math-ph/0411040 ; Commun.Math.Phys. 257 (2005) 725-771

armonicSpinors [26] Hitchin, Nigel J. (1974), "Harmonic spinors", Advances in Mathematics 14: 155, MR 358873

Mehta [27] M.L. Mehta, *Random Matrices*

Michel-1 [28] L. Michel and B.I. Zhilinskii, "Symmetry, Invariants, Topology: Basic Tools," Physics Reports **341**(2001), pp. 11-84

Kitaev [29] A. Kitaev, "Periodic table for topological insulators and superconductors," AIP Conf. Proc. **1134** (2009) 22-30; arXiv:0901.2686

Michel-2 [30] L. Michel, "Fundamental Concepts for the Study of Crystal Symmetry," Physics Reports **341**(2001), pp. 265-336

MilnorMorse [31] J. Milnor, *Morse Theory*, Ann. Math. Studies, **51**, Princeton (1963)

SaintOtt [32] G. Moore, Lecture notes at http://www.physics.rutgers.edu/ gmoore/QuantumSymmetryKTheory-Part1.pdf

SCGPLecture [33] G. Moore, Lecture at http://www.physics.rutgers.edu/ gmoore/SCGP-TWISTEDMATTER-2013D.pdf

UsersManual [34] G. Moore, "Linear ALgebra User's Manual," Lecture notes from Applied Group Theory Course, Spring 2013.

PS-LoopGroups [35] A. Pressley and G. Segal, *Loop Groups*, OUP

SegalLectures [36] R. Carter, G. Segal, and I. Macdonald, *Lectures on Lie Groups and Lie Algebras*, London Math Soc. Student Texts 32.

SimonQD [37] B. Simon, "Quantum Dynamics: From Automorphism to Hamiltonian."

Stone [38] M. Stone, C-K. Chiu, and A. Roy, "Symmetries, dimensions and topological insulators: the mechanism behind the face of the Bott clock," arXiv:1005:3213.

TwoElementary [39] R. Simon, N. Mukinda, S. Chaturvedi, and V. Srinivasan, "Two elementary proofs of the Wigner theorem on symmetry in quantum mechanics," arXiv:0808.0779

Wall [40] C.T.C. Wall, "Graded Brauer Groups," Journal für die reine und angewandte Mathematik, **213** (1963/1964), 187-199

Weinberg [41] S. Weinberg, *Quantum Field Theory, vol. 1*, pp.

Witten:1998cd [42] E. Witten, "D-branes and K theory," JHEP **9812**, 019 (1998) [hep-th/9810188].

Zirnbauer1 [43] M.R. Zirnbauer, "Symmetry classes in random matrix theory," arXiv:math-ph/0404058

Zirnbauer2 [44] M.R. Zirnbauer, "Symmetry Classes," arXiv:1001.0722