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Quantum Symmetries and K-Theory

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1. Lecture 1: Quantum Symmetries and the 3-fold way

1.1 Introduction

These lecture notes are a truncated version of an more extended set of notes which can be found in [43].

The notes are meant to accompany a set of lectures given at the Erwin Schrödinger Institute in Vienna, August 18-21, 2014. They constitute a review of material most of which has been drawn from the paper [26]. That paper, in turn, was largely meant as an exposition, in formal mathematical terms, of some beautiful results of C. Kane et. al., A. Kitaev, A. Ludwig et. al., and M. Zirnbauer et. al.

There is very little here which has not been published previously. There are only two novelties: One is a stress on a presentation of the 10-fold classification of gapped phases which follows only from general properties of quantum mechanics. The presentation deliberately avoids the mention of free fermions. It in principle could apply to systems of fermions or bosons, interacting or not. The second, rather minor, novelty is the use of localization techniques in equivariant K-theory to demonstrate some of the refined K-theory invariants in the theory of topological band structure which appear when one works equivariantly with respect to a crystallographic group.

For a more extensive introduction describing material to be covered see the first section of [43]. The URL is

http://www.physics.rutgers.edu/ gmoore/QuantumSymmetryBook.pdf

The present notes can be found at

http://www.physics.rutgers.edu/ gmoore/ViennaLecturesF.pdf

The lectures are an extended version of a talk at the SCGP:

http://www.physics.rutgers.edu/ gmoore/SCGP-TWISTEDMATTER-2013D.pdf

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1.2 Quantum Automorphisms

1.2.1 States, operators and probabilities

Let us recall the basic tenets of quantum mechanics (these are the most important of the Dirac-von Neumann axioms)

1. To a physical system we associate a complex Hilbert space $\mathcal{H}$ such that

2. Physical states are identified with traceclass positive operators ρ of trace one. They are usually called *density matrices*. We denote the space of physical states by $\mathcal{S}$.

3. Physical observables are identified with self-adjoint operators. We denote the set of self-adjoint operators by $\mathcal{O}$. [1]

4. *Born rule*: When measuring the observable O in a state ρ the probability of measuring value $e \in E \subset \mathbb{R}$, where E is a Borel-measurable subset of $\mathbb{R}$, is

$$P_{\rho,O}(E) = \mathrm{Tr} P_O(E)\rho. \tag{1.1}$$

   Here P_O is the projection-valued-measure associated to the self-adjoint operator O by the spectral theorem.

[1] We are skating over many subtleties here.

Now, the set of states $\mathcal{S}$ is a convex set and hence the extremal points, known as the *pure states* are distinguished. These are the dimension one projection operators. They are often identified with rays in Hilbert space for the following reason:

If $\psi \in \mathcal{H}$ is a nonzero vector then it determines a line

$$\ell_\psi := \{z\psi | z \in \mathbb{C}\} := \psi\mathbb{C} \tag{1.2}$$

Note that the line does not depend on the normalization or phase of ψ, that is, $\ell_\psi = \ell_{z\psi}$ for any nonzero complex number z. Put differently, the space of such lines is projective Hilbert space

$$\mathbb{P}\mathcal{H} := (\mathcal{H} - \{0\})/\mathbb{C}^* \tag{1.3}$$

Equivalently, this can be identified with the space of rank one projection operators. Indeed, given any line $\ell \subset \mathcal{H}$ we can write, in Dirac's bra-ket notation: [2]

$$P_\ell = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \tag{1.4}$$

where ψ is any nonzero vector in the line ℓ.

1.2.2 Automorphisms of a quantum system

Now we state the formal notion of a general "symmetry" in quantum mechanics:

Definition An *automorphism* of a quantum system is a pair of bijective maps $s_1 : \mathcal{O} \to \mathcal{O}$ and $s_2 : \mathcal{S} \to \mathcal{S}$ where s_1 is real linear on $\mathcal{O}$ such that (s_1, s_2) preserves probability measures:

$$P_{s_1(O), s_2(\rho)} = P_{O,\rho} \tag{1.5}$$

This set of mappings forms a group which we will call the group of *quantum automorphisms*.

♣Need to state some appropriate continuity properties. ♣

One can show that that $s = s_2$ must take extreme states to extreme states, and hence induces a single map

$$s : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}. \tag{1.6}$$

Moreover, the preservation of probabilities, restricted to the case of self-adjoint operators given by rank one projectors and pure states (also given by rank one projectors) means that the function

$$\mathfrak{o} : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0, 1] \tag{1.7}$$

defined by

$$\mathfrak{o}(\ell_1, \ell_2) := \mathrm{Tr} P_{\ell_1} P_{\ell_2} \tag{1.8}$$

must be invariant under s:

$$\mathfrak{o}(s(\ell_1), s(\ell_2)) = \mathfrak{o}(\ell_1, \ell_2) \tag{1.9}$$

[2] We generally denote inner products in Hilbert space by $(x_1, x_2) \in \mathbb{C}$ where $x_1, x_2 \in \mathcal{H}$. Our convention is that it is complex-linear in the second argument. However, we sometimes write equations in Dirac's bra-ket notation because it is very popular. In this case, identify x with $|x\rangle$. Using the Hermitian structure there is a unique anti-linear isomorphism of $\mathcal{H}$ with $\mathcal{H}^*$ which we denote $x \mapsto \langle x|$. Sometimes we denote vectors by Greek letters $\psi, \chi, \dots$, and scalars by Latin letters $z, w, \dots$. But sometimes we denote vectors by Latin letters, $x, w, \dots$ and scalars by Greek letters, $\alpha, \beta, \dots$.

Definition The function defined by (1.7) and (1.8) is known as the *overlap function*.

Remarks

1. The upshot of our arguments above is that the quantum automorphism group of a system with Hilbert space $\mathcal{H}$ can be identified with the group of (suitably continuous) maps (1.6) such that (1.9) holds for all lines ℓ_1, ℓ_2. We denote the group of such maps by $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

2. The reason for the name "overlap function" or "transition probability" which is also used, is that if we choose representative vectors $\psi_1 \in \ell_1$ and $\psi_2 \in \ell_2$ we obtain the - perhaps more familiar - expression:

$$\mathrm{Tr} P_{\ell_1} P_{\ell_2} = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle} \tag{1.10}$$

1.2.3 Overlap function and the Fubini-Study distance

If $\mathcal{H}$ is finite dimensional then we can identify it as $\mathcal{H} \cong \mathbb{C}^N$ with the standard hermitian metric. Then $\mathbb{P}\mathcal{H} = \mathbb{C}P^{N-1}$ and there is a well-known metric on $\mathbb{CP}^{N-1}$ known as the "Fubini-Study metric" from which one can define a minimal geodesic distance $d(\ell_1, \ell_2)$ between two lines (or projection operators). When the FS metric is suitably normalized the overlap function $\mathfrak{o}$ is nicely related to the Fubini-Study distance d by

$$\mathfrak{o}(\ell_1, \ell_2) = \left(\cos \frac{d(\ell_1, \ell_2)}{2}\right)^2 \tag{1.11}$$

Let us first check this for the case $N = 2$. Then we claim that for the case

$$\mathbb{P}\mathcal{H}^2 = \mathbb{C}P^1 \cong S^2 \tag{1.12}$$

d is just the usual round metric on the sphere and the proper normalization will be unit radius. Let us first check this:

First we write the most general general density matrix in two dimensions. Any 2×2 Hermitian matrix is of the form $a + \vec{b} \cdot \vec{\sigma}$ where $\vec{\sigma}$ is the vector of "Pauli matrices":

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{1.13}$$

$a \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^3$. Now a density matrix ρ must have trace one, and therefore $a = \frac{1}{2}$. Then the eigenvalues are $\frac{1}{2} \pm |\vec{b}|$ so positivity means it must have the form

$$\rho = \frac{1}{2}(1 + \vec{x} \cdot \vec{\sigma}) \tag{1.14}$$

where $\vec{x} \in \mathbb{R}^3$ with $\vec{x}^2 \leq 1$.

The extremal states, corresponding to the rank one projection operators are therefore of the form

$$P_{\vec{n}} = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \tag{1.15}$$

where $\vec{n}$ is a unit vector. This gives the explicit identification of the pure states with elements of S^2. Moreover, we can easily compute:

$$\mathrm{Tr} P_{\vec{n}_1} P_{\vec{n}_2} = \frac{1}{2}(1 + \vec{n}_1 \cdot \vec{n}_2) \tag{1.16}$$

and $\vec{n}_1 \cdot \vec{n}_2 = \cos(\theta_1 - \theta_2)$ where $|\theta_1 - \theta_2|$ (with θ's chosen so this is between 0 and π) is the geodesic distance between the two points on the unit sphere. Thus we obtain (1.11).

There is another viewpoint which is useful. Nonzero vectors in $\mathbb{C}^2$ can be normalized to be in the unit sphere S^3. Then the association of projector to state given by

$$|\psi\rangle \to |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \tag{1.17}$$

defines a map $\pi : S^3 \to S^2$ known as the *Hopf fibration*.

The unit sphere is a principal homogeneous space for $SU(2)$ and we may coordinatize $SU(2)$ by the Euler angles:

$$u = e^{-i\frac{\phi}{2}\sigma^3} e^{-i\frac{\theta}{2}\sigma^2} e^{-i\frac{\psi}{2}\sigma^3} \tag{1.18}$$

with range $0 \leq \theta \leq \pi$ and identifications:

$$(\phi, \psi) \sim (\phi + 4\pi, \psi) \sim (\phi, \psi + 4\pi) \sim (\phi + 2\pi, \psi + 2\pi) \tag{1.19}$$

We can make an identification with the unit sphere in $\mathbb{C}^2$ by viewing it as a homogeneous space:

$$\psi = \begin{pmatrix} e^{-i\frac{\psi+\phi}{2}} \cos\theta/2 \\ e^{-i\frac{\psi-\phi}{2}} \sin\theta/2 \end{pmatrix} = u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.20}$$

The projector onto the line through this space is

$$P_{\ell_\psi} = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \tag{1.21}$$

with $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ as usual. Alternatively, we could map $\pi : S^3 \to S^2$ by $\pi(\psi) = [\psi_1 : \psi_2] \cong \mathbb{C}P^1$, and this will correspond to the point in S^2 by the usual stereographic projection. ♣from which pole? ♣

In any case, for the case $N = 2$ we see that $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ is just the group of isometries of S^2 with its round metric. This group is well known to be the orthogonal group $O(3)$.

It is known that the FS metric on $\mathbb{C}P^N$ has the property that the submanifolds $\mathbb{C}P^k \to \mathbb{C}P^N$ embedded by $[z_1 : \cdots : z_{k+1}] \to [z_1 : \cdots : z_{N+1}]$ are *totally geodesic submanifolds*.

Definition If (M, g) is a Riemannian manifold a submanifold $M_1 \subset M$ is said to be *totally geodesic* if the geodesics between any two points in M_1 with respect to the induced metric (the pullback of g) are the same as the geodesics between those two points considered as points of M.

If M_1 is the fixed point set of an isometry of (M, g) then it is totally geodesic. Now note that the submanifolds $\mathbb{C}P^k$ are fixed points of the isometry ♣simple proof or ref? ♣

$$[z_1 : \cdots : z_{N+1}] \to [z_1 : \cdots : z_{k+1} : -z_{k+2} : \cdots : -z_{N+1}] \tag{1.22}$$

Another way to see this from the viewpoint of homogeneous spaces is that if we exponentiate a Lie algebra element in $\mathfrak{p}$ to give a geodesic in $U(N+1)$ and project to the homogeneous space we get all geodesics on the homogeneous space. But for any $t \in \mathfrak{p}$ we can put it into a $U(2)$ subalgebra.

Now, any two lines ℓ_1, ℓ_2 span a two-dimensional sub-Hilbert space of $\mathcal{H}$, so, thanks to the totally geodesic property of the FS metric, our discussion for $\mathcal{H} \cong \mathbb{C}^2$ suffices to check (1.11) in general.

1.2.4 From (anti-) linear maps to quantum automorphisms

Now, there is one fairly obvious way to make elements of $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. Suppose $u \in U(\mathcal{H})$ is a unitary operator. Then it certainly takes lines to lines and hence can be used to define a map (which we also denote by u) $u : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$. For example if we identify ℓ as ℓ_ψ for some nonzero vector ψ then we can define

$$u(\ell_\psi) := \ell_{u(\psi)} \tag{1.23}$$

One checks that which vector ψ we use does not matter and hence the map is well-defined. In terms of projection operators:

$$u : P \mapsto uPu^\dagger \tag{1.24}$$

and, since u is unitary, the overlaps $\mathrm{Tr}(P_1 P_2)$ are preserved.

Now - very importantly - this is not the *only* way to make elements of $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

We call a map $a : \mathcal{H} \to \mathcal{H}$ *anti-linear* if

$$a(\psi_1 + \psi_2) = a(\psi_1) + a(\psi_2) \tag{1.25}$$

but

$$a(z\psi) = z^* a(\psi) \tag{1.26}$$

where z is a complex scalar. It is in addition called *anti-unitary* if it is norm-preserving:

$$\| a(\psi) \|^2 = \| \psi \|^2 \tag{1.27}$$

Now, anti-unitary maps also can be used to define quantum automorphisms. If we try to define $a(\ell)$, $\ell \in \mathbb{C}\mathcal{H}$ by

$$a(\ell_\psi) = \ell_{a(\psi)} \tag{1.28}$$

1.2.5 Wigner's theorem

In the previous subsection we showed how unitary and antiunitary operators on Hilbert space induce quantum automorphisms. Are there other ways of making quantum automorphisms? Wigner's theorem says no:

Theorem: Every quantum automorphism $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ is induced by a unitary or antiunitary operator on Hilbert space, as above.

In addition to Wigner's own argument the paper [49] cites 26 references with alternative proofs! (And there are others, for examples [52][24].) One proof, which follows Bargmann's paper, is in the notes [43]. Reference [24] gives two proofs. One is fairly elementary and quick: It uses the identification of the tangent space to $\mathbb{P}\mathcal{H}$ at a line ℓ as $\mathrm{Hom}(\ell,\ell^{\perp})$. WLOG one can assume the isometry takes ℓ to ℓ and then such an isometry induces one on the tangent space. This is *a priori* only a real linear operator, but then one can prove that it must be complex linear or anti-linear. Then the complex linear or anti-linear isometry of $\ell^{\perp}$ can be extended to $\mathcal{H} = \ell \oplus \ell^{\perp}$ as a complex linear or anti-linear isometry.

1.3 A little bit about group extensions

If N, G, and Q are three groups and ι and π are homomorphisms such that

$$1 \to N \stackrel{\iota}{\to} G \stackrel{\pi}{\to} Q \to 1 \tag{1.29}$$

is exact at N, G and Q then the sequence is called a *short exact sequence* and we say that G is an *extension of Q by N* .

Note that since ι is injective we can identify N with its image in G. Then, N is a kernel of a homomorphism (namely π) and is hence a *normal* or *invariant* subgroup (hence the notation). Then it is well-known that G/N is a group and is in fact isomorphic to the image of π. That group Q is thus a quotient of G (hence the notation).

There is a notion of *homomorphism of two group extensions*

$$1 \to N \stackrel{\iota_1}{\to} G_1 \stackrel{\pi_1}{\to} Q \to 1 \tag{1.30}$$

$$1 \to N \stackrel{\iota_2}{\to} G_2 \stackrel{\pi_2}{\to} Q \to 1 \tag{1.31}$$

This means that there is a group homomorphism $\varphi : G_1 \to G_2$ so that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "\iota_1"] \arrow[d, equal] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\iota_2"] & G_2 \arrow[r, "\pi_2"] & Q \arrow[r] & 1
\end{tikzcd} \tag{1.32}$$

When there is a homomorphism of group extensions based on $\psi : G_2 \to G_1$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity then the group extensions are said to be *isomorphic extensions*.

Given group N and Q it can certainly happen that there is more than one nonisomorphic extension of Q by N. Classifying all extensions of Q by N is a difficult problem. We

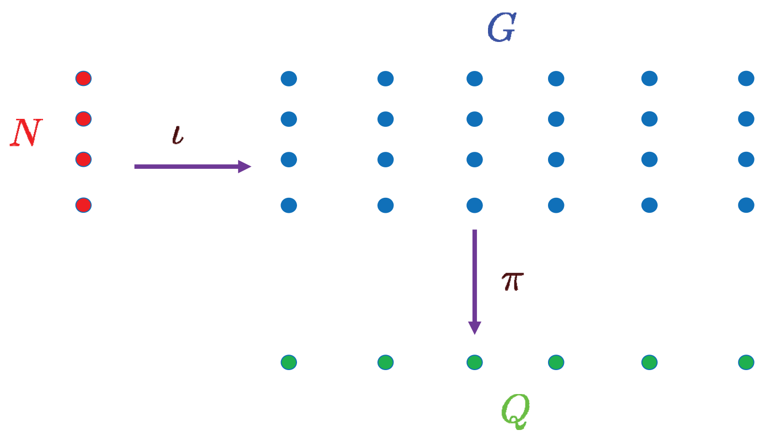

Figure 1: Illustration of a group extension $1 \to N \to G \to Q \to 1$ as an N-bundle over Q. The fiber over $q \in Q$ is just the preimage under π.

will just mention one simple result here, since it introduces a concept that will be quite useful later.

We would encourage the reader to think geometrically about this problem, even in the case when Q and N are finite groups, as in Figure 1. In particular we will use the important notion of a *section*, that is, a right-inverse to π: It is a map $s : Q \to G$ such that $\pi(s(q)) = q$ for all $q \in Q$. Such sections always exist.[3] Note that in general $s(\pi(g)) \neq g$. This is obvious from Figure 1: The map π projects the entire "fiber over q" to q. The section s chooses just one point above q in that fiber.

Now, given an extension and a choice of section s we define a map

$$\omega : Q \to \mathrm{Aut}(N) \tag{1.33}$$

$$q \mapsto \omega_q \tag{1.34}$$

The definition is given by

$$\iota(\omega_q(n)) = s(q)\iota(n)s(q)^{-1} \tag{1.35}$$

Because $\iota(N)$ is normal the RHS is again in $\iota(N)$. Because ι is injective $\omega_q(n)$ is well-defined. Moreover, for each q the reader should check that indeed $\omega_q(n_1 n_2) = \omega_q(n_1)\omega_q(n_2)$, therefore we really have homomorphism $N \to N$. Moreover ω_q is invertible (show this!) and hence it is an automorphism.

[3] By the axiom of choice. For continuous groups such as Lie groups there might or might not be continuous sections.

Remark: Clearly the ι is a bit of a nuisance and leads to clutter and it can be safely dropped if we consider N simply to be a subgroup of G. The confident reader is encouraged to do this. The formulae will be a little cleaner. However, we will be pedantic and retain the ι in most of our formulae.

Let us stress that the map $\omega : Q \to \mathrm{Aut(N)}$ *in general is not a homomorphism* and *in general depends on the choice of section* s. Let us see how close ω comes to being a group homomorphism:

$$\begin{aligned}\iota\left(\omega_{q_1} \circ \omega_{q_2}(n)\right) &= s(q_1)\iota(\omega_{q_2}(n))s(q_1)^{-1} \\ &= s(q_1)s(q_2)\iota(n)(s(q_1)s(q_2))^{-1}\end{aligned} \tag{1.36}$$

In general the section is not a homomorphism, but clearly something nice happens when it is:

Definition: We say an extension *splits* if there is a section $s : Q \to G$ which is *also* a *group homomorphism*.

Theorem: An extension is isomorphic to a semidirect product iff there is a splitting.

♣If you discuss central extensions should explain cocycle and H^2 here. Otherwise, drop the $\mathbb{Z}_p$ by $\mathbb{Z}_p$ example. ♣

1.3.1 Example 1: $SU(2)$ and $SO(3)$

Returning to (1.38) there is a standard homomorphism

$$\pi : SU(2) \to SO(3) \tag{1.37}$$

defined by $\pi(u) = R$ where

$$u\vec{x}\cdot\vec{\sigma}u^{-1} = (R\vec{x})\cdot\vec{\sigma} \tag{1.38}$$

Note that:

1. Every proper rotation R comes from some $u \in SU(2)$: This follows from the Euler angle parametrization.

2. $\ker(\pi) = \{\pm 1\}$. To prove this we write the general $SU(2)$ element as $\cos\chi + \sin\chi\vec{n}\cdot\vec{\sigma}$. This only commutes with all the σ^i if $\sin\chi = 0$ so $\cos\chi = \pm 1$.

Thus we have the extremely important extension:

$$1 \to \mathbb{Z}_2 \quad \stackrel{\iota}{\to} \quad SU(2) \quad \stackrel{\pi}{\to} \quad SO(3) \to 1 \tag{1.39}$$

The $\mathbb{Z}_2$ is embedded as the subgroup $\{\pm 1\} \subset SU(2)$, so this is a central extension. Note that there is no continuous splitting. Such a splitting $\pi s = Id$ would imply that $\pi_* s_* = 1$ on the first homotopy group of $SO(3)$. But that is impossible since it would have factor through $\pi_1(SU(2)) = 1$.

Remarks

1. The extension (1.39) generalizes to

$$1 \to \mathbb{Z}_2 \quad \overset{\iota}{\to} \quad \mathrm{Spin}(d) \quad \overset{\pi}{\to} \quad SO(d) \to 1 \tag{1.40}$$

as well as the two Pin groups which extend $O(d)$:

$$1 \to \mathbb{Z}_2 \quad \overset{\iota}{\to} \quad \mathrm{Pin}^{\pm}(d) \quad \overset{\pi}{\to} \quad O(d) \to 1 \tag{1.41}$$

we discuss these in Section *** below.

1.3.2 Example 2: The isometry group of affine Euclidean space $\mathbb{E}^d$

Definition Let V be a vector space. Then an *affine space modeled on* V is a principal homogeneous space for V. That is, a space with a transitive action of V (as an abelian group) with trivial stabilizer.

The point of the notion of an affine space is that it has no natural origin. A good example is the space of connections on a topologically nontrivial principal bundle.

Let $\mathbb{E}^d$ be the affine space modeled on $\mathbb{R}^d$ with Euclidean metric. The isometries are the 1-1 transformations $f : \mathbb{E}^d \to \mathbb{E}^d$ such that

$$\parallel f(p_1) - f(p_2) \parallel = \parallel p_1 - p_2 \parallel \tag{1.42}$$

for all $p_1, p_2 \in \mathbb{E}^d$. These transformations form a group $\mathrm{Euc}(d)$.

The translations act naturally on the affine space. Given $v \in \mathbb{R}^d$ we define the isometry:

$$T_v(p) := p + v \tag{1.43}$$

so $T_{v_1+v_2} = T_{v_1} + T_{v_2}$ and hence $v \mapsto T_v$ defines a subgroup of $\mathrm{Euc}(d)$ isomorphic to $\mathbb{R}^d$.

One can show that there is a short exact sequence:

$$1 \to \mathbb{R}^d \to \mathrm{Euc}(d) \to O(d) \to 1 \tag{1.44}$$

The rotation-reflections $O(d)$ do *not* act naturally on affine space. In order to define such an action one needs to *choose an origin* of the affine space.

If we do choose an origin then we can identify $\mathbb{E}^d \cong \mathbb{R}^d$ and then to a pair $R \in O(d)$ and $v \in \mathbb{R}^d$ we can associate the isometry: [4]

$$\{R|v\} : x \mapsto Rx + v \tag{1.45}$$

In this notation -known as the *Seitz notation* - the group multiplication law is

$$\{R_1|v_1\}\{R_2|v_2\} = \{R_1R_2|v_1 + R_1v_2\} \tag{1.46}$$

which makes clear that

1. There is a nontrivial automorphism used to construct the semidirect product: $O(d)$:

$$\{R|v\}\{1|w\}\{R|v\}^{-1} = \{1|Rw\} \tag{1.47}$$

[4] Logically, since we operate with R first and then translate by v the notation should have been $\{v|R\}$, but unfortunately the notation used here is the standard one.

and $\pi : \{R|v\} \to R$ is a surjective homomorphism $\mathrm{Euc}(d) \to O(d)$.

2. Thus, although $\mathbb{R}^d$ is abelian, the extension is *not* a central extension.

3. On the other hand, having chosen an origin, the sequence is split. We can choose a splitting $s : O(d) \to \mathrm{Euc}(d)$ by

$$s : R \mapsto \{R|0\} \tag{1.48}$$

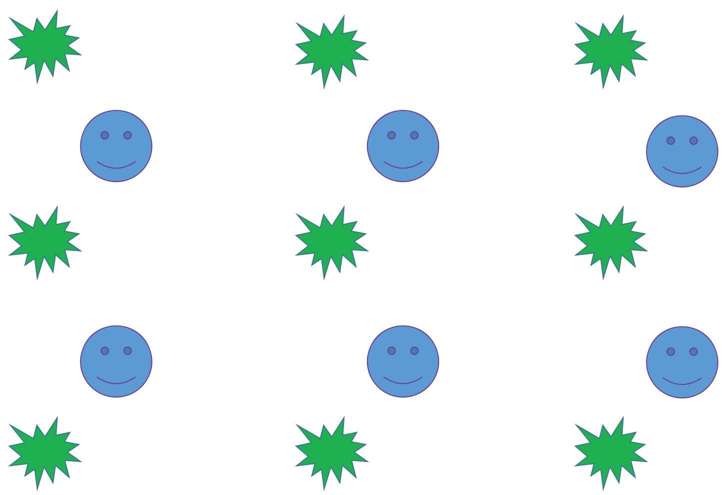

Figure 2: A portion of a crystal in the two-dimensional plane.

1.3.3 Example 3: Crystallographic groups

A crystal should be distinguished from a lattice. The term "lattice" has several related but slightly different meanings in the literature.

Definition An *embedded lattice* is a subgroup $L \subset V$ where V is a vector space with a nondegenerate symmetric bilinear quadratic form b such that $L \cong \mathbb{Z}^d$ as an abelian group.

Now there are several notions of equivalence of embedded lattices used in crystallography. The most obvious one is that L_1 is equivalent to L_2 if there is an element of the orthogonal group $O(b)$ of V taking L_1 to L_2.

Sometimes it is important not to choose an origin, so we can also have the definition:

Definitions Let L be an embedded lattice in Euclidean space $\mathbb{R}^d$. Then:

a.) A *crystal* is a subset $C \subset \mathbb{E}^d$ invariant under translations by a rank d lattice $L(C) \subset \mathbb{R}^n \subset \mathrm{Euc}(n)$.

b.) The *space group* $G(C)$ of a crystal C is the subgroup of $\mathrm{Euc}(d)$ taking $C \to C$.

c.) Recall that there is a projection $\pi : G(C) \to O(d)$. The *point group* $P(C)$ of $G(C)$ is the image of the projection of $G(C)$. Thus, $G(C)$ sits in a group extension:

$$1 \to L(C) \to G(C) \to P(C) \to 1 \tag{1.49}$$

and $P(C) \cong G(C)/L(C)$.

d.) A *crystallographic group* is a discrete subgroup of $\mathrm{Euc}(d)$ which acts properly discontinuously on $\mathbb{E}^d$ and has a subgroup isomorphic to an embedded d-dimensional lattice in the translation subgroup. It therefore sits in a sequence of the form (1.49).

e.) If the group extension (1.49) splits the crystal is said to be *symmorphic*. Similarly, for a crystallographic group G if the corresponding sequence splits it is said to be a *symmorphic group*.

Remark: An important variation on this is the case where the crystal C is endowed with some degrees of freedom, such as spins, which transform under time reversal. The *magnetic space group* is the subgroup of $\mathrm{Euc}(d) \times \mathbb{Z}_2$ which preserves C with its degrees of freedom and the *magnetic point group* is the subgroup of $O(d) \times \mathbb{Z}_2$ which is the image under the projection.

Example: An example of a two-dimensional crystal is shown in Figure 2 where the group of lattice translations is $\mathbb{Z}^2$. The point group is trivial. If we replace the starbursts and smiley faces by points then the point group is a subgroup of $O(2)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to reflection in the two axes. If the starbursts are replaced by the points $\mathbb{Z}^2$ and the smiley faces are replaced by points $\mathbb{Z}^2 + (\delta, \frac{1}{2})$ with $0 < \delta < \frac{1}{2}$ then $G(C)$ is not split. Indeed the group is generated by

$$g_1 : (x^1, x^2) \to (-x^1 + \delta, x^2 + \frac{1}{2}) \tag{1.50}$$

and

$$g_2 : (x^1, x^2) \to (x^1, -x^2) \tag{1.51}$$

These project to reflections in $O(2)$, but no lift of $\pi(g_1)$ will square to one because g_1^2 is a translation by $(0, 1)$.

1.4 Restatement of Wigner's theorem

Now that we have the language of group extensions it is instructive to give simple and concise formulation of Wigner's theorem.

Let us begin by introducing a new group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. This is the group whose elements are unitary *and* anti-unitary transformations on $\mathcal{H}$. The unitary operators $U(\mathcal{H})$ form a subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. If u is unitary and a is anti-unitary then ua and au are also anti-unitary, but if a_1, a_2 are antiunitary, then $a_1 a_2$ is unitary. Thus the set of all unitary and anti-unitary operators on $\mathcal{H}$ form a group, which we will denote as $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. Thus we have the exact sequence

$$1 \to U(\mathcal{H}) \stackrel{\iota}{\to} \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \stackrel{\phi}{\to} \mathbb{Z}_2 \to 1 \tag{1.52}$$

where ϕ is the homomorphism:

$$\phi(S) := \begin{cases} +1 & S \quad \text{unitary} \\ -1 & S \quad \text{anti}-\text{unitary} \end{cases} \tag{1.53}$$

In these notes the group $\mathbb{Z}_2$ will almost always be understood as the multiplicative group $\{\pm 1\}$.

Now, in Section §1.2.4 above we defined a homomorphism $\pi : \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ by $\pi(S)(\ell) = \ell_{S(\psi)}$ if $\ell = \ell_\psi$. (Check it is indeed a homomorphism.) Now we recognize the state of Wigner's theorem as the simple statement that π is surjective. What is the kernel? We also showed that $\ker(\pi) \cong U(1)$ where $U(1)$ is the group of unitary transformations:

$$\psi \mapsto z\psi \tag{1.54}$$

with $|z| = 1$. We will often denote this unitary transformation simply by z. Thus, we have the exact sequence

$$1 \to U(1) \quad \stackrel{\iota}{\to} \quad \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \quad \stackrel{\pi}{\to} \quad \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \to 1 \tag{1.55}$$

Remarks:

1. For $S \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}))$ we have

$$Sz = z^{\phi(S)}S = \begin{cases} zS & \phi(S) = +1 \\ \bar{z}S & \phi(S) = -1 \end{cases} \tag{1.56}$$

   So the sequence (1.55) is not central!

2. If we restrict the sequence (1.55) to $\ker(\phi)$ then we get (taking $\dim\mathcal{H} = N$ here, but it also holds in infinite dimensions):

$$1 \to U(1) \quad \stackrel{\iota}{\to} \quad U(N) \quad \stackrel{\pi}{\to} \quad PU(N) \to 1 \tag{1.57}$$

   which is a central extension, but it is not split. This is in fact the source of interesting things like anomalies in quantum mechanics.

3. The group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ has two connected components, measured by the homomorphism ϕ used in (1.52). This homomorphism "factors through" a homomorphism $\phi' : \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \to \mathbb{Z}_2$ which likewise detects the connected component of this two-component group. The phrase "factors through" means that ϕ and ϕ' fit into the diagram:

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] \arrow[dr, "\phi"] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] \arrow[d, "\phi'"] & 1 \\ & & & \mathbb{Z}_2 & \end{tikzcd} \tag{1.58}$$

Example: Again let us take $\mathcal{H} \cong \mathbb{C}^2$. As we saw,

$$\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) = O(3) = SO(3) \amalg P \cdot SO(3), \tag{1.59}$$

where P is any reflection. [5] Similarly, if we choose a basis for $\mathcal{H}$ then we can identify

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \cong U(2) \amalg \mathcal{C} \cdot U(2) \tag{1.60}$$

where $\mathcal{C}$ is complex conjugation with respect to that basis so that $\mathcal{C}u = u^*\mathcal{C}$. (Note that $\mathcal{C}$ does not have a 2×2 matrix representation.) Now

$$PU(2) := U(2)/U(1) \cong SU(2)/\mathbb{Z}_2 \cong SO(3) \tag{1.61}$$

1.5 ϕ-twisted extensions

So far we have discussed the group of all potential automorphisms of a quantum system $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. However, when we include dynamics, and hence Hamiltonians, a given quantum system will in general only have a subgroup of symmetries. If a physical system has a symmetry group G then we should have a homomorphism $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

In terms of diagrams we have

$$\begin{tikzcd} & & & G \arrow[d, "\rho"] & \\ 1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] & 1 \end{tikzcd} \tag{1.62}$$

The question we now want to address is:

How are G-symmetries represented on Hilbert space $\mathcal{H}$?

Note that each operation $\rho(g)$ in the group of quantum automorphisms has an entire circle of possible lifts in $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. These operators will form a group of operators which is a certain extension of G. What extension to we get?

1.5.1 ϕ-twisted extensions

Now, let us return to the situation of (1.62) we define a group G^{tw} that fits in the diagram:

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{\mathrm{tw}} \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho^{\mathrm{tw}}"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\ 1 \arrow[r] & U(1) \arrow[r, "\iota"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, "\pi"] & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \arrow[r] & 1 \end{tikzcd} \tag{1.63}$$

The definition is simple:

$$G^{\mathrm{tw}} := \{(S, g) | \pi(S) = \rho(g)\} \subset \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \times G \tag{1.64}$$

This is known as the *pullback construction.*

That is, the group of operators representing the G-symmetries of a quantum system form an extension, G^{tw}, of G by $U(1)$.

[5] Please do not confuse this with the notation $PGL(n)$, $PU(n)$ etc!

What kind of extension is it?

This motivates two definitions. First

Definition: A $\mathbb{Z}_2$-*graded group* is a pair (G, ϕ) where G is a group and $\phi : G \to \mathbb{Z}_2$ is a homomorphism.

When we have such a group of course we have an extension of $\mathbb{Z}_2$ by G. Our examples above show that in general it does not split. The group is a disjoint union $G_0 \amalg G_1$ of elements which are even and odd under ϕ and we have the $\mathbb{Z}_2$-graded multiplications:

$$\begin{aligned} G_0 \times G_0 &\to G_0 \\ G_0 \times G_1 &\to G_1 \\ G_1 \times G_0 &\to G_1 \\ G_1 \times G_1 &\to G_0 \end{aligned} \tag{1.65}$$

This is just saying that ϕ is a homomorphism.

Next we have the

Definition Given a $\mathbb{Z}_2$-graded group (G, ϕ) we define a *ϕ-twisted extension of* G to be an extension of the form

$$1 \longrightarrow U(1) \longrightarrow G^{\mathrm{tw}} \xrightarrow{\pi} G \longrightarrow 1 \tag{1.66}$$

where G^{tw} is a group such that

$$\tilde{g}z = z^{\phi(g)}\tilde{g} = \begin{cases} z\tilde{g} & \phi(g) = 1 \\ \bar{z}\tilde{g} & \phi(g) = -1 \end{cases} \tag{1.67}$$

where $\tilde{g}$ is any lift of $g \in G$, and $|z| = 1$ is any phase. Put differently, if we define $\phi^{\mathrm{tw}} := \phi \circ \pi$ then

$$\tilde{g}z = z^{\phi^{\mathrm{tw}}(\tilde{g})}\tilde{g} \qquad\qquad \forall \tilde{g} \in G^{\mathrm{tw}} \tag{1.68}$$

Example

Take $G = \mathbb{Z}_2$ It will be convenient to denote $M_2 = \{1, \bar{T}\}$, with $\bar{T}^2 = 1$. Of course, $M_2 \cong \mathbb{Z}_2$. We take the $\mathbb{Z}_2$ grading to be $\phi(\bar{T}) = -1$, that is, $\phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity homomorphism. There are two inequivalent ϕ-twisted extensions:

$$1 \longrightarrow U(1) \longrightarrow M_2^{\mathrm{tw}} \xrightarrow{\tilde{\pi}} M_2 \longrightarrow 1 \tag{1.69}$$

Choose a lift $\tilde{T}$ of $\bar{T}$. Then $\pi(\tilde{T}^2) = 1$, so $\tilde{T}^2 = z \in U(1)$. But, then

$$\tilde{T}z = \tilde{T}\tilde{T}^2 = \tilde{T}^2\tilde{T} = z\tilde{T} \tag{1.70}$$

on the other hand, $\phi(\bar{T}) = -1$ so

$$\tilde{T}z = z^{-1}\tilde{T} \tag{1.71}$$

Therefore $z^2 = 1$, so $z = \pm 1$, and therefore $\tilde{T}^2 = \pm 1$. Thus the two groups are

$$M_2^{\pm} = \{z\tilde{T} | z\tilde{T} = \tilde{T}z^{-1} \qquad \& \qquad \tilde{T}^2 = \pm 1\} \tag{1.72}$$

These possibilities are really distinct: If $\tilde{T}'$ is another lift of $\bar{T}$ then $\tilde{T}' = \mu\tilde{T}$ for some $\mu \in U(1)$ and so

$$(\tilde{T}')^2 = (\mu\tilde{T})^2 = \mu\bar{\mu}\tilde{T}^2 = \tilde{T}^2 \tag{1.73}$$

Remarks

1. It turns out that $M_2^{\pm}$ is also a double cover of $O(2)$ and in fact these turn out to be isomorphic to the Pin-groups $\mathrm{Pin}^{\pm}(2)$.

♣Simplify this to one brief statement! ♣

1.6 Real, complex, and quaternionic vector spaces

1.6.1 Complex structure on a real vector space

Definition Let V be a real vector space. A *complex structure* on V is a linear map $I : V \to V$ such that $I^2 = -1$.

Choose a squareroot of -1 and denote it i. If V is a real vector space with a complex structure I, then we can define an associated complex vector space (V, I). We take (V, I) to be identical with V, *as sets*, but define the scalar multiplication of a complex number $z \in \mathbb{C}$ on a vector v by

$$z \cdot v := x \cdot v + I(y \cdot v) = x \cdot v + y \cdot I(v) \tag{1.74}$$

where $z = x + iy$ with $x, y \in \mathbb{R}$.

If V is finite dimensional and has a complex structure its dimension (as a real vector space) is even. The dimension of (V, I) as a complex vector space is

$$\dim_{\mathbb{C}}(V, I) = \frac{1}{2}\dim_{\mathbb{R}} V \tag{1.75}$$

This is based on the simple

♣Note that in several of the later chapters our basepoint complex structure is $-I_0$. Need to straighten out this convention! ♣

Lemma If I is any $2n \times 2n$ *real* matrix which squares to -1_{2n} then there is $S \in GL(2n, \mathbb{R})$ such that

$$SIS^{-1} = I_0 := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \tag{1.76}$$

It follows from our Lemma above that the space of all complex structures on $\mathbb{R}^{2n}$ is a homogeneous space for $GL(2n, \mathbb{R})$. The stabilizer of I_0 is the set of $GL(2n, \mathbb{R})$ matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = A \otimes 1_2 + iB \otimes \sigma^2 \tag{1.77}$$

and since σ^2 is conjugate to σ^3, over the complex numbers this can be conjugated to

$$\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \tag{1.78}$$

The determinant is clearly $|\det(A+iB)|^2$ and hence $A+iB \in GL(n,\mathbb{C})$. Therefore, the stabilizer of I_0 is a group isomorphic to $GL(n,\mathbb{C})$ and hence we have proven:

Proposition: *The space of complex structures on $\mathbb{R}^{2n}$ is:*

$$\boxed{\mathrm{CplxStr}(\mathbb{R}^{2n}) = GL(2n,\mathbb{R})/GL(n,\mathbb{C})} \tag{1.79}$$

If we introduce a metric g on V then we can say that *a complex structure I is compatible with g if*

$$g(Iv, Iv') = g(v,v') \tag{1.80}$$

So, when expressed relative to an ON basis for g the matrix I is orthogonal: $I^{tr} = I^{-1}$. But $I^{-1} = -I$, and hence I is anti-symmetric. Then it is well known that there is a matrix $S \in O(2n)$ so that

$$SIS^{-1} = I_0 \tag{1.81}$$

Now the stabilizer of I_0 in $O(2n)$ is of the form (1.77) and can therefore be conjugated to (1.78). But now $A+iB$ must be a unitary matrix so

The space of complex structures on $\mathbb{R}^{2n}$ compatible with the Euclidean metric a homogeneous space isomorphic to

$$\boxed{\mathrm{CmptCplxStr}(\mathbb{R}^{2n}) \cong O(2n)/U(n)} \tag{1.82}$$

where $A+iB \in U(n)$ with A,B real is embedded into $O(2n)$ as in (1.77).

♣Put here a discussion of the two components and the relation between I and orientation. ♣

1.6.2 Real structure on a complex vector space

Definition Suppose V is a complex vector space. Then a *real structure* on V is an antilinear map $\mathcal{C} : V \to V$ such that $\mathcal{C}^2 = +1$.

If $\mathcal{C}$ is a real structure on a complex vector space V then we can define *real vectors* to be those such that

$$\mathcal{C}(v) = v \tag{1.83}$$

Let us call the set of such real vectors V_+. This set is a real vector space, but it is not a complex vector space, because $\mathcal{C}$ is antilinear. Indeed, if $\mathcal{C}(v) = +v$ then $\mathcal{C}(iv) = -iv$. If we let V_- be the *imaginary vectors*, for which $\mathcal{C}(v) = -v$ then we claim

$$V_{\mathbb{R}} = V_+ \oplus V_- \tag{1.84}$$

The proof is simply the isomorphism

$$v \mapsto \left(\frac{v+\mathcal{C}(v)}{2}\right) \oplus \left(\frac{v-\mathcal{C}(v)}{2}\right) \tag{1.85}$$

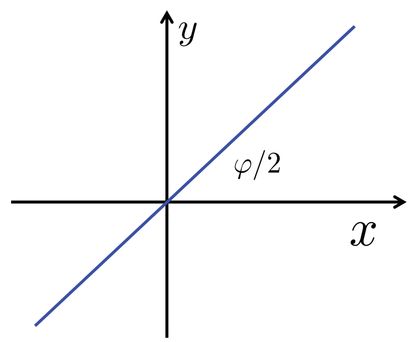

Figure 3: The real structure $\mathcal{C}$ has fixed vectors given by the blue line. This is a real vector space determined by the real structure $\mathcal{C}$.

Moreover multiplication by i defines an isomorphism of real vector spaces: $V_+ \cong V_-$. Thus we have

$$\dim_{\mathbb{R}} V_+ = \dim_{\mathbb{C}} V \tag{1.86}$$

Example $V = \mathbb{C}$,

$$\mathcal{C} : x + iy \to e^{i\varphi}(x - iy) \tag{1.87}$$

The fixed vectors under $\mathcal{C}$ consist of the real line at angle $\varphi/2$ to the x-axis as shown in Figure 3.

In general, if V is a finite dimensional complex vector space, if we choose any basis (over $\mathbb{C}$) $\{v_i\}$ for V then we can define a real structure:

$$\mathcal{C}(\sum_i z_i v_i) = \sum_i \bar{z}_i v_i \tag{1.88}$$

and thus

$$V_+ = \{\sum a_i v_i | a_i \in \mathbb{R}\} \tag{1.89}$$

The space of real structures on $\mathbb{C}^n$ is

$$\boxed{\mathrm{RealStr}(\mathbb{C}^n) \cong GL(n,\mathbb{C})/GL(n,\mathbb{R})} \tag{1.90}$$

$GL(n,\mathbb{C})/GL(n,\mathbb{R})$.

Remarks:

1. We introduced a group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$. This is the automorphisms of $\mathcal{H}$ as a Hilbert space which are real-linear. It should be distinguished from $\mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$ which would be a much larger group of automorphisms of a real inner product space $\mathcal{H}_{\mathbb{R}}$.

2. Suppose W is a real vector space with complex structure I giving us a complex vector space (W, I). An antilinear map $\mathcal{T} : (W, I) \to (W, I)$ is the same thing as a real linear transformation $T : W \to W$ such that

$$TI + IT = 0 \tag{1.91}$$

1.6.3 The quaternions and quaternionic vector spaces

Definition An algebra $\mathcal{A}$ over a field κ is a κ-vector space together with a κ-bilinear map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Usually we just write $a \cdot b$ for $\mu(a, b)$. The product satisfies:

1. $a \cdot (b + c) = a \cdot b + a \cdot c$,
2. $(b + c) \cdot a = b \cdot a + c \cdot a$,
3. $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$, for $\alpha \in \kappa$.

Remarks

1. If there is a multiplicative unit $\mathcal{A}$ is called *unital.*
2. If $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ then $\mathcal{A}$ is called associative.
3. A good example of a unital associative algebra over κ is $\mathrm{End}(V)$ where V is a vector space over κ. Choosing a basis we can identify this with the set of $n \times n$ matrices over κ. We will see many more examples.
4. Given any algebra $\mathcal{A}$ we can define the *opposite algebra* $\mathcal{A}^{\mathrm{opp}}$ by

$$\mu^{\mathrm{opp}}(a, b) := \mu(b, a) \tag{1.92}$$

Definition The *quaternion algebra* $\mathbb{H}$ is the algebra over $\mathbb{R}$ with generators $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfying the relations

$$\mathfrak{i}^2 = -1 \qquad \mathfrak{j}^2 = -1 \qquad \mathfrak{k}^2 = -1 \tag{1.93}$$

$$\mathfrak{i}\mathfrak{j} + \mathfrak{j}\mathfrak{i} = \mathfrak{i}\mathfrak{k} + \mathfrak{k}\mathfrak{i} = \mathfrak{j}\mathfrak{k} + \mathfrak{k}\mathfrak{j} = 0 \tag{1.94}$$

The quaternions form a four-dimensional algebra over $\mathbb{R}$, as a vector space we can write

$$\mathbb{H} = \mathbb{R}\mathfrak{i} \oplus \mathbb{R}\mathfrak{j} \oplus \mathbb{R}\mathfrak{k} \oplus \mathbb{R} \cong \mathbb{R}^4 \tag{1.95}$$

The algebra is associative, but noncommutative. It has a rich and colorful history, which we will not recount here. Note that if we denote a generic quaternion by

$$q = x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} + x_4 \tag{1.96}$$

then we can define the conjugate quaternion by the equation

$$\bar{q} := -x_1\mathfrak{i} - x_2\mathfrak{j} - x_3\mathfrak{k} + x_4 \tag{1.97}$$

and

$$q\bar{q} = \bar{q}q = x_\mu x_\mu \tag{1.98}$$

One fact about the quaternions that will be quite useful is the following. There is a left- and right-action of the quaternions on itself. If $\mathfrak{q}$ is a quaternion then we can define $L(\mathfrak{q}) : \mathbb{H} \to \mathbb{H}$ by

$$L(\mathfrak{q}) : \mathfrak{q}' \mapsto \mathfrak{q}\mathfrak{q}' \tag{1.99}$$

and similarly there is a right-action

$$R(\mathfrak{q}) : \mathfrak{q}' \mapsto \mathfrak{q}'\mathfrak{q} \tag{1.100}$$

The algebra of operators $L(\mathfrak{q})$ is isomorphic to $\mathbb{H}$ and the algebra of operators $R(\mathfrak{q})$ is isomorphic to $\mathbb{H}^{\rm opp}$, which in turn is isomorphic to $\mathbb{H}$ itself. Now $\mathbb{H}$ is a four-dimensional real vector space and $L(\mathfrak{q})$ and $R(\mathfrak{q})$ are commuting real-linear operators. Therefore there is an inclusion

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\rm opp} \hookrightarrow {\rm End}_{\mathbb{R}}(\mathbb{H}) \cong {\rm End}(\mathbb{R}^4) \tag{1.101}$$

Since $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\rm opp}$ has real dimension 16 this is isomorphism of algebras over $\mathbb{R}$.

Definition: A *quaternionic vector space* is a vector space V over $\kappa = \mathbb{R}$ together with three real linear operators $I, J, K \in {\rm End}(V)$ satisfying the quaternion relations. In other words, it is a real vector space which is a module for the quaternion algebra.

Example: The canonical example is given by taking a complex vector space V and forming

$$W = V \oplus \bar{V} \tag{1.102}$$

The underlying real vector space $W_{\mathbb{R}}$ has quaternion actions:

$$I : (v_1, \overline{v_2}) \mapsto (iv_1, i\overline{v_2}) = (iv_1, -\overline{iv_2}) \tag{1.103}$$

$$J : (v_1, \overline{v_2}) \mapsto (-v_2, \overline{v_1}) \tag{1.104}$$

$$K : (v_1, \overline{v_2}) \mapsto (-iv_2, -\overline{iv_1}) \tag{1.105}$$

Just as we can have a complex structure on a real vector space, so we can have a *quaternionic structure* on a complex vector space V. This is a $\mathbb{C}$-anti-linear operator K on V which squares to -1. Once we have $K^2 = -1$ we can combine with the operator I which is just multiplication by $\sqrt{-1}$, to produce $J = KI$ and then we can check the quaternion relations. The underlying real space $V_{\mathbb{R}}$ is then a quaternionic vector space.

It is possible to put a quaternionic Hermitian structure on a quaternionic vector space and thereby define the quaternionic unitary group. Alternatively, we can define $U(n, \mathbb{H})$ as the group of $n \times n$ matrices over $\mathbb{H}$ such that $uu^\dagger = u^\dagger u = 1$. In order to define the conjugate-transpose matrix we use the quaternionic conjugation $q \to \bar{q}$ defined above.

Exercise

Show that $\mathbb{H}^{\rm opp} \cong \mathbb{H}$.

Exercise

a.) Show that

$$\mathfrak{i} \to \sqrt{-1}\sigma^1 \qquad \mathfrak{j} \to -\sqrt{-1}\sigma^2 \qquad \mathfrak{k} \to \sqrt{-1}\sigma^3 \tag{1.106}$$

defines a set of 2×2 complex matrices satisfying the quaternion algebra. Under this mapping a quaternion q is identified with a 2×2 complex matrix

$$q \to \rho(q) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \tag{1.107}$$

with $z = x_4 + ix_3$ and $w = x_2 + ix_1$.

b.)

Show that $U(1, \mathbb{H}) \cong SU(2)$

c.) Show that $\det(\rho(q)) = q\bar{q} = x_\mu x_\mu$ and use this to define a homomorphism $SU(2) \times SU(2) \to SO(4)$.

d.) Show that the algebra $\mathrm{Mat}_n(\mathbb{H})$ of $n \times n$ matrices with quaternionic entries can be identified as the subalgebra of $\mathrm{Mat}_{2n}(\mathbb{C})$ of matrices A such that

$$A^* = JAJ^{-1} \qquad\qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.108}$$

e.) Show that the unitary group over $\mathbb{H}$:

$$U(n, \mathbb{H}) := \{u \in \mathrm{Mat}_n(\mathbb{H}) | u^\dagger u = 1\} \tag{1.109}$$

is isomorphic to

$$USp(2n) := \{u \in U(2n, \mathbb{C}) | u^* = JuJ^{-1}\} \tag{1.110}$$

Exercise *Complex structures on* $\mathbb{R}^4$

a.) Show that the complex structures on $\mathbb{R}^4$ compatible with the Euclidean metric can be identified as the maps

$$q \mapsto nq \qquad\qquad n^2 = -1 \tag{1.111}$$

OR

$$q \mapsto qn \qquad\qquad n^2 = -1 \tag{1.112}$$

b.) Use this to show that the space of such complex structures is $S^2 \amalg S^2$.

c.) Explain the relation to $O(4)/U(2)$.

Exercise *A natural sphere of complex structures*

Show that if V is a quaternionic vector space with complex structures I, J, K then there is a natural sphere of complex structures give by

$$\mathcal{I} = x_1 I + x_2 J + x_3 K \qquad x_1^2 + x_2^2 + x_3^2 = 1 \tag{1.113}$$

Exercise *Regular representation*

♣This is too important to be an exercise, and is used heavily later. ♣

Compute the left and right regular representations of $\mathbb{H}$ on itself Choose a real basis for $\mathbb{H}$ with $v_1 = \mathfrak{i}, v_2 = \mathfrak{j}, v_3 = \mathfrak{k}, v_4 = 1$. Let $L(q)$ denote left-multiplication by a quaternion q and $R(q)$ right-multiplciation by q. Then the representation matrices are:

$$L(\mathfrak{q})v_a := \mathfrak{q} \cdot v_a := L(\mathfrak{q})_{ba} v_b \tag{1.114}$$

$$R(\mathfrak{q})v_a := v_a \cdot \mathfrak{q} := R(\mathfrak{q})_{ba} v_b \tag{1.115}$$

a.) Show that:

$$L(\mathfrak{i}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{1.116}$$

$$L(\mathfrak{j}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{1.117}$$

$$L(\mathfrak{k}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.118}$$

$$R(\mathfrak{i}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{1.119}$$

$$R(\mathfrak{j}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{1.120}$$

$$R(\mathfrak{k}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.121}$$

b.) Show that these matrices generate the full 16-dimensional algebra $M_4(\mathbb{R})$. This is the content of the statement that

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{opp}} \cong \mathrm{End}(\mathbb{R}^4) \tag{1.122}$$

Exercise *'t Hooft symbols and the regular representation of* $\mathbb{H}$

The famous 't Hooft symbols, introduced by 't Hooft in his work on instantons in gauge theory are defined by

$$\alpha^{\pm,i}_{\mu\nu} := \frac{1}{2}(\pm\delta_{i\mu}\delta_{\nu 4} \mp \delta_{i\nu}\delta_{\mu 4} + \epsilon_{i\mu\nu}) \tag{1.123}$$

where $1 \leq \mu, \nu \leq 4$

a.) Show that

$$\alpha^{+,1} = \frac{1}{2}R(\mathfrak{i}) \qquad \alpha^{+,2} = \frac{1}{2}R(\mathfrak{j}) \qquad \alpha^{+,3} = \frac{1}{2}R(\mathfrak{k}) \tag{1.124}$$

$$\alpha^{-,1} = -\frac{1}{2}L(\mathfrak{i}) \qquad \alpha^{-,2} = -\frac{1}{2}L(\mathfrak{j}) \qquad \alpha^{-,3} = -\frac{1}{2}L(\mathfrak{k}) \tag{1.125}$$

b.) Verify the relations

$$\begin{aligned} [\alpha^{\pm,i}, \alpha^{\pm,j}] &= -\epsilon^{ijk}\alpha^{\pm,k} \\ [\alpha^{\pm,i}, \alpha^{\mp,j}] &= 0 \\ \{\alpha^{\pm,i}, \alpha^{\pm,j}\} &= -\frac{1}{2}\delta^{ij} \end{aligned} \tag{1.126}$$

So

$$\begin{aligned} \alpha^{+,i}\alpha^{+,j} &= -\frac{1}{4}\delta^{ij} - \frac{1}{2}\epsilon^{ijk}\alpha^{+,k} \\ \alpha^{-,i}\alpha^{-,j} &= -\frac{1}{4}\delta^{ij} - \frac{1}{2}\epsilon^{ijk}\alpha^{-,k} \end{aligned} \tag{1.127}$$

1.6.4 Quaternionic Structure On Complex Vector Space

$$QuatStr(\mathbb{C}^{2n}) \cong U(2n)/USp(2n) \tag{1.128}$$

1.6.5 Complex Structure On Quaternionic Vector Space

$$CplxStr(\mathbb{H}^n) \cong USp(2n)/U(n) \tag{1.129}$$

1.6.6 Summary

To summarize we have described three basic structures we can put on vector spaces:

♣Actually, there are four. We can have a complex structure on a quaternionic space. We should also derive the the moduli spaces of all four cases as recorded in (A.14) to (A.17). These are used later. ♣

1. A *complex structure* on a real vector space W is a real linear map $I : W \to W$ with $I^2 = -1$.
2. A *real structure* on a complex vector space V is a $\mathbb{C}$-anti-linear map $K : V \to V$ with $K^2 = +1$.
3. A *quaternionic structure* on a complex vector space V is a $\mathbb{C}$-anti-linear map $K : V \to V$ with $K^2 = -1$.

1.7 ϕ-twisted representations

Wigner's theorem is the source of the importance of group representation theory in physics. In these notes we are emphasizing the extra details coming from the fact that in general some symmetry operators are represented as $\mathbb{C}$-antilinear operators. In this section we summarize a few of the differences from standard representation theory.

1.7.1 Some definitions

There are some fairly straightforward definitions generalizing the usual definitions of group representation theory.

Definitions:

1. A *ϕ-representation* (or *ϕ-rep* for short) of a $\mathbb{Z}_2$-graded group (G, ϕ) is a complex vector space V together with a homomorphism

$$\rho : G \to \mathrm{End}(V_{\mathbb{R}}) \tag{1.130}$$

such that

$$\rho(g) = \begin{cases} \mathbb{C} - \text{linear} & \phi(g) = +1 \\ \mathbb{C} - \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \tag{1.131}$$

2. An *intertwiner* or *morphism* between two ϕ-reps (ρ_1, V_1) and (ρ_2, V_2) is a $\mathbb{C}$-linear map $\mathcal{O} : V_1 \to V_2$, i.e., $\mathcal{O} \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, which commutes with the G-action:

$$\mathcal{O}\rho_1(g) = \rho_2(g)\mathcal{O} \qquad \forall g \in G \tag{1.132}$$

We write $\mathrm{Hom}_{\mathbb{C}}^{G}(V_1, V_2)$ for the set of all intertwiners.

3. An *isomorphism of ϕ-reps* is an intertwiner $\mathcal{O}$ which is an isomorphism of complex vector spaces.
4. A ϕ-rep is said to be *ϕ-unitary* if V has a nondegenerate sesquilinear pairing such that $\rho(g)$ is an isometry for all g. That is, it is unitary or anti-unitary according to whether $\phi(g) = +1$ or $\phi(g) = -1$, respectively. If G is compact we can always choose the representation to by ϕ-unitary.

5. A ϕ-rep (ρ, V) is said to be *reducible* if there is a proper (i.e. nontrivial) ϕ-subrepresentation. That is, if there is a complex vector subspace $W \subset V$, with W not $\{0\}$ or V which is G-invariant. If it is not reducible it is said to be *irreducible.*

Remarks:

1. In our language, then, what we learn from Wigner's theorem is that if we have a quantum symmetry group $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ then there is a $\mathbb{Z}_2$-graded extension (G^{tw}, ϕ) and the Hilbert space is a ϕ-representation of (G^{tw}, ϕ). In general we will refer to a ϕ-representation of some extension (G^{tw}, ϕ) of (G, ϕ) as a *ϕ-twisted representation of G.*

2. In the older literature of Wigner and Dyson the term "corepresentation" for a ϕ-unitary representation is used, but in modern parlance the name "corepresentation" has several inappropriate connotations, so we avoid it. The term "ϕ-representation" is not standard, but it should be.

3. An important point below will be that $\mathrm{Hom}^G_{\mathbb{C}}(V_1, V_2)$ is, *a priori* only a *real* vector space. If $\mathcal{O}$ is an intertwiner the $i\mathcal{O}$ certainly makes sense as a linear map from V_1 to V_2 but if any of the $\rho(g)$ are anti-linear then $i\mathcal{O}$ will *not* be an intertwiner. Of course, if the $\mathbb{Z}_2$-grading ϕ of G is trivial and $\phi(g) = 1$ for all g then $\mathrm{Hom}^G_{\mathbb{C}}(V_1, V_2)$ admits a natural complex structure, namely $\mathcal{O} \to i\mathcal{O}$.

Example: Let us consider the ϕ-twisted representations of $M_2 = \{1, \bar{T}\}$ where $\phi(\bar{T}) = -1$. We showed above that there are precisely two ϕ-twisted extensions $M_2^{\pm}$. First, let us suppose $\mathcal{H}$ is a ϕ-rep of M_2^+. Then set

$$K = \rho(\tilde{T}). \tag{1.133}$$

This operator is anti-linear and squares to $+1$. Therefore K is a real structure on $\mathcal{H}$. On the other hand, if the ϕ-twisted extension of M_2 is M_2^- then $K^2 = -1$. Therefore we have a quaternionic structure on $\mathcal{H}$. Thus we conclude: *The ϕ-twisted representations of (M_2, ϕ), with $\phi(\bar{T}) = -1$ are the complex vector spaces with a real structure (for M_2^+) union the complex vector spaces with a quaternionic structure (for M_2^-).*

1.7.2 Schur's Lemma for ϕ-reps

While many of the standard notions and constructions of representation theory carry over straightforwardly to the theory of ϕ-reps, sometimes they come with very interesting new twists. A good example of this is Schur's lemma.

One very important fact for us below will be the analog of Schur's lemma. To state it correctly we recall a basic definition:

Definition An *associative division algebra* over a field κ is an associative unital algebra $\mathcal{A}$ over κ such that for every nonzero $a \in \mathcal{A}$ there is a multiplicative inverse $a^{-1} \in \mathcal{A}$, i.e. $aa^{-1} = a^{-1}a = 1$.

Then we have

Theorem [Schur's Lemma].

a.) If A is an intertwiner between two irreducible ϕ-reps (ρ, V) and (ρ', V') then either $A = 0$ or A is an isomorphism.

♣Use V_1 and V_2 rather than V and V' for consistency of notation. ♣

b.) Suppose (ρ, V) is an irreducible ϕ-representation of (G, ϕ). Then the commutant, that is, the set of all intertwiners A of (ρ, V) with itself:

$$Z(\rho, V) := \mathrm{Hom}^G_{\mathbb{C}}(V, V) = \{A \in \mathrm{End}_{\mathbb{C}}(V) | \forall g \in G \qquad A\rho(g) = \rho(g)A\} \tag{1.134}$$

is a real associative division algebra.

Schur's lemma for ϕ-representations naturally raises the question of finding examples of real division algebras. In fact, there are only three. This is the very beautiful theorem of Frobenius:

Theorem: If $\mathcal{A}$ is a finite dimensional [6] real associative division algebra then one of three possibilities holds:

- $\mathcal{A} \cong \mathbb{R}$
- $\mathcal{A} \cong \mathbb{C}$
- $\mathcal{A} \cong \mathbb{H}$

Examples

1. Let $G = M_2^+$ with $\phi(\bar{T}) = -1$. Take $V = \mathbb{C}$, $\rho(\tilde{T}) = \mathcal{C} \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ given by complex conjugation $\mathcal{C}(z) = \bar{z}$. Then $Z(\rho, V) = \mathbb{R}$.

2. Let $G = U(1)$ with $\phi = 1$, so the grading is trivial (all even). Let $V = \mathbb{C}$ and $\rho(z)v = zv$. Then $Z(\rho, V) = \mathbb{C}$. Notice we could replace G with any subgroup of multiplicative n^{th} roots of 1 in this example, so long as $n > 2$.

3. Let $G = M_2^-$, with $\phi(\tilde{T}) = -1$. Take $V = \mathbb{C}^2$ and represent

$$\rho(e^{i\theta}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i\theta} z_1 \\ e^{i\theta} z_2 \end{pmatrix} \tag{1.135}$$

$$\rho(\tilde{T}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{1.136}$$

One checks that these indeed define a ϕ-representation of M_2^-. We claim that in this case $Z(\rho, V) \cong \mathbb{H}$.

To prove the claim let us map $\mathbb{C}^2 \to \mathbb{H}$ by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_1 + z_2\mathrm{j} = (x_1 + \mathrm{i}y_1) + (x_2 + \mathrm{i}y_2)\mathrm{j} \tag{1.137}$$

[6] The Gelfand-Mazur theorem asserts that *any* unital Banach algebra over $\mathbb{R}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Thus, if we think of the ϕ-twisted representation as acting on the quaternions then we have:

$$\rho(e^{i\theta}) = \cos\theta + \sin\theta L(\mathfrak{i}) \tag{1.138}$$

and T is represented by

$$\rho(\tilde{T}) = L(\mathfrak{j}) \tag{1.139}$$

Of course, the commutant algebra $Z(\rho, V)$ must commute with the real algebra generated by all the elements $\rho(g)$. Therefore, it must commute with left-multiplication by arbitrary quaternions. From this it easily follows that $Z(\rho, V)$ is the algebra of *right-multiplication* by arbitrary quaternions. In fact this identifies $Z(\rho, V) \cong \mathbb{H}^{\rm opp}$ but as in an exercise above $\mathbb{H}^{\rm opp} \cong \mathbb{H}$ as a real algebra.

♣Perhaps put this general remark earlier. ♣

so the algebra of operators generated by $\rho(g)$ is the quaternion algebra acting on $V = \mathbb{R}^4$ as the left regular representation. The commutant of these operations is therefore *right*-multiplication by any quaterion $R(q)$, and hence $Z \cong \mathbb{H}$.

1.7.3 Complete Reducibility

A very important theorem in ordinary representation theory is the complete reducibility of representations of compact groups. This extends more or less directly to ϕ-reps.

If (G, ϕ) is a $\mathbb{Z}_2$-graded group then $(G, \phi)^\vee$, known as the "dual," is the set of inequivalent irreducible ϕ-representations of G. For each element of $\lambda \in (G, \phi)^\vee$ we select a representative irrep V_λ. Thanks to Schur's lemma it is unique up to isomorphism.

Theorem: If (ρ, V) is a finite-dimensional ϕ-unitary rep of (G, ϕ) then V is isomorphic to a representation of the form

$$\oplus_{\lambda \in (G,\phi)^\vee} W_\lambda \tag{1.140}$$

where, for each λ, W_λ is itself (noncanonically) isomorphic to a direct sum of representations V_λ:

$$W_\lambda \cong {\rm Hom}^G_{\mathbb{C}}(V_\lambda, V) \otimes_{D_\lambda} V_\lambda \cong \underbrace{V_\lambda \oplus \cdots \oplus V_\lambda}_{s_\lambda \quad \text{times}} \tag{1.141}$$

(If $s_\lambda = 0$ this is the zero vector space. The second isomorphism is noncanonical.)

Proof: The proof proceeds by choosing a sub-rep and observing that the orthogonal complement is again a subrep. $\diamondsuit$

Remarks

1. The isomorphism of a representation with (1.140), (1.141) is known as an *isotypical decomposition.* The nonnegative integers s_λ are known as degeneracies. ♣Drop this? ♣

2. Concretely the theorem means that we can choose a "block-diagonal" basis for V so that relative to this basis the matrix representation of $\rho(g)$ has the form

$$\rho(g) \sim \begin{pmatrix} \ddots & & \\ & 1_{s_\lambda} \otimes \rho_\lambda(g) & \\ & & \ddots \end{pmatrix} \tag{1.142}$$

We need to be careful about how to interpret $\rho_\lambda(g)$ because anti-linear operators don't have a matrix representation over the complex numbers. If we are working with ordinary representations over $\mathbb{C}$ and $\dim_{\mathbb{C}} V_\lambda = t_\lambda$ then $1_{s_\lambda} \otimes \rho_\lambda(g)$ means a matrix of the form

$$1_{s_\lambda} \otimes \rho_\lambda(g) = \begin{pmatrix} \rho_\lambda(g) & 0 & 0 \cdots & 0 \\ 0 & \rho_\lambda(g) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & \rho_\lambda(g) \end{pmatrix} \tag{1.143}$$

where $\rho_\lambda(g)$ and each of the 0's above is a $t_\lambda \times t_\lambda$ matrix and there is an $s_\lambda \times s_\lambda$ matrix of such blocks. On the other hand, if $\rho(g)$ is anti-linear then it does not have a matrix representation over the complex numbers. If we wish to work with matrix representations what we must do is work with $(V_{\mathbb{R}}, I)$ where I is a complex structure on $V_{\mathbb{R}}$, and similarly for the irreps $(V_{\lambda,\mathbb{R}}, I_\lambda)$. Then $\rho_\lambda(g)$ means a real representation matrix which is $2t_\lambda \times 2t_\lambda$ and anticommutes with I_λ. See the beginning of §1.7.4 for a specific way to do this.

3. Now if we combine the isotypical decomposition with the second part of Schur's lemma to compute the real algebra of self-endomorphisms $\mathrm{End}^G_{\mathbb{C}}(V)$. To lighten the notation let $S_\lambda := \mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, V)$ and let D_λ be the real division algebra over $\mathbb{R}$ of self-intertwiners of V_λ. (That is $D_\lambda := Z(\rho_\lambda, V_\lambda)$.) Then we compute:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(V,V) &\cong V^* \otimes_{\mathbb{C}} V \\ &\cong \oplus_{\lambda,\lambda'} (S^*_\lambda \otimes_{\mathbb{R}} S_{\lambda'}) \otimes_{\mathbb{R}} (V^*_\lambda \otimes_{\mathbb{C}} V_{\lambda'}) \\ &\cong \oplus_{\lambda,\lambda'} \mathrm{Hom}(S_\lambda, S_{\lambda'}) \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\lambda'}) \end{aligned} \tag{1.144}$$

Now G acts trivially on the $\mathrm{Hom}(S_\lambda, S_{\lambda'})$ factors and in the natural way on $\mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\lambda'})$. Therefore, taking the G-invariant part to get the intertwiners we invoke Schur's lemma

$$\mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, V_{\lambda'}) = \delta_{\lambda,\lambda'} D_\lambda \tag{1.145}$$

and hence

$$\mathrm{Hom}^G_{\mathbb{C}}(V,V) \cong \oplus_\lambda \mathrm{End}(S_\lambda) \otimes_{\mathbb{R}} D_\lambda \tag{1.146}$$

Of course, $\mathrm{End}(S_\lambda)$ is isomorphic to the algebra of real matrices $Mat_{s_\lambda}(\mathbb{R})$ upon choosing a basis and therefore

$$\mathrm{End}^G_{\mathbb{C}}(V) \cong \oplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda) \tag{1.147}$$

is a direct sum of matrix algebras over real division algebras.

♣Drop this subsection on algebras? ♣

1.7.4 Complete Reducibility in terms of algebras

The complete reducibility and commutant subalgebra can also be expressed nicely in terms of the group algebra $\mathbb{R}[G]$. We work with $V_{\mathbb{R}}$ with complex structure I with operators

$\rho_{\mathbb{R}}(g)$ commuting or anticommuting with I according to $\phi(g)$. This defines a subalgebra of $\mathrm{End}(V_{\mathbb{R}})$. If G is compact this algebra can be shown to be semisimple and therefore, by a theorem of Wedderburn all representations are matrix representations by matrices over a division algebra over $\mathbb{R}$. See Appendix **??** for background on semisimple algebras.

It is useful to be explicit and make a choice of basis. Therefore, we choose a basis to identify $V \cong \mathbb{C}^N$. Then we identify $V_{\mathbb{R}} \cong \mathbb{R}^{2N}$ by mapping each coordinate

$$z \to \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.148}$$

The complex structure on $\mathbb{R}^{2N}$ is therefore

$$I_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.149}$$

While the real structure of conjugation with respect to this basis is the operation

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.150}$$

Having chosen a basis $V \cong \mathbb{C}^N$ the $\mathbb{C}$-linear operators $\rho(g)$ with $\phi(g) = 1$ can identified with $N \times N$ complex matrices and then they are promoted to $2N \times 2N$ real matrices by replacing each complex matrix element by

$$z_{ij} \to \begin{pmatrix} x_{ij} & -y_{ij} \\ y_{ij} & x_{ij} \end{pmatrix} \tag{1.151}$$

The operators with $\phi(g) = -1$ must be represented by $\mathcal{C}$ times a matrix of the above type.

Now we want to describe the algebra $\rho(G)$ over $\mathbb{R}$ generated by the real $2N \times 2N$ matrices $\rho(g)$ together with I_0. To do this let us introduce some notation: If K is any algebra then we let $K[m]$ or $K(m)$ denote the algebra of all $m \times m$ matrices whose elements are in K. The notation mK will denote the subalgebra of $m \times m$ matrices over K of the specific form $\mathrm{Diag}\{k, k, \dots, k\}$. Thus mK and K are isomorphic as abstract algebras. Note that $m(K[n])$ and $(mK)[n]$ are canonically isomorphic so we just write $mK[n]$ when we combine the two constructions. Finally, with this notation we can state the:

Theorem The algebra $\mathcal{A}(\rho(G), I) \subset \mathrm{End}(V_{\mathbb{R}})$ generated over $\mathbb{R}$ by the operators $\rho(g)$ and I is equivalent to

$$\mathcal{A}(\rho(G), I) \cong \oplus_\lambda s_\lambda D_\lambda[\tau_\lambda] \tag{1.152}$$

and the commutant $Z(\rho, V)$ is equivalent to

$$Z(\rho, V) \cong \oplus_\lambda \tau_\lambda D_\lambda^{\mathrm{opp}}[s_\lambda] \tag{1.153}$$

Note that the dimensions τ_λ are slightly different from the complex dimensions t_λ of V_λ in general. Let us denote the real dimension of D_λ by $d_\lambda = 1, 2, 4$ according to $D_\lambda = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then

$$\tau_\lambda = \frac{2}{d_\lambda} t_\lambda \tag{1.154}$$

Recall that when $D_\lambda \cong \mathbb{H}$ there must be an action of $\mathbb{H}$ on V and hence t_λ must be even, so τ_λ is always an integer, as it must be.

Definition: A real semisimple algebra has the form (1.152). We say that it has *Wedderburn type* $\{D_\lambda\}$, that is, we have an unordered list of the real division algebras determining the simple summands.

1.8 Groups Compatible With Quantum Dynamics

With the possible exception of exotic situations in which quantum gravity is important, physics takes place in space and time. Except in unusual situations associated with nontrivial gravitational fields we can assume our spacetime is time-orientable. Then, any physical symmetry group G must be equipped with a homomorphism

$$\tau : G \to \mathbb{Z}_2 \tag{1.155}$$

telling us whether the symmetry operations preserve or reverse the orientation of time. That is $\tau(g) = +1$ are symmetries which preserve the orientation of time while $\tau(g) = -1$ are symmetries which reverse it.

Now, suppose that G is a symmetry of a quantum system. Then Wigner's theorem gives G another grading $\phi : G \to \mathbb{Z}_2$. Thus, on very general grounds, a symmetry of a quantum system should be *bigraded* by a pair of homomorphisms (ϕ, τ), or what is the same, a homomorphism to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is natural to ask whether ϕ and τ are related. A natural way to try to relate them is to study the dynamical evolution.

In quantum mechanics time evolution is described by unitary evolution of states. That is, there should be a family of unitary operators $U(t_1, t_2)$, strongly continuous in both variables and satisfying composition laws $U(t_1, t_3) = U(t_1, t_2)U(t_2, t_3)$ so that

$$\rho(t_1) = U(t_1, t_2)\rho(t_2)U(t_2, t_1) \tag{1.156}$$

Let us - for simplicity - make the assumption that our physical system has time-translation invariance so that $U(t_1, t_2) = U(t_1 - t_2)$ is a strongly continuous group of unitary transformations. [7]

By Stone's theorem, $U(t)$ has a self-adjoint generator H, the Hamiltonian, so that we may write

♣There is an obvious generalization of this statement for $U(t_1, t_2)$. Is it proved rigorously somewhere? ♣

$$U(t) = \exp\left(-\frac{it}{\hbar}H\right) \tag{1.157}$$

Now, we say a quantum symmetry $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ lifting to $\rho^{\mathrm{tw}} : G^{\mathrm{tw}} \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ is a *symmetry of the dynamics* if for all $g \in G^{\mathrm{tw}}$:

$$\rho^{\mathrm{tw}}(g)U(t)\rho^{\mathrm{tw}}(g)^{-1} = U(\tau(g)t) \tag{1.158}$$

[7] In the more general case we would need an analog of Stone's theorem to assert that there is a family of self-adjoint operators with $U(t_1, t_2) = \mathrm{Pexp}[-\frac{i}{\hbar}\int_{t_1}^{t_2} H(t')dt']$. Then, the argument we give below would lead to $\rho^{\mathrm{tw}}(g)H(t)\rho^{\mathrm{tw}}(g)^{-1} = \phi(g)\tau(g)H(t)$ for all t.

where $\tau : G^{\mathrm{tw}} \to \mathbb{Z}_2$ is inherited from the analogous homomorphism on G.

Now, substituting (1.157) and paying proper attention to ϕ we learn that the condition for a symmetry of the dynamics (1.158) is equivalent to

$$\phi(g)\rho^{\mathrm{tw}}(g)H\rho^{\mathrm{tw}}(g)^{-1} = \tau(g)H \tag{1.159}$$

in other words,

$$\rho^{\mathrm{tw}}(g)H\rho^{\mathrm{tw}}(g)^{-1} = \phi(g)\tau(g)H \tag{1.160}$$

Thus, the answer to our question is that ϕ and τ are *unrelated* in general. We should therefore define a third homomorphism $\chi : G \to \mathbb{Z}_2$

$$\chi(g) := \phi(g)\tau(g) \in \{\pm 1\} \tag{1.161}$$

Note that

$$\phi \cdot \tau \cdot \chi = 1 \tag{1.162}$$

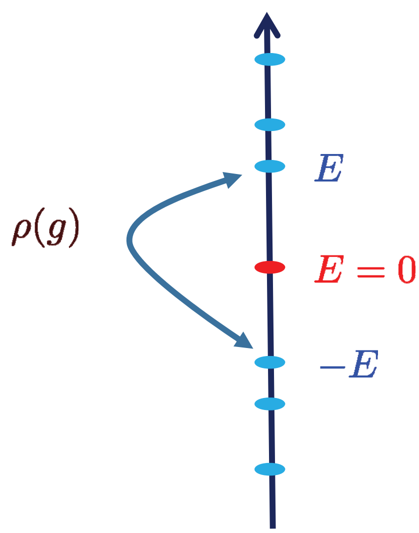

Figure 4: If a symmetry operation has $\chi(g) = -1$ then the spectrum of the Hamiltonian must be symmetric around zero.

Remarks

1. We should stress that in general a system can have time-orientation reversing symmetries but the simple transformation $t \to -t$ is not a symmetry. Rather, it must be accompanied by other transformations. Put differently, the exact sequence

$$1 \to \ker(\tau) \to G \to \mathbb{Z}_2 \to 1 \tag{1.163}$$

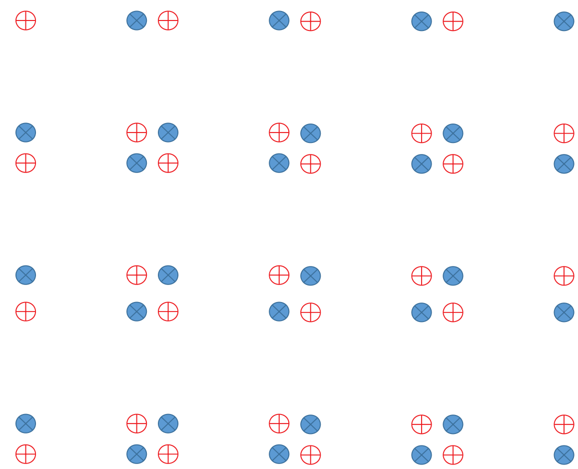

Figure 5: In this figure the blue crosses represent an atom with a local magnetic moment pointing up while the red crosses represent an atom with a local magnetic moment pointing down. The magnetic point group is isomorphic to D_4 but the homomorphism τ to $\mathbb{Z}_2$ has a kernel $\mathbb{Z}_2 \times \mathbb{Z}_2$ (generated by π rotation around a lattice point together with a reflection in a diagonal). Since D_4 is nonabelian the sequence $1 \to \widehat{P}_0 \to \widehat{P} \xrightarrow{\tau} \mathbb{Z}_2 \to 1$ plainly does not split.

in general *does not split*. Many authors assume it does, and that we can always write $G = G_0 \times \mathbb{Z}_2$ where G_0 is a group of time-orientation-preserving symmetries. However, when considering, for example, the magnetic space groups the sequence typically does not split. As a simple example consider a crystal

$$C = \left(\mathbb{Z}^2 + (\delta_1, \delta_2)\right) \amalg \left(\mathbb{Z}^2 + (-\delta_2, \delta_1)\right) \amalg \left(\mathbb{Z}^2 + (-\delta_1, -\delta_2)\right) \amalg \left(\mathbb{Z}^2 + (\delta_2, -\delta_1)\right) \quad (1.164)$$

and suppose there is a dipole moment, or spin S on points in the sub-crystal

$$C_+ = (\mathbb{Z} + (\delta_1, \delta_2)) \amalg (\mathbb{Z} + (-\delta_1, -\delta_2)) \quad (1.165)$$

but a spin $-S$ at the complementary sub-crystal

$$C_- = (\mathbb{Z} + (-\delta_2, \delta_1)) \amalg (\mathbb{Z} + (\delta_2, -\delta_1)) \quad (1.166)$$

such that reversal of time orientation exchanges S with $-S$. Then the time-orientation-reversing symmetries must be accompanied by a $\pi/2$ or $3\pi/2$ rotation around some integer point or a reflection in some diagonal. See Figure 5. Therefore, the extension of the point group is our friend:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1 \quad (1.167)$$

which does not split.

2. It is very unusual to have a nontrivial homomorphism χ. Note that

$$\rho^{\mathrm{tw}}(g)H\rho^{\mathrm{tw}}(g)^{-1} = \chi(g)H \tag{1.168}$$

implies that if any group element has $\chi(g) = -1$ then the spectrum of H must be symmetric around zero as shown in Figure 4. In many problems, e.g. in the standard Schrödinger problem with potentials which are bounded below, or in relativistic QFT with H bounded below we must have $\chi(g) = 1$ for all g and hence $\phi(g) = \tau(g)$, which is what one reads in virtually every physics textbook: A symmetry is anti-unitary iff it reverses the orientation of time.

3. However, there *are* physical examples where $\chi(g)$ can be non-trivial, that is, there can be symmetries which are both anti-unitary and time-orientation preserving. An example are the so-called "particle-hole" symmetries in free fermion systems. We will discuss those later.

4. It is often stated that the only physical systems with $\chi \neq 1$ are free fermionic systems. While free fermionic systems are a very natural set of examples, they are probably not the only ones. Some other examples are [8]

   - Models on bipartite lattices: Ising and interacting fermions. ♣MORE DETAILS! ♣
   - Photonic crystals [G. De Nittis and M. Lein] [9]
   - One can engineer phonon bandstructures with the desired $E \to -E$ symmetry
   - The Hilbert-Polya Hamiltonian has the $E \to -E$ symmetry.

5. The transformations with $\chi(g) = -1$ are sometimes called "charge-conjugation symmetries" and are sometimes called "particle-hole symmetries." The CMT literature is inconsistent about whether we should allow "symmetry groups" with $\chi \neq 1$ and about whether "particle-hole symmetry" should be a $\mathbb{C}$-linear or a $\mathbb{C}$-anti-linear operation. So we have deliberately avoided using the term "particle-hole symmetry" and "charge conjugation" associated with $\chi(g)$.

1.9 Dyson's 3-fold way

Often in physics we begin with a Hamiltonian (or action) and then find the symmetries of the physical system in question. However there are cases when the dynamics are very complicated. A good example is in the theory of nuclear interactions. The basic idea has been applied to many physical systems in which one can identify a set of quantum states corresponding to a large but finite-dimensional Hilbert space. Wigner had the beautiful idea that one could understand much about such a physical system by assuming the Hamiltonian of the system is randomly selected from an *ensemble* of Hamiltonians with a

[8] I thank Matt Hastings, Karin Rabe, and Shankar for useful discussions about this question

[9] This claim has been strongly criticized and is controversial.

probability distribution on the ensemble. In particular one could still make useful predictions of expected results based on averages over the ensemble.

So, suppose $\mathcal{E}$ is an ensemble of Hamiltonians with a probability measure $d\mu$. Then if $\mathcal{O}$ is some attribute of the Hamiltonians (such as the lowest eigenvalue, or the typical eigenvalue spacing) then we might *expect* our complicated system to have the attribute $\mathcal{O}$ close to the expectation value:

$$\langle \mathcal{O} \rangle := \int_{\mathcal{E}} d\mu \mathcal{O}. \tag{1.169}$$

Of course, for this approach to be sensible there should be some natural or canonical measure on the ensemble $\mathcal{E}$, justified by some *a priori* physically reasonable principles. For example, if we take the space of *all* Hermitian operators on some (say, finite-dimensional) Hilbert space $\mathbb{C}^N$ then any probability distribution which is

- Invariant under unitary transformation.
- Statistically independent for H_{ii} and $\mathrm{Re}(H_{ij})$ and $\mathrm{Im}(H_{ij})$ for $i < j$

can be shown [34] to be of the form

$$d\mu = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-a\mathrm{Tr}(H^2)+b\mathrm{Tr}(H)+c} \tag{1.170}$$

The specific choice

$$d\mu = \frac{1}{Z} \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-\frac{N}{2}\mathrm{Tr} H^2} \tag{1.171}$$

where Z is a constant chosen so that $\int d\mu = 1$ defines what is known as the *Gaussian unitary ensemble.*

Now sometimes we know *a priori* that the system under study has a certain kind of symmetry. Dyson pointed out in [20] that such symmetries can constrain the ensemble in ways that affect the probability distribution $d\mu$ in important ways.

1.10 The Dyson problem

Now we can formulate the main problem which was addressed in [20]:

Given a $\mathbb{Z}_2$-graded group (G, ϕ) and a ϕ-unitary rep $(\rho, \mathcal{H})$, what is the ensemble of commuting Hamiltonians? That is: What is the set of self-adjoint operators commuting with $\rho(g)$ for all g?

Note that the statement of the problem presumes that $\chi(g) = 1$. In Section §2.5 below we generalize the problem to allow for $\chi \neq 1$.

The solution to Dyson's problem follows readily from the machinery we have developed. We assume that we can write the isotypical decomposition of $\mathcal{H}$ as

$$\mathcal{H} \cong \oplus_\lambda S_\lambda \otimes_{\mathbb{R}} V_\lambda \tag{1.172}$$

This will always be correct if G is compact. Moreover, $\mathcal{H}$ is a Hilbert space and there are Hermitian structures on S_λ and V_λ so that V_λ a ϕ-unitary rep and we have an isomorphism of ϕ-unitary reps.

Now, if $\chi(g) = 1$ then any Hamiltonian H on $\mathcal{H}$ must commute with the symmetry operators $\rho(g)$ and hence must be in $\mathrm{End}^G_{\mathbb{C}}(\mathcal{H})$. But we have computed this commutant above. Choosing an ON basis for S_λ we have

$$Z(\rho, \mathcal{H}) \cong \oplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda) \tag{1.173}$$

The subset of matrices $\mathrm{Mat}_{s_\lambda}(D_\lambda)$ which are Hermitian is

$$\mathrm{Herm}_{s_\lambda}(D_\lambda) = \begin{cases} \text{Real} \quad \text{symmetric} & D_\lambda = \mathbb{R} \\ \text{Complex} \quad \text{Hermitian} & D_\lambda = \mathbb{C} \\ \text{Quaternion} \quad \text{Hermitian} & D_\lambda = \mathbb{H} \end{cases} \tag{1.174}$$

where quaternion Hermitian means that the matrix elements H_{ij} of H are quaternions and $\overline{H_{ij}} = H_{ji}$. (In particular, the diagonal elements are real.)

In conclusion, the answer to the Dyson problem is the ensemble:

$$\mathcal{E} = \prod_\lambda \mathrm{Herm}_{s_\lambda}(D_\lambda) \tag{1.175}$$

Each ensemble $\mathrm{Herm}_N(D)$ has a natural probability measure invariant under the unitary groups

$$U(N, D) := \begin{cases} O(N; \mathbb{R}) & D = \mathbb{R} \\ U(N) & D = \mathbb{C} \\ Sp(N) \cong USp(2N; \mathbb{C}) & D = \mathbb{H} \end{cases} \tag{1.176}$$

such that the matrix elements (not related by symmetry) are statistically independent. These are:

$$d\mu_{GOE} = \frac{1}{Z_{GOE}} \prod_{i=1}^N dH_{ii} \prod_{i<j} dH_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr} H^2} \tag{1.177}$$

where $H \in \mathrm{Herm}_N(\mathbb{R})$ is real symmetric.

$$d\mu_{GUE} = \frac{1}{Z_{GUE}} \prod_{i=1}^N dH_{ii} \prod_{i<j} d^2 H_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr} H^2} \tag{1.178}$$

where $H \in \mathrm{Herm}_N(\mathbb{C})$ is complex Hermitian.

$$d\mu_{GSE} = \frac{1}{Z_{GSE}} \prod_{i=1}^N dH_{ii} \prod_{i<j} d^4 H_{ij} e^{-\frac{N}{2\sigma^2}\mathrm{Tr} H^2} \tag{1.179}$$

where $H \in \mathrm{Herm}_N(\mathbb{H})$ is quaternionic Hermitian.

They lead to very different eigenvalue distributions

$$d\mu(\lambda_1, \dots, \lambda_N) = \frac{1}{Z} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp[-\frac{N}{2\sigma^2} \sum_i \lambda_i^2] \tag{1.180}$$

with $\beta = 1, 2, 4$. ♣CHECK!! ♣

2. Lecture 2: Phases of gapped systems: The 10-fold way

2.1 Gapped systems and the notion of phases

An active area of current [10] research in condensed matter theory is the "classification of phases of matter." There are physical systems, such as the quantum Hall states, "topological insulators" and "topological superconductors" which are thought to be "topologically distinct" from "ordinary phases of matter." We put quotation marks around all these phrases because they are never defined with any great precision, although it is quite clear that precise definitions in principle must exist.

One way to define a "phase of matter" is to consider *gapped systems*.

Definition By a *gapped system* we mean a pair of a Hilbert space $\mathcal{H}$ with a self-adjoint Hamiltonian H where $1/H$ exists as a bounded operator. (In particular, 0 is not in the spectrum of the Hamiltonian.)

♣There was a lot of push-back on this definition in the talk. Use another name? Discuss relation to what other people mean? ♣

Now suppose we have a *continuous family* of quantum systems. Defining this notion precisely is not completely trivial. See Appendix D of [26] for details. Roughly speaking, we have a family of Hilbert spaces $\mathcal{H}_s$ and Hamiltonians H_s varying continuously with parameters s in some topological space $\mathcal{S}$. [11]

Suppose we are given a continuous family of quantum systems $(\mathcal{H}_s, H_s)_{s\in\mathcal{S}}$ which is generically, but not always gapped. In general there will be a subspace $\mathcal{D} \subset \mathcal{S}$ of Hamiltonians for which $0 \in \mathrm{Spec}(H)$ is a generically real codimension one subset of $\mathcal{S}$. It could be very complicated and very singular in places. Nevertheless, if it is real codimension one then $\mathcal{S} - \mathcal{D}$ will have well-defined connected components.

Definition Given a continuous family of quantum systems $(\mathcal{H}_s, H_s)_{s\in\mathcal{S}}$ we define a *phase* of the system to be a connected component of $\mathcal{S} - \mathcal{D}$.

Another way to define the same thing is to say that two quantum systems $(\mathcal{H}_0, H_0)$ and $(\mathcal{H}_1, H_1)$ are *homotopic* if there is a continuous family of systems $(\mathcal{H}_s, H_s)$ interpolating between them. [12] Phases are then homotopy classes of quantum systems in the set of all gapped systems.

[10] 2007-2013

[11] To be slightly more precise: We use the compact-open topology to define a bundle of Hilbert spaces over $\mathcal{S}$ and we use this topology for the representations of topological groups. The map $s \to H_s$ should be such that $(t, s) \to \exp[-itH_s]$ is continuous from $\mathbb{R} \times \mathcal{S} \to U(\mathcal{H})_{\mathrm{c.o.}}$ where we use the compact-open topology on the unitary group.

[12] Strictly speaking, we should allow for an isomorphism between the endpoint systems and the given $(\mathcal{H}_0, H_0)$ and $(\mathcal{H}_1, H_1)$ so that homotopy is an equivalence relation on isomorphism classes of quantum systems.

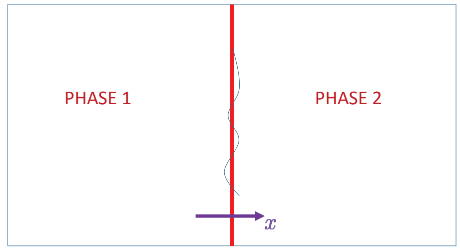

Figure 6: A domain wall between two phases. The wavy line is meant to suggest a localized low energy mode trapped on the domain wall.

Remark: A common construction in this subject is to consider a domain wall between two phases as shown in Figure 6. The domain wall has a thickness and the Hamiltonian is presumed to be sufficiently local that we can choose a transverse coordinate x to the domain wall and the Hamiltonian for the local degrees of freedom is a family H_x. (Thus, x serves both as a coordinate in space and as a parameter for a family of Hamiltonians.) Then if the domain wall separates two phases by definition the Hamiltonian must fail to be gapped for at least one value $x = x_0$ within the domain wall. This suggests that there will be massless degrees of freedom confined to the wall. That indeed happens in some nice examples of domain walls between phases of gapped systems.

The focus of these notes is on the generalization of this classification idea to continuous families of quantum systems with a symmetry. Thus we assume now that there is a group G acting as a symmetry group of the quantum system: $\rho : G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$. As we have seen that G is naturally $\mathbb{Z}_2$-graded by a homomorphism ϕ, there is a ϕ-twisted extension G^{tw} and a ϕ-representation of G^{tw} on $\mathcal{H}$. Now, as we have also seen, if we have a symmetry of the dynamics then there are also homomorphisms $\tau : G^{\mathrm{tw}} \to \mathbb{Z}_2$ and $\chi : G^{\mathrm{tw}} \to \mathbb{Z}_2$ with $\phi(g)\tau(g)\chi(g) = 1$. When we combine this with the assumption that H is gapped we see that we can define a $\mathbb{Z}_2$-grading on the Hilbert space given by the sign of the Hamiltonian. That is, we can decompose:

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \tag{2.1}$$

where $\mathcal{H}^0$ is the subspace on which $H > 0$ and $\mathcal{H}^1$ is the subspace on which $H < 0$. Put differently, since H is gapped we can define $\Pi = \mathrm{sign}(H)$. Then $\Pi^2 = 1$ and Π serves as the grading operator defining the $\mathbb{Z}_2$ grading (2.1). From this viewpoint the equation (1.158), written as

$$\rho^{\mathrm{tw}}(g)H = \chi(g)H\rho^{\mathrm{tw}}(g) \tag{2.2}$$

means that the operators $\rho^{\mathrm{tw}}(g)$ have a definite $\mathbb{Z}_2$-grading: They are *even* if $\chi(g) = +1$. That means they preserve the sign of the energy and hence take $\mathcal{H}^0 \to \mathcal{H}^0$ and $\mathcal{H}^1 \to \mathcal{H}^1$ while they are *odd* if $\chi(g) = -1$ and exchange $\mathcal{H}^0$ with $\mathcal{H}^1$. See §2.2 below for a summary of $\mathbb{Z}_2$-graded linear algebra.

This motivates the following definition:

Definition Suppose G is a bigraded group, that is, it has a homomorphism $G \to \mathbb{Z}_2 \times \mathbb{Z}_2$ or, what is the same thing, a pair of homomorphisms (ϕ, χ) from G to $\mathbb{Z}_2$. Then we define a (ϕ, χ)-*representation of* G to be a complex $\mathbb{Z}_2$-graded vector space $V = V^0 \oplus V^1$ and a homomorphism $\rho : G \to \mathrm{End}(V_{\mathbb{R}})$ such that

$$\rho(g) = \begin{cases} \mathbb{C} - \text{linear} & \phi(g) = +1 \\ \mathbb{C} - \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \qquad \textit{and} \qquad \rho(g) = \begin{cases} \text{even} & \chi(g) = +1 \\ \text{odd} & \chi(g) = -1 \end{cases} \tag{2.3}$$

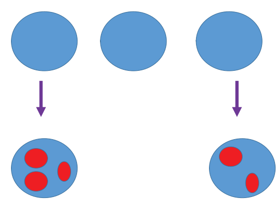

Figure 7: The blue regions in the top row represent different phases of a family of gapped Hamiltonians. The red regions in the bottom row represent different phases with a specified symmetry. Some of the original phases might not have the symmetry at all. Some of the connected components of the original phases might break up into several components with a fixed symmetry.

In terms of this concept we see that if G is a symmetry of a gapped quantum system then there is a (ϕ, χ)-representation of G^{tw}. We can again speak of continuous families of quantum systems with G-symmetry. This means that we have $(\mathcal{H}_s, H_s, \rho_s)$ where the representation ρ_s is a symmetry of the dynamics of H_s which also varies continuously with $s \in \mathcal{S}$. If we have a continuous family of gapped systems then we have a continuous family of (ϕ, χ)-representations. Again we can define phases with G-symmetry to be the connected components of $\mathcal{S} - \mathcal{D}$. This can lead to an interesting refinement of the classification of phases without symmetry, as explained in Figure 7. We will denote the set of phases by

$$\mathcal{TP}(G^{\mathrm{tw}}, \phi, \chi, \mathcal{S}) \tag{2.4}$$

We have been led rather naturally to the notion of $\mathbb{Z}_2$-graded linear algebra. Therefore in the next section §2.2 we very briefly recall a few relevant facts and definitions.

Remarks

1. Except in quantum theories of gravity one is always free to add a constant to the Hamiltonian of any closed quantum system. Typically, though not always, the constant is chosen so that $E = 0$ lies between the ground state and the first excited state.

For example, if we were studying the Schrodinger Hamiltonian for a single electron in the Hydrogen atom instead of the usual operator $H_a = \frac{p^2}{2m} - \frac{Ze^2}{r}$ we might choose $H_a + 12eV$ so that the groundstate would be at $-1.6eV$ and the continuum would begin at $E_c = 12eV$.

2. Again we stress that situations with $\chi \neq 1$ are somewhat rare. However, they do *not* just include the case of noninteracting fermions. Any time a Hamiltonian has some set of energy levels in a range $[E_1, E_2]$ with a symmetry of the spectrum about some value $E_0 \in [E_1, E_2]$ (and E_0 is not in the spectrum) there is an effective system with $\chi \neq 1$.

♣Explore the possibilities of BEC. Phonons ♣

3. In general $\mathcal{TP}(G^{\rm tw}, \phi, \chi, \mathcal{S})$ is just a set. In some nice examples that set turns out to be related to an abelian group which in turn ends up being a twisted equivariant K-theory group. An example of how this refinement is relevant to condensed matter physics is that in topological band structure we can consider families of one-electron Hamiltonians which respect a given (magnetic) space-group. Then there is an interesting refinement of the usual K-theoretic classification of band structures [26] which will be discussed in Chapter 4.5.

2.2 $\mathbb{Z}_2$-graded, or super-, linear algebra

In this section "super" is merely a synonym for "$\mathbb{Z}_2$-graded." Super linear algebra is extremely useful in studying supersymmetry and supersymmetric quantum theories, but its applications are much broader than that and the name is thus a little unfortunate.

Superlinear algebra is very similar to linear algebra, but there are some crucial differences: *It's all about signs.*

2.2.1 Super vector spaces

It is often useful to add the structure of a $\mathbb{Z}_2$-*grading* to a vector space. A $\mathbb{Z}_2$-graded vector space over a field κ is a vector space over κ which, moreover, is written as a direct sum

$$V = V^0 \oplus V^1. \tag{2.5}$$

The vector spaces V^0, V^1 are called the even and the odd subspaces, respectively. We may think of these as eigenspaces of a "parity operator" P_V which satisfies $P_V^2 = 1$ and is $+1$ on V^0 and -1 on V^1. If V^0 and V^1 are finite dimensional, of dimensions m, n respectively we say the super-vector space has graded-dimension or superdimension $(m|n)$.

A vector $v \in V$ is called *homogeneous* if it is an eigenvector of P_V. If $v \in V^0$ it is called *even* and if $v \in V^1$ it is called *odd*. We may define a *degree* or *parity* of homogeneous vectors by setting $\deg(v) = \bar{0}$ if v is even and $\deg(v) = \bar{1}$ if v is odd. Here we regard $\bar{0}, \bar{1}$ in the additive abelian group $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Note that if v, v' are homogeneous vectors of the same degree then

$$\deg(\alpha v + \beta v') = \deg(v) = \deg(v') \tag{2.6}$$

for all $\alpha, \beta \in \kappa$. We can also say that $P_V v = (-1)^{\deg(v)} v$ acting on homogeneous vectors. For brevity we will also use the notation $|v| := \deg(v)$. Note that $\deg(v)$ is *not defined* for general vectors in V.

Mathematicians define the *category of super vector spaces* so that a *morphism* from $V \to W$ is a linear transformation which *preserves grading*. We will denote the space of morphisms from V to W by $\underline{\mathrm{Hom}}(V,W)$. The underline is there to distinguish from the space of linear transformations from V to W discussed below. The space of morphisms $\underline{\mathrm{Hom}}(V,W)$ is just the set of ungraded linear transformations of ungraded vector spaces, $T : V \to W$, which commute with the parity operator $TP_V = P_W T$. One reason for this definition is that only then can we say the *super-dimension*

$$\mathrm{sdim}V := (\mathrm{dim}V^0|\mathrm{dim}V^1) \tag{2.7}$$

is an isomorphism invariant.

If κ is any field we let $\kappa^{p|q}$ denote the supervector space over κ of superdimension $(p|q)$. That is:

$$\kappa^{p|q} = \underbrace{\kappa^p}_{\text{even}} \oplus \underbrace{\kappa^q}_{\text{odd}} \tag{2.8}$$

So far, there is no big difference from, say, a $\mathbb{Z}$-graded vector space. However, important differences arise when we consider *tensor products*.

Put differently: we defined a category of supervector spaces, and now we will make it into a *tensor category*. (See definition below.)

The tensor product of two $\mathbb{Z}_2$ graded spaces V and W is $V \otimes W$ as vector spaces over κ, but the $\mathbb{Z}_2$-grading is defined by the rule:

$$\begin{aligned} \left(V \widehat{\otimes} W\right)^0 &:= V^0 \otimes W^0 \oplus V^1 \otimes W^1 \\ \left(V \widehat{\otimes} W\right)^1 &:= V^1 \otimes W^0 \oplus V^0 \otimes W^1 \end{aligned} \tag{2.9}$$

Thus, under tensor product the degree is additive on homogeneous vectors:

$$\deg(v \otimes w) = \deg(v) + \deg(w) \tag{2.10}$$

Thus, for example:

$$\mathbb{R}^{n_e|n_o} \widehat{\otimes} \mathbb{R}^{n'_e|n'_o} \cong \mathbb{R}^{n_e n'_e + n_o n'_o | n_e n'_o + n_o n'_e} \tag{2.11}$$

For example:

$$\mathbb{R}^{8|8} \widehat{\otimes} \mathbb{R}^{8|8} = \mathbb{R}^{128|128} \tag{2.12}$$

Now, in fact we have a *braided tensor category*:

In ordinary linear algebra there is an isomorphism of tensor products

$$c_{V,W} : V \widehat{\otimes} W \to W \widehat{\otimes} V \tag{2.13}$$

given by $c_{V,W} : v \widehat{\otimes} w \mapsto w \widehat{\otimes} v$. In the category of super vector spaces there is also an isomorphism (2.13) defined by taking

$$c_{V,W} : v \widehat{\otimes} w \to (-1)^{|v| \cdot |w|} w \widehat{\otimes} v \tag{2.14}$$

on homogeneous objects, and extending by linearity.

Let us pause to make two remarks:

1. Note that in (2.14) we are now viewing $\mathbb{Z}/2\mathbb{Z}$ as a ring, not just as an abelian group. Do not confuse $\deg v + \deg w$ with $\deg v \deg w$! In computer science language $\deg v + \deg w$ corresponds to *XOR*, while $\deg v \deg w$ corresponds to *AND*.

2. It is useful to make a general rule: In equations where the degree appears it is understood that all quantities are homogeneous. Then we extend the formula to general elements by linearity. Equation (2.14) is our first example of another general rule: In the super world, commuting any object of homogeneous degree A with any object of homogeneous degree B results in an "extra" sign $(-1)^{AB}$. This is sometimes called the "*Koszul sign rule.*"

Remark: (Reversal of parity)

Introduce an operation which switches the parity of a supervector space: $(\Pi V)^0 = V^1$ and $(\Pi V)^1 = V^0$. In the category of finite-dimensional supervector spaces V and ΠV isomorphic iff $n = m$.

2.2.2 Linear transformations between supervector spaces

If the ground field κ is taken to have degree 0 then the dual space $V^\vee$ in the category of supervector spaces is the supervector space:

$$\begin{aligned} (V^\vee)^0 &:= (V^0)^\vee \\ (V^\vee)^1 &:= (V^1)^\vee \end{aligned} \tag{2.15}$$

Thus, we can say that $(V^\vee)^\epsilon$ are the linear functionals $V \to \kappa$ which vanish on $V^{1+\epsilon}$.

Taking our cue from the natural isomorphism in the ungraded theory:

$$\mathrm{Hom}(V, W) \cong V^\vee \widehat{\otimes} W \tag{2.16}$$

we use the same definition so that the space of linear transformations between two $\mathbb{Z}_2$-graded spaces becomes $\mathbb{Z}_2$ graded. We also write $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

In particular, a linear transformation is an even linear transformation between two $\mathbb{Z}_2$-graded spaces iff $T : V^0 \to W^0$ and $V^1 \to W^1$, and it is odd iff $T : V^0 \to W^1$ and $V^1 \to W^0$. Put differently:

$$\begin{aligned} \mathrm{Hom}(V, W)^0 &\cong \mathrm{Hom}(V^0, W^0) \oplus \mathrm{Hom}(V^1, W^1) \\ \mathrm{Hom}(V, W)^1 &\cong \mathrm{Hom}(V^0, W^1) \oplus \mathrm{Hom}(V^1, W^0) \end{aligned} \tag{2.17}$$

The general linear transformation is neither even nor odd.

If we choose a basis for V made of vectors of homogeneous degree then with respect to such a basis even transformations have block diagonal form

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{2.18}$$

while odd transformations have block diagonal form

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{2.19}$$

As in the ungraded case, $\mathrm{End}(V)$ is a ring, but now it is a $\mathbb{Z}_2$-graded ring under composition: $T_1T_2 := T_1 \circ T_2$. That is if $T_1, T_2 \in \mathrm{End}(V)$ are homogeneous then $\deg(T_1T_2) = \deg(T_1) + \deg(T_2)$, as one can easily check using the above block matrices. These operators are said to *graded-commute*, or *supercommute* if

$$T_1T_2 = (-1)^{\deg T_1 \deg T_2} T_2T_1 \tag{2.20}$$

2.2.3 Superalgebras

The set of linear transformations $\mathrm{End}(V)$ of a supervector space is an example of a superalgebra. In general we have:

Definition

a.) A *superalgebra* $\mathcal{A}$ is a supervector space over a field κ together with a morphism

$$\mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A} \tag{2.21}$$

of supervector spaces. [13] We denote the product as $a\widehat{\otimes}a' \mapsto aa'$. Note this implies that

$$\deg(aa') = \deg(a) + \deg(a'). \tag{2.22}$$

We assume our superalgebras to be unital so there is a $1_{\mathcal{A}}$ with $1_{\mathcal{A}}a = a1_{\mathcal{A}} = a$. Henceforth we simply write 1 for $1_{\mathcal{A}}$.

b.) Two elements a, a' in a superalgebra are said to *graded-commute*, or *super-commute* provided

$$aa' = (-1)^{|a||a'|}a'a \tag{2.23}$$

If every pair of elements a, a' in a superalgebra graded-commmute then the superalgebra is called *graded-commutative* or *supercommutative.*

c.) The *supercenter*, or $\mathbb{Z}_2$*-graded center* of an algebra, denoted $Z_s(\mathcal{A})$, is the subsuperalgebra of $\mathcal{A}$ such that all homogeneous elements $a \in Z_s(\mathcal{A})$ satisfy

$$ab = (-1)^{|a||b|}ba \tag{2.24}$$

for all homogeneous $b \in \mathcal{A}$.

Example 1: *Matrix superalgebras.* If V is a supervector space then $\mathrm{End}(V)$ as described above is a matrix superalgebra. As an exercise, show that the supercenter is isomorphic to κ, consisting of the transformations $v \to \alpha v$, for $\alpha \in \kappa$. So in this case the center and super-center coincide.

[13] Warning!! We are here using $\widehat{\otimes}$ as a tensor product of supervector spaces, not yet a tensor product of superalgebras!

Definition Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. The *graded tensor product* $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is the superalgebra which is the graded tensor product as a vector space and the multiplication of homogeneous elements satisfies

$$(a_1\widehat{\otimes}b_1)\cdot(a_2\widehat{\otimes}b_2) = (-1)^{|b_1||a_2|}(a_1a_2)\widehat{\otimes}(b_1b_2) \tag{2.25}$$

Example For matrix superalgebras we have $\mathrm{End}(V)\widehat{\otimes}\mathrm{End}(V') \cong \mathrm{End}(V\otimes V')$, and in particular:

$$\mathrm{End}(\mathbb{C}^{n_e|n_o})\widehat{\otimes}\mathrm{End}(\mathbb{C}^{n_e'|n_o'}) \cong \mathrm{End}(\mathbb{C}^{n_e|n_o}\widehat{\otimes}\mathbb{C}^{n_e'|n_o'}) \cong \mathrm{End}(\mathbb{C}^{n_en_e'+n_on_o'|n_en_o'+n_on_e'}) \tag{2.26}$$

Remarks

1. Every $\mathbb{Z}_2$-graded algebra is also an ungraded algebra: We just forget the grading. However this can lead to some confusions:

2. An algebra can be $\mathbb{Z}_2$-graded-commutative and not ungraded-commutative: The Grassmann algebras are an example of that. We can also have algebras which are ungraded commutative but not $\mathbb{Z}_2$-graded commutative. The Clifford algebras $C\ell_{\pm 1}$ described below provide examples of that.

3. The $\mathbb{Z}_2$-graded-center of an algebra can be different from the center of an algebra as an ungraded algebra. Again, the Clifford algebras $C\ell_{\pm 1}$ described below provide examples.

4. One implication of (2.25) is that when writing matrix representations of graded algebras we do *not* get a matrix representation of the graded tensor product just by taking the tensor product of the matrix representations. This is important when discussing reps of Clifford algebras, as we will stress below.

5. As for ungraded algebras, there is a notion of *simple, semi-simple, and central superalgebras*. These are discussed in Appendix A of [43]. A key fact is that the Clifford algebras over $\kappa = \mathbb{R},\mathbb{C}$ are all central simple superalgebras over $\kappa = \mathbb{R},\mathbb{C}$ respectively.

2.2.4 Modules over superalgebras

Definition A *super-module M over a super-algebra $\mathcal{A}$* (where $\mathcal{A}$ is itself a superalgebra over a field κ) is a supervector space M over κ together with a κ-linear map $\mathcal{A}\times M \to M$ defining a left-action or a right-action. That is, it is a left-module if, denoting the map by $L : \mathcal{A}\times M \to M$ we have

$$L(a, L(b,m)) = L(ab,m) \tag{2.27}$$

and it is a right-module if, denoting the map by $R : \mathcal{A}\times M \to M$ we have

$$R(a, R(b,m)) = R(ba,m) \tag{2.28}$$

In either case:

$$\deg(R(a,m)) = \deg(L(a,m)) = \deg(a) + \deg(m) \tag{2.29}$$

The notations $L(a,m)$ and $R(a,m)$ are somewhat cumbersome and instead we write $L(a,m) = am$ and $R(a,m) = ma$ so that $(ab)m = a(bm)$ and $m(ab) = (ma)b$. We also sometimes refer to a super-module over a super-algebra $\mathcal{A}$ just as a *representation of* $\mathcal{A}$.

Definition A *linear transformation* between two super-modules M, N over $\mathcal{A}$ is a κ-linear transformation of supervector spaces such that if T is homogeneous and M is a left $\mathcal{A}$-module then $T(am) = (-1)^{|T||a|} aT(m)$ while if M is a right $\mathcal{A}$-module then $T(ma) = T(m)a$. We denote the space of such linear transformations by $\mathrm{Hom}_{\mathcal{A}}(M,N)$. If N is a left $\mathcal{A}$-module then $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is a left $\mathcal{A}$-module with $(a \cdot T)(m) := a \cdot (T(m))$. If N is a right $\mathcal{A}$-module then $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is a right $\mathcal{A}$-module with $(T \cdot a)(m) := (-1)^{|a||m|} T(m)a$. When $M = N$ we denote the module of linear transformations by $\mathrm{End}_{\mathcal{A}}(M)$.

Just as in the case of supervector spaces, we must be careful about the definition of a morphism:

Definition A *morphism* in the category of $\mathcal{A}$-modules is a morphism T of supervector spaces which commutes with the $\mathcal{A}$-action.

2.2.5 Star-structures and super-Hilbert spaces

There are at least *three* notions of a real structure on a complex superalgebra which one will encounter in the literature:

♣Use a symbol for structure 2 which is more easily distinguished from structure 3. Star with circle? ♣

1. It is a $\mathbb{C}$-antilinear involutive *automorphism* $a \mapsto a^{\bigstar}$. Hence $\deg(a^{\bigstar}) = \deg(a)$ and $(ab)^{\bigstar} = a^{\bigstar} b^{\bigstar}$.

2. It is a $\mathbb{C}$-antilinear involutive *anti-automorphism*. Thus $\deg(a^*) = \deg(a)$ but

$$(ab)^* = (-1)^{|a||b|} b^* a^* \tag{2.30}$$

3. It is a $\mathbb{C}$-antilinear involutive *anti-automorphism*. Thus $\deg(a^\star) = \deg(a)$ but

$$(ab)^\star = b^\star a^\star \tag{2.31}$$

If $\mathcal{A}$ is a supercommutative complex superalgebra then structures 1 and 2 coincide: $a \to a^{\bigstar}$ is the same as $a \to a^*$. See remarks below for the relation of 2 and 3.

Definition A sesquilinear form h on a complex supervector space $\mathcal{H}$ is a map $h : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

1. It is even, so that $h(v,w) = 0$ if v and w have opposite parity
2. It is $\mathbb{C}$-linear in the second variable and $\mathbb{C}$-antilinear in the first variable
3. An *Hermitian form* on a supervector space is a sesquilinear form which moreover satisfies the symmetry property:

$$(h(v,w))^* = (-1)^{|v||w|} h(w,v) \tag{2.32}$$

4. If in addition for all nonzero $v \in \mathcal{H}^0$

$$h(v, v) > 0 \tag{2.33}$$

while for all nonzero $v \in \mathcal{H}^1$

$$i^{-1}h(v, v) > 0, \tag{2.34}$$

then h is super-positive definite. If every Cauchy sequence in $\mathcal{H}$ converges to an element of $\mathcal{H}$ it is a *super-Hilbert space.*

For bounded operators we define the adjoint of a homogeneous linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$h(T^*v, w) = (-1)^{|T||v|}h(v, Tw) \tag{2.35}$$

The spectral theorem is essentially the same as in the ungraded case with one strange modification. For even Hermitian operators the spectrum is real. However, for odd Hermitian operators the point spectrum sits in a real subspace of the complex plane which is *not* the real line! It lies on the line passing through $e^{i\pi/4} = (1 + i)/\sqrt{2}$ in the complex plane, as shown in Figure 8

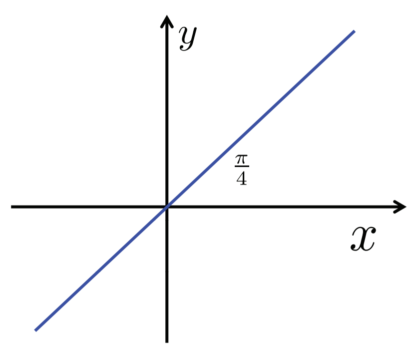

Figure 8: When the Koszul rule is consistently implemented odd super-Hermitian operators have a spectrum which lies along the line through the origin which runs through $1 + i$.

Remark:

In general star-structures 2 and 3 above are actually closely related. Indeed, given a structure $a \to a^*$ of type 2 we can define a structure of type 3 by defining either

$$a^\star = \begin{cases} a^* & |a| = 0 \\ ia^* & |a| = 1 \end{cases} \tag{2.36}$$

or

$$a^\star = \begin{cases} a^* & |a| = 0 \\ -ia^* & |a| = 1 \end{cases} \tag{2.37}$$

It is very unfortunate that in most of the physics literature the definition of a star structure is that used in item 3 above. For example a typical formula used in manipulations in superspace is

$$\overline{\theta_1\theta_2} = \bar{\theta}_2\bar{\theta}_1 \tag{2.38}$$

and the fermion kinetic energy

$$\int dt i\bar{\psi}\frac{d}{dt}\psi \tag{2.39}$$

is only "real" with the third convention.

2.3 Clifford Algebras and Their Modules

2.3.1 The real and complex Clifford algebras

Clifford algebras are defined for a general nondegenerate symmetric quadratic form Q on a vector space V over κ. They are officially defined as a quotient of the tensor algebra of V by the ideal generated by the set of elements of TV of the form $v_1\otimes v_2 + v_2\otimes v_1 - 2Q(v_1,v_2)\cdot 1$ for any $v_1, v_2 \in V$. A more intuitive definition is that $C\ell(Q)$ is the $\mathbb{Z}_2$ graded algebra over κ which has a set of *odd* generators $\{e_i\}$ in one-one correspondence with a basis, also denoted $\{e_i\}$, for the vector space V. The only relations on the generators are given by

♣Poor choice of notation since e_{ij} is an element of the Clifford algebra but also our notation for matrix units... ♣

$$\{e_i, e_j\} = 2Q_{ij}\cdot 1 \tag{2.40}$$

where $Q_{ij} \in \kappa$ is the matrix of Q with respect to a basis $\{e_i\}$ of V, and $1 \in C\ell(Q)$ is the multiplicative identity. Henceforth we will usually identify κ with $\kappa\cdot 1$ and drop the explicit 1.

Because e_i are odd and 1 is even, the algebra $C\ell(Q)$ does not admit a $\mathbb{Z}$-grading. However, every expression in the relations on the generators is even so the algebra admits a $\mathbb{Z}_2$ grading:

$$C\ell(Q) = C\ell(Q)^0 \oplus C\ell(Q)^1 \tag{2.41}$$

Of course, one is always free to regard $C\ell(Q)$ as an ordinary ungraded algebra, and this is what is done in much of the physics literature. However, as we will show below, comparing the graded and ungraded algebras leads to a lot of insight.

Suppose we can choose a basis $\{e_i\}$ for V so that Q_{ij} is diagonal. Then $e_i^2 = q_i \neq 0$. It follows that $C\ell(Q)$ is *not supercommutative*, because an odd element must square to zero in a supercommutative algebra. Henceforth we assume Q_{ij} has been diagonalized, so that e_i anticommutes with e_j for $i \neq j$. Thus, we have the basic Clifford relations:

$$e_ie_j + e_je_i = 2q_i\delta_{ij} \tag{2.42}$$

When $\{i_1,\dots,i_p\}$ are all distinct is useful to define the notation

$$e_{i_1\cdots i_p} := e_{i_1}\cdots e_{i_p} \tag{2.43}$$

Of course, this expression is totally antisymmetric in the indices, and a moment's thought shows that it forms a basis for $C\ell(Q)$ as a vector space and so we have

$$\begin{aligned} C\ell(Q) &\cong \Lambda^* V \\ &= \Lambda^{\mathrm{ev}}V \oplus \Lambda^{\mathrm{odd}}V \end{aligned} \tag{2.44}$$

We stress that (2.44) is only an isomorphism of *super-vector spaces*. If V is finite-dimensional with $d = \dim_\kappa V$ then the graded dimension is easily seen to be

$$s - \dim_\kappa C\ell(Q) = (2^{d-1}|2^{d-1}) \tag{2.45}$$

We must also stress that while the left and right hand sides (2.44) are both algebras over κ the equation is completely false as an isomorphism of algebras. The right hand side of (2.44) is a Grassmann algebra, which is supercommutative and as we have noted $C\ell(Q)$ is not supercommutative.

If we take the case of a real vector space $\mathbb{R}^d$ then WLOG we can diagonalize Q to the form

$$Q = \begin{pmatrix} +1_r & 0 \\ 0 & -1_s \end{pmatrix} \tag{2.46}$$

For such a quadratic form on a real vector space we denote the real Clifford algebra $C\ell(Q)$ by $C\ell_{r+,s-}$. We will see that $C\ell_{-s,r}$ and $C\ell_{-r,s}$ are in general inequivalent.

♣It is probably better to use the notation $C\ell_{r,-s}$ where r, s are always understood to be nonnegative integers. ♣

In general, one can show that $C\ell(Q)^{\rm opp} \cong C\ell(-Q)$. So they are opposite superalgebras.

We can similarly discuss the complex Clifford algebras $\mathbb{C}\ell_n$. Note that over the complex numbers if $e^2 = +1$ then $(ie)^2 = -1$ so we do not need to account for the signature, and WLOG we can just consider $\mathbb{C}\ell_n$ for $n \geq 0$.

The even subalgebra

The even subalgebra is an ungraded algebra and is isomorphic, as an ungraded algebra, to another Clifford algebra. For example, if $d \geq 1$ then

$$\mathbb{C}\ell_d^0 \cong \mathbb{C}\ell_{d-1} \qquad \textit{ungraded algebras.} \tag{2.47}$$

Relations by tensor products

One important advantage of regarding $C\ell(Q)$ as a superalgebra, rather than just an algebra is that if $Q_1 \oplus Q_2$ is a quadratic form on $V_1 \oplus V_2$ then

$$C\ell(Q_1 \oplus Q_2) \cong C\ell(Q_1)\widehat{\otimes}C\ell(Q_2) \tag{2.48}$$

As we will see below, this is completely false if we regard the Clifford algebras as ungraded algebras.

From (2.48) we have some useful identities: First, note that for $n > 0$:

$$C\ell_n \cong \underbrace{C\ell_1\widehat{\otimes}\cdots\widehat{\otimes}C\ell_1}_{\text{n times}} \tag{2.49}$$

$$C\ell_{-n} \cong \underbrace{C\ell_{-1}\widehat{\otimes}\cdots\widehat{\otimes}C\ell_{-1}}_{\text{n times}} \tag{2.50}$$

More generally we have

$$C\ell_{r+,s-} = \underbrace{C\ell_1\widehat{\otimes}\cdots\widehat{\otimes}C\ell_1}_{\text{r times}}\widehat{\otimes}\underbrace{C\ell_{-1}\widehat{\otimes}\cdots\widehat{\otimes}C\ell_{-1}}_{\text{s times}} \tag{2.51}$$

Moreover

$$\mathbb{C}\ell_n \cong \underbrace{\mathbb{C}\ell_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathbb{C}\ell_1}_{\text{n times}} \tag{2.52}$$

If we view the Clifford algebras as ungraded algebras then the tensor product relations are a bit more complicated.

It is important to stress that when producing matrix representations we use the ungraded tensor product, so that the structure of the algebras is more intricate than the above simple identities would suggest.

<u>The Clifford volume element</u>

A key object in discussing the structure of Clifford algebras is the *Clifford volume element*. When V is provided with an orientation this is the canonical element in $C\ell(Q)$ defined by

$$\omega := e_1 \cdots e_d \tag{2.53}$$

where $d = \dim_\kappa V$ and $e_1 \wedge \cdots \wedge e_d$ is the orientation of V. Since there are two orientations there are really two volume elements.

Note that:

Remarks

1. For d even, ω is even and anti-commutes with the generators $e_i\omega = -\omega e_i$. Therefore it is neither in the center nor in the ungraded center of $C\ell(Q)$. It is in the ungraded center of the ungraded algebra $C\ell(Q)^0$.

2. For d odd, ω is odd and $e_i\omega = +\omega e_i$. Therefore it is in the *ungraded* center $Z(C\ell(Q))$ but, because it is odd, it is not in the graded center $Z_s(C\ell(Q))$.

3. Thus, ω is never in the supercenter of $C\ell(Q)$. In fact, the super-center of $C\ell_{r,s}$ is $\mathbb{R}$ and the super-center of $\mathbb{C}\ell_d$ is $\mathbb{C}$.

4. ω^2 is always ± 1 (independent of the orientation). Here is the way to remember the result: The sign only depends on the value of $r_+ - s_-$ modulo 4. Therefore we can reduce the question to $C\ell_n$ and the result only depends on n modulo four. For $n = 0 \bmod 4$ the sign is clearly $+1$. For $n = 2 \bmod 4$ it is clearly -1, because $(e_1e_2)^2 = -e_1^2e_2^2 = -1$ as long as e_1^2 and e_2^2 have the same sign. For $C\ell_{+1}$ and $C\ell_{-1}$ it is obviously $+1$ and -1, respectively.

2.3.2 Clifford algebras and modules over $\kappa = \mathbb{C}$

Let us begin by considering the low-dimensional examples. We will contrast both the graded and ungraded structures, to highlight the differences.

Of course $\mathbb{C}\ell_0 \cong \mathbb{C}$ is purely even. Nevertheless, as a superalgebra it has two inequivalent irreducible graded modules $M_0^+ \cong \mathbb{C}^{1|0}$ and $M_0^- \cong \mathbb{C}^{0|1}$. As an ungraded algebra it has one irreducible module - the regular representation $N_0 \cong \mathbb{C}$.

On the other hand, it is easy to show that $\mathbb{C}\ell_1$ has a unique graded irreducible module. We can take it to be $M_1 \cong \mathbb{C}^{1|1}$ with $\rho(e) = \sigma^1$. Note that $\mathbb{C}\ell_1$ cannot be a matrix superalgebra.

By contrast, for $\mathbb{C}\ell_2$ we can take two generators $\rho(e_1) = \sigma^1$ and $\rho(e^2) = \pm\sigma^2$. Now we find that for either of these two irreps $M_2^\pm$ the Clifford algebra is the full supermatrix algebra $\mathrm{End}(\mathbb{C}^{1|1})$.

What about dimensons $n > 2$? Now we can use tensor products to get the general structure.

First, in the superalgebra case we can invoke (2.52). We know how to multiply matrix superalgebras by (2.26). As we have stressed, $\mathbb{C}\ell_1$ is not a matrix superalgebra, but $\mathbb{C}\ell_2$ is. Therefore, since $\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \widehat{\otimes} \mathbb{C}\ell_2$ we have the key fact

$$\boxed{\mathbb{C}\ell_{n+2} \cong \mathrm{End}(\mathbb{C}^{1|1}) \widehat{\otimes} \mathbb{C}\ell_n} \tag{2.54}$$

Therefore, one can show inductively that

♣BLACKBOARD TECHNIQUE!! Don't write twice. Build the table. ♣

$$\mathbb{C}\ell_{2k} \cong \mathrm{End}(\mathbb{C}^{2^{k-1}|2^{k-1}}) \tag{2.55}$$

and hence

$$\mathbb{C}\ell_{2k+1} \cong \mathrm{End}(\mathbb{C}^{2^{k-1}|2^{k-1}}) \widehat{\otimes} \mathbb{C}\ell_1 \tag{2.56}$$

In both cases they are central simple superalgebras.

If we define

$$\omega_c := \begin{cases} \omega & n = 0, 1 \mathrm{mod} 4 \\ i\omega & n = 2, 3 \mathrm{mod} 4 \end{cases} \tag{2.57}$$

then $\omega_c^2 = 1$

The irreducible modules are:

Clifford Algebra	$\mathbb{C}\ell_{2k}$	$\mathbb{C}\ell_{2k+1}$
Graded algebra	$\mathrm{End}(\mathbb{C}^{2^{k-1}\|2^{k-1}})$	$\mathrm{End}(\mathbb{C}^{2^{k-1}\|2^{k-1}}) \widehat{\otimes} \mathbb{C}\ell_1$
Ungraded algebra	$\mathbb{C}(2^k)$	$\mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$
Graded irreps	$M_{2k}^\pm \cong \mathbb{C}^{2^{k-1}\|2^{k-1}}$, $\rho(\omega_c)\vert_{M_{2k}^{\pm,0}} = \pm 1$	$M_{2k+1} \cong \mathbb{C}^{2^k\|2^k}$
Ungraded irreps	$N_{2k} \cong \mathbb{C}^{2^k}$	$N_{2k+1}^\pm \cong \mathbb{C}^{2^k}$, $\rho(\omega_c) = \pm 1$

2.3.3 Morita equivalence of algebras

There is a very useful equivalence relation on (super)-algebras known as *Morita equivalence*.

The basic idea of Morita equivalence is that, to algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent if their "representation theory is the same." More technically, if we consider the categories $\mathrm{Mod}^L(\mathcal{A}_i)$ of left $\mathcal{A}_i$-modules then the categories are equivalent.

Example 1: $\mathcal{A}_1 = \mathbb{C}$ and $\mathcal{A}_2 = M_n(\mathbb{C}) = \mathbb{C}(n)$ are Morita equivalent ungraded algebras. The general representation of $\mathcal{A}_1$ is a sum of n copies of its irrep $\mathbb{C}$. So the general left $\mathcal{A}_1$-module M is isomorphic to

$$M \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m \quad \text{times}} \tag{2.58}$$

for some positive integer m. On the other hand, the general representation N of $\mathcal{A}_2$ can similarly be written

$$N \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{m \quad \text{times}} \tag{2.59}$$

again, for some positive integer m. Now, $\mathbb{C}^n$ is a left $\mathcal{A}_2$-module, but is also a right $\mathcal{A}_1$-module. Then, if M is a general left $\mathcal{A}_1$-module we can form $\mathbb{C}^n \otimes_{\mathcal{A}_1} M$ which is now a left $\mathcal{A}_2$-module. Conversely, given a left $\mathcal{A}_2$-module N we can recover a left $\mathcal{A}_1$-module from

$$M = \mathrm{Hom}_{\mathcal{A}_2}(\mathbb{C}^n, N) \tag{2.60}$$

Example 2: More generally, if $\mathcal{A}$ is a unital algebra then $\mathcal{A}$ and $M_n(\mathcal{A})$ are Morita equivalent, by considerations similar to those above.

In more general terms, a criterion for Morita equivalence is based on the notion of *bimodules*. An $\mathcal{A}_1 - \mathcal{A}_2$ bimodule $\mathcal{E}$ is a vector space which is simultaneously a left $\mathcal{A}_1$ module and a right $\mathcal{A}_2$ module (so that the actions of $\mathcal{A}_1$ and $\mathcal{A}_2$ therefore commute).

Now, given an $\mathcal{A}_1 - \mathcal{A}_2$ bimodule $\mathcal{E}$ we can define two functors:

$$F : \mathrm{Mod}^L(\mathcal{A}_1) \to \mathrm{Mod}^L(\mathcal{A}_2) \tag{2.61}$$

$$G : \mathrm{Mod}^L(\mathcal{A}_2) \to \mathrm{Mod}^L(\mathcal{A}_1) \tag{2.62}$$

as follows: For $M \in \mathrm{Mod}^L(\mathcal{A}_1)$ we define

$$F(M) := \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, M) \tag{2.63}$$

Note that this is in fact a *left* $\mathcal{A}_2$ module. To see that suppose that $a \in \mathcal{A}_2$ and $T : \mathcal{E} \to M$ commutes with the left $\mathcal{A}_1$-action. Then we define $(a \cdot T)(p) := T(pa)$ for $p \in \mathcal{E}$. Then we compute

$$\begin{aligned} (a_1 \cdot (a_2 \cdot T))(p) &= (a_2 \cdot T)(pa_1) \\ &= T(pa_1 a_2) \\ &= ((a_1 a_2) \cdot T)(p) \end{aligned} \tag{2.64}$$

On the other hand, given a left $\mathcal{A}_2$-module N we can define a left $\mathcal{A}_1$-module by

$$G(N) = \mathcal{E} \otimes_{\mathcal{A}_2} N \tag{2.65}$$

For Morita equivalence we would like F, G to define equivalences of categories so there must be natural identifications of

$$M \cong \mathcal{E} \otimes_{\mathcal{A}_2} \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, M) \cong \left(\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee\right) \otimes_{\mathcal{A}_1} M \tag{2.66}$$

$$N \cong \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}_2} N) \cong \left(\mathcal{E}^\vee \otimes_{\mathcal{A}_1} \mathcal{E}\right) \otimes_{\mathcal{A}_2} N \tag{2.67}$$

Therefore, for Morita equivalence $\mathcal{E}$ must be *invertible* in the sense that there is an $\mathcal{A}_2 - \mathcal{A}_1$ bimodule $\mathcal{E}^\vee$ with

$$\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee \cong \mathcal{A}_1 \tag{2.68}$$

as $\mathcal{A}_1 - \mathcal{A}_1$ *bimodules* together with

$$\mathcal{E}^\vee \otimes_{\mathcal{A}_1} \mathcal{E} \cong \mathcal{A}_2 \tag{2.69}$$

as $\mathcal{A}_2 - \mathcal{A}_2$ *bimodules.* In fact we can recover one algebra from the other

$$\mathcal{A}_2 \cong \mathrm{End}_{\mathcal{A}_1}(\mathcal{E}) \tag{2.70}$$

$$\mathcal{A}_1 \cong \mathrm{End}_{\mathcal{A}_2}(\mathcal{E}) \tag{2.71}$$

and within the algebra of κ-linear transformations $\mathrm{End}(\mathcal{E})$ we have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are each others commutant: $\mathcal{A}_1' = \mathcal{A}_2$.

Moreover, $\mathcal{E}$ determines $\mathcal{E}^\vee$ by saying

$$\mathcal{E}^\vee \cong \mathrm{Hom}_{\mathcal{A}_1}(\mathcal{E}, \mathcal{A}_1) \qquad \text{as} \quad \text{left} \quad \mathcal{A}_2 \quad \text{module}, \tag{2.72}$$

$$\mathcal{E}^\vee \cong \mathrm{Hom}_{\mathcal{A}_2}(\mathcal{E}, \mathcal{A}_2) \qquad \text{as} \quad \text{right} \quad \mathcal{A}_1 \quad \text{module}. \tag{2.73}$$

Another useful characterization of Morita equivalent algebras is that there exists a full idempotent [14] $e \in \mathcal{A}_1$ and a positive integer n so that

$$\mathcal{A}_2 \cong e M_n(\mathcal{A}_1) e. \tag{2.74}$$

Example $\mathcal{A}_1 = M_n(\kappa)$ is Morita equivalent to $\mathcal{A}_2 = M_m(\kappa)$ by the bimodule $\mathcal{E}$ of all $n \times m$ matrices over κ. Indeed, one easily checks that

$$\mathcal{E} \otimes_{\mathcal{A}_2} \mathcal{E}^\vee \cong \mathcal{A}_1 \tag{2.75}$$

(Exercise: Explain why the dimensions match.) and

$$\mathcal{E}^\vee \otimes_{\mathcal{A}_2} \mathcal{E} \cong \mathcal{A}_2 \tag{2.76}$$

Similarly, we can check the other identities above.

[14] If R is a ring then an idempotent $e \in R$ satisfies $e^2 = e$. It is a *full idempotent* if $ReR = R$.

Remark: One reason Morita equivalence is important is that many aspects of representation theory are "the same." In particular, one approach to K-theory emphasizes algebras. Roughly speaking, $K_0(\mathcal{A})$ is defined to be the Grothendieck group [15] or group completion of the monoid of finite-dimensional projective left $\mathcal{A}$-modules. The K-theories of two Morita equivalent algebras are isomorphic abelian groups.

The above discussion generalizes to superalgebras. A sufficient condition (and the only condition we will need) for two superalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ to be Morita equivalent is that there is a matrix superalgebra $\mathrm{End}(V)$ such that

$$\mathcal{A}_1 \cong \mathcal{A}_2 \widehat{\otimes} \mathrm{End}(V) \tag{2.77}$$

or the other way around.

Tensor product induces a multiplication structure on Morita equivalence classes of (super) algebras.

$$[\mathcal{A}] \cdot [\mathcal{B}] := [\mathcal{A} \widehat{\otimes}_\kappa \mathcal{B}] \tag{2.78}$$

If we take the algebra consisting of the ground field κ itself then we have an identity element $[\kappa] \cdot [\mathcal{A}] = [\mathcal{A}]$ for all algebras over κ.

If, moreover, $\mathcal{A}$ is central simple then there is an isomorphism

$$\mathcal{A} \widehat{\otimes} \mathcal{A}^{\mathrm{opp}} \cong \mathrm{End}_\kappa(\mathcal{A}) \tag{2.79}$$

where on the RHS we mean the algebra of linear transformations of $\mathcal{A}$ as a κ vector space. In fact, this property is equivalent to $\mathcal{A}$ being central simple. For the real Clifford algebras,

$$C\ell_{+1} \widehat{\otimes} C\ell_{-1} \cong \mathrm{End}(\mathbb{R}^{1|1}) \tag{2.80}$$

so by this criterion all the $C\ell_n$ are central simple superalgebras. Since $\mathcal{A}$ is assumed finite dimensional the RHS is isomorphic to a matrix (super-)algebra over κ and hence Morita equivalent to κ itself. Therefore the above product defines a group operation and not just a monoid. If we speak of ordinary algebras then this group is known as the *Brauer group* of κ, and if we speak of superalgebras we get the *graded Brauer group* of κ.

2.3.4 Morita equivalence of Clifford algebras and the complex K-theory of a point

Equation (2.54) shows that the Morita equivalence classes of complex Clifford algebras have a *mod two periodicity*:

$$[\mathbb{C}\ell_{n+2}] = [\mathbb{C}\ell_n] \tag{2.81}$$

As explained above there is a group structure on Morita equivalence classes

$$[\mathbb{C}\ell_n] \cdot [\mathbb{C}\ell_m] := [\mathbb{C}\ell_n \widehat{\otimes} \mathbb{C}\ell_m] = [\mathbb{C}\ell_{n+m}] = [\mathbb{C}\ell_{(n+m)\mathrm{mod}2}] \tag{2.82}$$

Therefore, the graded Brauer group of $\mathbb{C}$ is the group $\mathbb{Z}_2$.

♣Need to distinguish Morita equivalence classes from notation for K-theory classes below. ♣

[15] See Remark 1 at the end of this section

At this point we are at the threshhold of the subject of K-theory. This is a generalization of the cohomology groups of topological spaces. At this point we are only equipped to discuss the "cohomology groups" of a point, but even this involves some interesting ideas.

Definition

a.) $\mathcal{M}_n^c$ is the abelian monoid of finite-dimensional complex graded modules for $\mathbb{C}\ell_n$. The monoid operation is direct sum and the identity is the 0 vector space.

b.) $\mathcal{M}_n^{c,\mathrm{triv}}$ is the submonoid of modules M that admit an *invertible odd operator* $T \in \mathrm{End}(M)$ which graded-commutes with the $\mathbb{C}\ell_n$-action. [16]

c.) We define

$$K^n(pt) := \mathcal{M}_n^c / \mathcal{M}_n^{c,\mathrm{triv}}. \tag{2.83}$$

Note that since $\mathbb{C}\ell_n = \mathbb{C}\ell_{-n}$ it follows that $K^n(pt) = K^{-n}(pt)$ so this makes sense for $n \in \mathbb{Z}$.

There is a well-defined sum on equivalence classes:

$$[M_1] \oplus [M_2] := [M_1 \oplus M_2] \tag{2.84}$$

and in the quotient monoid there are additive inverses. The reason is that

$$[M] \oplus [\Pi M] = [M \oplus \Pi M] = 0 \tag{2.85}$$

Therefore, $K^n(pt)$ is an abelian group.

Example 1: $\mathcal{M}_0^c$ is just the abelian monoid of finite-dimensional complex super-vector-spaces. It is generated by $\mathbb{C}^{0|1}$ and $\mathbb{C}^{1|0}$ and is just $\mathbb{Z}_+ \oplus \mathbb{Z}_+$. The submonoid $\mathcal{M}_0^{c,\mathrm{triv}}$ requires the existence of an isomorphism $t : V^0 \to V^1$. In this case we can take

$$T = \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} \tag{2.86}$$

So the submonoid of trivial modules is generated by $\mathbb{C}^{1|1}$. Therefore

$$K^0(pt) := \mathcal{M}_0^c / \mathcal{M}_0^{c,\mathrm{triv}} \cong \mathbb{Z} \tag{2.87}$$

The homomorphism

$$[V] \mapsto n_e - n_o \tag{2.88}$$

is in fact an isomorphism.

Example 2: $K^1(pt)$. Then there is a unique irreducible module M_1 for $\mathbb{C}\ell_1$. We can take $M_1 \cong \mathbb{C}^{1|1}$ with, say, $\rho(e) = \sigma^1$. Then we can introduce the odd invertible operator $T = i\sigma^2$ which graded commutes with $\rho(e)$. Therefore $M_1 \in \mathcal{M}_1^{\mathrm{triv}}$ and since $\mathbb{C}\ell_1$ is a super-simple algebra all the modules are direct sums of M_1. Therefore $\mathcal{M}_1^{\mathrm{triv}} = \mathcal{M}_1$ and hence $K^{-1}(pt) \cong 0$.

[16] We can put an Hermitian structure on M and require that T be odd anti-Hermitian. This is useful for generalizations from the K-theory of a point.

It is easy to show, using the above table, that

$$K^n(pt) \cong \begin{cases} \mathbb{Z} & n = 0(2) \\ 0 & n = 1(2) \end{cases} \tag{2.89}$$

Remarks

1. In general, given an abelian monoid $\mathcal{M}$ there are two general methods to produce an associated abelian group. One, the method adopted here, is to define a submonoid $\mathcal{M}^{\rm triv}$ so that the quotient $\mathcal{M}/\mathcal{M}^{\rm triv}$ admits inverses and hence is a group. A second method, known as the *Grothendieck group* is to consider the produce $\mathcal{M} \times \mathcal{M}$ and divide by an equivalence relation. We say that (a, b) is equivalent to (c, d) if there is an $e \in \mathcal{M}$ with
$$a + d + e = c + b + e \tag{2.90}$$
The idea is that if we could cancel then this would say $a - b = c - d$. Now it is easy to see that the set of equivalence classes $[(a, b)]$ is an abelian group, with $[(a, b)] = -[(b, a)]$. A standard example is that the Grothendieck group of $\mathcal{M} = \mathbb{Z}_+$ produces the integers. Note that if we took $\mathcal{M} = \mathbb{Z}_+ \cup \{\infty\}$ then the Grothendieck group would be the trivial group. This idea actually generalizes to additive categories where we have a notion of sum of objects. In that case (2.90) should be understood to mean that there exists an isomorphism between $a + d + e$ and $c + b + e$. Then one takes the monoid of isomorphism classes of objects to the *Grothendieck group of the category.*

♣Explain the Grothendieck group approach to $K^{-n}(pt)$? ♣

2. There are very many ways to introduce and discuss K-theory. In the original approach of Atiyah and Hirzebruch [8], $K^{-n}(pt)$ was defined in terms of stable isomorphism classes of complex vector bundles on S^n. One of the main points of [7] was the reformulation in terms of Clifford modules, an approach which culminated in the beautiful paper of Atiyah and Singer [11]. We have chosen this approach because it is the one closest to the way K-theory appears in physics. For more about this see the remark in Section §2.3.8 below.

3. A third method uses algebraic K-theory of $\mathbb{C}^*$-algebras. If we identify $K^{-n}(pt) \cong K_0(\mathbb{C}\ell_n)$ then the results on Morita equivalence of the Clifford algebras implies the Bott periodicity of the K-theories.

2.3.5 Real Clifford algebras and Clifford modules of low dimension

In this section we consider the real Clifford algebras $C\ell_n$ for $n \in \mathbb{Z}$. We also describe their irreducible modules and hence the abelian monoid $\mathcal{M}_n$ of isomorphism classes of $\mathbb{Z}_2$-graded representations.

♣Should go back to previous section and change notation to $\mathcal{M}_n^c$. etc. ♣

Here is a summary of the graded and ungraded irreps of the low dimensional real Clifford algebras. (Here $\varepsilon_\pm$ is odd and $\varepsilon_\pm^2 = \pm 1$.):

Clifford Algebra	Ungraded algebra	Graded algebra	Ungraded irreps	Graded irreps
$C\ell_{+4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1})\widehat{\otimes}\mathbb{H}$	$\mathbb{H}^2$	$\tilde{\mu}^{\pm}$
$C\ell_{+3}$	$\mathbb{C}(2)$	$\mathbb{R}[\varepsilon_-]\widehat{\otimes}\mathbb{H}$	$\mathbb{C}^2$	$\tilde{\eta}^3$
$C\ell_{+2}$	$\mathbb{R}(2)$	$\mathbb{C}[\varepsilon_+]$, $z\varepsilon_+ = \varepsilon_+\bar{z}$	$\mathbb{R}^2$	$\tilde{\eta}^2$
$C\ell_{+1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}[\varepsilon_+]$	$\mathbb{R}_{\pm}, \rho(e) = \pm 1$	$\tilde{\eta}$
$C\ell_0$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^{1\|0}, \mathbb{R}^{0\|1}$
$C\ell_{-1}$	$\mathbb{C}$	$\mathbb{R}[\varepsilon_-]$	$\mathbb{C}$	η
$C\ell_{-2}$	$\mathbb{H}$	$\mathbb{C}[\varepsilon_-]$, $z\varepsilon_- = \varepsilon_-\bar{z}$	$\mathbb{H}$	η^2
$C\ell_{-3}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{R}[\varepsilon_+]\widehat{\otimes}\mathbb{H}$	$\mathbb{H}_{\pm}$, $\rho(e_1e_2e_3) = \pm 1$	η^3
$C\ell_{-4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1})\widehat{\otimes}\mathbb{H}$	$\mathbb{H}^2$	$\mu^{\pm}$

Explicit models for the modules $\eta, \mu^{\pm}$ and so forth can be found explained in detail in [43].

2.3.6 The periodicity theorem for Clifford algebras

The fact that $C\ell_{\pm 4}$ are the same, and are matrix superalgebras over the even division algebra $\mathbb{H}$ is very significant.

Note that since left and right multiplication of the quaternions acting on $\mathbb{H}$ we have

$$\mathbb{H}\otimes\mathbb{H}^{\mathrm{opp}} \cong \mathbb{R}(4) \tag{2.91}$$

is an ungraded matrix algebra. Thus,

$$C\ell_{\pm 8} \cong \mathrm{End}(\mathbb{R}^{8|8}). \tag{2.92}$$

It immediately follows that there is a mod-eight periodicity of the Morita equivalence class of the graded real Clifford algebras $C\ell_n$ for all positive n and, separately, for all negative n.

Moreover, one can also show that

$$C\ell_{\pm 3} \cong C\ell_{\mp 1}\widehat{\otimes}\mathbb{H} \tag{2.93}$$

Using this identity we can

1. Build the remaining $C\ell_n$ for $5 \le |n| \le 8$ by taking graded tensor products with the $C\ell_n$ for $|n| \le 4$. The result is shown in Table **** .

2. Establish mod-eight periodicity of the Morita classes of $C\ell_n$ for *all* integers, both positive and negative. For example, note that, at the level of Morita equivalence we have
$$[C\ell_{\pm 1}] = [C\ell_{\mp 7}] \qquad [C\ell_{\pm 2}] = [C\ell_{\mp 6}] \qquad [C\ell_{\pm 3}] = [C\ell_{\mp 5}] \tag{2.94}$$

Clifford Algebra	Ungraded Algebra	$M_{r\|s} \otimes D^s_\alpha$
$C\ell_{+8}$	$\mathbb{R}(16)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{8\|8})\hat{\otimes} D^s_0$
$C\ell_{+7}$	$\mathbb{C}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{4\|4})\hat{\otimes} D^s_{-1}$
$C\ell_{+6}$	$\mathbb{H}(4)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2\|2})\hat{\otimes} D^s_{-2}$
$C\ell_{+5}$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\|1})\hat{\otimes} D^s_{-3}$
$C\ell_{+4}$	$\mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\|1})\hat{\otimes} D^s_4$
$C\ell_{+3}$	$\mathbb{C}(2)$	D^s_3
$C\ell_{+2}$	$\mathbb{R}(2)$	D^s_2
$C\ell_{+1}$	$\mathbb{R} \oplus \mathbb{R}$	D^s_1
$C\ell_0$	$\mathbb{R}$	D^s_0
$C\ell_{-1}$	$\mathbb{C}$	D^s_{-1}
$C\ell_{-2}$	$\mathbb{H}$	D^s_{-2}
$C\ell_{-3}$	$\mathbb{H} \oplus \mathbb{H}$	D^s_{-3}
$C\ell_{-4}$	$\mathbb{H}(2)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\|1})\hat{\otimes} D^s_4$
$C\ell_{-5}$	$\mathbb{C}(4)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{1\|1})\hat{\otimes} D^s_{+3}$
$C\ell_{-6}$	$\mathbb{R}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2\|2})\hat{\otimes} D^s_{+2}$
$C\ell_{-7}$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{4\|4})\hat{\otimes} D^s_{+1}$
$C\ell_{-8}$	$\mathbb{R}(16)$	$\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{8\|8})\hat{\otimes} D^s_0$

The upshot is that if we define the following 8 basic superalgebras:

$$\begin{aligned} D^s_0 &:= \mathbb{R} \\ D^s_{\pm 1} &:= C\ell_{\pm 1} \\ D^s_{\pm 2} &:= C\ell_{\pm 2} \\ D^s_{\pm 3} &:= C\ell_{\pm 3} \\ D^s_4 = D^s_{-4} &:= \mathbb{H} \end{aligned} \tag{2.95}$$

where D^s_0 and D^s_4 are purely even, then all the Clifford algebras are matrix superalgebras over the D^s_α: The graded Morita equivalence class of $[C\ell_n]$ where $n \in \mathbb{Z}$ is positive or negative is determined by the residue $\alpha = n \mathrm{mod} 8$, and we have:

$$[C\ell_n] = [D^s_\alpha] \tag{2.96}$$

and moreover, the multiplication on Morita equivalence classes is just given by

$$[D^s_\alpha] \cdot [D^s_\beta] = [D^s_{\alpha+\beta}] \tag{2.97}$$

Thus the real graded Brauer group over $\mathbb{R}$ is $\mathbb{Z}/8\mathbb{Z}$.

The Wedderburn type of the ungraded algebras is now easily determined from the graded ones by using the explicit determination we gave above for the basic cases $C\ell_n$ with $|n| \leq 4$. Notice that there is a basic genetic code in this subject

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{R}, \ldots \tag{2.98}$$

We will meet it again and again. One would do well to memorize this sequence. It is illustrated in Figure 9.

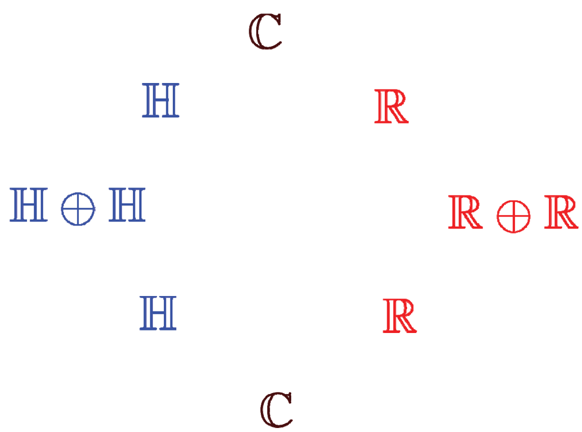

Figure 9: An illustration of the "Bott clock": For $C\ell_n$ with decreasing n read it clockwise (= decreasing phase) and with increasing n read it counterclockwise (= increasing phase).

Finally, we can now easily determine the structure of $C\ell_{r_+,s_-}$ for all r, s. The Morita class is determined by:

$$[C\ell_{r_+,s_-}] = [D^s_{r-s}] \tag{2.99}$$

and hence, lifting $\alpha = (r-s)\mathrm{mod}8$ to $|\alpha| \leq 4$

$$C\ell_{r_+,s_-} \cong \mathrm{End}(\mathbb{R}^{2^n|2^n})\widehat{\otimes} D^s_\alpha \tag{2.100}$$

for an n which can be computed by matching dimensions.

2.3.7 KO-theory of a point

Now in this section we describe the real KO-theory ring of a point along the lines we discussed in Section §2.3.4. In order to complete the story we need to name the irreducible representations of $C\ell_{\pm 8}$. These are supermatrix algebras and so we have simply $\lambda^{\pm} \cong \mathbb{R}^{8|8}$ for $C\ell_{-8}$ and $\tilde{\lambda}^{\pm} \cong \mathbb{R}^{8|8}$ for $C\ell_{+8}$. The superscript $\pm$ refers to the sign of the volume form on the even subspace.

Now let us consider $KO^{-n}(pt)$ along the above lines. A useful viewpoint is that we are considering real algebras and modules as fixed points of a real structure on the complex modules and algebras. Recall we described $\mathcal{M}^{\mathrm{triv,c}}_n$ (where the extra c in the superscript reminds us that we are talking about complex modules of complex Clifford algebras) as those modules which admit an odd invertible operator which graded commutes with the Clifford action. In order to speak of real structures we can take our complex modules to have an Hermitian structure. Then the conjugation will act as $T \to \pm T^\dagger$ where the $\pm$ is a choice of convention. We will choose the convention $T \to -T^\dagger$. The other convention leads to an equivalent ring, after switching signs on the degrees. ♣check ♣

Note that we have introduced an Hermitian structure into this discussion. If one strictly applies the the Koszul rule to the definition of Hermitian structures and adjoints in the $\mathbb{Z}_2$-graded case then some unusual signs and factors of $\sqrt{-1}$ appear. See Section §2.2.5 above. We will use a standard Hermitian structure on $\mathbb{R}^{n|m}$ and $\mathbb{C}^{n|m}$ such that the even and odd subspaces are orthogonal and the standard notion of adjoint. Since we introduce the structure the question arises whether the groups we define below depend on that choice. It can be shown that these groups do not depend on that choice, and the main ingredient in the proof is the fact that the space of Hermitian structures is a contractible space.

This motivates the following definitions:

Definition

a.) For $n \in \mathbb{Z}$, $\mathcal{M}_n$ is the abelian monoid of modules for $C\ell_n$ under direct sum.

b.) For $n \in \mathbb{Z}$, $\mathcal{M}_n^{\rm triv}$ is the submonoid of $\mathcal{M}_n$ consisting of those modules which admit an odd invertible anti-hermitian operator T which graded-commutes with the $C\ell_n$ action.

c.)

$$KO^n(pt) := \mathcal{M}_n/\mathcal{M}_n^{\rm triv} \tag{2.101}$$

We now compute the $KO^n(pt)$ groups for low values of n:

1. Of course $KO^0(pt) \cong \mathbb{Z}$, with the isomorphism given by the superdimension.

2. Now consider $KO^1(pt)$. In our model for $\tilde{\eta}$ we had $\rho(e) = \sigma^1$. Therefore we could introduce $T = \epsilon$. Thus $[\tilde{\eta}] = 0$ in KO-theory and $KO^1(pt) = 0$.

3. Next consider $KO^{-1}(pt)$. In our model for η we had $\rho(e) = \epsilon$. Now we cannot introduce an antisymmetric operator which graded commutes with ϵ. Thus, η is a nontrivial class. However, we encounter a new phenomenon relative to the complex case. Consider $2\eta = \eta \oplus \eta$. As a vector space this is $\mathbb{R}^{2|2}$ and as usual taking an ordered bases with even elements first we have

$$\rho(e) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \epsilon \otimes 1 \tag{2.102}$$

We can therefore introduce $T = \sigma^1 \otimes \epsilon$ which is odd, anticommutes with $\rho(e)$, and squares to -1. Therefore, $KO^{-1}(pt) \cong \mathbb{Z}_2$ with generator $[\eta]$.

Now, using the periodicity of the Clifford algebras we conclude that:

Theorem

$KO^n(pt)$ is mod-eight periodic in n and the groups $KO^{-n}(pt)$ for $1 \leq n \leq 8$ are given by [17]

$$\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \tag{2.103}$$

2.3.8 Digression: A hint of the relation to topology

The original Atiyah-Hirzebruch definition of $K(X)$ used the Grothendieck construction applied to the abelian monoid $\mathrm{Vect}(X)$ of complex vector bundles over a topological space X Thus, we consider equivalence classes $[(E_1, E_2)]$ where $[(E_1, E_2)] = [(F_1, F_2)]$ if there exists a G with

$$E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G \tag{2.104}$$

Intuitively, we think of $[(E_1, E_2)]$ as a difference $E_1 - E_2$.

Example: If we consider from this viewpoint the K-theory of a point $K^0(pt)$ then we obtain the abelian group $\mathbb{Z}$, the isomorphism being $[(E_1, E_2)] \to \mathrm{dim}E_1 - \mathrm{dim}E_2$.

For vector bundles the Grothendieck construction can be considerably simplified thanks to the Serre-Swan theorem:

Theorem[Serre; Swan] Any vector bundle [18] has a complementary bundle so that $E \oplus E^{\perp} \cong \theta_N$ is a trivial rank N bundle for some N. Equivalently, every bundle is a subbundle of a trivial bundle defined by a continuous family of projection operators.

This leads to the notion of *stable equivalence* of vector bundles: Two bundles E_1, E_2 are *stably equivalent* if there exist trivial bundles θ_s of rank s so that

$$E_1 \oplus \theta_{s_1} \cong E_2 \oplus \theta_{s_2} \tag{2.105}$$

Example: A very nice example, in the category of real vector bundles, is the tangent bundle of S^2. The real rank two bundle TS^2 is topologically nontrivial. You can't comb the hair on a sphere. However, if we consider $S^2 \subset \mathbb{R}^3$ the normal bundle is a real rank one bundle and is trivial. But that means $TS^2 \oplus \theta_1 \cong \theta_3$. So TS^2 is stably trivial.

Returning to the general discussion. In the difference $E_1 - E_2$ we can add and subtract the complementary bundle to get $(E_1 \oplus E_2^{\perp}) - \theta_N$ for some N. If we restrict the bundle to any point we get an element of $K^0(pt)$. By continuity, it does not matter what point we choose, provided X is connected.

In other words, there is a homomorphism

$$K^0(X) \to K^0(pt) \tag{2.106}$$

The kernel of this homomorphism is, by definition, $\tilde{K}^0(X)$. We can represent it by formal differences of the form $E - \theta_N$ where $N = \mathrm{rank}(E)$.

[17] This is easily memorized using the "Bott song." Sing the names of the groups to the tune of "Ah! vous dirai-je, Maman," aka "Twinkle, twinkle, little star."

[18] over a suitable nice topological space

There is another construction of $K(X)$ called the *difference-bundle* approach which is more flexible and useful for generalizations.

As in our discussion using Clifford modules we consider an abelian monoid and divide by a submonoid of "trivial" elements. As we mentioned, the latter viewpoint is closer to the physics. The abelian monoid consists of isomoprhism classes of pairs (E, T) where E is a $\mathbb{Z}_2$-graded complex (or real) vector bundle over X equipped with an Hermitian structure, and T is an odd anti-Hermitian operator (that operator could well be zero). The trivial submonoid will be those pairs (E, T) where T is *invertible.*

Very roughly speaking the difference $E^0 - E^1$ in the Grothendieck construction is to be compared with a $\mathbb{Z}_2$-graded bundle $\mathcal{E} = E^0 \oplus E^1$ with an odd operator $T = 0$. Note that in the Grothendieck construction we could also have used

$$E^0 - E^1 = (E^0 \oplus F^0) - (E^1 \oplus F^1) \tag{2.107}$$

if there is an isomorphism of $t : F^0 \to F^1$. This would correspond to to the $\mathbb{Z}_2$-graded bundle with even part $\mathcal{E}^0 = E^0 \oplus F^0$ and odd part $\mathcal{E}^1 = E^1 \oplus F^1$ with odd anti-hermitian operator (in an ordered basis for the fibers of E^0, E^1, F^0, F^1, in that order):

$$T = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t^* & 0 & 0 & 0 \end{pmatrix} \tag{2.108}$$

Now $T \in \mathrm{End}(\mathcal{E})$ is odd, and $E^0 \cong \mathrm{ker}T$ while $E^1 \cong \mathrm{ker}T^\dagger \cong \mathrm{cok}T$. Therefore:

$$\mathcal{E}^0 - \mathcal{E}^1 \cong (\mathrm{ker}T|_{\mathcal{E}^0} \oplus (\mathrm{ker}T|_{\mathcal{E}^0})^\perp) - (\mathrm{ker}T^\dagger|_{\mathcal{E}^1} \oplus (\mathrm{ker}T^\dagger|_{\mathcal{E}^1})^\perp). \tag{2.109}$$

Now T provides a bundle isomorphism between $(\mathrm{ker}T|_{\mathcal{E}^0})^\perp$ and $(\mathrm{ker}T^\dagger|_{\mathcal{E}^1})^\perp$, so these can be canceled.

Remark: In string theory, T turns out to be the classical value of a tachyon field [53]. The invertibility of the tachyon field has an interpretation in terms of being able to undergo renormalization group flow to its true vacuum value. See [39] for more about this viewpoint. In the applications to topological phases of matter T is related to "topologically trivial pairing of particles and holes". See, e.g. discussion in [36, 48].

Now let us briefly consider the relation to generalized cohomology theory. We would like to define $K^{-n}(X)$.

In some treatments of K-theory one will find K^{-n} defined in terms of suspension:

♣Careful. Need to define $\tilde{K}$ and be careful about basepoints and suspension vs. reduced suspension. ♣

$$\tilde{K}^{-n}(X) := \tilde{K}^0(S^n X) \tag{2.110}$$

However, another way to introduce the degree is enhance our $\mathbb{Z}_2$-graded bundles $\mathcal{E}$ with odd antihermitian endomorphism T by including a left graded $\mathbb{C}\ell_n$ action that T

graded commutes with T. This is actually an example of *twisting of K-theory* by a bundle of (Clifford) algebras. (For more about why that is, see [25].)

The relation between the old definition using suspension (which makes clear the relation to generalized cohomology theory) and the definition of $K^{-n}(X)$ using Clifford algebras is provided by the Thom isomorphism of K theory together with a nice model for the Thom class constructed using Clifford modules.

The Thom isomorphism in K-theory states that if $\pi : E \to X$ is a real rank m vector bundle (with some orientation data) then

$$K^n(X) \cong K^{n+m}_{cpt-vert}(E) \tag{2.111}$$

with the isomorphism given by $[V] \to \pi^*[V] \otimes \Phi$ for a K-theory class Φ called the *Thom class*. This is a very general statement in generalized cohomology theory. See Section 12 of [7] for a very nice discussion.

Now, in order to explain the model for the Thom class Φ in real and complex K-theory we need to look at some interesting ways that Clifford algebras are related to the homotopy groups of the unitary groups.

To explain this let us begin with the following standard result from the theory of fiber bundles:

Theorem. If $d > 1$ and G is connected then principal G-bundles on S^d are topologically classified by $\pi_{d-1}(G)$, i.e. there is an isomorphism of *sets*:

$$\mathrm{Prin}_G(S^d) \cong \pi_{d-1}(G) \tag{2.112}$$

The isomorphism is very easily understood from choosing north and southern hemispheres as patches and thinking about the gluing function for the fibers at the equator.

For complex vector bundles of rank N the relevant structure group is $GL(N, \mathbb{C})$. We can always put an Hermitian structure on the vector bundle and take the structure group to be $U(N)$. Now we recall Bott's result for the stable homotopy groups:

$$\pi_{2p-1}(U(N)) = \mathbb{Z} \qquad N \geq p \tag{2.113}$$

$$\pi_{2p}(U(N)) = 0 \qquad N > p \tag{2.114}$$

Note that these equations say that for N sufficiently large, the homotopy groups do not depend on N. These are called the *stable homotopy groups* of the unitary groups and can be denoted $\pi_k(\mathbf{U})$. The mod two periodicity of $\pi_k(\mathbf{U})$ as a function of k is known as Bott periodicity. We will prove these statements using Clifford algebras and Morse theory in Lecture 3.

It follows from this theorem that, for $N > d/2$ we have

$$\mathrm{Vect}^N_{\mathbb{C}}(S^d) \cong \begin{cases} \mathbb{Z} & d = 0(2) \\ 0 & d = 1(2) \end{cases} \tag{2.115}$$

where $\mathrm{Vect}^N_{\mathbb{C}}(S^d)$ is the set of isomorphism classes of rank N complex vector bundles over S^d.

Remark

1. One way to measure the integer is via a characteristic class known as the Chern character $\mathrm{ch}(E) \in H^{2*}(X;\mathbb{Q})$. If we put a connection on the bundle then we can write an explicit representative for the image of $\mathrm{ch}(E)$ in DeRham cohomology. Locally the connection is an anti-hermitian matrix-valued 1-form A. It transforms under gauge transformations like
$$(d + A) \to g^{-1}(d + A)g \tag{2.116}$$
The fieldstrength is
$$F = dA + A^2 \tag{2.117}$$
and is locally an anti-hermitian matrix-valued 2-form transforming as $F \to g^{-1}Fg$. Then, in DeRham cohomology
$$\mathrm{ch}(E) = [\mathrm{Trexp}\left(\frac{F}{2\pi \mathrm{i}}\right)] \tag{2.118}$$
and the topological invariant is measured by
$$\int_{S^d} \mathrm{ch}(E) \tag{2.119}$$
Note that since $\mathrm{ch}(E)$ has even degree this only has a chance of being nonzero for d even. On a bundle with transition function g on the equator we can take $A = rg^{-1}dg$ on the northern hemisphere, where $g(x)$ is a function only of the "angular coordinates" on the hemisphere and $A = 0$ on the southern hemisphere. Note that thanks to the factor of r, which vanishes at the north pole this defines a first-order differentiable connection. For this connection the fieldstrength is
$$F = drg^{-1}dg - r(1-r)(g^{-1}dg)^2 \tag{2.120}$$
and hence if $d = 2\ell$
$$\begin{aligned} \int_{S^{2\ell}} \mathrm{ch}(E) &= (-1)^{\ell-1}\frac{1}{(2\pi \mathrm{i})^\ell(\ell-1)!}\int_0^1 (r(1-r))^{\ell-1}dr \int_{S^{2\ell-1}} \mathrm{Tr}(g^{-1}dg)^{2\ell-1} \\ &= (-1)^{\ell-1}\frac{(\ell-1)!}{(2\pi \mathrm{i})^\ell(2\ell-1)!}\int_{S^{2\ell-1}} \mathrm{Tr}(g^{-1}dg)^{2\ell-1} \end{aligned} \tag{2.121}$$
The integral of the Maurer-Cartan form over the equator measures the homotopy class of the transition function g. It is not at all obvious that this integral will be an integer, but for $U(N)$ and the trace in the N it is. This is a consequence of the Atiyah-Singer index theorem.

Note that in the equations (2.112) and (2.115) there is no obvious abelian group operation despite the fact that in these isomorphism of sets the RHS has a structure of an abelian group. We can of course take direct sum, but this operation changes the rank. If we want to make an abelian group we might take a direct sum in the case of $\mathrm{Vect}^N_{\mathbb{C}}(X)$, but this changes the rank. This is one place where K-theory is useful. We can form the abelian monoid obtained by taking the direct sum

$$\mathrm{Vect}(S^d) := \oplus_{N\geq 0}\mathrm{Vect}^N_{\mathbb{C}}(S^d). \tag{2.122}$$

then we can take, e.g., the Grothendieck construction.

For spheres, we will get, from this result:

$$\tilde{K}^0(S^d) = \begin{cases} \mathbb{Z} & d = 0(2) \\ 0 & d = 1(2) \end{cases} \tag{2.123}$$

and this is the abelian group which is to be compared with the group (2.89) defined above.

♣Explain a little more that ch is a homomorphism to deduce this isomorphism of groups. ♣

Now, let us finally relate all this to the Clifford algebras. We begin with projected bundles which have fibers given by the spin representation.

Therefore, let us begin with an ungraded irreducible representation of $\mathbb{C}\ell_{2k+1}$ by *Hermitian* matrices $\mathbf{\Gamma}^\mu$, $\mu = 1, \dots, 2k+1$. We consider the unit sphere $S^{2k} \subset \mathbb{R}^{2k+1}$ defined by

$$\sum_{\mu=1}^{2k+1} X^\mu X^\mu = 1 \tag{2.124}$$

Therefore, restricted to the sphere $(X^\mu \mathbf{\Gamma}^\mu)^{\mathbf{2}} = \mathbf{1}$ and hence

$$P_\pm(X) := \frac{1}{2}(1 + X^\mu \mathbf{\Gamma}^\mu) \tag{2.125}$$

are a family of projection operators, with $X \in S^{2k}$, acting on the fixed vector space $V = \mathbb{C}^{2^k}$.

We then form nontrivial *projected* bundles by taking $E = S^{2k} \times V$ to be the trivial bundle and letting

$$E_\pm|_X := P_\pm(X)V \tag{2.126}$$

These will be nontrivial bundles with nontrivial Chern characters. Indeed, to see this let us choose a trivialization and compute the transition function.

We consider northern and southern hemispheres as patches and hence divide up the coordinates as

$$X^\mu = (x^a, x^{2k}, y) \tag{2.127}$$

where $a = 1, \ldots, 2k-1$. Moreover, we can choose our ungraded representation of $\mathbb{C}\ell_{2k+1}$ so that

$$\begin{aligned}
\mathbf{\Gamma^a} &= \gamma^a \otimes \sigma^1 = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix} \qquad a = 1, \ldots, 2k-1 \\
\mathbf{\Gamma^{2k}} &= 1_{2^{k-1}} \otimes \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\mathbf{\Gamma^{2k+1}} &= 1_{2^{k-1}} \otimes \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{2.128}$$

where γ^a is an ungraded representation of $\mathbb{C}\ell_{2k-1}$ by Hermitian matrices.

Choose a basis v_α, $\alpha = 1, \ldots, 2^{k-1}$ for the irrep of $\mathbb{C}\ell_{2k-1}$ on which γ^a is an ungraded representation. Then

$$\begin{pmatrix} v_\alpha \\ 0 \end{pmatrix} \tag{2.129}$$

is a trivialization of the bundle V_+ at the north pole $y = 1$. Indeed:

$$u^+_\alpha := P_+ \begin{pmatrix} v_\alpha \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+y)v_\alpha \\ (\gamma^a x^a + i x^{2k}) v_\alpha \end{pmatrix} \tag{2.130}$$

Similarly,

$$\begin{pmatrix} 0 \\ v_\alpha \end{pmatrix} \tag{2.131}$$

provides a trivialization at the south pole $y = -1$ and

$$u^-_\alpha := P_+ \begin{pmatrix} 0 \\ v_\alpha \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\gamma^a x^a - i x^{2k}) v_\alpha \\ (1-y)v_\alpha \end{pmatrix} \tag{2.132}$$

The transition function at the equator S^{2k-1}, $y = 0$ is

$$u^-_\alpha = -i v(x) u^+_\alpha \tag{2.133}$$

where

$$v(x) = x^{2k} + \sum_{a=1}^{2k-1} (i\gamma^a) x^a \tag{2.134}$$

is a *unitary* matrix.

We claim that this unitary matrix actually generates the homotopy group π_{2k-1} of the unitary group. This construction generalizes the standard constructions of the magnetic monopole and instanton bundles on S^2 and S^4, respectively. Indeed, choosing the trivial connection on E we can form projected connections on $E^\pm$. The projected connections on $E^\pm$ define the basic (anti)monopole and (anti)instanton connections.

The above construction provides motivation for the following discussion:

Consider an ungraded representation (not necessarily irreducible) of $\mathbb{C}\ell_d$ by anti-Hermitian gamma matrices on a vector space (with basis) V where Γ^μ are such that $\{\Gamma^\mu, \Gamma^\nu\} = -2\delta^{\mu\nu}$, where $\mu = 1, \dots, d$. [19] Let $\dim_{\mathbb{C}} V = L$.

Suppose x_0, x_μ, $\mu = 1, \dots, d$ are functions on the unit sphere S^d embedded in $\mathbb{R}^{d+1}$, so

$$x_0^2 + x_\mu x_\mu = 1 \tag{2.135}$$

Consider the matrix-valued function

$$\upsilon(x) := x_0 1 + x_\mu \Gamma^\mu \tag{2.136}$$

Note that

$$\upsilon(x)\upsilon(x)^\dagger = 1 \tag{2.137}$$

and therefore $\upsilon(x)$ is a unitary matrix for every $(x_0, x_\mu) \in S^d$. We can view $\upsilon(x)$ as describing a continuous map $\upsilon : S^d \to U(L)$. Therefore it defines an element of the homotopy group $[\upsilon] \in \pi_d(U(L))$. The following examples show that the homotopy class of the map can be nontrivial:

Example 1: If $d = 1$ then we could take either of the ungraded irreducible representations $V = \mathbb{C}$ and $\Gamma = \pm i$. If $x_0^2 + x_1^2 = 1$ then

$$\upsilon^\pm(x) = x_0 \pm i x_1 \tag{2.138}$$

and, for either choice of sign, $[\upsilon^\pm]$ is a generator of $\pi_1(U(1)) = \mathbb{Z}$.

Example 2: If $d = 3$ then we may choose either of the ungraded representations $V = \mathbb{C}^2$ and $\Gamma^i = \pm\sqrt{-1}\sigma^i$ and then

$$\upsilon(x) = x_0 + x_i \Gamma^i \tag{2.139}$$

is one way to parametrize $SU(2)$. Thus the map $\upsilon : S^3 \to SU(2)$ is the identity map (with the appropriate orientation on S^3). If we fix a an orientation on S^3 we get winding number ± 1 and hence $[\upsilon^\pm]$ is a generator of $\pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Here is one easy criterion for triviality of $[\upsilon]$: Suppose we can introduce another anti-Hermitian $L \times L$ gamma matrix on V, call it Γ, so that $\Gamma^2 = -1$ and $\{\Gamma, \Gamma^\mu\} = 0$. Now consider the unit sphere

♣Make the connection between Γ and the odd invertible element υ that graded commutes with $\mathbb{C}\ell_n$ we discussed above? But we are working with ungraded reps so it is not clear what "odd" means. ♣

$$S^{d+1} = \{(x_0, x_\mu, y) | x_0^2 + \sum_{\mu=1}^{d} x_\mu x_\mu + y^2 = 1\} \subset \mathbb{R}^{d+2} \tag{2.140}$$

Then we can define

$$\tilde{\upsilon}(x, y) = x_0 + x_\mu \Gamma^\mu + y\Gamma \tag{2.141}$$

When restricted to $S^{d+1} \subset \mathbb{R}^{d+2}$, $\tilde{\upsilon}$ is also unitary and maps $S^{d+1} \to U(L)$. Moreover $\tilde{\upsilon}(x, 0) = \upsilon(x)$ while $\tilde{\upsilon}(0, 1) = \Gamma$. Thus $\tilde{\upsilon}(x, y)$ provides an explicit homotopy of $\upsilon(x)$ to the constant map.

[19] Note that the notation μ has changed from the previous discussion.

Figure 10: The map on the equator extends to the northern hemisphere, and is therefore homotopically trivial.

Thus, *if the representation V of $\mathbb{C}\ell_d$ is the restriction of a representation of $\mathbb{C}\ell_{d+1}$ then $v(x)$ is automatically homotopically trivial.*

Let us see what this means if we combine it with what we learned above about the irreducible ungraded representations of $\mathbb{C}\ell_d$.

1. If $d = 2p$ we have irrep $N_{2p} \cong \mathbb{C}^{2^p}$. It is indeed the restriction of $N^{\pm}_{2p+1} \cong \mathbb{C}^{2^p}$ and hence, $v(x)$ must define a trivial element of $\pi_{2p}(U(L))$, with $L = 2^p$.

2. On the other hand, if $d = 2p+1$ then $N^{\pm}_{2p+1} \cong \mathbb{C}^{2^p}$ is *not* the restriction of $N_{2p+2} \cong \mathbb{C}^{2^{p+1}}$. All we can conclude from what we have said above is that $v(x)$ *might* define a homotopically nontrivial element of $\pi_{2p+1}(U(L))$ with $L = 2^p$. On the other hand, if we had used $V = N^{+}_{2p+1} \oplus N^{-}_{2p+1}$ then since V is the restriction of the representation N_{2p+2} and $v = v^{+} \oplus v^{-}$, it follows that the homotopy classes satisfy $[v^-] = -[v^+]$.

Now, a nontrivial result of [7] is:[20]

Theorem[Atiyah, Bott, Shapiro]. If V is an irreducible ungraded representation of $\mathbb{C}\ell_d$ then $[v]$ generates $\pi_d(U(L))$.

It therefore follows that $\pi_{2p}(U(L)) = 0$ and $\pi_{2p+1}(U(L)) \cong \mathbb{Z}$, with generator $[v^+]$ or $[v^-]$.

Now, finally, we can use the above constructions to provide an explicit model for the Thom class.

The simplest case to state is when $V \to X$ is a real, rank 8, oriented bundle with a spin structure. Then we need to construct a K-theory class on V compactly supported in the fiber directions. To do this we give V a metric so that we can form a bundle of Clifford algebras $Cl(V) \to X$. Let $S^{\pm}$ be the associated spin bundles, pulled back to V itself. Then we take the difference bundle $T : S^+ \to S^-$ where T is just $T(v) = v \cdot \gamma$ given by Clifford multiplication by the vector v in the fiber. This is invertible away from $v = 0$, the zero section and represents the Thom class, as described in [7]. This construction can be generalized to arbitrary twisted equivariant vector bundles, as described in the papers of Freed-Hopkins-Teleman, and as used extensively in my work with Distler and Freed on orientifolds. See [25] for some further discussion.

[20] More precisely, they used the above $v(x)$ to define a K-theoretic Thom class. Then the result we have stated follows from the relation of K-theory to homotopy theory.

2.4 The 10 Real Super-division Algebras

Definition An associative unital superalgebra over a field κ is an associative *super-division algebra* if every nonzero *homogeneous* element is invertible.

Example 1: We claim that $\mathbb{C}\ell_1$ is a superdivision algebra over $\kappa = \mathbb{C}$ (and hence a superdivision algebra over $\mathbb{R}$). Elements in this superalgebra are of the form $x + ye$ with $x, y \in \mathbb{C}$. Homogeneous elements are therefore of the form x or ye, and are obviously invertible, if nonzero. Note that it is *not true* that every nonzero element is invertible! For example $1 + e$ is a nontrivial zero-divisor since $(1 + e)(1 - e) = 0$. Thus, $\mathbb{C}\ell_1$ is not a division algebra, as an ungraded algebra.

Example 2: We also claim that the 8 superalgebras D^s_α, with $\alpha \in \mathbb{Z}/8\mathbb{Z}$ defined in (2.95) are real super-division algebras. The argument of Example 1 show that $C\ell_{\pm 1}$ are super-division algebras. For $C\ell_{\pm 2}$ the even subaglebra is isomorphic to the complex numbers, which is a division algebra. It follows that $C\ell_{\pm 2}$ are superdivision algebras. To spell this out in more detail: For $C\ell_{+2}$ we note that for even elements we can write

$$(x + ye_{12})(x - ye_{12}) = x^2 + y^2 \tag{2.142}$$

and for odd elements we can write

$$(xe_1 + ye_2)^2 = x^2 + y^2 \tag{2.143}$$

where $x, y \in \mathbb{R}$. Thus the nonzero homogeneous elements are invertible. For $C\ell_{-2}$ the equation (2.142) holds and (2.143) simply has a sign change on the RHS. So this too is a superdivision algebra. More conceptually, note that $C\ell^0_{\pm 2}$ is isomorphic to $\mathbb{C}$, which is a division algebra, and $C\ell^1_{\pm 2}$ is related to $C\ell^0_{\pm 2}$ by multiplying with an invertible element. We can now apply this strategy to $C\ell_{\pm 3}$: The even subalgebra is isomorphic to the quaternion algebra, which is a division algebra and the odd subspace is related to the even subspace by multiplication with an invertible odd element. Hence $C\ell_{\pm 3}$ is a superdivision algebra.

Note well that $C\ell_{+1,-1}$ being a matrix superalgebra is definitely *not* a superdivision algebra! For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.144}$$

is even and is a nontrivial zerodivisor. By the same token, $C\ell_{\pm 4} \cong \mathrm{End}(\mathbb{R}^{1|1})\widehat{\otimes}\mathbb{H}$ is also not a superdivision algebra.

The key result we need is really a corollary of Wall's theorem classifying central simple superalgebras. For a summary of Wall's result see Appendix **??**.

Theorem There are 10 superdivision algebras over the real numbers: The three purely even algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, together with the 7 superalgebras $\mathbb{C}\ell_1, C\ell_{\pm 1}, C\ell_{\pm 2}, C\ell_{\pm 3}$.

2.5 The 10-fold way for gapped quantum systems

We are now in a position to describe the generalization of Dyson's 3-fold way to a 10-fold way, valid for gapped quantum systems.

♣NEED TO REWRITE: ASSUME AN *a priori* (G, ϕ, χ) rep on a super-Hilbert space $\mathcal{H}$. ♣

Recall from our discussion of a general symmetry of dynamics (§1.8) that if G is a symmetry of the dynamics of a gapped quantum system then there are two independent homomorphisms $(\phi, \chi) : G \to \mathbb{Z}_2$. In the Dyson problem one explicitly assumes that $\chi = 1$. Nevertheless, as we saw when discussing phases of gapped systems in Section §2.1, there is a natural $\mathbb{Z}_2$-grading of the Hilbert space so that, if $\chi \neq 1$ then the Hilbert space is a (ϕ, χ)-representation of G. (See Definition (2.3).) Therefore we can state the

Generalized Dyson Problem: Let G be a bigraded compact group, bigraded by (ϕ, χ), and $\mathcal{H}$ a $\mathbb{Z}_2$-graded (ϕ, χ)-representation $\mathcal{H}$ of G. What is the ensemble of gapped Hamiltonians H such that G is a symmetry of the dynamics?

Remark: One must use caution here when thinking about the $\mathbb{Z}_2$-grading induced by the Hamiltonians in the ensemble. If we interpret the Hamiltonian-reversing property of group elements $\rho(g)$ with $\chi(g) = -1$ as a graded commutation relation, then H must be considered odd. On the other hand, if $\mathrm{sign}(H)$ is a grading then H must be even! It is best to consider a (ϕ, χ)-graded representation and ask what are the compatible gapped Hamiltonians. Then the Hamiltonians in the ensemble each induce different gradings on the Hilbert space.

We can proceed to answer this along lines closely analogous to those for Dyson's 3-fold way.

First, we imitate the definitions of Section §1.7.1 for ϕ-representations:

Definitions:

1. If G is a bigraded group by (ϕ, χ) then a (ϕ, χ)-representation is defined in (2.3).
2. An *intertwiner* or *morphism* between two (ϕ, χ)-reps (ρ_1, V_1) and (ρ_2, V_2) is a $\mathbb{C}$-linear map $T : V_1 \to V_2$, which is a morphism of super-vector spaces: $T \in \underline{\mathrm{Hom}}_{\mathbb{C}}(V_1, V_2)$, which commutes with the G-action:
$$T\rho_1(g) = \rho_2(g)T \qquad \forall g \in G \tag{2.145}$$
We write $\underline{\mathrm{Hom}}_{\mathbb{C}}^{G}(V_1, V_2)$ for the set of all intertwiners.
3. An *isomorphism of (ϕ, χ)-reps* is an intertwiner T which is an isomorphism of complex supervector spaces.

4. A (ϕ, χ)-rep is said to be *ϕ-unitary* if V has a nondegenerate even Hermitian structure [21] such that $\rho(g)$ is an isometry for all g. That is, it is unitary or anti-unitary according to whether $\phi(g) = +1$ or $\phi(g) = -1$, respectively.

5. A (ϕ, χ)-rep (ρ, V) is said to be *reducible* if there is a nontrivial proper (ϕ, χ)-sub-representation. That is, if there is a complex super-vector subspace $W \subset V$, (and hence $W^0 \subset V^0$ and $W^1 \subset V^1$) with W not $\{0\}$ or V which is G-invariant. If it is not reducible it is said to be *irreducible*.

As before, if G is compact and (ρ, V) is a (ϕ, χ)-rep then WLOG we can assume that the rep is unitary, by averaging. Then if W is a sub-rep the orthogonal complement is another (ϕ, χ)-rep, and hence we have complete reducibility.

Now we need to deal with a subtle point. In addition to intertwiners we needed to consider the *graded intertwiners* $\mathrm{Hom}^G_{\mathbb{C}}(V_1, V_2)$ between two (ϕ, χ)-representations. These are super-linear transformations T such that if we decompose $T = T^0 + T^1$ into even and odd transformations then $T^0 \in \underline{\mathrm{Hom}}^G_{\mathbb{C}}(V_1, V_2)$ but T^1 instead satisfies

♣Be consistent in notation about V, V' vs. V_1, V_2. ♣

$$T^1 \rho_1(g) = \chi(g)\rho_2(g)T^1 \qquad \forall g \in G \tag{2.146}$$

Two irreducible representations can be distinct as (ϕ, χ)-representations but can be graded-isomorphic. The simplest example is $G = \{1\}$ which has graded irreps $\mathbb{C}^{1|0}$ and $\mathbb{C}^{0|1}$.

Let $\{V_\lambda\}$ be a set of representatives of the distinct graded-isomorphism classes of irreducible (ϕ, χ)-representations. We then obtain the isotypical decomposition of (ϕ, χ)-representations:

$$\mathcal{H} \cong \oplus_\lambda \mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, \mathcal{H}) \widehat{\otimes} V_\lambda \tag{2.147}$$

Note that $\mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, \mathcal{H})$ is not an even vector space in general. This will be more convenient to us because of the nature of the super-Schur lemma:

♣Careful. Then decomposition is not unique. ♣

Lemma[Super-Schur] Let G be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded group, graded by the pair of homomorphisms (ϕ, χ).

a.) If T is a graded intertwiner between two irreducible (ϕ, χ)-representations (ρ, V) and (ρ', V') then either $T = 0$ or there is an isomorphism of (ρ, V) and (ρ', V').

b.) If (ρ, V) is an irreducible (ϕ, χ)-representation then the super-commutant $Z_s(\rho, V)$, namely, the set of graded intertwiners of (ρ, V) with itself is a super-division algebra.

Now we can now proceed as before to derive the analog of Dyson's ensembles. We consider the isotypical decomposition (2.147) of $\mathcal{H}$. Let $S_\lambda := \mathrm{Hom}^G_{\mathbb{C}}(V_\lambda, \mathcal{H})$. It is a real super-vector space of degeneracies. Now we compute the set of superlinear transformations:

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}, \mathcal{H}) \cong \oplus_{\lambda,\mu}(S^*_\lambda \widehat{\otimes} S_\mu) \widehat{\otimes} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\mu) \tag{2.148}$$

[21] See Section §2.2.5 above.

Now we take the graded G-invariants and apply the super-Schur lemma to get

$$Z_s(\rho, \mathcal{H}) = \mathrm{Hom}^G_{\mathbb{C}}(V, V) \cong \oplus_\lambda \mathrm{End}(S_\lambda) \widehat{\otimes} D^s_\lambda \tag{2.149}$$

where to each isomorphism class of graded-irreducible (ϕ, χ)-rep λ we associate a unique real super-division algebra D^s_λ. Meanwhile $\mathrm{End}(S_\lambda) \cong \mathrm{End}(\mathbb{R}^{s^0_\lambda|s^1_\lambda})$ is a matrix superalgebra.

Finally, let us apply this to the generalized Dyson problem. If G is a symmetry of the dynamics determined by H then

$$H\rho(g) = \chi(g)\rho(g)H \tag{2.150}$$

and hence the $\mathbb{C}$-linear operator H is in the graded-commutant of the given (ϕ, χ) representation $\mathcal{H}$. Therefore, H is in the space (2.149). For each irreducible representation λ there is a corresponding super-division algebra D^s_λ and this gives the 10-fold classification.

To write the ensemble of Hamiltonians more explicitly we recall that H must be a self-adjoint element of $Z_s(\rho, \mathcal{H})$. There is a natural $*$ structure on the superdivision algebras since the Clifford generators can be represented as Hermitian or anti-Hermitian operators. That is, we take $e_i^* = \pm e_i$ with the sign determined by $e_i^* = e_i^3$. We then extend this to be an anti-automorphism, and for $\mathrm{End}(S_\lambda)\widehat{\otimes} D^s_\lambda$ we take the tensor product of the natural $*$-structures. H must be a self-adjoint element of this superalgebra.

Moreover, if $\chi(g)$ is nontrivial for any g then H must be in the odd element of the superdivision algebra.

Thus, the 10-fold way is the following:

1. If the (ϕ, χ) representation has $\chi = 1$ then the generalized Dyson problem is identical to the original Dyson problem, and there are three possible ensembles.
2. But if χ is nontrivial then there are new ensembles not allowed in the Dyson classification. In these cases, D^s_λ is one of the 7 superalgebras which are not purely even and H is an odd element of the superalgebra $\mathrm{End}(S_\lambda)\widehat{\otimes} D^s_\lambda$.

Remarks:

1. It was easy to give examples of the three classes in Dyson's 3-fold way. Below we will give examples using the 10 bigraded "CT-groups" discussed in Section §2.6 below.
2. The above is, strictly speaking, a new result, although it is really a simple corollary of [26]. However, it should be stressed that the result is just a *very general statement about quantum mechanics*. No mention has been made of bosons vs. fermions, or interacting vs. noninteracting, or one-body vs. many-body Hamiltonians.
3. A key point we want to stress is that the 10-fold way is usually viewed as $10 = 2 + 8$, where 2 and 8 are the periodicities in complex and real K-theory. And then the K-theory classification of topological phases is criticized because it only applies to free systems. However, we believe this viewpoint is slightly misguided. The unifying concept is really that of a real super-division algebra, and there are 10 such. They can be parceled into $10 = 8 + 2$ but they can also equally naturally be parceled into $10 = 7 + 3$ (with the 3 referring to the purely even superdivision algebras).

♣Go back to Fidkowski-Kitaev and interpret their interacting models in this framework. They should be getting some of the 7 non-Dyson classes. ♣

4. The Altland-Zirnbauer classification discussed below makes explicit reference to free fermions.

5. As a curious digression we note that in Dyson's original paper on ϕ-representations [20] he in fact had a 10-fold classification of irreducible ϕ-representations! (Assuming $\phi \neq 1$.) Conjecturally, it is related to the 10 real superdivision algebras, although this has never been demonstrated in a totally satisfactory way. See [43] for more discussion.

2.6 Realizing the 10 classes using the CT groups

To make contact with some of the literature on topological insulators we describe here the 10 "CT groups." (This is a nonstandard term used in [26].) This is a set of 10 bigraded groups which we now define.

To motivate the 10 CT groups note that in some disordered systems, (sometimes well-described by free fermions), the only symmetries we might know about *a priori* are the presence or absence of "time-reversal" and "particle-hole" symmetry. Thus it is interesting to consider the various ϕ-twisted extensions of the group

$$M_{2,2} = \langle \bar{T}, \bar{C} | \bar{T}^2 = \bar{C}^2 = \bar{T}\bar{C}\bar{T}\bar{C} = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{2.151}$$

or of its subgroups.

A second motivation is that if G is a bigraded group by (ϕ, χ) then there is a sequence

$$1 \to G_0 \to G \to M_{2,2} \to 1 \tag{2.152}$$

It is easy to show that, WLOG, if the bigrading (ϕ, χ) is surjective then we can choose generators of $M_{2,2}$ so that:

$$\begin{aligned} \phi(\bar{T}) &= -1 & \qquad \phi(\bar{C}) &= -1 \\ \chi(\bar{T}) &= +1 & \qquad \chi(\bar{C}) &= -1 \\ \tau(\bar{T}) &= -1 & \qquad \tau(\bar{C}) &= +1 \end{aligned} \tag{2.153}$$

where we have defined τ from ϕ and χ so that $\tau \cdot \phi \cdot \chi = 1$.

In general we should allow for the possibility that one or two of the gradings ϕ, τ, χ is trivial and this motivates us to study *subgroups* of the group $M_{2,2}$.

Therefore, let us consider the ϕ-twisted extensions of $M_{2,2}$ and its subgroups. This is a simple generalization of the example we discussed in Section §1.5, equation (1.69). First, let us note that there are 5 subgroups of $M_{2,2}$ depending on whether $\bar{T}, \bar{C}$ or $\bar{T}\bar{C}$ is in the group. See Figure 11.

As in the example (1.69) the isomorphism class of the extension is completely determined by whether the lift $\tilde{T}$ and/or $\tilde{C}$ of $\bar{T}$ and/or $\bar{C}$ squares to ± 1. After a few simple considerations discussed in the exercises below it follows that one has the table of 10 ϕ-twisted extensions of the subgroups of $M_{2,2}$:

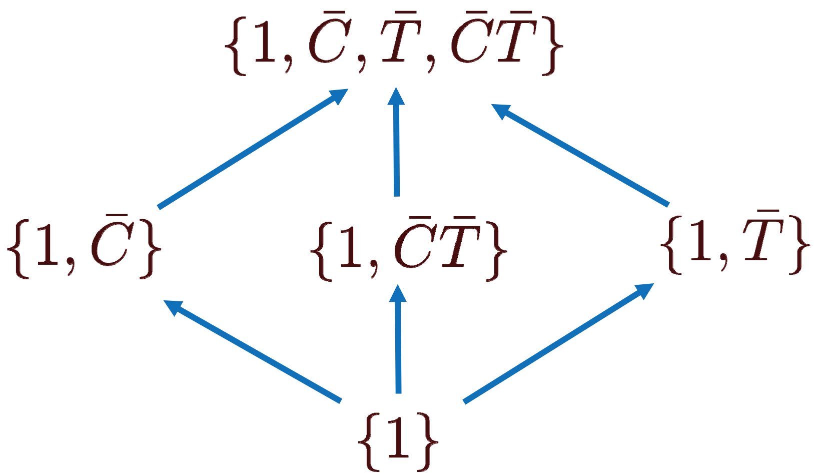

Figure 11: The 5 subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Subgroup $U \subset M_{2,2}$	$\tilde{T}^2$	$\tilde{C}^2$	[Clifford]
$\{1\}$			$[\mathbb{C}\ell_0] = [\mathbb{C}]$
$\{1, \bar{S}\}$			$[\mathbb{C}\ell_1]$
$\{1, \bar{T}\}$	$+1$		$[C\ell_0] = [\mathbb{R}]$
$M_{2,2}$	$+1$	-1	$[C\ell_{-1}]$
$\{1, \bar{C}\}$		-1	$[C\ell_{-2}]$
$M_{2,2}$	-1	-1	$[C\ell_{-3}]$
$\{1, \bar{T}\}$	-1		$[C\ell_4] = [\mathbb{H}]$
$M_{2,2}$	-1	$+1$	$[C\ell_{+3}]$
$\{1, \bar{C}\}$		$+1$	$[C\ell_{+2}]$
$M_{2,2}$	$+1$	$+1$	$[C\ell_{+1}]$

Now, we can generalize the remark near the example of Section §1.7.1. Recall that we could identify ϕ-representations of ϕ-twisted extensions of M_2 with real and quaternionic vector spaces. If we consider subgroups of M_2 then for the trivial subgroup we also get complex vector spaces. This trichotomy is generalized to a decachotomy for the CT groups:

Theorem There is a one-one correspondence, given in the table above, between the ten CT groups and the ten real super-division algebras (equivalently, the 10 Morita classes of the real and complex Clifford algebras) such that there is an equivalence of categories between the (ϕ,χ)-representations of the CT group and the graded representations of the corresponding Clifford algebra.

Sketch of Proof:

We systematically consider the ten cases beginning with a (ϕ,χ)-representation of a CT group and producing a corresponding representation of a Clifford algebra. Then we show how the inverse functor is constructed.

1. First, consider the subgroup $U = \{1\}$. A (ϕ,χ) representation W is simply a $\mathbb{Z}_2$-graded complex vector space, so $V = W$ is a graded $\mathbb{C}\ell_0$-module.

2. Now consider $U = \{1, \bar{S}\}$. There is a unique central extension and $S = \tilde{C}T$ acts on W as an odd operator which, WLOG, we can take to square to one. Moreover, S is $\mathbb{C}$-linear. Therefore, we can take $V = W$ and identify S with an odd generator of $\mathbb{C}\ell_1$.

3. Now consider $U = \{1, \bar{C}\}$. On the representation W of $U^{\rm tw}$ we have two odd anti-linear operators $\tilde{C}$ and $i\tilde{C}$. Note that

$$(i\tilde{C})^2 = \tilde{C}^2 \qquad\qquad \{i\tilde{C},\tilde{C}\} = 0 \tag{2.154}$$

   since $\tilde{C}$ is antilinear. Therefore, we can define a graded Clifford module $V = W$ with $e_1 = \tilde{C}$ and $e_2 = i\tilde{C}$. It is a Clifford module for a real Clifford algebra, again because $\tilde{C}$ is anti-linear. The Clifford algebra is $C\ell_{+2}$ if $\tilde{C}^2 = +1$ and $C\ell_{-2}$ if $\tilde{C}^2 = -1$.

4. Next, consider $U = \{1, \bar{T}\}$. The lift $\tilde{T}$ to $U^{\rm tw}$ acts on a (ϕ,χ) representation W as an even, $\mathbb{C}$-antilinear operator. It is therefore a real structure if $\tilde{T}^2 = +1$ and a quaternionic structure if $\tilde{T}^2 = -1$. In the first case, the fixed points of $\tilde{T}$ define a real $\mathbb{Z}_2$-graded vector space $V = W|_{\tilde{T}=+1}$ which is thus a graded module for $C\ell_0$. In the second case, $\tilde{T}$ defines a quaternionic structure on $V = W_{\mathbb{R}}$. As we have seen, $C\ell_4$ is Morita equivalent to $\mathbb{H}$, and in fact the $C\ell_4$ module is $V \oplus V$. (Recall equation (**??**) above.)

5. Now consider $U = M_{2,2}$. This breaks up into 4 cases:

6. If $T^2 = +1$ then, as we have just discussed T defines a real structure. As shown in the exercises, WLOG we can choose the lift of C so that $CT = TC$. Therefore, C acts as an odd operator on the real vector space of $T = +1$ eigenstates: $V = W|_{T=+1}$. Then V is the corresponding module for $C\ell_{\pm 1}$ according to whether $C^2 = \pm 1$.

7. If $T^2 = -1$, then, as we just discussed, T defines a quaternionic structure on $V = W_{\mathbb{R}}$. Then C, iC, and iCT are odd endomorphisms of $W_{\mathbb{R}}$ and one checks they generate a $C\ell_{+3}$ action if $C^2 = +1$ and a $C\ell_{-3}$ action if $C^2 = -1$.

$\diamondsuit$

Now, in order to give our application to the generalized Dyson problem we note a key:

Proposition: Let U^{tw} be one of the 10 bigraded CT groups and let D be the associated real superdivision algebra. Let (ρ, W) be an irreducible (ϕ, χ)-rep of U^{tw}. Then, the graded commutant $Z_s(\rho, W)$ is a real superdivision algebra isomorphic to D^{opp}.

This is proved in detail in [43]

We can now give examples of all 10 generalized Dyson classes. If U^{tw} corresponds to one of the even superdivision algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ then there are two irreps $W_\pm$. The general rep of U^{tw} is isomorphic to $\mathcal{H} = W_+^{\oplus n_+} \oplus W_-^{\oplus n_-}$. Then the graded commutant is

$$Z_s(\rho, \mathcal{H}) = \mathrm{End}(\mathbb{R}^{n_+|n_-})\widehat{\otimes} D^{\mathrm{opp}} \tag{2.155}$$

In these cases the group U^{tw} (which is isomorphic to $\mathrm{Pin}^\pm(1)$, see below) is purely even so the Hamiltonian can be even or odd or a sum of even and odd. We can therefore forget about the grading and we recover precisely Dyson's 3 cases. If U^{tw} corresponds to one of the remaining 7 superdivision algebras (those which are not even) then there is a unique graded irrep W and up to isomorphism $\mathcal{H} = W^{\oplus n}$ so again

$$Z_s(\rho, \mathcal{H}) = \mathrm{End}(\mathbb{R}^n)\widehat{\otimes} D^{\mathrm{opp}} \tag{2.156}$$

As discussed above we can impose Hermiticity conditions on the graded commutant to get the relevant ensembles of Hamiltonians.

Example 1: Let $U = \{1, \bar{S}\} \cong M_2$ with $\phi(\bar{S}) = +1$ and $\chi(\bar{S}) = -1$. Since $\phi = 1$ the extension of M_2 must be trivial and $U^{\mathrm{tw}} = U(1) \times M_2$ and there is a single graded irrep $W = \mathbb{C}^{1|1}$ with

$$S : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} \tag{2.157}$$

Every (ϕ, χ) representation is of the form $W^{\oplus n} \cong \mathbb{C}^{n|n}$. In a basis of (even,odd) the general commutant consists of complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{2.158}$$

The Hermitian ensemble consists of such matrices with $\alpha^\dagger = \alpha$ and $\beta^\dagger = -\beta$. Since the Hamiltonian must be odd the ensemble is

$$\mathcal{E}(\mathbb{C}\ell_1) = \{ \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} | \beta \in M_n(\mathbb{C}) \quad \& \quad \beta^\dagger = -\beta \} \tag{2.159}$$

and we further require that 0 is not in the spectrum, so β is gappped. ♣just say $\beta \in GL(n, \mathbb{C})$? ♣

How does this differ from the Dyson ensemble? If we compare with M_2 acting $\mathbb{C}$-linearly but evenly then there are two irreps and we have $\mathbb{C}^{n_1} \otimes 1 \oplus \mathbb{C}^{n_2} \otimes \epsilon$. The ensembles are $\mathrm{Herm}_{n_1}(\mathbb{C}) \times \mathrm{Herm}_{n_2}(\mathbb{C})$.

Example 2: If $U = \{1, \bar{C}\}$ with $\phi(\bar{C}) = \chi(\bar{C}) = -1$ then the extension U^{tw} is characterized by a sign $C^2 = \xi \in \{\pm 1\}$. The bigraded group U^{tw} has a single (ϕ, χ)-irrep, $W \cong \mathbb{C}^{1|1}$ with action

$$C : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} \xi \bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{2.160}$$

The most general super-linear transformation which graded commutes with C is

$$A = \begin{pmatrix} \alpha & \beta \\ -\xi\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{2.161}$$

the algebra of these matrices is $C\ell_{\mp 2}$ for $\xi = \pm 1$. The general (ϕ, χ) representation can be written as $W^{\oplus n}$ and the graded commutant consists of matrices of the form (2.161) where now α, β are $n \times n$ complex matrices and $\bar{\alpha}$ means the complex conjugate. To pass to the ensemble of compatible Hamiltonians we require that H be Hermitian and odd. Therefore, again $\alpha = 0$ and $\beta^\dagger = -\xi\bar{\beta}$, that is, $\beta^{tr} = -\xi\beta$, and again β must be gapped, and so, labeling the ensemble by the superdivision algebra of the graded commutant:

$$\mathcal{E}(C\ell_{\mp 2}) = \{\begin{pmatrix} 0 & \beta \\ -\xi\bar{\beta} & 0 \end{pmatrix} | \beta \in M_n(\mathbb{C}) \quad \& \quad \beta^{tr} = -\xi\beta\} \tag{2.162}$$

Example 3: If $U = M_{2,2}$ then we again have four cases to consider. Suppose first that $T^2 = +1$. Then $C^2 = \xi = \pm 1$ corresponds to the case $D = C\ell_{\pm 1}$. There is a unique (ϕ, χ) irrep $W = \mathbb{C}^{1|1}$ of U^{tw} with actions:

$$C : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} \xi \bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{2.163}$$

$$T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \tag{2.164}$$

The supercommutant for the representation $W^{\oplus n}$ is given by matrices of the form (2.161) but now with α, β real, so that it commutes with T. Therefore

$$\mathcal{E}(C\ell_{\mp 1}) = \{\begin{pmatrix} 0 & \beta \\ -\xi\beta & 0 \end{pmatrix} | \beta \in M_n(\mathbb{R}) \quad \& \quad \beta^{tr} = -\xi\beta\} \tag{2.165}$$

Example 4: Finally, for $T^2 = -1$ and $C^2 = \xi$ there is a unique (ϕ, χ)-irrep of U^{tw} and we can take $W = \mathbb{H}^{1|1}$ with complex structure $I = L(\mathfrak{i})$. Then

$$T : \begin{pmatrix} \mathfrak{q}_1 \\ \mathfrak{q}_2 \end{pmatrix} \to \begin{pmatrix} L(\mathfrak{j})\mathfrak{q}_1 \\ L(\mathfrak{j})\mathfrak{q}_2 \end{pmatrix} \tag{2.166}$$

$$C : \begin{pmatrix} \mathfrak{q}_1 \\ \mathfrak{q}_2 \end{pmatrix} \to \begin{pmatrix} -\xi L(\mathfrak{j})\mathfrak{q}_2 \\ L(\mathfrak{j})\mathfrak{q}_1 \end{pmatrix} \tag{2.167}$$

so that $C^2 = \xi = \pm 1$, is odd, and commutes with T. The odd elements generating the graded commutant are

$$\mathcal{R}(\mathfrak{q}) : \begin{pmatrix} \mathfrak{q}_1 \\ \mathfrak{q}_2 \end{pmatrix} \to \begin{pmatrix} \xi \mathfrak{q}_2 \mathfrak{q} \\ \mathfrak{q}_1 \mathfrak{q} \end{pmatrix} \tag{2.168}$$

Taking $\mathcal{R}(\mathfrak{i}), \mathcal{R}(\mathfrak{j}), \mathcal{R}(\mathfrak{k})$ we get 3 odd generators generating $C\ell_{\pm 3}$. It is useful to write the commutant in terms of 2×2 complex matrices. To that end, we identify $\mathbb{H} \cong \mathbb{C}^2$ via $\mathfrak{q} = z_1 + z_2\mathfrak{j}$ where $z_1 = x_1 + \mathfrak{i}y_1$ and $z_2 = x_2 + \mathfrak{i}y_2$ with x_i, y_i real. Then

$$L(\mathfrak{j}) : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} \tag{2.169}$$

and right-multiplication by a quaternion is $\mathbb{C}$-linear and hence represented by a 2×2 matrix. The general (ϕ, χ)-representation of $U^{\rm tw}$ is $W^{\oplus n}$. It has real superdimension $(4n|4n)$ and, identifying $\mathbb{H} \cong \mathbb{C}^2$ as above the general graded intertwiner takes the form

$$\begin{pmatrix} \alpha & \beta \\ \xi\beta & \alpha \end{pmatrix} \tag{2.170}$$

where α, β are $n \times n$ matrices whose entries are right-multiplication by quaterions. In other words, we can consider them to be $2n \times 2n$ complex matrices which satisfy $J\beta J^{-1} = \beta^*$ and $J\alpha^* J^{-1} = \alpha$, where $J = 1_n \otimes \epsilon$, as usual. Thus, for the case $C^2 = \xi = \pm 1$,

$$\mathcal{E}(C\ell_{\mp 3}) = \{ \begin{pmatrix} 0 & \beta \\ \xi\beta & 0 \end{pmatrix} | \beta \in M_n(\mathbb{H}) \quad \& \quad \beta^\dagger = \xi\beta \} \tag{2.171}$$

Remark: We motivated the study of $M_{2,2}$ and its subgroups using the example of disordered systems. Unfortunately, in the literature on this subject it is often assumed that given a pair of homomorphisms

$$(\phi, \chi) : G \to M_{2,2} \tag{2.172}$$

such that $\tau \cdot \chi = \phi$, we will always have $G \cong G_0 \times U$, where U is a subgroup of $M_{2,2}$ and G_0 is $\ker(\phi) \cap \ker(\chi)$. This is *not true* in general! There is a sequence

$$1 \to G_0 \to G \to U \to 1 \tag{2.173}$$

and in general it will not split, let alone be a direct product.

♣Should make the point better by taking an example of such a bigraded group G and a (ϕ, χ)-rep and working out the ensemble of Hamiltonians. ♣

3. Lecture 3: Free Fermions, the Altland-Zirnbauer classification, and Bott Periodicity

3.1 Fermions and the Spin Representation

We now return to quantum mechanics.

The central motivation for this chapter, in the context of these notes, is that important examples of the 10-fold way described above are provided by free fermions. They also appear in the Altland-Zirnbauer classification, and in applications to topological band structure.

Of course, the basic mathematics of free fermion quantization is very broadly applicable. In this chapter we give a summary of that quantization and comment on the relation to the Spin group and spin representations.

3.1.1 Finite dimensional fermionic systems

A finite dimensional fermionic system (FDFS) is a quantum system based on a certain kind of operator algebra and its representation:

Definition: A finite dimensional fermionic system is the following data:

1. A finite-dimensional real vector space $\mathcal{M} \cong \mathbb{R}^N$, called the *mode space* with a positive symmetric bilinear form Q.

2. An extension of the complex Clifford algebra

$$\mathcal{A} = \mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C} \tag{3.1}$$

   to a $*$-algebra.

3. A choice of Hilbert space $\mathcal{H}_F$ together with a $*$ homomorphism of $\mathcal{A}$ into the algebra of $\mathbb{C}$-linear operators on $\mathcal{H}_F$. [22]

Here are a number of remarks about this definition:

1. As an algebra $\mathcal{A}$ is the complex Clifford algebra of $V := \mathcal{M} \otimes \mathbb{C}$ with Q extended $\mathbb{C}$-linearly.

2. From $(\mathcal{M}, Q)$ we can make the real Clifford algebra $\mathrm{Cliff}(\mathcal{M}, Q)$. In quantum mechanics we will want a $*$-algebra of operators and the observables will be the operators fixed by the $*$-action. For us the $*$-algebra structure on

$$\mathcal{A} := \mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C} \tag{3.2}$$

   is $\beta \otimes \mathcal{C}$, where β is the canonical anti-automorphism of $\mathrm{Cliff}(\mathcal{M}, Q)$ and $\mathcal{C}$ is complex conjugation on $\mathbb{C}$. Thus $*$ fixes $\mathcal{M}$ and is an anti-automorphism. (These conditions uniquely determine $*$.) Axioms of quantum mechanics would simply give us *some* $*$-algebra without extra structure. The fermionic system gives us the extra data $(\mathcal{M}, Q)$.

♣Use font for $*$ consistently with discussion above. ♣

[22] The subscript "F" is for "Fock."

3. Since we have a $*$ structure on a $\mathbb{Z}_2$-graded algebra we must deal with a convention issue. Here we are taking the convention that $(ab)^* = b^* a^*$ for any a, b because this is the convention almost universally adopted in the physics literature. However, a systematic application of the Koszul sign rule in the definition of $*$ would require $(ab)^* = (-1)^{|a|\cdot|b|} b^* a^*$. One can freely pass between these two conventions and, if used consistently, the final results are the same. See Section §2.2.5 above for more discussion.

4. If Q is positive definite then we can diagonalize it to the unit matrix. If e_i is a choice of basis in which Q is δ_{ij} then the usual Clifford relations

$$e_i e_j + e_j e_i = 2\delta_{ij} \qquad\qquad i, j = 1, \dots, N \tag{3.3}$$

are known in this context as the *fermionic canonical commutation relations*. Because of our choice of $*$-structure we have $e_i^* = e_i$. Of course, the choice of basis is far from unique. Different choices are related by $O(N)$ transformations. Those transformations commute with the $*$ structure. The e_i are known in the literature as *real fermions* or *Majorana fermions*. In terms of the e_i the most general quantum observable is

$$\mathcal{O} = \mathcal{O}_0 + \sum_{k=1}^{d} \mathcal{O}_{i_1 \dots i_k} e_{i_1 \dots i_k} \tag{3.4}$$

where the coefficients are totally antisymmetric tensors such that $\mathcal{O}_0 \in \mathbb{R}$ and

$$\mathcal{O}^*_{i_1 \dots i_k} = (-1)^{\frac{1}{2}k(k-1)} \mathcal{O}_{i_1 \dots i_k}. \tag{3.5}$$

5. In quantum mechanics we must also have a Hilbert space representation of the $*$-algebra of operators so that $*$ corresponds to Hermitian conjugation in the Hilbert space representation. That is, we have an algebra homomorphism

$$\rho_F : \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}_F) \tag{3.6}$$

to the $\mathbb{C}$-linear operators on the Hilbert space $\mathcal{H}_F$. The is a $*$-homomorphism in the sense that

$$(\rho_F(a))^\dagger = \rho_F(a^*) \tag{3.7}$$

♣Here a is a general element of the algebra. Not a fermion oscillator. Bad notation. ♣

In the fermionic system we are assuming that $\mathcal{H}_F$ is a choice of an *irreducible* module for $\mathcal{A}$. We will describe explicit models for $\mathcal{H}_F$ in great detail below. (Of course, we have already discussed them at great length - up to isomorphism.)

6. The notation N is meant to suggest some large integer, since this is a typical case in the cond-matt applications. But we will not make specific use of that property.

3.1.2 Spin representations from complex isotropic subspaces

If $v \in \mathbb{R}^{r,s}$ then we can define the contraction operator on the exterior algebra of V by:

$$\mathfrak{i}(v)(v_{i_1}\wedge\cdots\wedge v_{i_k}) := \sum_{s=1}^{k}(-1)^{s-1}Q(v,v_{i_s})v_{i_1}\wedge\cdots\wedge\widehat{v_{i_s}}\wedge\cdots\wedge v_{i_k} \tag{3.8}$$

where the hat superscript $\widehat{v}$ means we omit that factor. Similarly, we can define the wedge operator by

$$\mathfrak{w}(v)(v_{i_1}\wedge\cdots\wedge v_{i_k}) := v\wedge v_{i_1}\wedge\cdots\wedge v_{i_k} \tag{3.9}$$

These operators are easily shown to satisfy the algebra:

$$\begin{aligned}\{\mathfrak{i}(v_1),\mathfrak{i}(v_2)\} &= 0\\ \{\mathfrak{w}(v_1),\mathfrak{w}(v_2)\} &= 0\\ \{\mathfrak{i}(v_1),\mathfrak{w}(v_2)\} &= Q(v_1,v_2)\end{aligned} \tag{3.10}$$

Let us assume $\dim_{\mathbb{R}}\mathcal{M}$ is even so that $N=2n$. The standard finite-dimensional fermionic Fock space construction begins by *choosing* a complex structure I on $\mathcal{M}$. As shown in (**??**) above we automatically have

♣Put remark below on the odd case. ♣

$$\mathcal{M}\otimes\mathbb{C} = V \cong W\oplus\overline{W} \tag{3.11}$$

given by the projection operators $P_{\pm}=\frac{1}{2}(1\pm I\otimes i)$. Here we take

$$W := P_{-}\mathcal{M}\otimes\mathbb{C} = \mathrm{Span}_{\mathbb{C}}\{e-iIe|e\in\mathcal{M}\} \tag{3.12}$$

$$\overline{W}\cong P_{+}\mathcal{M}\otimes\mathbb{C} = \mathrm{Span}_{\mathbb{C}}\{e+iIe|e\in\mathcal{M}\} \tag{3.13}$$

If a vector $P_{-}v=v$ then $Iv=iv$, so W is the $(1,0)$ space of V.

Now, we henceforth assume that I is compatible with the quadratic form Q so

$$Q(Iv_1,Iv_2)=Q(v_1,v_2) \tag{3.14}$$

We then extend Q to be a symmetric $\mathbb{C}$-*linear* form on V. Note that W is a maximal dimension isotropic complex subspace of V. For if $w_1,w_2\in W$ then

$$Q(w_1,w_2)=Q(Iw_1,Iw_2)=Q(\mathrm{i}w_1,\mathrm{i}w_2)=\mathrm{i}^2Q(w_1,w_2)=-Q(w_1,w_2) \tag{3.15}$$

and hence $Q(w_1,w_2)=0$. Note we have crucially used the fact that the extension of Q is $\mathbb{C}$-linear.

Remark: Recall that the space of complex structures compatible with Q is a homogeneous space $\mathrm{CmptCplxStr}(\mathcal{M},Q)$ isomorphic to $O(2n)/U(n)$. (See (1.82) above.) Once we have extended Q in this $\mathbb{C}$-linear fashion we can also understand the space of complex structures as the Grassmannian of maximal dimension complex isotropic subspaces in V. This interpretation is sometimes quite useful, especially in giving a geometrical interpretation of the spin representation in Section **??** below. We denote this Grassmannian by $\mathcal{G}(V,Q)$.

Now, given the decomposition $V \cong W \oplus \bar{W}$ it is fairly evident how to take a "square-root" of

$$\Lambda^* V \cong \Lambda^* W \otimes \Lambda^* \overline{W} \tag{3.16}$$

We could, for example, consider the vector space

$$\Lambda^* W = \oplus_{k=0}^{n} \Lambda^k W \tag{3.17}$$

We can make this vector space into an irreducible Clifford module for $\mathrm{Cliff}(V, Q)$ by similarly taking "half" of the representation (**??**):

1. For $w \in W$ we define $\rho_{F,W}(w) := \mathfrak{w}(w)$
2. For $\bar{w} \in \overline{W}$ we define $\rho_{F,W}(\bar{w}) = \mathfrak{i}(\bar{w})$.
3. Now define ρ_F on V by extending the above equations $\mathbb{C}$-linearly:

$$\rho_{F,W}(w_1 \oplus \overline{w_2}) := \rho_{F,W}(w_1) + \rho_{F,W}(\overline{w_2}). \tag{3.18}$$

Now one checks that indeed the Clifford relations are satisfied and $\rho_{F,W}$ defines the structure of a Clifford module on $\Lambda^* W$. We will often denote this module as

$$\mathcal{H}_{F,W} := \Lambda^* W \tag{3.19}$$

and to lighten the notation we sometimes abbreviate $\rho_{F,W}$ by ρ_F if W is understood or drop it altogether if the context is clear.

In fact $(\rho_{F,W}, \mathcal{H}_{F,W})$ is naturally a graded representation with

$$\mathcal{H}^0_{F,W} \cong \Lambda^{\mathrm{even}} W := \oplus_{k=0(2)} \Lambda^k W \tag{3.20}$$

$$\mathcal{H}^1_{F,W} \cong \Lambda^{\mathrm{odd}} W := \oplus_{k=1(2)} \Lambda^k W \tag{3.21}$$

Now if we think of $\mathrm{Spin}(2n)$ as a group of invertible elements in

$$\mathrm{Spin}(2n) \subset \mathrm{Cliff}(\mathcal{M}, Q) \subset \mathrm{Cliff}(V, Q) \tag{3.22}$$

then through ρ_F the group $\mathrm{Spin}(2n)$ acts on $\mathcal{H}_{F,W}$, but not irreducibly. The operations of contraction and wedging with a vector change the parity of k, but $\mathrm{Spin}(2n)$ involves the action of an even number of vectors so we see that, as a representation of $\mathrm{Spin}(2n)$, $\mathcal{H}_{F,W} \cong S_c$ and this decomposes into:

$$S_c^+ \cong \Lambda^{\mathrm{even}} W \tag{3.23}$$

$$S_c^- \cong \Lambda^{\mathrm{odd}} W \tag{3.24}$$

♣Need to check conventions for chiral vs. antichiral! ♣

In the physical applications it is important to note that we can put an Hermitian structure on V by defining the sesquilinear form

$$h(v_1, v_2) := Q(\overline{v}_1, v_2) \tag{3.25}$$

where $\overline{v}$ is defined from the decomposition $W \oplus \overline{W}$. Note that $V = W \oplus \bar{W}$ is an orthogonal Hilbert space decomposition: W and $\bar{W}$ are separately Hilbert spaces and are orthogonal. To prove this note that orthogonality follows since W and $\bar{W}$ are isotropic with respect to Q. Then since W is maximal isotropic and Q is nondegenerate the sesquilinear form restricted to W must be nondegenerate. Moreover, since $Q > 0$, this defines a Hilbert space inner product on V.

The Fock space $\mathcal{H}_{F,W}$ now inherits a Hilbert space structure since we can define

$$h(w_1 \wedge \cdots \wedge w_k, w_1' \wedge \cdots \wedge w_\ell') := \delta_{k,\ell} \mathrm{det} h(w_i, w_j') \tag{3.26}$$

for $k, \ell > 0$. We extend this to $\Lambda^0 W$ by declaring it orthogonal to the subspaces $\Lambda^k W$ with $k > 0$ and normalizing:

$$h(1,1) := 1 \tag{3.27}$$

Note that $\rho_{F,W}(e_i)$ are self-adjoint operators so that $\rho_{F,W}$ is indeed a $*$ homomorphism, as desired. Moreover, $\rho_{F,W}(e_{ij})$ are skew-adjoint. Therefore with this Hilbert space structure $\mathcal{H}_{F,W}$ is a *unitary* representation of $\mathrm{Spin}(2n)$. Indeed, the operators representing the group $\mathrm{Spin}(2n)$ are of the form $\exp[\frac{1}{2}\omega^{ij}\rho_{F,W}(e_{ij})]$ with real ω^{ij}.

The upshot of this discussion is the theorem:

Theorem: There is a bundle of $\mathbb{Z}_2$-graded finite dimensional Hilbert spaces over

$$\mathrm{CmptCplxStr}(\mathcal{M}, Q) \cong \mathcal{G}(V, Q) \cong O(2n)/U(n) \tag{3.28}$$

whose fiber at a complex maximal isotropic subspace $W \subset V$ is the fermionic Fock space

$$\mathcal{H}_{F,W} \cong \Lambda^{\mathrm{even}} W \oplus \Lambda^{\mathrm{odd}} W \tag{3.29}$$

The homogeneous subspaces in the fibers are naturally unitary chiral representations of the spin group $\mathrm{Spin}(2n)$.

Remarks:

1. The reader might well be wondering: "Why not choose $\Lambda^* \overline{W}$?" Indeed that works too. Exchanging I and $-I$ is equivalent to exchanging W and $\overline{W}$. So, with our construction, $\rho_{F,\overline{W}}$ is simply the module we would get from complex structure $-I$. The space of complex structures $O(2n)/U(n)$ has two connected components. These can be distinguished by the extra data of a choice of orientation of $\mathcal{M}$. For example, we could associate to any basis in which I is of the form

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.30}$$

the orientation $e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}$. Then, thanks to (1.78) above, this association is well-defined.

♣Maybe this discussion should be in Section 1.6.1. ♣

2. More generally, what happens if we change the choice of W? This is discussed in Section 3.1.4 below.

3.1.3 Fermionic Oscillators

Now let us connnect the construction of the spin representation in 3.1.2 to the usual discussion in the physics literature using fermionic harmonic oscillators. In particular, we would like to justify the terminology "fermionic Fock space" for $\mathcal{H}_{F,W}$.

Given a complex structure I on $\mathcal{M}$ compatible with Q we can find an ON basis e_i for $\mathcal{M}$ such that

$$\begin{aligned} Ie_{2j-1} &= -e_{2j} \\ Ie_{2j} &= e_{2j-1}, \qquad\qquad j = 1, \dots, n \end{aligned} \tag{3.31}$$

Put differently, the ordered basis:

$$\{\mathfrak{e}\}_{\alpha=1}^{2n} := \{e_1, e_3, \dots, e_{2n-1}, e_2, e_4, \dots, e_{2n}\}, \tag{3.32}$$

is a basis in which

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.33}$$

Once again: The choice of such a basis is far from unique. Different choices are related by a subgroup of $O(2n)$ isomorphic to $U(n)$, as described in Section §1.6.1. We will explore this in detail below. ♣Opposite sign from I_0 in Section 1.6.1. ♣

Then applying projection operators gives us a basis for W and $\bar{W}$, respectively:

$$\begin{aligned} \bar{a}_j &= P_- e_{2j-1} = \frac{1}{2}(e_{2j-1} + \mathrm{i} e_{2j}) \\ a_j &= P_+ e_{2j-1} = \frac{1}{2}(e_{2j-1} - \mathrm{i} e_{2j}) \end{aligned} \tag{3.34}$$

$$\begin{aligned} e_{2j-1} &= a_j + \bar{a}_j \\ e_{2j} &= \mathrm{i}(a_j - \bar{a}_j) \end{aligned} \tag{3.35}$$

We easily compute the fermionic CCR's in this basis to get the usual fermionic harmonic oscillator algebra:

$$\begin{aligned} \{a_j, a_k\} &= \{\bar{a}_j, \bar{a}_k\} = 0 \\ \{a_j, \bar{a}_k\} &= \delta_{j,k} \end{aligned} \tag{3.36}$$

The space $\Lambda^* W$ has a natural basis $1, \bar{a}_j, \dots$ where the general basis element is given by $\bar{a}_{j_1} \cdots \bar{a}_{j_\ell}$ for $j_1 < \cdots < j_\ell$. In particular, note that [23]

$$\rho_{F,W}(a_i) \cdot 1 = 0 \tag{3.37}$$

where $1 \in \Lambda^0 W \cong \mathbb{C}$. We build up the other basis vectors by acting with $\rho_{F,W}(\bar{a}_j)$ on 1.

The transcription to physics notation should now be clear. The *vacuum line* is the complex vector space $\Lambda^0 W \cong \mathbb{C}$. Physicists usually choose an element of that line and

[23] Note that, if we drop the $\rho_{F,W}$ then the equation would be wrong!

denote it $|0\rangle$. Moreover, they drop the heavy notation $\rho_{F,W}$, so, in an irreducible module we have just

$$a_i|0\rangle = 0. \tag{3.38}$$

The state $|0\rangle$ is variously called the *Dirac vacuum*, the *Fermi sea*, or the *Clifford vacuum*. However, irrespective of whose name you wish to name the state after, it must be stressed that these equations only determine a line, not an actual vector, and, when considering families of representations this can be important. Indeed, some families of quantum field theories are inconsistent because there is no way to assign an unambiguous vacuum vector to every element in the family which varies with sufficient regularity.

In our case we have a canonical choice $1 \in \Lambda^0 W \leftrightarrow |0\rangle \in \mathcal{H}_F$, where $\mathcal{H}_F$ is our notation for the fermionic Fock space. Then, $\Lambda^k W$ is the same as the subspace spanned by $\bar{a}_{j_1} \cdots \bar{a}_{j_k}|0\rangle$.

In physical interpretations $\Lambda^k W$ is a subspace of a Fock space describing states with k-particle excitations above the vacuum $|0\rangle$. It is very convenient to introduce the *fermion number operator*

$$\mathcal{F} := \sum_{i=1}^{n} \bar{a}_i a_i = \frac{n}{2} - \frac{\mathrm{i}}{4} \sum_{\alpha,\beta} e_\alpha I_{\alpha\beta} e_\beta \tag{3.39}$$

so that $\Lambda^k W$ is the subspace of "fermion number k."

The operator $(-1)^{\mathcal{F}}$ commutes with the spin group and decomposes the Fock space into even and odd subspaces. That is, the eigenspaces $(-1)^{\mathcal{F}} = \pm 1$ are isomorphic to the chiral spin representations.

Finally, consider the Hilbert space structure. With respect to the Hilbert space structure (3.26) we find that indeed

$$\rho_{F,W}(\bar{a}_i) = \rho_{F,w}(a_i)^\dagger, \tag{3.40}$$

so in physics we would just write $\bar{a}_i \to a_i^\dagger$. The normalization condition (3.27) is written in physics notation as

$$\langle 0|0\rangle = 1. \tag{3.41}$$

3.1.4 Bogoliubov transformations

We now return to the fact that we had to choose a complex structure to construct an irreducible spin representation in Section §3.1.2. However, as we saw in equation (1.82) above there is a whole family of complex structures I which we can use to effect the construction. On the other hand, the irreducible spin representations $S_c^\pm$ are unique up to isomorphism. Therefore there must be an isomorphism between the constructions: this isomorphism is known in physics as a "Bogoliubov transformation." It can have nontrivial physical consequences.

To a mathematician, there is just one isomorphism class of chiral spin representation S_c^+ or S_c^- (distinguished, invariantly, by the volume element). However, in physics, the fermionic oscillators represent physical degrees of freedom: Nature chooses a vacuum, and if, as a function of some control parameters a new vacuum becomes preferred when those parameters are varied then the Bogoliubov transformation has very important physical implications. A good example of this is superconductivity.

Returning to mathematics, suppose we choose one complex structure I_1 with a compatible basis $\{e_\alpha\}$ satisfying (3.31) for I_1. Next, we consider a different complex structure I_2 with corresponding basis $\{f_\alpha\}$ satisfying (3.31) for I_2.

♣Explain somewhere what a polarization is. ♣

♣We need to explain that although Spin$(2n)$ acts on $\mathcal{H}_{F,W}$ for fixed W we can also make it (through the map to $O(2n)$) act on the space of complex structures. So we could ask how the actions on different complex structures is defined. Thus, we could also ask about whether the bundle $\mathcal{H}_F \to \mathcal{G}(V,Q)$ can be interpreted as an equivariant bundle. ♣

With the different basis $\{f_\alpha\}$ we can form fermionic oscillators according to (3.34):

$$\begin{aligned}\bar{b}_j &= \frac{1}{2}(f_{2j-1} + \mathrm{i} f_{2j})\\ b_j &= \frac{1}{2}(f_{2j-1} - \mathrm{i} f_{2j})\end{aligned} \tag{3.42}$$

Now we must have a transformation of the form

$$\begin{aligned}\bar{b}_i &= A_{ji}\bar{a}_j + C_{ji}a_j\\ b_i &= B_{ji}\bar{a}_j + D_{ji}a_j \qquad\qquad 1 \le i,j \le n\end{aligned} \tag{3.43}$$

Observation: For a general transformation of the form (3.43), with complex $n\times n$ matrices A,B,C,D the fermionic CCR's are preserved iff the matrix

$$g = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \tag{3.44}$$

satisfies:

$$g^{tr}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} g = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \tag{3.45}$$

That is

$$\begin{aligned}A^{tr}D + C^{tr}B &= 1\\ A^{tr}C &= -(A^{tr}C)^{tr}\\ D^{tr}B &= -(D^{tr}B)^{tr}\end{aligned} \tag{3.46}$$

Proof: The proof is a straightforward computation. $\diamondsuit$

The proposition characterizes the general matrices which preserve the CCR's. We recognize (3.45) as the definition of the complex orthogonal group for the quadratic form

$$q = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \tag{3.47}$$

So, Bogoliubov transformations can be effected by complex orthogonal transformations

$$O(q;\mathbb{C}) := \{g \in GL(2n;\mathbb{C}) | g^{tr} q g = q\} \tag{3.48}$$

The form q has signature (n,n) over the real numbers but is, of course, equivalent to the standard Euclidean form over the complex numbers.

3.2 Free fermion dynamics and their symmetries

3.2.1 FDFS with symmetry

Finally, let us define formally what it means for a FDFS to have a symmetry.

Definition: Let (G, ϕ) be a $\mathbb{Z}_2$-graded group with $\phi : G \to \mathbb{Z}_2$. We will say that (G, ϕ) *acts as a group of symmetries of the FDFS* if

1. There is a homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}_F)$ of $\mathbb{Z}_2$-graded groups. That is, $\mathcal{H}_F$ is a ϕ-rep of G. (See Section §1.7.1.)

2. There is a compatible automorphism α of the $*$-algebra $\mathcal{A}$ so that $\mathcal{A}$ is a ϕ-representation of G. That is, $\alpha(g)$ is $\mathbb{C}$-linear or anti-linear according to ϕ and ρ and α are compatible in the sense that:
$$\rho(g)\rho_F(a)\rho(g)^{-1} = \rho_F(\alpha(g) \cdot a) \tag{3.49}$$

♣Again a for generic element of $\mathcal{A}$ is bad notation. ♣

3. The automorphism preserves the real subspace $\mathcal{M} \subset \mathcal{A}$, and hence we have a group homomorphism: $\alpha : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{M}, Q) = O(\mathcal{M}, Q) \cong O(N)$.

Remarks:

1. Assuming ρ_F is faithful and surjective (as happens for example if N is even and we choose an irreducible Clifford module for $\mathcal{H}_F$) the map $a \mapsto a'$ defined by
$$\rho(g)\rho_F(a)\rho(g)^{-1} = \rho_F(a') \tag{3.50}$$
defines the automorphism of $\mathcal{A}$. When $\mathcal{A}$ is a central simple algebra it must be inner. The condition (3) above puts a further restriction on what elements we can conjugate by.

2. We put condition (3) because we want the symmetry to preserve the notion of a fermionic field. The mode space $\mathcal{M}$ is the space of real fermionic fields. It should then preserve Q because we want it to preserve the canonical commutation relations. In terms of operators on $\mathcal{H}_F$:
$$\rho(g)\rho_F(e_j)\rho(g)^{-1} = \sum_m S_{mj}\rho_F(e_m) \tag{3.51}$$
where $g \mapsto S(g) \in O(N)$ is a representation of G by orthogonal matrices.

3. When constructing examples it is natural to start with a homomorphism $\alpha : G \to O(N)$. We then automatically have an extension to an automorphism of $\mathrm{Cliff}(\mathcal{M}, Q)$. There is no *a priori* extension to an automorphism of $\mathcal{A}$. The data of the ϕ-representation determines that extension because $a \mapsto \rho_F(a)$ is $\mathbb{C}$-linear. It follows that $\rho(g)$ is conjugate linear iff $\alpha(g)$ is conjugate linear. This tells us how to extend α to $\mathrm{Aut}_{\mathbb{R}}(\mathcal{A})$.

Examples

1. By its very construction, the group $G = \mathrm{Pin}^+(N)$ with $\phi = 1$ is a symmetry group of the FDFS generated by $(\mathcal{M}, Q)$ for $\mathcal{M}$ of dimension N. We can simply take $\rho = \rho_F$. This forces us to take $\alpha = \mathrm{Ad}$. [24]

2. What about $G = \mathrm{Pin}^-(N)$? In fact we can make $G = \mathrm{Pin}^c(N)$ (which contains both $\mathrm{Pin}^\pm(N)$ as subgroups) act. We think of generators of $\mathrm{Pin}^c(N)$ as ζe_i where $|\zeta| = 1$ is in $U(1)$. Then $\rho(\zeta e_i) = \zeta \rho_F(e_i)$ and $\alpha(\zeta e_i) = \mathrm{Ad}(e_i)$. Again we take $\phi = 1$ in this example.

3. Now we can ask what $\mathbb{Z}_2$-gradings we can give, say, $G = \mathrm{Pin}^+(N)$. Since we take ϕ to be continuous $\phi = 1$ on the connected component of the identity. Then if we take $\phi(v) = -1$ for some norm-one vector then if v' is any other norm-one vector $vv' \in \mathrm{Spin}(N)$ and hence $\phi(vv') = 1$ so $\phi(v') = -1$. Therefore the only nontrivial $\mathbb{Z}_2$-grading is given by the determinant representation described in (**??**) above. If we use this then in general there is no consistent action of (G, ϕ) on the N-dimensional FDFS.

♣Maybe when $C\ell_N$ has real reps it is ok? ♣

4. To give a very simple example with $\phi \neq 1$ consider $N = 2$, hence a single oscillator $a, \bar{a}$ and let $G = \mathbb{Z}_4 = \langle T | T^4 = 1 \rangle$. Then, in the explicit representation of **??** take

$$\begin{aligned} \rho(T)|+\rangle &= -|-\rangle \\ \rho(T)|-\rangle &= |+\rangle \end{aligned} \tag{3.52}$$

and extend by linearity for $\phi(T) = +1$, and by anti-linearity for $\phi(T) = -1$, to define $\rho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}_F)$. In either case $\alpha(T) \cdot e_1 = -e_1$, but a small computation shows that

$$\alpha(T) \cdot e_2 = \begin{cases} e_2 & \phi(T) = +1 \\ -e_2 & \phi(T) = -1 \end{cases} \tag{3.53}$$

Note that

$$\begin{aligned} \alpha(T) \cdot a &= -\bar{a} \\ \alpha(T) \cdot \bar{a} &= -a \end{aligned} \tag{3.54}$$

in *both cases* $\phi(T) = \pm 1$.

5. Inside the real Clifford algebra generated by e_i is a *group* $\mathcal{E}_N$ generated by e_i. This group is discrete, has 2^{N+1} elements and is a nonabelian extension of $\mathbb{Z}_2^N$ with cocycle determined from $e_i e_j e_i^{-1} e_j^{-1} = -1$ for $i \neq j$. $\mathcal{E}_N$ is known as an *extraspecial group*. Suppose T_i, $i = 1, \ldots, k$ with $2k \leq N$ generate an extraspecial group $\mathcal{E}_k$ of order 2^{k+1}. Thus, $T_i^2 = 1$ and $T_i T_j T_i^{-1} T_j^{-1} = -1$ for $i \neq j$. Then there are many $\mathbb{Z}_2$ gradings ϕ of $\mathcal{E}_k$ because we can choose the sign of $\phi(T_i)$ independently for each generator. For each such choice $(\mathcal{E}_k, \phi)$ acts as a symmetry group of the N-dimensional FDFS.

[24]Note that it is Ad and not $\widetilde{\mathrm{Ad}}$. This leads to some important signs below.

Using the basis for the explicit representation of **??** we can take $\rho(T_i) = \rho_F(e_{2i-1})$. Since the latter matrix is real the operators $\rho(T_i)$ can be consistently anti-linearly extended in the basis of **??**. A small computation shows that

$$\alpha(T_i) \cdot e_{2j} = \begin{cases} -e_{2j} & \phi(T_i) = +1 \\ e_{2j} & \phi(T_i) = -1 \end{cases} \tag{3.55}$$

but

$$\alpha(T_i) \cdot \bar{a}_j = \begin{cases} a_i & j = i \\ -a_j & j \neq i \end{cases} \tag{3.56}$$

$$\alpha(T_i) \cdot a_j = \begin{cases} \bar{a}_i & j = i \\ -\bar{a}_j & j \neq i \end{cases} \tag{3.57}$$

independent of the choice of ϕ.

♣Now comment on possibilities for ϕ and what the Dyson classes would be. ♣

3.2.2 Free fermion dynamics

♣Put general comments on dynamics here? ♣

In general, the Hamiltonian is a self-adjoint element of the operator $*$-algebra and thus has the form (3.4). We will distinguish a $*$-invariant element $h \in \mathcal{A}$ from the Fock space Hamiltonian $H := \rho_F(h)$.

Usually, for reasons of rotational invariance, physicists restrict attention to Hamiltonians in the even part of the Clifford algebra, so then

$$h = h_0 + \sum_{k=0(2)} h_{i_1 \dots i_k} e_{i_1 \dots i_k} \tag{3.58}$$

with $h_0 \in \mathbb{R}$ and $h^*_{i_1 \dots i_k} = (-1)^{k/2} h_{i_1 \dots i_k}$. These elements generate a one-parameter group of automorphisms $\mathrm{Ad}(u(t))$ on $\mathcal{A}$ where $u(t) = e^{-\mathrm{i}th}$. Related to this is a one-parameter group of unitary operators

$$U(t) = \rho_F(u(t)) = e^{-\mathrm{i}tH} \tag{3.59}$$

on $\mathcal{H}_F$ representing time evolution in the Schrödinger picture.

In the Heisenberg picture $\mathrm{Ad}(u(t))$ induces a one-parameter group of automorphisms of the algebra of operators and in particular the fermions themselves evolve according to

$$u(t)^{-1} e_i u(t) = e_i + \sqrt{-1} t \sum_I h_I [e_I, e_i] + \mathcal{O}(t^2) \tag{3.60}$$

where we have denoted a multi-index $I = \{i_1 < \cdots < i_k\}$. Terms with $k > 2$ will clearly not preserve the subspace $\mathcal{M}$ in $\mathcal{A}$.

By definition, a *free fermion dynamics* is generated by a Hamiltonian h such that $\mathrm{Ad}(u(t))$ preserves the subspace $\mathcal{M}$. (Note well, when expressed in terms of harmonic oscillators relative to some complex structure it might or might not commute with $\mathcal{F}$.) The most general Hamiltonian defining free fermion dynamics is a self-adjoint element of

$\mathcal{A} = \text{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C}$ which can be written with at most two generators. Therefore, the general free fermion Hamiltonian is

$$h = h_0 + \frac{\sqrt{-1}}{4} \sum_{i,j} A_{jk} e_j e_k \tag{3.61}$$

where $A_{ij} = -A_{ji}$ is a *real antisymmetric matrix.*

Remarks

1. Note well that A_{ij} is an element of the real Lie algebra $so(N)$ and indeed

$$\frac{1}{4} \sum_{j,k} A_{jk} e_j e_k \tag{3.62}$$

   is the corresponding element of $\text{spin}(N) \cong so(N)$.

2. As we remarked, there are two Hilbert spaces associated to the fermionic system. In the Fock space $\mathcal{H}_F$ we have Hamiltonian

$$H = h_0 + \frac{\sqrt{-1}}{4} \sum_{i,j} A_{jk} \rho_F(e_j e_k) \tag{3.63}$$

   and, up to a trivial evolution by $e^{-ih_0 t}$, the free fermion dynamics is the action of a one-parameter subgroup $U(t)$ of $\text{Spin}(2N)$ acting on the spin representation, in the Schrödinger picture. In the Heisenberg picture the corresponding dynamical evolution preserves the real subspace $\mathcal{M} \subset \mathcal{A}$ is given by the real vector representation: $\widetilde{\text{Ad}}(u(t))$.

3. Upon choosing a complex structure we have a second Hilbert space, the Dirac-Nambu Hilbert space $\mathcal{H}_{DN} := V \cong W \oplus \bar{W}$ and, (only in the case of free fermion dynamics) $U(t)$ induces an action on V. This is simply $\widetilde{\text{Ad}}(u(t))$ on $\mathcal{M}$ extended $\mathbb{C}$-linearly to $V = \mathcal{M} \otimes \mathbb{C}$. The "Dirac-Nambu Hamiltonian" is therefore just $\rho_{DN}(h) := \text{Ad}(h)$ acting on V, thought of as a subspace of $\text{Cliff}(V, Q)$.

4. Any real antisymmetric matrix can be skew-diagonalized by an orthogonal transformation. That is, given A_{ij} there is an orthogonal transformation R so that

$$RAR^{tr} = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \tag{3.64}$$

   The Bogoliubov transformation corresponding to R can be implemented unitarily and hence if h_0 is zero then the spectrum of $\widehat{H}$ must be symmetric about zero. Therefore this is a system in which it is possible to have symmetries with $\chi \neq 0$. In this basis we simply have (with $h_0 = 0$)

$$h = \sum \lambda_j \bar{a}_j a_j - \frac{1}{2}(\lambda_1 + \cdots + \lambda_n) \tag{3.65}$$

   The spectrum of the Hamiltonian on $\mathcal{H}_{DN}$ is $\{\pm\lambda_j\}$ and on $\mathcal{H}_F$ is $\{\frac{1}{2}\sum_i \epsilon_i \lambda_i\}$ where $\epsilon_i \in \{\pm 1\}$.

Exercise

Compute the time evolution on $\mathcal{M}$ of the one-parameter subgroup generated by the self-adjoint operator e_i. [25]

3.2.3 Symmetries of free fermion systems

Now suppose we have a $\mathbb{Z}_2$-graded group (G, ϕ) acting as a group of symmetries of a finite dimensional fermion system. We therefore have the following data: $(\mathcal{M}, Q)$ together with a $*$-representation of $\mathcal{A} = \mathrm{Cliff}(V; Q)$ on the Hilbert space $\mathcal{H}_F$ together with the homomorphisms α and ρ satisfying (3.49).

Suppose furthermore that we have a free fermionic system, hence a Hamiltonian of the form (3.61).

Definition: We say that G is acting as a group of symmetries of the dynamics of the free fermionic system if

$$\rho(g)U(s)\rho(g) = U(s)^{\tau(g)} \tag{3.66}$$

for *some* homomorphism $\tau : G \to \mathbb{Z}_2$. Here $U(s) = \exp[-isH/\hbar]$ is the one-parameter time evolution operator. If (3.66) holds then we declare g with $\tau(g) = -1$ to be time-reversing symmetries.

1. The above definition looks like a repeat of our previous definition of a symmetry of the dynamics from Section §1.8. The data $(\mathcal{M}, Q, \mathcal{H}_F, G, \phi, \alpha, \rho, H)$ determine $\rho(g)H\rho(g)^{-1}$. With our logical setup here, a symmetry of the fermionic system is a symmetry of the dynamics if there is *some* homomorphism $\chi : G \to \mathbb{Z}_2$ so that

$$\rho(g)H\rho(g)^{-1} = \chi(g)H \tag{3.67}$$

   Then because general quantum mechanics requires $\phi\tau\chi = 1$, we will *declare* g to be time-orientation preserving or reversing according to $\tau(g) := \phi(g)\chi(g)$. This logic is reversed from our standard approach where we consider ϕ determined by an *a priori* given homomorphism $G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ together with an *a priori* given homomorphism τ determined by an *a priori* action on spacetime.

2. There will be physical situations, e.g. a single electron moving in a crystal where there is an *a priori* notion of what time-reversing symmetries should be and how they should act on fermion fields.

3. Let us see what the above definition implies for the transformation of the oscillators under $\widetilde{\mathrm{Ad}}$. Choose an ON basis for $(\mathcal{M}, Q)$ satisfying (3.3). Then, in terms of operators on $\mathcal{H}_F$:

$$\rho(g)\rho_F(e_j)\rho(g)^{-1} = \sum_m S_{mj}\rho_F(e_m) \tag{3.68}$$

♣Maybe S_{mj} should be $\alpha(g)_{mj}$. ♣

[25] *Answer*: $e_j(t) = \cos(2t)e_j + \mathrm{i}\sin(2t)e_ie_j$ for $j \neq i$ and $e_i(t) = e_i$.

Or, equivalently:

$$\alpha(g) \cdot e_j = \sum_m S_{mj} e_m \tag{3.69}$$

so

$$\begin{aligned}\rho(g) H \rho(g)^{-1} &= h_0 + \phi(g) \frac{\mathrm{i}}{4} \sum_{m,n} (SAS^{tr})_{mn} \rho_F(e_m e_n) \\ &= \chi(g) H\end{aligned} \tag{3.70}$$

This shows that

1. If $\chi(g) = -1$ for any $g \in G$ then $h_0 = 0$.
2. The matrix A must satisfy

$$\boxed{S(g) A S(g)^{tr} = \tau(g) A} \tag{3.71}$$

for all $g \in G$, where $\tau(g)$ is either prescribed, or deduced from $\tau = \phi \cdot \chi$, depending on what logical viewpoint we are taking.

The condition (3.71) can be expressed more invariantly: Given $\alpha : G \to O(\mathcal{M}, Q)$ there is an induced action $\mathrm{Ad}_{\alpha(g)}$ on $o(\mathcal{M}, Q)$, and we are requiring that

$$\mathrm{Ad}_{\alpha(g)} A = \tau(g) A. \tag{3.72}$$

3.2.4 The free fermion Dyson problem and the Altland-Zirnbauer classification

There is a natural analog of the Dyson problem suggested by the symmetries of free fermionic systems:

Given a finite dimensional fermionic system $(\mathcal{M}, Q, \mathcal{H}_F, \rho_F)$ and a $\mathbb{Z}_2$-graded group (G, ϕ) acting as a symmetry on the FDFS via (α, ρ), what is the ensemble of free Hamiltonians for the FDFS such that (G, ϕ) is a symmetry of the dynamics?

Note well! We have changed the Dyson problem for the ϕ-rep $\mathcal{H}_F$ of G in a crucial way by restricting the ensemble to free fermion Hamiltonians.

Our analysis above which led to (3.71) above shows that the answer, at one level, is immediate from (3.71): We have the subspace in $o(Q; \mathbb{R})$ satisfying (3.71). Somewhat surprisingly, this answer depends only on α and τ as is evident from (3.72). For a given τ there can be more than one choice for ϕ and χ.

However, the answer can be organized in a very nice way as noticed by Altland and Zirnbauer [4]: Such free fermion ensembles can be identified with the tangent space at the origin of classical Cartan symmetric spaces. This result was proved more formally in a subsequent paper of Heinzner, Huckleberry, and Zirnbauer [31]. We explain the main idea:

Let us consider two subspaces of $o(2n; \mathbb{R})$:

$$\mathfrak{k} := \{A | Ad_{\alpha(g)}(A) = A\} \tag{3.73}$$

$$\mathfrak{p} := \{A | Ad_{\alpha(g)}(A) = \tau(g) A\} \tag{3.74}$$

$\mathfrak{p}$ is of course the ensemble we want to understand. If $\tau = 1$ it is identical to $\mathfrak{k}$ but in general, when $\tau \neq 1$ it is *not* a Lie subalgebra of $o(2n;\mathbb{R})$ because the Lie bracket of two elements in $\mathfrak{p}$ is in $\mathfrak{k}$. This motivates us to define a Lie algebra structure on

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.75}$$

by

$$[k_1 \oplus p_1, k_2 \oplus p_2] := ([k_1, k_2] + [p_1, p_2]) \oplus ([p_1, k_2] + [k_1, p_2]) \tag{3.76}$$

One can check this satisfies the Jacobi relation.

Note that we have an automorphism of the Lie algebra which is $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$, so this is a Cartan decomposition.

Both $\mathfrak{k}$ and $\mathfrak{g}$ are classical Lie algebras: This means that they are Lie subalgebras of matrix Lie algebras over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ preserving a bilinear or sesquilinear form.

To prove this for $\mathfrak{k}$: Note that we have a representation of G on $o(\mathcal{M};Q) \cong o(2n;\mathbb{R})$. If G is compact this representation must decompose into irreducible representations. The group algebra is therefore a direct sum of algebras of the form $n\mathbb{K}(m)$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. By the Weyl duality theorem (1.152),(1.153) the commutant is $m\mathbb{K}(n)$. Since $\mathfrak{k}$ is, by definition, the commutant, when restricted to each irreducible representation $\exp[\mathfrak{k}]$ must generate a matrix algebra over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Therefore, $\mathfrak{k}$ is a classical Lie algebra.

A similar argument works to show that $\mathfrak{g}$ is a classical Lie algebra. There is a Lie algebra homomorphism

$$\mathfrak{g} \to o(\mathcal{M}, Q) \oplus o(\mathcal{M}, Q) \tag{3.77}$$

given by

$$k \oplus p \to (k + p) \oplus (k - p) \tag{3.78}$$

Now, we can characterize $\mathfrak{g}$ as the commutant of a representation of G on $\mathcal{M} \oplus \mathcal{M}$ given by

♣Should $\alpha(g)$ be denoted $S(g)$? ♣

$$g \mapsto \begin{pmatrix} \alpha(g) & 0 \\ 0 & \alpha(g) \end{pmatrix} \qquad \tau(g) = 1 \tag{3.79}$$

$$g \mapsto \begin{pmatrix} 0 & \alpha(g) \\ \alpha(g) & 0 \end{pmatrix} \qquad \tau(g) = -1 \tag{3.80}$$

We embed $\mathfrak{g}$ into $o(\mathcal{M}) \oplus o(\mathcal{M})$. The matrices in the commutant of the form $x \otimes 1_2$ is isomorphic to $\mathfrak{k}$ and the matrices in the commutant of the image of G which are of the form $x \otimes \sigma^3$ is isomorphic to $\mathfrak{p}$. Hence $\mathfrak{k}$ and $\mathfrak{g}$ are both classical real Lie algebras.

♣explain why we don't need to worry about other kinds of matrices in the commutant. ♣

Next, note that the Killing form of $o(\mathcal{M};Q)$ restricts to a Killing form on $\mathfrak{k}$ and on $\mathfrak{g}$. It is therefore negative definite. Hence the real Lie algebras $\mathfrak{k}$ and $\mathfrak{g}$ are of compact type.

This proves the theorem of [31]:

Theorem: The ensemble $\mathfrak{p}$ of free fermion Hamiltonians in $\mathrm{Cliff}(\mathcal{M}, Q) \otimes \mathbb{C}$ compatible with (α, τ) is the tangent space at the identity of a classical compact symmetric space G/K.

Remarks

1. See Appendix A for background material on symmetric spaces.

2. Cartan classified the compact symmetric spaces. They are of the form G/K where G and K are Lie groups. There are some exceptional cases and then there are several infinite series analogous to the infinite series A, B, C, D of simple Lie algebras. These can be naturally organized into a series of 10 distinct classical symmetric spaces. Thus, the Altland-Zirnbauer argument provides a 10-fold classification of ensembles of free fermionic Hamiltonians. This gives yet another 10-fold way! We will relate it to the 10 Morita equivalence classes of Clifford algebras (and thereby implicitly to the 10 real super-division algebras) below. That relation will involve K-theory.

3. It is possible to give many examples of this classification scheme. See [43] for some of them. Using the description of the 10 classes given in (A.7) - (A.17) one can give a description of the 10 AZ classes along the following lines. Recalling (**??**) we can, with a suitable choice of complex structure as basepoint write the free fermion hamiltonian as

$$h = \sum_{i,j} W_{ij}\bar{a}_i a_j + \frac{1}{2}\sum_{i,j}\left(Z_{ij}\bar{a}_i\bar{a}_j + \bar{Z}_{ij}a_j a_i\right) \tag{3.81}$$

where W_{ij} is hermitian and Z_{ij} is a complex antisymmetric matrix. Then the 10 cases correspond to various restrictions on W_{ij} and Z_{ij}. See Table 1 of [55]. We will give a uniform construction of 10 examples using some mathematics related to Bott periodicity below.

♣Should probably reproduce that table here and explain the entries in detail. Actually, his table is confusing because in some cases Z_{ij} is not antisymmetric, so they are not all FF Hamiltonians in an oscillator basis. Clear this up. ♣

3.2.5 Analog for bosons

<u>Free Hamiltonians</u>

The quantization of a bosonic system is very similar. The mode space $\mathcal{M}$ is now a real symplectic vector space.

Up to a constant the general free boson Hamiltonian is an element of $\mathcal{A}$ of the form

$$h = h^{ij}e_i e_j \tag{3.82}$$

This should be $*$ invariant and hence h^{ij} must be a real symmetric matrix. Now, notice that from (**??**) that we can therefore identify hJ with an element of the symplectic Lie algebra. Thus,

The space of free boson Hamiltonians is naturally identified with $sp(2n;\mathbb{R})$.

<u>Analog of the AZ classification of free bosonic Hamiltonians</u>

Now we define a symmetry of the bosonic dynamics to be a group G with $\rho : G \rightarrow \mathrm{End}(\mathcal{H}_F)$ such that *****

An argument completely analogous to that for (3.71) applies. The symmetry operators act by

$$\rho(g)\rho_F(v_j)\rho(g)^{-1} = \sum_m S_{mj}(g)\rho_F(v_m) \tag{3.83}$$

where now $S(g) \in Sp(2n;\mathbb{R})$. The result is that the symmetry condition is just that $A = hJ \in sp(2n;\mathbb{R})$ is in the space

$$\mathfrak{p} := \{A \in sp(2n;\mathbb{R}) | S(g)AS(g)^{-1} = \chi(g)A\} \tag{3.84}$$

For bosons the Hamiltonian will have an infinite spectrum. It is natural to assume that the Hamiltonian is bounded below, in which case $\chi = 1$. From a purely mathematical viewpoint one could certainly consider quadratic forms with Hamiltonian unbounded from above or below. Consider, e.g., the upside down harmonic oscillator. Thus, one could still contemplate systems with $\chi \neq 1$, although they are a bit unphysical.

3.3 Symmetric Spaces and Classifying Spaces

3.3.1 The Bott song and the 10 classical Cartan symmetric spaces

Now we will give an elegant description of how the 10 classical symmetric spaces arise directly from the representations of Clifford algebras. This follows a treatment by Milnor [38]. Then, thanks to a paper of Atiyah and Singer [11] we get a connection to the classifying spaces of K-theory. Milnor's construction was discussed in the context of topological insulators by Stone et. al. in [48].

We begin by considering the complex Clifford algebra $\mathbb{C}\ell_{2d}$. In this section we will be considering the algebras and representations as *ungraded*. [26] Thus, there is a unique irreducible representation with carrier space $\mathcal{S}_c = \mathbb{C}^{2^d}$. Give it the standard Hermitian structure. We can then take the representation of the generators $J_i = \rho(e_i)$ so that $J_i^2 = -1$, $J_i^\dagger = -J_i$ and hence J_i are both in the unitary group and its Lie algebra.

Then we define a sequence of groups

$$G_0 \supset G_1 \supset G_2 \supset \cdots \tag{3.85}$$

We take $G_0 = U(2r)$ where we have denoted $2^d = 2r$ and we define

$$G_k = \{g \in G_0 | gJ_s = J_s g \qquad s = 1, \ldots, k\} \tag{3.86}$$

We claim that $G_1 \cong U(r) \times U(r)$.

One very explicit way to see this is to note that we could represent

$$\rho(e_1) = J_1 = i \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix} \tag{3.87}$$

and hence the matrices which commute with it are of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{3.88}$$

by a unitary transformation we can bring this to the form

$$\begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} \tag{3.89}$$

hence such matrices are unitary iff $(A \pm B)$ are unitary. So the group of such unitary matrices is isomorphic to $U(r) \times U(r)$, as claimed.

---

[26] It would be interesting to give a parallel discussion from the $\mathbb{Z}_2$-graded viewpoint.

While G_1 is isomorphic to $U(r) \times U(r)$ it is embedded in $U(2r)$ slightly nontrivially. Now, at the next step, we can take

$$\rho(e_2) = J_2 = \begin{pmatrix} 0 & -1_r \\ 1_r & 0 \end{pmatrix} \tag{3.90}$$

Then clearly G_2 is the subgroup of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{3.91}$$

Now, further Clifford generators will act within the 11 and 22 blocks, so the process repeats: This is the mod-two periodicity.

The above line of reasoning would be a little tedious in the case of the real Clifford algebras, so we rederive the above with a new strategy (that will generalize nicely):

1. Use the representation theory of Clifford algebras to characterize the Wedderburn type of the image $\rho(\mathbb{C}\ell_k)$.

2. Then use Weyl's commutant theory to characterize the Wedderburn type of $Z(\rho(\mathbb{C}\ell_k)) = \rho(\mathbb{C}\ell_k)'$.

3. Then read off the group $G_k = \mathrm{Aut}(\mathcal{S}) \cap \rho(\mathbb{C}\ell_k)'$.

For example, let us rederive G_1 this way: As an ungraded algebra $\mathbb{C}\ell_1$ has two irreps and so we can write $\mathcal{S}_c$ as a sum of ungraded irreps of $\mathbb{C}\ell_1$: $\mathcal{S} = r_+ N_1^+ + r_- N_1^-$. The projector to the isotypical components is $P_\pm = \frac{1}{2}(1 + \rho(e_1))$ but the trace of $\rho(e_1)$ is zero, so

$$\mathcal{S}_c \cong rN_1^+ \oplus rN_1^- \tag{3.92}$$

and therefore the algebra $\rho(\mathbb{C}\ell_1)$ has Wedderburn type

$$r\mathbb{C} \oplus r\mathbb{C} \tag{3.93}$$

so the commutant must have Wedderburn type

$$\mathbb{C}(r) \oplus \mathbb{C}(r) \tag{3.94}$$

and the intersection with $\mathrm{Aut}(\mathcal{S}_c)$, which gives precisely G_1, must be

$$G_1 \cong U(r) \times U(r) \tag{3.95}$$

Next for G_2, $\mathbb{C}\ell_2 \cong M_2(\mathbb{C})$, so $\rho(\mathbb{C}\ell_2)$ has Wedderburn type $r\mathbb{C}(2)$ and hence the commutant is $2\mathbb{C}(r)$ so the group G_2 is isomorphic to $U(r)$. As r is a power of 2 we clearly have periodicity so that our sequence of groups is isomorphic to

$$U(2r) \supset U(r) \times U(r) \supset U(r) \supset \cdots \tag{3.96}$$

The successive quotients give the two kinds of symmetric spaces $U(2r)/(U(r) \times U(r))$ and $(U(r) \times U(r))/U(r)$.

♣Don't get Grassmannians $Gr_{n,m}$ with $n \neq m$. ♣

Now let us move on to the real Clifford algebra $C\ell_{-8d}$. We choose a real graded irreducible representation, $\mathbb{R}^{2N}$, with $2N = 2^{4d}$. It is convenient to define an integer r by

$2N = 16r$. Again, we will regard the Clifford algebras as ungraded and the representation $\mathcal{S} \cong \mathbb{R}^{2N}$. Denote the representations of the generators $J_i = \rho(e_i)$, so of course

$$J_s J_t + J_t J_s = -2\delta_{s,t}. \tag{3.97}$$

We can give $\mathcal{S}$ a Euclidean metric so that the representation of $\mathrm{Pin}^-(8d)$ is orthogonal. Therefore, $J_i^\dagger = -J_i$, so $J_i^{tr} = -J_i$, and hence $J_i \in o(2N)$. However, since $J_i^2 = -1$ we have $J_i^{tr} = J_i^{-1}$ and hence we also have $J_i \in O(2N)$.

Now we define a sequence of groups

$$O(2N) := G_0 \supset G_1 \supset G_2 \supset \cdots \tag{3.98}$$

These are defined for $k > 0$ again by taking the commutant with $C\ell_{-k}$:

$$G_k = \{g \in G_0 | g J_s = J_s g \qquad s = 1, \ldots, k\} = \mathrm{Aut}(\mathcal{S}) \cap Z(\rho(C\ell_{-k})). \tag{3.99}$$

Now, we claim that the series of groups is isomorphic to

$$\begin{aligned} O(16r) \supset U(8r) \supset Sp(4r) \supset Sp(2r) \times Sp(2r) \supset Sp(2r) \supset \\ \supset U(2r) \supset O(2r) \supset O(r) \times O(r) \supset O(r) \supset \cdots \end{aligned} \tag{3.100}$$

We will show that this follows easily from the basic Bott genetic code:

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{R}, \cdots \tag{3.101}$$

The argument proceeds along the lines of the three steps outlined above for the complex case:

1. Determine the Wedderburn type of $\rho(C\ell_{-k})$: Decompose $\mathcal{S}$ in terms of ungraded irreps of $C\ell_{-k}$. For $k \neq 3\mathrm{mod}4$ there is a unique irrep N_k up to isomorphism, and for $k = 3\mathrm{mod}4$ there are two $N_k^\pm$. Therefore, $\mathcal{S} \cong N_k^{\oplus s_k}$ for $k \neq 3\mathrm{mod}4$ and $\mathcal{S} \cong (N_k^+)^{\oplus s_k} \oplus (N_k^-)^{\oplus s_k}$ for $k = 3\mathrm{mod}4$. The number of summands is the same $N_k^\pm$ for $k = 3\mathrm{mod}4$ because the decomposition is effected by the projection operator using the volume form $P_\pm = \frac{1}{2}(1 \pm \omega_k)$ and $\mathrm{Tr}_{\mathcal{S}}(\omega_k) = 0$ for all k. Now, the image of the Clifford algebra in $\mathrm{End}(\mathcal{S})$ (as an ungraded algebra) will have be isomorphic to $s_k \mathbb{K}(t_k)$ for $k \neq 3\mathrm{mod}4$ and $s_k \mathbb{K}(t_k) \oplus s_k \mathbb{K}(t_k)$ for $k = 3\mathrm{mod}4$.

2. Apply the Weyl commutant theorem: $Z(\rho(C\ell_{-k}))$ will be isomorphic to $t_k \mathbb{K}(s_k)$ for $k \neq 3\mathrm{mod}4$ and $t_k \mathbb{K}(s_k) \oplus t_k \mathbb{K}(s_k)$ for $k = 3\mathrm{mod}4$.

3. Determine the group which can generate this commutant: When we intersect $Z(\rho(C\ell_{-k}))$ with $\mathrm{Aut}(\mathcal{S}) \cong O(16r)$ we get the group G_k. In this way we determine the following table

k	Bott clock	$\rho(C\ell_{-k})$	$Z(\rho(C\ell_{-k}))$	G_k
0	$\mathbb{R}$	$16r\mathbb{R}$	$\mathbb{R}(16r)$	$O(16r)$
1	$\mathbb{C}$	$8r\mathbb{C}$	$\mathbb{C}(8r)$	$U(8r)$
2	$\mathbb{H}$	$4r\mathbb{H}$	$\mathbb{H}^{\rm opp}(4r)$	$Sp(4r)$
3	$\mathbb{H}\oplus\mathbb{H}$	$2r\mathbb{H}\oplus 2r\mathbb{H}$	$\mathbb{H}^{\rm opp}(2r)\oplus\mathbb{H}^{\rm opp}(2r)$	$Sp(2r)\times Sp(2r)$
4	$\mathbb{H}$	$2r\mathbb{H}(2)$	$2\mathbb{H}^{\rm opp}(2r)$	$Sp(2r)$
5	$\mathbb{C}$	$2r\mathbb{C}(4)$	$4\mathbb{C}(2r)$	$U(2r)$
6	$\mathbb{R}$	$2r\mathbb{R}(8)$	$8\mathbb{R}(2r)$	$O(2r)$
7	$\mathbb{R}\oplus\mathbb{R}$	$r\mathbb{R}(8)\oplus r\mathbb{R}(8)$	$8\mathbb{R}(r)\oplus 8\mathbb{R}(r)$	$O(r)\times O(r)$
8	$\mathbb{R}$	$r\mathbb{R}(16)$	$16\mathbb{R}(r)$	$O(r)$

We should stress that the entries for $\rho(C\ell_{-k})$, $Z(\rho(C\ell_{-k}))$, and G_k just give the isomorphism type. Of course, r is some power of 2 and for large d we can repeat the periodic sequence down many steps.

♣NEED TO DO COMPLEX CASE OF CARTAN EMBEDDING ♣

The series of homogeneous spaces G_k/G_{k+1} for $k\geq 0$ provide examples of the Cartan symmetric spaces (for ranks which are a power of two!). Note that the tangent space at $1\cdot G_{k+1}$ has an elegant description. First define $\mathfrak{g}_0 := T_1 G_0 = so(2N)$. Now for $k>0$ define:

♣Use $so(2N)$ for the Lie algebra to distinguish it clearly from the group $O(2N)$. ♣

$$\mathfrak{g}_k := T_1 G_k = \{a\in so(2N)|aJ_s = J_s a \qquad s=1,\dots,k\}. \tag{3.102}$$

Observe that, for $k\geq 0$, the map $\theta_k(a) = J_{k+1}aJ_{k+1}^{-1}$ acts as an involution on $\mathfrak{g}_k$ and that the eigenspace with $\theta_k=+1$ is just $\mathfrak{g}_{k+1}$. Therefore we can identify

$$\mathfrak{p}_k := T_{G_{k+1}} G_k/G_{k+1} = \{a\in so(2N)|aJ_s = J_s a \qquad s=1,\dots,k \quad \& \quad aJ_{k+1} = -J_{k+1}a\} \tag{3.103}$$

so that

$$\mathfrak{g}_k = \mathfrak{g}_{k+1}\oplus\mathfrak{p}_k \tag{3.104}$$

3.3.2 Cartan embedding of the symmetric spaces

The involution θ_k described above extends to a global involution $\theta_k : G_k\to G_k$ defined by conjugation with J_{k+1}:

$$\theta_k(g) = J_{k+1}gJ_{k+1}^{-1} \tag{3.105}$$

Of course, the fixed subgroup of θ_k in G_k is G_{k+1} so the Cartan symmetric space is G_k/G_{k+1}. Moreover, the Cartan embedding of this symmetric space is just

$$\mathcal{O}_k = \{g\in G_k|\theta_k(g) = g^{-1}\}\subset G_k\subset O(2N) \qquad k\geq 0. \tag{3.106}$$

Let us unpack this definition: The condition $\theta_k(g) = g^{-1}$ is equivalent to the condition $(J_{k+1}g)^2 = -1$. Therefore, writing $\tilde{g} = J_{k+1}g$ we can also write

$$\tilde{\mathcal{O}}_k := J_{k+1}\mathcal{O}_k = \{\tilde{g}\in O(2N)|\tilde{g}^2 = -1 \qquad \{\tilde{g},J_s\} = 0 \quad s=1,\dots,k\} \tag{3.107}$$

The map $g \mapsto \tilde{g} = J_{k+1} g$ is a simple diffeomorphism so $\tilde{\mathcal{O}}_k \cong G_k/G_{k+1}$, and $\tilde{\mathcal{O}}_k$ is also embedded in $O(2N)$. This manifestation of the homogeneous space will be more convenient to work with. Note that:

$$\tilde{\mathcal{O}}_0 = \{\tilde{g} \in O(2N) | \tilde{g}^2 = -1\} \tag{3.108}$$

and

$$\tilde{\mathcal{O}}_0 \supset \tilde{\mathcal{O}}_1 \supset \tilde{\mathcal{O}}_2 \supset \cdots \tag{3.109}$$

When we wish to emphasize the dependence on N we will write $\tilde{\mathcal{O}}_k(N)$. It will also be convenient to define $\tilde{\mathcal{O}}_{-1} := O(2N)$.

Since $g = \pm 1$ is in $\mathcal{O}_k$ we have $\pm J_{k+1} \in \tilde{\mathcal{O}}_k$, as is immediately verified from the definition. (Note that ± 1 are *not* elements of $\tilde{\mathcal{O}}_k$.) Let us compute the tangent space to $\tilde{\mathcal{O}}_k$ at J_{k+1}. A path through J_{k+1} must be of the form $J_{k+1} e^{ta}$ where $a \in T_1 \mathcal{O}_k = \mathfrak{p}_k$. Therefore there is an isomorphism $T_1 \mathcal{O}_k \leftrightarrow T_{J_{k+1}} \tilde{\mathcal{O}}_k$ given simply by left-multiplication by J_{k+1}. Now $a \in T_1 \mathcal{O}_k$ iff $a^{tr} = -a$, $[a, J_s] = 0$ for $s = 1, \dots, k$ and $\{a, J_{k+1}\} = 0$ and therefore

$$\tilde{\mathfrak{p}}_k := T_{J_{k+1}} \tilde{\mathcal{O}}_k = \{\tilde{a} \in so(2N) | \{\tilde{a}, J_s\} = 0, \qquad s = 1, \dots, k+1\} \tag{3.110}$$

(A slightly tricky point here is that $J_{k+1} a$ is itself an anti-symmetric matrix, and hence in the Lie algebra $so(2N)$.)

3.3.3 Application: Uniform realization of the Altland-Zirnbauer classes

The characterization (3.107) of $\tilde{\mathfrak{p}}_k$ is nicely suited to a realization of 8 of the 10 AZ classes of free fermion Hamiltonians. We take a FDFS based on $\mathcal{M} = \mathbb{R}^{2N}$ with Q the Euclidean metric. We take as our symmetry group $G = \mathrm{Pin}^-(k+1)$ with Clifford generators T_i. We choose the nontrivial option for τ on G, thus $\tau(T_i) = -1$ for $i = 1, \dots, k+1$. For α we choose the embedding of G into $O(2N)$ using $\alpha(T_i) = \mathrm{Ad}(e_i)$ (not $\widetilde{\mathrm{Ad}}$) acting on $\mathcal{M} \subset C\ell_{-8d}$. Comparing the definitions (3.73) and (3.74) we find that we have precisely

$$\begin{aligned} \mathfrak{k} &= \mathfrak{g}_k \\ \mathfrak{p} &= \tilde{\mathfrak{p}}_k \end{aligned} \tag{3.111}$$

thus neatly exhibiting examples of 8 of the AZ 10 classes.

The remaining two AZ classes follow from completely analogous manipulations for the series $U(2r) \supset U(r) \times U(r) \supset U(r) \supset \cdots$.

Remarks:

1. Note that our fermionic oscillators are a basis for the spin representation of $\mathrm{Spin}(8d)$. So *their* Hilbert space will be a representation of the much larger group $\mathrm{Spin}(2N)$ of dimension $2^N = 2^{2^{8r}} = 2^{2^{4d-1}}$.

2. This example can be extended to compute the 3- and 10-fold classes on $\mathcal{H}_{DN}$ and $\mathcal{H}_F$. Again there are two options $(\phi(T_i) = +1, \chi(T_i) = -1)$ and $(\phi(T_i) = -1, \chi(T_i) = +1)$. Representing the J_i by real matrices on $\mathcal{H}_F$ we can take $\rho = \rho_F$ restricted to $C\ell_{-k-1}$.

♣Finish by analyzing the classes with this choice. ♣

3.3.4 Relation to Morse theory and loop spaces

The homogeneous spaces $\tilde{\mathcal{O}}_k$ have a further beautiful significance when we bring in some ideas from Morse theory.

We consider the quantum mechanics of a particle moving on these manifolds using the action

$$S[q] = -\int dt \mathrm{Tr}(q^{-1}\frac{dq}{dt})^2 \tag{3.112}$$

where $q(t)$ is a path in the orthogonal group or one of the $\tilde{\mathcal{O}}_k$.

We begin with quantum mechanics on $SO(2N)$. We choose boundary conditions and define $\mathcal{P}_{-1}$ to be the space of (continuously differentiable) paths $q : [0,1] \to SO(2N)$ such that $q(0) = +1_{2N}$ and $q(1) = -1_{2N}$. We are particularly interested in the *minimal action paths*. Such paths will be geodesics in the left-right-invariant metric. The geodesics are well known to be of the form $q(t) = \exp[\pi t A]$ with $A \in so(2N)$. We can always conjugate A to the form

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & a_N \\ -a_N & 0 \end{pmatrix} \tag{3.113}$$

where $a_i \in \mathbb{R}$. This has action $2\pi^2 \sum a_i^2$ and the boundary conditions imply that a_i are odd integers. Therefore the minimal action paths have $a_i = \pm 1$ and hence the space of minimal action paths is precisely given by the conjugacy class of $A \in o(2N)$ with $A^2 = -1$. Moreover, such paths have a very simple form:

$$q(t) = \cos \pi t + A \sin \pi t \tag{3.114}$$

Now, notice a trivial but significant fact:

1. If $g \in O(2N)$ is an orthogonal matrix and $g^2 = -1$ then $g \in so(2N)$ is also in the Lie algebra. Proof: $g^2 = -1$ and $g \in O(2N)$ implies $g^{tr} = -g$.
2. If $A \in so(2N)$ is in the Lie algebra and $A^2 = -1$ then $A \in O(2N)$ is also in the Lie group. Proof: $A^{tr} = -A$ and $A^2 = -1$ implies $A^{tr} = A^{-1}$.

Therefore, the space of minimal action paths in $\mathcal{P}_{-1}$ is naturally identified with

$$\tilde{\mathcal{O}}_0 := \{g \in O(2N) | g^2 = -1\} \subset O(2N). \tag{3.115}$$

Of course $\tilde{\mathcal{O}}_0$ is $J_1\mathcal{O}_0$ where $\mathcal{O}_0$ is the Cartan embedding of $G_0/G_1 = O(2N)/U(N)$.

Now let us consider the quantum mechanics of a particle on the orbit $\tilde{\mathcal{O}}_0$, again with the action (3.112). We choose boundary conditions so that $\mathcal{P}_0$ consists of maps $q : [0,1] \to \tilde{\mathcal{O}}_0$ such that $q(0) = J_1$ and $q(1) = -J_1$. The solutions to the equations of motion are of the form [27] $q(t) = J_1 \exp[\pi t A]$ where now $A \in \mathfrak{p}_0$ implies $\{A, J_1\} = 0$, which guarantees that the path indeed remains in $\tilde{\mathcal{O}}_0$. Again, the boundary conditions together with the minimal action criterion implies that $A^2 = -1$, so we can write:

$$q(t) = J_1 \exp[\pi t A] = J_1 \cos \pi t + (J_1 A) \sin \pi t = J_1 \cos \pi t + \tilde{A} \sin \pi t \tag{3.116}$$

[27] We use the fact that $\tilde{\mathcal{O}}_0$ is totally geodesic.

Because $A \in \mathfrak{p}_0$ both A and $\tilde{A} = J_1 A$ are both antisymmetric and square to -1: $A^2 = -1 = \tilde{A}^2$. We can therefore consider A and $\tilde{A}$ to be in $O(2N)$ and hence the minimal action paths on $\tilde{\mathcal{O}}_0$ are parametrized by A, or better by $\tilde{A}$, and hence the space of minimal actions paths is naturally identified, via the mapping $q(t) \mapsto \tilde{A}$ with $\tilde{\mathcal{O}}_1 \subset O(2N)$.

This stunningly beautiful pattern continues: We take

$$\mathcal{P}_k := \{q : [0,1] \to \tilde{\mathcal{O}}_k | q(0) = J_{k+1} \quad q(1) = -J_{k+1}\} \tag{3.117}$$

Since $\tilde{\mathcal{O}}_k$ is totally geodesic the solutions to the equations of motion are of the form $J_{k+1}\exp[\pi A t]$ with $A \in \mathfrak{p}_k$. The minimal action paths have $A^2 = -1$ and hence they are of the form

$$q(t) = J_{k+1}\exp[\pi t A] = J_{k+1}\cos\pi t + (J_{k+1}A)\sin\pi t = J_{k+1}\cos\pi t + \tilde{A}\sin\pi t \tag{3.118}$$

But now $\tilde{A}^2 = -1$ and $\tilde{A} \in \tilde{\mathcal{O}}_{k+1}$, so we can identify the space of minimal action paths in $\mathcal{P}_k$ with $\tilde{\mathcal{O}}_{k+1}$.

The space of minimal action paths in the set $\mathcal{P}_k$ of all smooth paths $[0,1] \to \tilde{\mathcal{O}}_k$ from J_{k+1} to $-J_{k+1}$ is naturally identified with $\tilde{\mathcal{O}}_{k+1}$ by equation (3.118).

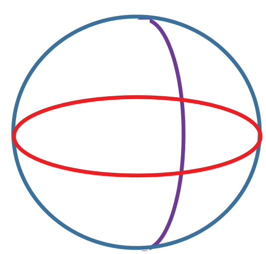

Figure 12: The minimal length geodesics on S^N from the north pole to the south pole are parametrized by S^{N-1}. Similarly, the geodesics in $\tilde{\mathcal{O}}_k$ from J_{k+1} to $-J_{k+1}$ are parametrized by $\tilde{\mathcal{O}}_{k+1}$.

Remark: A good analogy to keep in mind is the length of a path on the N-dimensional sphere. If we consider the paths on S^N from the north pole $\mathfrak{N}$ to the south pole $\mathfrak{S}$ then the minimal length paths are great circles and are hence parametrized by their intersection with the equator S^{N-1}. See Figure 12.

The great significance of this comes about through Morse theory. The action (3.112) for the paths is a (degenerate) Morse function on $\mathcal{P}_k$ and the critical manifolds allow us to describe the homotopy type of $\mathcal{P}_k$. One considers a series of "approximations" to $\mathcal{P}_k$ by looking at paths with bounded action:

$$\mathcal{P}_k^{\Lambda} := \{q \in \mathcal{P}_k | S[q] \leq \Lambda\} \tag{3.119}$$

As we have seen, the minimal action space is $\mathcal{P}_k^{\Lambda_{\min}} \cong \tilde{\mathcal{O}}_k(N) \subset O(2N)$. Now – it turns out – that the solutions of the equations of motion which are non-minimal have many unstable modes. The number of unstable modes is the "Morse index." The number of unstable modes is linear in N. The reason this is important is that in homotopy theory the way $\mathcal{P}_k^{\Lambda}$ changes as Λ crosses a critical value Λ_* is

$$\mathcal{P}_k^{\Lambda_*+\epsilon} \sim (\mathcal{P}_k^{\Lambda_*-\epsilon} \times D^{n_-})/\sim \tag{3.120}$$

where n_- is the number of unstable modes at the critical value Λ_* and D^{n_-} is a ball of dimension n_-. This operation does not change the homotopy groups π_j for $j < n_-$. Therefore, in this topological sense, $\tilde{\mathcal{O}}_k(N)$ gives a "good approximation" to $\mathcal{P}_k(N)$.

♣Need a separate subsubsection on how Morse function gives a cell decomposition. ♣

On the other hand, the spaces $\mathcal{P}_k$ have the same homotopy type as the *based loop spaces* $\Omega_*\tilde{\mathcal{O}}_k$. Indeed, choosing any standard path from $-J_{k+1}$ to J_{k+1} we can use it to convert any path in $\mathcal{P}_k$ to a loop $\Omega_*\tilde{\mathcal{O}}_k$ based, say, at J_{k+1} by composition. Conversely, composing the (inverse of) the standard path with any loop gives a path in $\mathcal{P}_k$.

Putting these remarks together we can say that

$$\Omega_*\tilde{\mathcal{O}}_k(\infty) \sim \tilde{\mathcal{O}}_{k+1}(\infty) \tag{3.121}$$

In the case of the complex Clifford algebras we have an even simpler slogan:

$$\boxed{\textit{A loop of projectors is a unitary; A loop of unitaries is a projector.}}$$

From this discussion we get two nice outcomes:

1. We get a nice proof of Bott periodicity from the periodicity of the Clifford algebras [38].

2. We thereby make a connection to generalized cohomology theory through the notion of a *spectrum.*

♣Both of these points require a lot more explanation. ♣

Definition: A sequence of spaces with basepoint, $\{\mathcal{E}_q\}_{q\in\mathbb{Z}}$, is called a *loop spectrum* if there are homotopy equivalences

$$\mathcal{E}_q \to \Omega_*\mathcal{E}_{q+1}. \tag{3.122}$$

This is an important concept in algebraic topology. Given a loop spectrum one can define generalized cohomology groups $E^q(X) := [X, \mathcal{E}_q]$, and conversely, given a generalized cohomology theory there is a corresponding loop spectrum.

We see that the sequence of spaces $\dots, U, (U \times U)/U, U, (U \times U)/U, \dots$ form a spectrum known as the *complex K-theory spectrum.* Similarly, the sequence of spaces

♣Do we want $BU \times \mathbb{Z}$. Comment on the index in $U(2r)/(U(r+x) \times U(r-x))$. Then what is the basepoint? ♣

$$\dots, O/U, U/Sp, Sp/Sp \times Sp, Sp \times Sp/Sp, Sp/U, U/O, O/O \times O, O \times O/O, \dots \tag{3.123}$$

forms the *real K-theory spectrum.* In the next section we will discuss a very nice realization of these spectra in terms of operators on Hilbert space.

3.3.5 Relation to classifying spaces of K-theory

The fact that the Morse index for the space of paths $\mathcal{P}_k(N)$ (where the N-dependence comes from the fact that the paths are in $\tilde{\mathcal{O}}_k(N) \subset O(2N)$) grows linearly in N suggests that it will be interesting to take the $N \to \infty$ limit. We can do this as follows:

We make a real Hilbert space by taking a countable direct sum of copies of simple modules of the real Clifford algebra $C\ell_{-(k+1)}$. Specifically we define, for $k \geq 0$, [28]

$$\mathcal{H}_R^k := \begin{cases} N_{k+1} \otimes \ell^2(\mathbb{R}) & k \neq 2(4) \\ (N_{k+1}^+ \oplus N_{k+1}^-) \otimes \ell^2(\mathbb{R}) & k = 2(4) \end{cases} \tag{3.124}$$

and for an integer n let $\mathcal{H}_R^k(n)$ be the sum of the first n representations N_{k+1} or $(N_{k+1}^+ \oplus N_{k+1}^-)$. Now define a subspace of the space of orthogonal operators $\Omega_k(n) \subset O(\mathcal{H}_R^k)$. These are operators which satisfy the following three conditions:

♣Atiyah-Singer just write $\mathcal{H}_R$ with no k since all separable Hilbert spaces are isomorphic, but the isomorphism between different k is slightly nontrivial... ♣

1. They preserve separately $\mathcal{H}_R^k(n)$ and $\mathcal{H}_R^k(n)^\perp$.
2. They are just given by $A = J_k$ on $\mathcal{H}_R^k(n)^\perp$
3. On $\mathcal{H}_R^k(n)$ they satisfy:

$$\begin{aligned} A^2 &= -1 \\ \{A, J_i\} &= 0 \qquad i = 1, \dots, k-1 \end{aligned} \tag{3.125}$$

We recognize that $\Omega_k(n) \cong \tilde{\mathcal{O}}_{k-1}(N)$ where N and n are linearly related. (We define $\tilde{\mathcal{O}}_{-1}(N) := O(2N)$ so this holds for $k \geq 0$.) Now, from this description it is easy to see that there are embeddings

$$\Omega_k(n) \hookrightarrow \Omega_k(n+1) \tag{3.126}$$

and we can take a suitable "$n \to \infty$ limit" and norm closure to produce a set of operators $\Omega_k(\infty)$ on $\mathcal{H}_R^k$. In [11] Atiyah and Singer show that this set of operators is closely related to a set of Fredholm operators on $\mathcal{H}_R^k$.

[28] Recall that N_k denote irreducible ungraded Clifford modules.

Define $\mathfrak{F}^0$ to be the set of all Fredholm operators on $\mathcal{H}_R^k$, and let $\mathfrak{F}^1 \subset \mathfrak{F}^0$ denote the subspace of skew-adjoint Fredholm operators: $A^{tr} = -A$. (Formally, this is the Lie algebra of $O(\mathcal{H}_R^k)$.) Now for $k \geq 2$ define $\mathfrak{F}^k \subset \mathfrak{F}^1$ to be the subspace such that [29]

$$TJ_i = -J_iT \qquad i = 1, \ldots, k-1 \tag{3.127}$$

Now, the space of Fredholm operators has a standard topology using the operator norm topology. Using this topology Atiyah and Singer prove

♣You are changing k's here. Need to clarify. ♣

1. $\mathfrak{F}^k \sim \Omega_{k-1}(\infty) \cong \tilde{\mathcal{O}}_{k-2}(\infty)$, $k \geq 1$, where $\sim$ denotes homotopy equivalence.
2. $\mathfrak{F}^{k+1} \sim \Omega\mathfrak{F}^k$, and in fact, the homotopy equivalence is given by

$$A \mapsto J_{k+1}\cos\pi t + A\sin\pi t \qquad 0 \leq t \leq 1 \tag{3.128}$$

which should of course be compared with (3.118).

The relation to Fredholm operators implies a relation to K-theory because one way of defining the real KO-theory groups of a topological space X is via the set of homotopy classes:

$$KO^{-k}(X) := [X, \mathfrak{F}^k] \qquad k \geq 0 \tag{3.129}$$

We summarize with a table

k	$\mathfrak{F}^k \sim G_{k-2}/G_{k-1}$	Cartan's Label
0	$(O/(O \times O)) \times \mathbb{Z}$	BDI
1	O	D
2	O/U	DIII
3	U/Sp	AII
4	$(Sp/(Sp \times Sp)) \times \mathbb{Z}$	CII
5	Sp	C
6	Sp/U	CI
7	U/O	AI

and the complex case is

[29] For $k = 3\text{mod}4$ the subspace of $\mathfrak{F}^1$ satisfying (3.127) in fact has three connected components in the norm topology. Two of these are contractible but one is topologically nontrivial and we take $\mathfrak{F}^k$ to be that component. In fact for T satisfying (3.127) one can show that $\omega_{k-1}T$ is self-adjoint, where $\omega_{k-1} = J_1 \cdots J_{k-1}$ is the volume form. The contractible components are those for which $\omega_{k-1}T$ is positive or negative - up to a compact operator.

k	$\mathfrak{F}_c^k \sim G_{k-2}/G_{k-1}$	Cartan's Label
0	$(U/(U \times U)) \times \mathbb{Z}$	AIII
1	U	A

where $\mathfrak{F}_c$ is the space of Fredholm operators on a complex separable Hilbert space, and is the classifying space for $K^0(X)$ and $\mathfrak{F}_c^1$ is the subspace of skew-adjoint Fredholm operators and is the classifying space for $K^{-1}(X)$.

Remark: We indicate how this discussion of $KO(X)$ is related to what we discussed in Section §2.3.7 above. We take $X = pt$. Then, $KO^0(pt) \cong \mathbb{Z}$. In terms of Fredholm operators T the isomorphism is given by $T \mapsto \text{Index}(T) := \text{dimker}T - \text{dimcok}T$. Thus, "invertible part of T cancels out." The idea that if T is invertible then it defines a trivial class was the essential idea in the definition in Section §2.3.7. It is also worth noting the Fredholm interpretation of $KO^{-1}(pt) \cong \mathbb{Z}_2$ in this context. For a skew-adjoint Fredholm operator $\ker(T) = \ker(T^\dagger)$ so the usual notion of index is just zero. However we can form the "mod-two index," which is defined to be $\text{dimker}T \text{mod} 2$. This is indeed continuous in the norm topology and provides the required isomorphism.

♣Improve this discussion by rephrasing the AS results in terms of $\mathbb{Z}_2$-graded Hilbert spaces. ♣

4. Lecture 4: K-theory classification and application to topological band structure

4.1 Reduced topological phases of a FDFS and twisted equivariant K-theory of a point

4.1.1 Definition of G-equivariant K-theory of a point

Now let G be a compact Lie group. (It could be a finite group.)

$K_G(pt)$ is the representation ring of G. It can be defined in two ways:

1. Group completion of the monoid of finite-dimensional complex representations: A typical element is a formal difference $R_1 - R_2$ with R_1, R_2, finite-dimensional representations on complex vector spaces.

2. Divide by trivial representations: Define $\text{Rep}_s(G)$ to be $\mathbb{Z}_2$-graded fin. dim. cplx reps (with even G-action) Define the submonoid: $\text{Triv}_s(G)$ to be those with an odd invertible superlinear transformation T with

$$T\rho(g) = \rho(g)T \tag{4.1}$$

Then we define:

$$K_G(pt) = \text{Rep}_s(G)/\text{Triv}_s(G) \tag{4.2}$$

An important point is that $K_G(pt)$ is not just a group but a *ring*, because we can take the tensor product of G-representations, and if we take a product of a representation with a representation in $\mathrm{Triv}_s(G)$ then the resulting representation is in $\mathrm{Triv}_s(G)$. We will need this in our discussion of localization below.

As an example, let us return to the automorphism groups of two-dimensional quantum systems. We have

$$1 \to U(1) \to U(2) \rtimes \mathbb{Z}_2 \to SO(3) \rtimes \mathbb{Z}_2 \to 1 \tag{4.3}$$

where the $\mathbb{Z}_2$ actions on $U(2)$ by complex conjugation and on $SO(3)$ by **** ♣work out z2 action on so3 ♣

It is interesting to compare the K theories of the groups, but to do that we must use pullback by the obvious homomorphisms to compare

$$\begin{aligned} \iota_1^* : K_{U(2)} &\hookrightarrow K_{SU(2)\times U(1)} \\ \iota_2^* : K_{SO(3)\times U(1)} &\hookrightarrow K_{SU(2)\times U(1)} \end{aligned} \tag{4.4}$$

Now, the representation ring of $SU(2) \times U(1)$ is generated by f, the two-dimensional representation of $SU(2)$ and $\delta^{\pm 1}$, the charge ± 1 representations of $U(1)$, so

$$K_{SU(2)\times U(1)} \cong \mathbb{Z}[f, \delta^{\pm 1}] \tag{4.5}$$

In these terms we can say that the images are:

$$\iota_2^* K_{SO(3)\times U(1)} \cong \mathbb{Z}[\delta^{\pm 1}, f^2 - 1] \tag{4.6}$$

$$\iota_1^* K_{U(2)} \cong \mathbb{Z}[\delta^{\pm 1}, f]^+ \tag{4.7}$$

where f represents the class of the two-dimensional fundamental representation of $SU(2)$, δ^n is the charge n representation of $U(1)$, and the superscript $+$ means we project onto the even elements under $f \to -f$ and $\delta \to -\delta$ (so that the subgroup generated by $(-1,-1)$ in $SU(2) \times U(1)$ acts trivially.

4.1.2 Definition of twisted G-equivariant K-theory of a point

There is a general notion of a "twisting" of a generalized cohomology theory. This can be defined in terms of some sophisticated topology (like using nontrivial bundles of spectra) but in practice it often amounts to introducing some extra signs of phases. This is not always the case: Degree shift in K-theory can be viewed as an example of twisting.

A simple example of a twisting of ordinary cohomology theory arises when one has a double cover $\pi : \tilde{X} \to X$. Then the "twisted cohomology" of X refers to using cocycles, coboundaries, etc. on $\tilde{X}$ that are odd under the deck transformation.

In the case of equivariant K-theory of a point, a "twisting of $K_G(pt)$" is an isomorphism class of a central extension of G by $\mathbb{C}^*$ (or by $U(1)$, if we include an Hermitian structure). See appendix §B.1 for a description of central extensions. It can be constructed using a $\mathbb{C}^*$- or $U(1)$-valued function $\lambda(g_1, g_2)$ on $G \times G$ satisfying the 2-cocycle condition

$$\lambda(g_2, g_1)\lambda(g_3, g_2 g_1) = \lambda(g_3, g_2)\lambda(g_3 g_2, g_1) \tag{4.8}$$

Isomorphic central extensions induce a suitable equivalence such that equivalence class of λ are classified by $H^2(G, U(1))$. In general, isomorphism classes of twistings of K-theory are classified by certain cohomology groups.

Let τ denote such a class of central extensions. A "twisted G-bundle over a point with twisting τ" is a representation of a corresponding central extension G^τ. The τ-twisted G-equiviariant K-theory of a point is then just the G^τ-equivariant K-theory of a point:

$$K_G^\tau(pt) = K_{G^\tau}(pt) \tag{4.9}$$

Now, when we add the other ingredients that we encountered from a general realization of a quantum symmetry with gapped Hamiltonians we have a pair of homomorphisms $(\phi, \chi) : G \to M_{2,2}$:

1. The data of ϕ indicates that we take not central extensions but ϕ-twisted extensions. For these the cocycle relation is modified to

$$\lambda(g_2, g_1)\lambda(g_3, g_2 g_1) = {}^{\phi(g_1)}\lambda(g_3, g_2)\lambda(g_3 g_2, g_1) \tag{4.10}$$

where λ are valued in $U(1)$ and ${}^{\phi(g)}\lambda$ is defined in (1.56). (If we take $\lambda \in \mathbb{C}^*$ then instead of $\bar{\lambda}$ use λ^{-1}.) Given an isomorphism class τ of such a ϕ-twisted extension we can define the monoid of representations, $\mathrm{Rep}_s(G^\tau, \phi, c)$ as usual.

2. The data of χ enters in modifying the definition of the submonoid of trivial representations. These are the ones with an odd automorphism P such that

$$P\rho^\tau(g) = \chi(g)\rho^\tau(g)P \tag{4.11}$$

We can now define the more general twisted-equivariant K-theories:

$${}^\phi K_G^{\tau;\chi}(pt) := \mathrm{Rep}_s(G^\tau, \phi, c)/\mathrm{Triv}_s(G^\tau, \phi, c) \tag{4.12}$$

In general this is not a ring. It will be a module for $K_{G_0}(pt)$ where G_0 is an untwisted quotient of G^τ.

Example Take $G = M_2 = \{1, \bar{T}\}$.

As we have seen when $\phi(\sigma) = -1$ there are two inequivalent ϕ-twisted extensions by $U(1)$:

$$1 \to U(1) \to \mathrm{Pin}^+(2) \to M_2 \to 1 \tag{4.13}$$

$$1 \to U(1) \to \mathrm{Pin}^-(2) \to M_2 \to 1 \tag{4.14}$$

Now, the untwisted K theory of $\mathbb{Z}_2$ is

$$K_{\mathbb{Z}_2}(pt) \cong \mathbb{Z}[\epsilon]/(\epsilon^2 - 1) \tag{4.15}$$

where ϵ is the sign representation. The two twisted versions are:

$${}^\phi K_{\mathbb{Z}_2}^{\tau_+}(pt) \cong \mathbb{Z}[f] \tag{4.16}$$

where $f \cong \mathbb{R}^2$ is a real representation, and

$${}^\phi K_{\mathbb{Z}_2}^{\tau_-}(pt) \cong \mathbb{Z}[q]^- \tag{4.17}$$

where $q = \mathbb{H}$ are the quaternions, the complex structure is $L(\mathfrak{i})$, we choose a lift T of $\bar{T}$ to act $L(\mathfrak{j})$ and the superscript means we must take odd powers. The reason is that T must act anti-linearly.

4.1.3 Appliction to FDFS: Reduced topological phases

Now let us return to our finite dimensional fermionic systems.

We take G to be a compact group and we restrict to a finite dimensional Hilbert space.

If we have not put any locality conditions on the Hamiltonians A_{ij} then we simply have

$$\mathcal{TP}_{\rm finite}(G^\tau, \phi, \chi) = {\rm Rep}_s(G^\tau, \phi, \chi) \tag{4.18}$$

The proof is simply that for gapped Hamiltonians we can always homotope to $H^2 = 1$ and (twisted) representations of compact groups are discrete.

Now comes an important point: We would like to put an abelian group structure on the set of phases. To do that, we need a monoid structure.

For free fermions we can use the monoid structure provided by the "single-particle" Dirac-Nambu Hilbert space:

If we combine the systems 1 and 2 then

$$\mathcal{H}^{DN}_{1+2} = \mathcal{H}^{DN}_1 \oplus \mathcal{H}^{DN}_2 \tag{4.19}$$

Then, given this monoid structure we can group-complete or quotient by a suitable notion of "topologically trivial systems." In this was we can get an abelian group of "reduced" topological phases $\mathcal{RTP}(G^\tau, \phi, \chi)$.

Remark: This raises the obvious question of what happens more generally when the Dirac-Nambu structure is not available. There is in fact a general monoid structure on quantum systems given by (graded) tensor product of Hilbert spaces:

$$\mathcal{H}_{1+2} = \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2 \tag{4.20}$$

The corresponding K-theory hasn't been much studied in the literature, and with good reason. Part of the problem is that the group completion can be fairly trivial. Consider for example the K-theory of a point with this monoid structure using group completion. Then $(V_1, V_2) \sim (V_3, V_4)$ if $V_1 \otimes V_4 \cong V_2 \otimes V_3$. Instead of producing the integers, we produce the abelian group of nonzero rational numbers under multiplication. This does not have torsion and the resulting K-theory groups will tend to be trivial.

♣Should do this for super vector spaces and take into account the super dimension. ♣

If we interpret the existence of an odd automorphism (4.11) as the possibility of a total pairing of "particles" and "holes" and hence of a trivial system then we can identify the abelian group of reduced topological phases with a twisted K-theory group:

$$\mathcal{RTP}(G^\tau, \phi, \chi) = {}^\phi K_G^{\tau,\chi}(pt) \tag{4.21}$$

We have not yet made contact with standard statements found in the works of Kitaev et. al. To do that we must consider the very special case that

$$G^{\rm tw} = U^{\rm tw} \times G_0 \tag{4.22}$$

where $G_0 = \ker(\tau, \chi)$. In this case we can invoke the theorem of 2.6 to identify twisted G-equivariance with G_0 equivariance together with having a Clifford module for the corresponding Clifford algebra.

♣Explain this point more fully! ♣

In this way we get the correspondence between the 10-possibilities for the CT group with the 10 possible K-theories

Subgroup $U \subset M_{2,2}$	T^2	C^2	${}^\phi K_G^{\tau,\chi}$
$\{1\}$			$K^0_{G_0}(pt)$
$\{1, \bar{S}\}$			$K^1_{G_0}(pt)$
$\{1, \bar{T}\}$	$+1$		$KO^0_{G_0}(pt)$
$M_{2,2}$	$+1$	-1	$KO^{-1}_{G_0}(pt)$
$\{1, \bar{C}\}$		-1	$KO^{-2}_{G_0}(pt)$
$M_{2,2}$	-1	-1	$KO^{-3}_{G_0}(pt)$
$\{1, \bar{T}\}$	-1		$KO^{-4}_{G_0}(pt) = KSp^4_{G_0}(pt)$
$M_{2,2}$	-1	$+1$	$KO^{-5}_{G_0}(pt)$
$\{1, \bar{C}\}$		$+1$	$KO^{-6}_{G_0}(pt)$
$M_{2,2}$	$+1$	$+1$	$KO^{-7}_{G_0}(pt)$

The above is the kind of table (usually without the G_0) that one finds in the papers.

♣Aren't we missing a twisting ν_0 of the G_0-equivariant K theory here? ♣

4.2 Motivation: Bloch theory

Next we would like to generalize to certain noncompact groups - the crystallographic groups - and we would also like to include some notion of locality so we can speak of fermions in different dimensions. To do that we need to develop a bit more mathematical machinery.

We consider noninteracting particles in an affine space E modeled on Euclidean space $\mathbb{R}^d$ subjected to a potential with the symmetry of a crystal $C \subset E$.

♣Change notation $E \to \mathbb{E}$? ♣

In the absence of electromagnetic fields the symmetry group of the physical system will be a magnetic crystallographic group $G(C) \subset \mathrm{Euc}(d) \times \mathbb{Z}_2$. The $\mathbb{Z}_2$ factor is for time-reversal, and hence $G(C)$ is a $\mathbb{Z}_2$-graded group, by restricting τ which is the projection onto the second factor of $\mathrm{Euc}(d) \times \mathbb{Z}_2$. As we have stressed many times, this does *not* imply that $G(C)$ is a product with $\mathbb{Z}_2$.

The Hilbert space of a single electron is

$$\mathcal{H} = L^2(E; W), \tag{4.23}$$

where W is a finite-dimensional Hilbert space accounting for internal degrees of freedom. Different choices of W will change the relevant group $G(C)$. For example, if we want to take into account the spin of the electron the modification is described below. We will take $W = \mathbb{C}$ for simplicity, and indicate the spin-generalization in remarks.

Now, according to the general principles, a ϕ-twisted extension $G(C)^{\rm tw}$ acts on the Hilbert space. In the absence of electromagnetic fields the subgroup of lattice translations is not extended so we have

$$1 \to L(C) \to G(C)^{\rm tw} \to P(C)^{\rm tw} \to 1 \tag{4.24}$$

where $P(C)^{\rm tw}$ is a ϕ-twisted extension of the magnetic point group $P(C) \subset O(d) \times \mathbb{Z}_2$, and we assume the grading ϕ of $G(C)$ is pulled back from that of $P(C)$.

A nice geometrical way to think about Bloch waves is to introduce a pair of dual tori T and $T^\vee$.

♣The notation T is heavily overused! Find another font. ♣

The torus T is the quotient of an affine space E modeled on $\mathbb{R}^d$ by the sublattice $L = L(C)$ or $\mathbb{R}^d$:

$$T := E/L \tag{4.25}$$

On the other hand, we also have the *Brillouin torus*

$$T^\vee := E^\vee / L^\vee \tag{4.26}$$

where $E^\vee$ is the dual space to Euclidean space. In physics it is the momentum space or reciprocal space and the dual torus (normalized so that $L^\vee := {\rm Hom}(L, \mathbb{Z})$) is the set of reciprocal lattice vectors. We should think of $T^\vee$ as the *set of characters of* L. The general characters are of the form:

$$v \mapsto e^{2\pi i k\cdot v} \qquad v \in L(C) \tag{4.27}$$

where $k \in E^\vee$ is a lift of $\bar{k}$. This only depends on the projection $\bar{k} \in T^\vee$ and we can denote this character $\lambda_{\bar{k}}$, or just λ.

Next, we introduce the *Poincaré line bundle*, a complex line bundle

$$\mathcal{L} \to T^\vee \times T \tag{4.28}$$

One way (albeit not the most conceptual way) to define it is to define the restriction $\mathcal{L}_{\bar{k}} \to T$ for each fixed $\bar{k}$. We regard T as the base of a principal L-bundle $L \to E \to T$, and we take the associated complex line bundle determined by the representation $\bar{k}$ of L. Then $\mathcal{L}_{\bar{k}}$ varies smoothly with $\bar{k}$ to define a line bundle over $T^\vee \times T$. The space of sections of the line bundle $\mathcal{L}_{\bar{k}} \to T$ is naturally identified with equivariant functions on the total space E,

$$T^\vee \times E \to \mathbb{C} \tag{4.29}$$

with equivariance determined by $\bar{k}$. This is a mathematical way of stating the familiar Bloch periodicity condition:

$$\psi(\bar{k}, x+v) = \chi_{\bar{k}}(v)\psi(\bar{k}, x) = e^{2\pi i k\cdot v}\psi(\bar{k}, x) \qquad \forall v \in L, x \in E \tag{4.30}$$

We can define a Hilbert space of sections

$$\mathcal{E}_{\bar{k}} = L^2(T; \mathcal{L}_{\bar{k}} \otimes W) \tag{4.31}$$

As $\bar{k}$ varies over $T^\vee$ the Hilbert spaces fit into a Hilbert bundle $\mathcal{E} \to T^\vee$, and we can identify

$$\mathcal{H} := L^2(E;W) \cong L^2(T^\vee;\mathcal{E}) \tag{4.32}$$

♣Maybe comment on Fourier transform D.19-D.22 ♣

We would like to understand the isomorphism (4.32) as an isomorphism of $G(C)^{\rm tw}$ representations. Taking $W = \mathbb{C}$, there is a canonical right action of $G(C)$ on $L^2(E;W)$:

$$(\{R|v\} \cdot \psi)(x) := \begin{cases} \psi(Rx+v) & \phi(R) = +1 \\ \psi^*(Rx+v) & \phi(R) = -1 \end{cases} \tag{4.33}$$

(If we had used the inverse transformation acting on x we would get a left-action.) We would like to understand this in terms of an action on $L^2(T^\vee;\mathcal{E})$.

♣This is treating ψ as a scalar. Discuss how it generalizes for a spinor and for the magnetic point group. That is necessary to get $T^2 = -1$. Also, with this definition $G(C)$ is realized on the Hilbert space without being twisted. ♣

The action of L is clear: It is just a multiplication operator on the fibers. Moreover, if $\psi_{\bar{k}}(x)$ is quasiperiodic as in (4.30), then it is easy to check that $\{R|v\} \cdot \psi_{\bar{k}}$ is quasiperiodic for $\bar{k}'$ with

$$\{R|v\} \cdot \bar{k} = \bar{k}' = \begin{cases} \overline{R^{-1} \cdot k} & \phi(R) = +1 \\ -\overline{R^{-1} \cdot k} & \phi(R) = -1 \end{cases} \tag{4.34}$$

where k is a lift of $\bar{k}$. The tricky part is in lifting this $P(C)$ action to the Hilbert space fibers to make an equivariant bundle. It turns out that $\mathcal{E}$ is a *twisted* equivariant $P(C)$ bundle over $T^\vee$. To explain this tricky point we will introduce some general mathematical ideas which allow us to interpret $\mathcal{E}$ as *twisted vector bundles over a groupoid* $T^\vee//P(C)$.

Remark: The bundle $\mathcal{E} \to T^\vee$ carries a canonical family of flat connections labeled by $\bar{x}_0 \in T$. In order to see this it suffices to describe the parallel transport along straight-line paths in $T^\vee$. Suppose $\bar{k} \in T^\vee$ and we choose a lift $k \in E^\vee$ and a path $k(t) = k + t\Delta k$, where $0 \le t \le 1$ and $\Delta k \in L^\vee$, so that $\overline{k(t)}$ is a closed loop in $T^\vee$. Then, for any $\psi_{\bar{k}} \in \mathcal{E}_{\bar{k}}$ we have to say how it is parallel transported along $\overline{k(t)}$. We interpret the fibers of $\mathcal{E}$ as spaces of quasiperiodic functions and define a family of quasiperiodic functions

$$(U(t) \cdot \psi_{\bar{k}})(x) := e^{2\pi i t \Delta k \cdot (x - x_0)} \psi_{\bar{k}}(x) \tag{4.35}$$

This has quasiperiodicity determined by $\overline{k(t)}$. Computation of the parallel transport around contractible loops shows that the connection is a flat connection: the curvature is zero. However, the holonomy around the closed loop γ represented by the path $\overline{k(t)}$, $0 \le t \le 1$ is clearly multiplication by the *periodic* function:

$$h(\gamma) = e^{2\pi i \Delta k \cdot (x - x_0)} \tag{4.36}$$

Note that it only depends on the projection $\bar{x}_0$ of x_0 to T.

Remark: Let us indicate how to include electron spin. The spin-orbit terms in the Schrodinger equation arise from the Foldy-Wouthuysen expansion of the Dirac equation. In QED the electron wavefunction transforms as a τ-representation of the $\mathbb{Z}_2$-graded group $(\mathrm{Pin}(-1,+3),\tau)$ which restricts to the 4-complex-dimensional pin representation on the components that do not reverse time orientation. In solid state physics we ignore the boosts and hence should define a double cover $\tilde{O}_m$ of $O(3)\times\mathbb{Z}_2$ by the pullback construction:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & \tilde{O}_m \arrow[r, "\tilde{\pi}"] \arrow[d] & O(3)\times\mathbb{Z}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\iota"] & \mathrm{Pin}(-1,3) \arrow[r, "\pi"] & O(1,3) \arrow[r] & 1
\end{tikzcd} \tag{4.37}$$

When pulled back to a representation of $\tilde{O}_m$ the representation decomposes as a sum of (identical) two-dimensional τ-twisted representations of $\tilde{O}_m$. This two-dimensional representation is defined by taking the lift of a reflection $R_{\hat{n}}$ in the plane orthogonal to a unit vector $\hat{n}$ to be

$$(\tilde{R}_{\hat{n}}\cdot\psi)(x,t) = (\hat{n}\cdot\sigma)\psi(R_{\hat{n}}x,t) \tag{4.38}$$

and the lift T of a time-reversal $\bar{T}\in O(3)\times\mathbb{Z}_2$ to be

$$(\tilde{T}\cdot\psi)(x,t) = i\sigma^2\overline{\psi(x,-t)} \tag{4.39}$$

These operators generate $\tilde{\mathcal{O}}_m$. Now, given a crystallographic group $G(C)$, the relevant group that is represented on $\mathcal{H}$ is the pullback $\widehat{G(C)}$ constructed as follows. We first pull back

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & \widehat{P(C)} \arrow[r, "\tilde{\pi}"] \arrow[d] & P(C) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\iota"] & \tilde{\mathcal{O}}_m \arrow[r, "\pi"] & O(3)\times\mathbb{Z}_2 \arrow[r] & 1
\end{tikzcd} \tag{4.40}$$

to construct the proper spin cover of the magnetic translation group acting on electron wavefunctions and then we pull back again

$$\begin{tikzcd}
1 \arrow[r] & L(C) \arrow[r] \arrow[d, equal] & \widehat{G(C)} \arrow[r, "\pi'"] \arrow[d] & \widehat{P(C)} \arrow[r] \arrow[d, "\tilde{\pi}"] & 1 \\
1 \arrow[r] & L(C) \arrow[r, "\iota"] & G(C) \arrow[r, "\pi"] & P(C) \arrow[r] & 1
\end{tikzcd} \tag{4.41}$$

Note that a corollary of this discussion is that, because it is experimentally important that $T^2=-1$ on the electron wavefunction, the relevant pin double cover of spatial reflections for electron wavefunctions in solid state physics is $\mathrm{Pin}^+(3)$ and not $\mathrm{Pin}^-(3)$.

4.2.1 Bands and insulators

The Hamiltonian of a noninteracting electron in a crystal is

$$H = -\frac{1}{2m}\nabla^2 + V + \mathrm{spin}-\mathrm{orbit} \tag{4.42}$$

acting on $L^2(E;W)$. Under this isomorphism it acts on $\mathcal{E}_{\bar{k}}$ with discrete spectrum, since it is an elliptic operator acting on wavefunctions on a compact space. The discrete spectrum varies continuously with $\bar{k}\in T^\vee$, producing a set of *bands*.

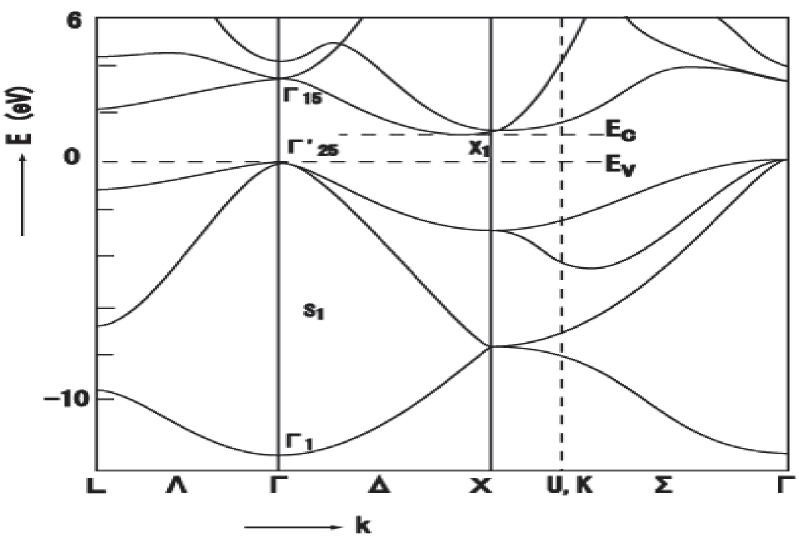

Figure 13: An example of band structure. The horizontal axis is a line of characters starting from the origin (labelled Γ) to the edge of the Brillouin zone. The material shown here is silicon. CHECK!!!

In materials the groundstate configuration of electrons fills up the lowest energy levels consistent with the Pauli principle. In insulators the ground state configuration of electrons fills all the bands up to some level but there is a gap between the filled (valence) and unfilled (conduction) bands, as shown in Figure 13. We choose the Fermi energy E_f to be within the gap.

It follows that in insulators there is a natural splitting of the Hilbert bundle

$$\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+ \tag{4.43}$$

given by the spectrum below and above E_f. In families of materials where the bands do not cross E_f we have a gapped system.

It is clear that $\mathcal{E}^-$ will be finite rank and $\mathcal{E}^+$ is infinite rank. In general $\mathcal{E}^-$ will be a topologically nontrivial vector bundle over the Brillouin torus in two and three dimensions.

Sometimes it is useful to focus on just a few bands near the Fermi level. In this case $\mathcal{E}^\pm$ will be finite rank. We thus have two natural cases to consider: $\mathcal{E}^+$ of infinite rank and $\mathcal{E}^+$ of finite rank.

Remark: Explain the family of flat connections on $\mathcal{E}$ and the "Berry connection"

4.3 Groupoids

4.3.1 Categories

Definition A *category* $\mathcal{C}$ consists of

a.) A set $\mathfrak{X}_0$ of *objects*

b.) A set $\mathfrak{X}_1$ of *morphisms* with two maps $p_0 : \mathfrak{X}_1 \to \mathfrak{X}_0$ and $p_1 : \mathfrak{X}_1 \to \mathfrak{X}_0$. The map $p_0(f) = x_1$ is the *range* map and $p_1(f) = x_0$ is the *domain* map. Morphisms are often denoted as arrows:

$$x_0 \quad \stackrel{f}{\to} \quad x_1 \tag{4.44}$$

The set of all morphisms from x_0 to x_1 is denoted $\hom(x_0, x_1)$.

c.) There is a composition law on the set of *composable morphisms*

$$\mathfrak{X}_2 = \{(f_1, f_2) \in \mathfrak{X}_1 \times \mathfrak{X}_1 | p_0(f_1) = p_1(f_2)\} \tag{4.45}$$

The composition law takes $\mathfrak{X}_2 \to \mathfrak{X}_1$ and may be pictured as the composition of arrows. If $f_1 : x_0 \to x_1$ and $f_2 : x_1 \to x_2$ then the composed arrow will be denoted $f_2 \circ f_1 : x_0 \to x_2$. The composition law satisfies the axioms

1. For all $x \in \mathfrak{X}_0$ there is an identity morphism in $\mathfrak{X}_1$, denoted 1_x, or Id_x, such that $1_{x_1} \circ f = f$ and $f \circ 1_{x_0} = f$.

2. The composition law is associative: If we consider 3 composable morphisms

$$\mathfrak{X}_3 = \{(f_1, f_2, f_3) | p_0(f_i) = p_1(f_{i+1})\} \tag{4.46}$$

then $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$.

Remarks:

1. When defining composition of arrows one needs to make an important notational decision. If $f : x_0 \to x_1$ and $g : x_1 \to x_2$ then the composed arrow is an arrow $x_0 \to x_2$. We will write $g \circ f$ when we want to think of f, g as functions and fg when we think of them as arrows.

2. It is possible to endow the data $\mathfrak{X}_0, \mathfrak{X}_1$ and p_0, p_1 with additional structures, such as topologies, and demand that p_0, p_1 have continuity or other properties.

3. A morphism $f \in \hom(x_0, x_1)$ is said to be *invertible* if there is a morphism $g \in \hom(x_1, x_0)$ such that $g \circ f = 1_{x_0}$ and $f \circ g = 1_{x_1}$. If x_0 and x_1 are objects with an invertible morphism between then then they are called *isomorphic objects*. One key reason to use the language of categories is that objects can have nontrivial automorphisms. That is, $\hom(x_0, x_0)$ can have more than just 1_{x_0} in it. When this is true then it is tricky to speak of "equality" of objects, and the language of categories becomes very helpful.

4. There is a geometrical picture associated with a category where objects are points, morphisms are one-simplices etc. See Figure 14. Elements of $\mathfrak{X}_2$ are associated with 2-simplices, elements of $\mathfrak{X}_3$ are associated with 3-simplices. See Figure 15. We can clearly continue this to $\mathfrak{X}_n$ for n-tuples of composable morphisms and create what is known as a *simplicial space*.

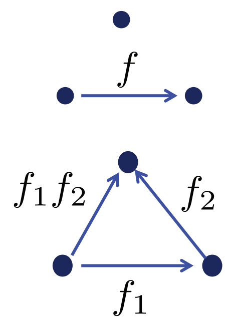

Figure 14: Elementary $0, 1, 2$ simplices in the simplicial space $|\mathcal{C}|$ of a category

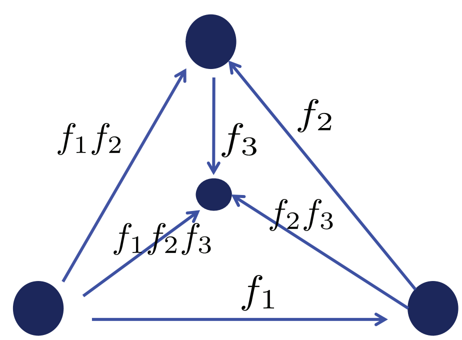

Figure 15: An elementary 3-simplex in the simplicial space $|\mathcal{C}|$ of a category

4.3.2 Groups and groupoids

Let us consider a category with only *one object*, but we allow the possibility that there are several morphisms. For such a category let us look carefully at the structure on morphisms $f \in \mathfrak{X}_1$. We know that there is a binary operation, with an identity 1 which is associative.

But this is just the definition of a monoid!

If we have in addition inverses then we get a group. Hence:

Definition A *group* is a category with one object, all of whose morphisms are invertible.

We should think of it geometrically in terms of Figure 14 and Figure 15. (In these figures identify all the objects to one point.)

Definition A *groupoid* $\mathfrak{G}$ is a category all of whose morphisms are invertible.

Note that for any object x in a groupoid, $\hom(x,x)$ is a group. It is called the *automorphism group* of the object x. It is useful to define a homomorphism μ of groupoids to be a functor so if f_1, f_2 are composable then $\mu(f_2 \circ f_1) = \mu(f_2) \circ \mu(f_1)$. In particular, we will use below the notion of a homomorphism of a groupoid $\mathfrak{G}$ to the group $\mathbb{Z}_2$ (thought of multiplicatively). Since a group has a single object this amounts to a map $\mu : \mathfrak{G}_1 \to \mathbb{Z}_2$ so that $\mu(f_2 \circ f_1) = \mu(f_2)\mu(f_1)$.

Example 1: The key example we will need is provided by the action of a group G on a space X. We can then form the a groupoid denoted $\mathfrak{G} = X/\!/G$. This groupoid has as objects and morphisms given by

$$\mathfrak{G}_0 := X \qquad\qquad \mathfrak{G}_1 := X \times G \tag{4.47}$$

and $p_1(x,g) = x$ while $p_0(x,g) = g \cdot x$ for a left-action or $x \cdot g$ for a right-action. The picture we associate with a morphism (x,g) is an arrow from x to gx (for a left action) or xg (for a right action). The groupoid $X/\!/G$ contains important information not available in the ordinary quotient X/G in that it contains the automorphism group of the action on x. We should think of the points x as having the isotropy group attached to them. When identifying a group with a category with one object we are really speaking of the groupoid $\mathfrak{G} = pt/\!/G$, where the group G acts on a single point. The main example of groupoids used later in this chapter is the groupoid $T^\vee/\!/P(C)$.

Example 2: Another useful example of a groupoid is provided by a manifold M with an open cover $\{\mathcal{U}_\alpha\}$ labeled by an ordered set. Then

$$\mathfrak{G}_0 = \amalg_\alpha \mathcal{U}_\alpha \tag{4.48}$$

$$\mathfrak{G}_1 = \amalg_{\alpha<\beta} \mathcal{U}_\alpha \cap \mathcal{U}_\beta \tag{4.49}$$

and then we take p_0, p_1 to be the inclusion maps into β, α, respectively.

4.3.3 Vector bundles on groupoids

By a complex vector bundle on a groupoid $\mathfrak{G}$ we mean:

1. V1: A complex vector bundle $E \to \mathfrak{G}_0$. Fibers will be denoted E_x for $x \in \mathfrak{G}_0$.
2. V2: An isomorphism $\psi : p_1^* E \to p_0^* E$. Concretely, if $(x_0 \xrightarrow{f} x_1) \in \mathfrak{G}_1$ then we have a $\mathbb{C}$-linear isomorphism
$$\psi_f : E_{x_0} \to E_{x_1}. \tag{4.50}$$
3. V3: Cocycle condition on $\mathfrak{G}_2$, if $(f,g) \in \mathfrak{G}_2$ then we must have
$$\psi_g \psi_f = \psi_{gf} : E_{x_0} \to E_{x_2} \tag{4.51}$$

Example 1: In the case $\mathfrak{G} = X//G$, a bundle on the groupoid is the same thing as a G-equivariant bundle on X. The maps ψ_f provide a lifting of the group action on X to the total space of the bundle E.

Example 2: In the case of the groupoid $\mathfrak{G}$ provided by an open cover $\{\mathcal{U}_\alpha\}$, the ψ_f are the clutching transformations, and the cocycle condition is the standard one of bundle theory.

4.4 Twisted equivariant K-theory of groupoids

4.4.1 Central extensions as line bundles over a group

Let us now return to the twisted equivariant K-theory of a point, but now incorporating the viewpoint of a group as a groupoid $\mathfrak{G} = pt//G$.

We will identify a central extension of G with a complex line bundle over $\mathfrak{G}_1$, the space of morphisms: For each group element g we have a line L_g and there is a multiplication

$$\lambda_{g_2,g_1} : L_{g_2} \otimes L_{g_1} \to L_{g_2 g_1} \tag{4.52}$$

The familiar cocycle is obtained by choosing basis vectors ℓ_g for each line L_g and then comparing basis vectors:

$$\lambda_{g_2,g_1}(\ell_{g_2} \otimes \ell_{g_1}) = \lambda(g_2, g_1)\ell_{g_2 g_1} \tag{4.53}$$

The line bundle is defined over $\mathfrak{G}_1$, the multiplication is defined over $\mathfrak{G}_2$, and there is a consistency condition defined for each element of $\mathfrak{G}_3$. This is the requirement that the different ways of using λ to multiply

$$L_{g_3} \otimes L_{g_2} \otimes L_{g_1} \to L_{g_3 g_2 g_1} \tag{4.54}$$

yield the same answer. Choosing a basis we recover the 2-cocycle condition (4.8). Making a change of basis ℓ_g induces a change of the 2-cocycle by a coboundary.

Of course, a given group will in general admit different isomorphism classes of such line bundles with multipliation. The isomorphism class is labeled by an an element of $H^2(BG; U(1)) \cong H^2_G(pt; U(1))$.

If we denote by τ the isomorphism class of the central extension then in this language a representation of G^τ can be thought of as a vector bundle over the groupoid $\mathfrak{G}$ together with isomorphisms defined on $\mathfrak{G}_1$

$$\rho_g : L_g \otimes V \to V \tag{4.55}$$

satisfying the consistency conditions defined on $\mathfrak{G}_2$. These are obtained by demanding that the two ways of multiplying

$$L_{g_2} \otimes L_{g_1} \otimes V \to V \tag{4.56}$$

yield the same answer. ♣Write out the commutative diagram ♣

4.4.2 ϕ-twisted extensions and representations of a graded group

Now for a $\mathbb{Z}_2$-graded group G with $\phi : G \to \mathbb{Z}_2$ we can similarly interpret ϕ-twisted extensions in terms of complex line bundles.

Now, in the categorical picture of a group, a $\mathbb{Z}_2$ graded group has a sign $\phi(g) = \pm 1$ attached to each morphism (one-simplex).

Let us introduce the useful notation

$$ {}^{\phi(g)}V := \begin{cases} V & \phi(g) = +1 \\ \overline{V} & \phi(g) = -1 \end{cases} \tag{4.57} $$

where V is a complex vector space.

Then, a ϕ-twisted extension of G is equivalent to a multiplication law

$$ \lambda_{g_2,g_1} : {}^{\phi(g_1)}L_{g_2} \otimes L_{g_1} \to L_{g_2 g_1} \tag{4.58} $$

Again we require associativity of the map

♣Check it is the same condition as that quoted above. ♣

$$ {}^{\phi(g_2 g_1)}L_{g_3} \otimes {}^{\phi(g_1)}L_{g_2} \otimes L_{g_1} \to L_{g_3 g_2 g_1} \tag{4.59} $$

If we choose basis vectors ℓ_g for each complex line L_g we obtain the ϕ-twisted co-cycle condition given in (4.10) above.

If, now G is a *bigraded group* with bigrading (ϕ, χ) then we can incorporate the idea of a (τ, ϕ, χ)-twisted representation of G:

First, each morphism in $\mathfrak{G}_1$ is bigraded by (ϕ, χ).

Next, we choose a complex line bundle over $\mathfrak{G}_1$, the space of morphisms, satisfying (4.58). This determines a ϕ-twisted extension τ. We now consider these lines to be $\mathbb{Z}_2$-graded with parity determined by $\chi(g)$.

Then, a (τ, ϕ, χ)-twisted representation of G is a vector bundle over the groupoid $\mathfrak{G} = pt//G$ with a compatible set of *even* and $\mathbb{C}$-*linear* isomorphisms

$$ \rho_g : {}^{\phi(g)}(L_g \otimes V) \to V \tag{4.60} $$

satisfying the consistency conditions defined on $\mathfrak{G}_2$. These are obtained by demanding that the two ways of multiplying

$$ {}^{\phi(g_2 g_1)}\left({}^{\phi(g_1)}L_{g_2} \otimes L_{g_1} \otimes V \right) \to V \tag{4.61} $$

yield the same answer.

♣Write out the commutative diagram ♣

Remark: Once again, ϕ-twisted extensions of $\mathfrak{G}$ are classified by cohomology, but now the cohomology is itself twisted cohomology. ***** EXPLAIN MORE ****

4.4.3 Twisting the K-theory of a groupoid

It is now straightforward to generalize to groupoids $\mathfrak{G}$ more general than $pt//G$.

Definition: Now suppose we have a bigraded groupoid $\mathfrak{G}$, that is, a groupoid with two homomorphisms $\phi, \chi : \mathfrak{G} \to \mathbb{Z}_2$. Then we define a (ϕ, χ)-*twisting of the K-theory of* $\mathfrak{G}$ to be

1. A collection of $\mathbb{Z}_2$-graded complex lines L_f, for all $f \in \mathfrak{G}_1$, graded by $\chi(f)$.

2. A collection of $\mathbb{C}$-linear even isomorphisms

$$\lambda_{f_2,f_1} : {}^{\phi(f_1)}L_{f_2} \otimes L_{f_1} \to L_{f_2 \circ f_1} \tag{4.62}$$

3. Such that the λ satisfies the cocycle condition which requires that for all 3-composable morphisms $(f_1, f_2, f_3) \in \mathfrak{G}_3$ the two ways of multiplying

$$ {}^{\phi(f_2 \circ f_1)}L_{f_3} \otimes {}^{\phi(f_1)}L_{f_2} \otimes L_{f_1} \to L_{f_3 \circ f_2 \circ f_1} \tag{4.63}$$

give the same map.

Remarks:

1. We define a twisting of the K-theory of $\mathfrak{G}$ *before* we define the K-theory of $\mathfrak{G}$!

2. These twistings have a nice generalization to a class of geometrical twistings determined by bundles of central simple superalgebras (over $\mathfrak{G}_0$) and invertible bimodules (over $\mathfrak{G}_1$). The isomorphism classes of such twistings, for $\mathfrak{G} = X//G$ are classified - as a set - by

$$H^0_G(X; \mathbb{Z}_2) \times H^1_G(X; \mathbb{Z}_2) \times H^3_G(X; \mathbb{Z}_\phi) \tag{4.64}$$

3. **** EXPLAIN THAT THE TWISTINGS WE ENCOUNTER ARE NONEQUIVARIANTLY TRIVIAL ****

4. There are yet more general twistings, but they have not yet had an application in physics. Not even in string theory.

4.4.4 Definition of a twisted bundle and twisted K-theory of a groupoid

Definition Given a (ϕ, τ, χ)-twisting of the K-theory of a groupoid $\mathfrak{G}$, a (ϕ, τ, χ)-twisted bundle over $\mathfrak{G}$ is

1. A complex $\mathbb{Z}_2$-graded bundle over $\mathfrak{G}_0$.

2. A collection of $\mathbb{C}$-linear, even isomorphisms over morphsms $f : x_0 \to x_1$ in $\mathfrak{G}_1$:

$$\rho_f : {}^{\phi(f)}\left(L_f \otimes V_{x_0}\right) \to V_{x_1} \tag{4.65}$$

3. Such that the $\rho_{x,f}$ satisfy the compatibility condition over $\mathfrak{G}_2$ obtained by requiring that the two ways of multiplying

$$ {}^{\phi(f_2 \circ f_1)}\left({}^{\phi(f_1)}L_{f_2} \otimes L_{f_1} \otimes V_{x_0}\right) \to V_{x_2} \tag{4.66}$$

give the same answer.

Finally, we can give a definition of the twisted equivariant K-theory over a groupoid $\mathfrak{G}$:

Let $\nu = (\phi, \tau, \chi)$ denote a twisting of the K-theory of $\mathfrak{G}$.

The isomorphisms of ν-twisted bundles form a monoid $\mathrm{Vect}_s^\nu(\mathfrak{G})$ under direct sum. (The subscript s reminds us that the bundles are $\mathbb{Z}_2$-graded.)

We define a sub-monoid of "trivial" elements, denoted $\mathrm{Triv}_s^\nu(\mathfrak{G})$ to be those with an odd invertible automorphism $T : V \to V$. Then we set

$$K^\nu(\mathfrak{G}) := \mathrm{Vect}_s^\nu(\mathfrak{G})/\mathrm{Triv}_s^\nu(\mathfrak{G}) \tag{4.67}$$

In the case that the groupoid is a global quotient $\mathfrak{G} = X//G$ we also use the notation

$$K^\nu(\mathfrak{G}) := {}^\phi K_G^{\tau,\chi}(X) \tag{4.68}$$

Remarks:

1. If we take τ and χ to be trivial and $G = \mathbb{Z}_2 \times G_0$ with $G_0 = \ker\phi$ then (4.68) reduces to G_0-equivariant KR theory.
2. For $X = pt$ we recover the previous description.

4.5 Applications to topological band structure

Now we will apply some of the above general ideas to the case of topological band structure.

4.5.1 The canonical twisting

We now consider the groupoid $T^\vee // P(C)$. We claim there is a canonical (ϕ, χ)-twisting of the K-theory of this groupoid.

The canonical twisting is actually a special case of a more general result discussed in [26], Section 9, and references therein. We will explain the the statement of the general result, omit the proof, but explain the upshot for the special case of interest for band structure.

The general result concerns extensions

$$1 \to N \to G \to Q \to 1 \tag{4.69}$$

Let X be the space of irreps of N. Then Q acts on X as follows:

Suppose $\rho_V : N \to \mathrm{Aut}(V)$ is an irrep of N. Then we can twist it by an element $q \in Q$ by choosing a lift $s(q) \in G$ and defining

$$\rho_V^q(n) := \rho_V(s(q)ns(q)^{-1}) \tag{4.70}$$

Note that since N is normal in G this makes sense: $s(q)ns(q)^{-1} \in N$, and ρ_V^q is a new representation of N on the vector space V. It clearly depends on the choice of section s, but the isomorphism class of of ρ_V^q does not depend on the choice of X. In general, the isomorphism class is distinct from ρ_V. Denoting the isomorphism class by $[V]$ for brevity, there is a *right*-action of Q on X,

$$[V] \to [V] \cdot q. \tag{4.71}$$

Example 1: Consider the permutation group S_3 as an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ with the $\mathbb{Z}_3$ normal subgroup identified with the subgroup A_3 of even permutations:

$$1 \to A_3 \to S_3 \to \mathbb{Z}_2 \to 1 \tag{4.72}$$

There are three characters of A_3 labeled by $\chi(1)$, a third root of unity. So, choosing a nontrivial third root of unity ω we can identify $X = \{1, \omega, \omega^2\}$. Now choose the lift of the nontrivial element q of $\mathbb{Z}_2$ to be $s(q) = (12)$. Then $(12)(123)(12) = (132)$ and hence the action of Q on X fixes 1 and exchanges ω and ω^2.

Example 2: In our crystallographic example, $N = L(C)$, so $X = T^\vee$ is the Brillouin torus. An isomorphism class is determined by a character $\chi_{\bar{k}}$ and $R \in P(C) \subset O(d)$ acts by

$$R : \chi_{\bar{k}} \to \chi_{\bar{k}'} \tag{4.73}$$

where, if k and k' are lifts of $\bar{k}$ and $\bar{k}'$ then

$$k' = R^{-1} k \tag{4.74}$$

Next, the statement of the theorem is that there is a canonical central extension ν of $\mathfrak{G} = X//Q$ such that a representation E of G is in one-one correspondence with ν-twisted bundles [30] over $\mathfrak{G}$. In fact, if G is compact this sets up an isomorphism

$$K_G(pt) \cong K_Q^\nu(X) \cong K^\nu(\mathfrak{G}). \tag{4.75}$$

The result can be extended to (ϕ, χ)-twisted bundles, but we will restrict to the simplest case here.

In the application to Bloch theory the extension of the groupoid is simply the following: Given $R \in P(C)$ and $\bar{k} \in T^\vee$ we need to produce a line $L_{\bar{k},R}$. We identify it with the vector space of equivariant functions from the fiber in $G(C)$ above R. That is $\pi^{-1}(R) \subset G(C)$ is the set of space group elements g projecting to R. A typical element will be $\{R|v_0\}$ where v_0 might or might not be in the lattice L. (If the group is non-symmorphic then for some R it will be impossible to take v_0 in L.) The set $\pi^{-1}(R)$ is a torsor for L. We consider the set of complex-valued functions

$$f : \pi^{-1}(R) \to \mathbb{C} \tag{4.76}$$

which satisfy the equivariance condition:

$$f(T_v g) = \chi_{\bar{k}}(v) f(g) \tag{4.77}$$

[30] actually, sheaves

for all $v \in L(C)$ and all $g \in \pi^{-1}(R)$. Note that the set of such functions (4.76) satisfying the equivariance constraint (4.77) form a vector space. In fact, it is a one-dimensional vector space: If we choose any lift $s(R) \in \pi^{-1}(R)$ then the value of the function f at $s(R)$ determines its values on all other points in the fiber.

The one-dimensional vector space of equivariant functions on the fiber $\pi^{-1}(R)$ is identified with the line bundle of the twisting $L_{\bar{k},R}$.

It remains to show that we can define a product

$$\lambda_{\bar{k},R_1,R_2} : L_{R_1^{-1}\bar{k},R_2} \otimes L_{\bar{k},R_1} \to L_{\bar{k},R_1R_2} \tag{4.78}$$

which satisfies the cocycle condition. To do this we consider the general element of $L_{\bar{k},R_1}$. ♣careful about left vs. right actions ♣ It is a function satisfying

$$f_1(\{R_1|v_0+v\}) = \chi_{\bar{k}}(v) f_1(\{R_1|v_0\}) \qquad \forall v \in L \tag{4.79}$$

where v_0 represents a choice of section so that $\{R_1|v_0\} \in G(C)$. Similarly,

$$f_2(\{R_2|v_0'+v\}) = \chi_{R_1^{-1}\bar{k}}(v) f_2(\{R_2|v_0'\}) \qquad \forall v \in L \tag{4.80}$$

Then, note that $\{R_1|v_0\} \cdot \{R_2|v_0'\} = \{R_1R_2|v_0+R_1v_0'\}$ so that we can define the product: $\lambda_{\bar{k},R_1,R_2}(f_1 \otimes f_2)$ to be the function f_{12} defined by:

$$f_{12}(\{R_1R_2|v_0+R_1v_0'+v\} := \chi_{\bar{k}}(v) f_1(\{R_1|v_0\}) f_2(\{R_2|v_0'\}) \tag{4.81}$$

where $v \in L$. Note that f_{12} is, by construction, an equivariant function on $\pi^{-1}(R_1R_2)$ and moreover it is bilinear in f_1 and f_2. Moreover, one can easily check that the definition of the product is in fact independent of the choice of section determined by v_0, v_0'. We leave the verification of the cocycle relation to the reader. This completes the construction of the canonical twisting ν^{can} for the K-theory of the groupoid $T^\vee//P(C)$ relevant for band structure.

It is interesting to consider the restriction of the canonical twisting ν^{can} to the subgroupoid of the fixed subgroup of a particular point $\bar{k} \in T^\vee$. That is, let $P(C,\bar{k}) \subset P(C)$ denote the subgroup of $P(C)$ which fixes $\bar{k} \in T^\vee$. Then there is a natural inclusion homomorphism

$$\bar{k}//P(C,\bar{k}) \hookrightarrow T^\vee//P(C) \tag{4.82}$$

and the twisting ν^{can} will pull back to a line bundle with multiplication over the group $P(C,\bar{k})$. This is just a central extension of $P(C,\bar{k})$ and we will now describe it. Since $P(C,\bar{k})$ is a finite group we can enumerate its elements as R_i, $i = 1,\dots,n$ for some integer n. Then we choose a section of $\pi : G(C) \to P(C)$ over $P(C,\bar{k})$, that is, for each i we choose an element $\{R_i|v^i\} \in G(C)$. Now we choose a basis for $L_{\bar{k},R_i}$ by declaring the basis vector to be the equivariant function determined by $f(\{R_i|v^i\}) = 1$. Now, for any pair of elements $R_i, R_{i'} \in P(C,\bar{k})$ there is an i'' so that $R_iR_{i'} = R_{i''}$. Then we can define vectors $v^{ii'} \in L$ by

$$\{R_i|v^i\}\{R_{i'}|v^{i'}\} = \{R_iR_{i'}|v^i + R_iv^{i'}\} = \{1|v^{ii'}\}\{R_{i''}|v^{i''}\} \tag{4.83}$$

That is:

$$v^{ii'} := v^i + R_i v^{i'} - v^{i''} \tag{4.84}$$

It follows from the definition of the cocycle that, with respect to this basis it is

$$\lambda(R_i, R_{i'}) = \chi_{\bar{k}}(v^{ii'}) \tag{4.85}$$

This determines the central extension fairly explicitly.

Remarks

1. If $G(C)$ is symmorphic (a split extension of $P(C)$ by $L(C)$) then we can choose $v^i = 0$ for all elements R_i and hence the central extension will be trivial.
2. The above central extension has a much more invariant and conceptual description, given in Proposition 9.31 of [26]. ♣Give it here? ♣

Example: Let us return to the two-dimensional crystallographic group $G(C)$ described by the generators (1.50) and (1.51). The character on $L = \mathbb{Z}^2$ given by $\lambda(n^1, n^2) = (-1)^{n^1+n^2}$ is clearly fixed by the full point group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by

$$R_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.86}$$

since it is unchanged if we change the sign of n^1 or n^2. A lift of $\lambda = \chi_{\bar{k}}$ is $k = (\frac{1}{2}, \frac{1}{2})$. The generators are $g_1 = \{R_1|(\delta, \frac{1}{2})\}$ and $g_2 = \{R_2|0\}$. Then we lift R_1R_2 to the element $g_1g_2 = \{R_1R_2|(\delta, \frac{1}{2})\}$ but then $g_2g_1 = \{R_1R_2|(\delta, -\frac{1}{2})\}$. It follows that the cocycle satisfies

$$\frac{\lambda(R_1, R_2)}{\lambda(R_2, R_1)} = \frac{\chi_{\bar{k}}(v^{12})}{\chi_{\bar{k}}(v^{21})} = -1 \tag{4.87}$$

In general, if the group $P(C, \bar{k})$ is abelian then the ratio

$$\frac{\lambda(R_i, R_{i'})}{\lambda(R_{i'}, R_i)} = \frac{\chi_{\bar{k}}(v^{ii'})}{\chi_{\bar{k}}(v^{i'i})} = \exp[2\pi i k \cdot \Big((1 - R_i)v^{i'} - (1 - R_{i'})v^i\Big)] \tag{4.88}$$

is gauge invariant and if it is nontrivial the central extension is nontrivial. In our example the central extension is D_4. ♣check! ♣

4.5.2 $\mathcal{E}$ as a $\nu^{\rm can}$-twisted bundle over $T^\vee//P(C)$

Returning to the general theorem (4.75) a representation E of G defines a vector bundle over X whose fiber at $[V]$ is

$$\mathcal{E}_{[V]} = \mathrm{Hom}_N(V, E) \tag{4.89}$$

That is, the degeneracy space in the isotypical decomposition of E as a representation of N. On the RHS we have used a specific representative V of an isomorphism class. In the application to band structure, this will just be the space $\mathcal{E}_{\bar{k}}$ corresponding to Bloch waves of momentum $\bar{k}$.

The required action

$$L_{\bar{k},R} \otimes \mathcal{E}_{\bar{k}} \to \mathcal{E}_{R^{-1}\bar{k}} \tag{4.90}$$

is defined by choosing a lift $\{R|v\}$ of R in $G(C)$ and setting

$$\rho_{\bar{k},R} : f \otimes \psi \to \psi' \tag{4.91}$$

where

$$\psi'(x) := f(\{R|v\})\psi(Rx + v) \tag{4.92}$$

♣CHECK! ♣

4.5.3 Modifications for including (ϕ, χ)

Now let us suppose that $G(C)$ is bigraded by (ϕ, χ) and show how the canonical twisting is modified.

Returning to the general situation, (4.69) we assume that the grading on G factors through a grading on Q. That is, there is a homomorphism $\phi : Q \to \mathbb{Z}_2$ defining a $\mathbb{Z}_2$ grading on Q and that on G is just the pullback. Then the action of Q on the space of irreps X is modified to

$$\rho_V^q(n) := \begin{cases} \rho_V(s(q)ns(q)^{-1}) & \phi(q) = +1 \\ \bar{\rho}_{\bar{V}}(s(q)ns(q)^{-1}) & \phi(q) = -1 \end{cases} \tag{4.93}$$

In the case of band structure, we are assuming that the grading on $G(C)$ is pulled back from a grading on the (magnetic) point group and then the modified action on $T^\vee$ is

$$R^{-1} \cdot \bar{k} = \begin{cases} \overline{R^{-1}k} & \phi(R) = +1 \\ -\overline{R^{-1}k} & \phi(R) = -1 \end{cases} \tag{4.94}$$

where k is a lift of $\bar{k}$ to $L^\vee$.

Now we must generalize to the multiplication of the form (4.58), that is

$$\lambda_{\bar{k},R_1,R_2} : {}^{\phi(R_1)}L_{R_1^{-1}\bar{k},R_2} \otimes L_{\bar{k},R_1} \to L_{\bar{k},R_1R_2} \tag{4.95}$$

This will be given by an equation of the form (4.81) but with a complex conjugation on f_2. ♣check ♣

Now, including both ϕ and χ the generalization of equations (4.90)-(4.92) for $W = \mathbb{C}$ isf

$$\psi'(x) := \begin{cases} f(\{R|v\})\psi(Rx + v) & \phi(R) = +1 \\ \overline{f(\{R|v\})}\ \overline{\psi(Rx + v)} & \phi(R) = -1 \end{cases} \tag{4.96}$$

GIVE GENERALIZATION WITH SPIN.

4.5.4 K-theory classification of phases in topological band theory

Having set up this formalism it is fairly straightforward to relate the classification of phases, and the reduced phases for the set of Hamiltonians of noninteracting electrons with $G(C)$-symmetry, where $G(C)$ is a bigraded group by (ϕ, χ) as follows:

1. If $\mathcal{E}^+$ is finite-rank, then the components $\mathcal{TP}$ are in 1-1 correspondence with the monoid of $\mathbb{Z}_2$-graded twisted equivariant bundles
$$\mathcal{TP} \cong \mathrm{Vect}_s^{\nu^{\mathrm{can}}}(T^\vee /\!/ P(C)) \tag{4.97}$$
2. If $\mathcal{E}^+$ is finite-rank and we define trivial phases by those such that $\mathcal{E}$ admits an invertible odd intertwiner then the abelian group $\mathcal{RTP}$ is a K-theory group
$$\mathcal{RTP} \cong K^{\nu^{\mathrm{can}}}(T^\vee /\!/ P(C)) \tag{4.98}$$
3. If $\mathcal{E}^+$ is infinite-rank, then there is a projection of the components $\mathcal{TP}$ onto the monoid of $\mathbb{Z}_2$-graded twisted equivariant bundles
$$\mathcal{TP} \to \mathrm{Vect}_s^{\nu^{\mathrm{can}}}(T^\vee /\!/ P(C)) \tag{4.99}$$

♣Make the arrow an onto symbol. ♣

4. If $\mathcal{E}^+$ is infinite-rank, and we define trivial phases by those such that $\mathcal{E}$ admits an invertible odd intertwiner then the abelian group $\mathcal{RTP}$ is a K-theory group
$$\mathcal{RTP} \cong K^{\nu^{\mathrm{can}}}(T^\vee /\!/ P(C)) \tag{4.100}$$

EXPLAIN A LITTLE BIT

In the very special case that the group $G(C)^\tau$ splits as $U^{\mathrm{tw}} \times G_0$ where $G_0 = \ker(\phi) \cap \ker(\chi)$ and U^{tw} is a CT-group then, using the main theorem of Section §2.6 and the relation between degree-shift and twisting by a Clifford algebra we can relate the above twisted equivariant K-theories to more standard K-theories.

For example, in the case that $\mathcal{E}^+$ is infinite-rank we necessarily have $\chi = 1$. Then we just have three cases, as we have seen repeatedly above. In this case we have

Subgroup $U \subset M_2$	T^2	$K^{\nu^{\mathrm{can}}}(T^\vee /\!/ P)$
$\{1\}$		$K_P^{\nu^{\mathrm{can}}}(T^\vee)$
$\{1, \bar{T}\}$	$+1$	$KR_{P_0}^{\nu_0}(T^\vee)$
$\{1, \bar{T}\}$	-1	$KR_{P_0}^{\nu_0 - 4}(T^\vee)$

where P_0 is the projection of G_0 into $O(d)$.

Finally, we compare with the literature. Essentially three special cases have been discussed:

1. Take $G(C) = \langle I \rangle \times L$, where I is inversion, and $\phi(I) = \chi(I) = 1$. Then the relevant K-theory group is $K^0_{\mathbb{Z}_2}(T^\vee)$. This was studied in [50].

2. Take $G(C) = \langle \bar{T} \rangle \times L$, where $\bar{T}$ is time-reversal, and $\phi(\bar{T}) = -1$ and $\chi(\bar{T}) = +1$. Because we are working with electrons the lift T of $\bar{T}$ has square $T^2 = -1$ determining τ. Then the relevant K-theory group is $KR^{-4}(T^\vee)$. This was studied in [33, 44].

3. Take $G(C) = \langle \bar{T}, \bar{I} \rangle \times L$, where $\bar{T}$ is time-reversal, and the bigrading is defined as above. Because we are working with electrons the lift T of $\bar{T}$ has square $T^2 = -1$, while $I^2 = +1$. This determines τ. Then the relevant K-theory group is $KO^{-4}_{\mathbb{Z}_2}(T^\vee)$. This was studied in [27, 28].

COMMENT ON KANE-MELE and CHERN-SIMONS INVARIANTS.

4.6 Computation of some equivariant K-theory invariants via localization

These equivariant K-theory groups are not easy to compute. What are some new invariants? One way to get a handle on some of them is via the use of "localization." This allows us to simplify the K-theory groups and still retain some nontrivial information.

4.6.1 Localization of rings and modules

If R is a ring, and $S \subset R$ a subset closed under multiplication then we can form the localized ring $S^{-1}R$ consisting of elements r/s, $r \in R, s \in S$ with equivalence relation $r_1/s_1 = r_2/s_2$ if $ur_1s_2 = ur_2s_1$ for some $u \in S$, and we define the ring structure:

$$\frac{r_1}{s_1}\frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2} \qquad \frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \tag{4.101}$$

Similarly, if M is a module over a ring R then we can localize the module $S^{-1}M$. This is a module over the ring $S^{-1}R$. $S^{-1}M$ consists of elements $\frac{m}{s}$ with $m \in M, s \in S$ with $m_1/s_1 = m_2/s_2$ if $um_1s_2 = um_2s_1$ for some $u \in S$.

The module $S^{-1}M$ is called *the localization of M at S*.

Remarks and Examples:

1. If there is an element $s_* \in S$ with $s_*M = 0$ then $S^{-1}M = 0$. Proof: $1 \cdot m = s_*^{-1} \cdot s_* \cdot m = s_*^{-1} \cdot 0 = 0$.

2. A common construction is the following: For any $r \in R$ we can form the multiplicative set $S_r := \{r^n\}_{n \in \mathbb{Z}}$. Then we denote $S_r^{-1}R := R_r$ and if M is an R-module, $M_r := S_r^{-1}M$. A key remark for what follows is the following. If M is a module over R and r annihilates M then $M_r = \{0\}$ is the 0-module, because

$$m = 1 \cdot m = \frac{1}{r} \cdot (rm) = 0 \tag{4.102}$$

for all $m \in M$.

3. In particular: For $G = \mathbb{Z}_2$ we have

$$R(\mathbb{Z}_2) = \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1) = \mathbb{Z} \oplus \mathbb{Z}\varepsilon.$$

where ε is the sign representation $\varepsilon(\sigma) = -1$, for σ the nontrivial element of $\mathbb{Z}_2$. Let us choose $r = 1 - \varepsilon$. Then we claim:

$$R(\mathbb{Z}_2)_{1-\varepsilon} \cong \mathbb{Z}[\frac{1}{2}] \tag{4.103}$$

To see this note that

$$n + m\varepsilon = \frac{1}{1-\varepsilon}(1-\varepsilon)(n + m\varepsilon) = \frac{1}{1-\varepsilon}(n-m)(1-\varepsilon) = n - m \tag{4.104}$$

Now note that $2 = (1+\varepsilon) + (1-\varepsilon)$ so, multiplying by $(1-\varepsilon)$ we derive

$$2\frac{1}{1-\varepsilon} = 1 \tag{4.105}$$

(Alternatively, divide the equation $(1-\varepsilon)^2 = 2(1-\varepsilon)$ by $(1-\varepsilon)^2$.) In any case, we have inverted $2 \in R(\mathbb{Z}_2)$, with $(1-\varepsilon)^{-1} = \frac{1}{2}$. So now we have the general element of the ring $S^{-1}R$ is

$$\frac{n + m\varepsilon}{(1-\varepsilon)^k} \cong \frac{n-m}{2^k} \tag{4.106}$$

where n, m, k are arbitrary integers. In this sense, inverting $1 - \varepsilon$ in $RO(\mathbb{Z}_2)$ is essentially the same thing as setting $\varepsilon = -1$:

4. One important construction is the following: Suppose $P \subset R$ is a *prime* ideal. Then $S := R - P$ is a multiplicative set: for if $s_1 s_2 \in P$ then, by definition s_1 or s_2 is in P. In this case the localization $S^{-1}R$ is called *the localization at* P and denoted R_P. Similarly, modules localized at P are denoted M_P. As an important example consider the representation ring $R(G)$, i.e. the character ring, of a compact group G. Let γ be a conjugacy class in G. The set of representations whose character vanishes on γ is a prime ideal. We can localize at this ideal. That is, we can take S to be the multiplicative set of virtual representations ρ whose character $\chi_\rho(\gamma) \neq 0$. This is used in Atiyah and Segal [10].

5. Good references are Eisenbud [21] ch.2 and Atiyah-MacDonald [9], ch. 3

4.6.2 The localization theorem

Segal's localization theorem for K-theory, nicely reviewed in Atiyah-Segal *****

Suppose a compact group G acts on a space X. Then $K_G(X)$ is a $K_G(pt) = R(G)$-module. If we choose a multiplicative set $S \subset R(G)$ we can localize $R(G)$ and the module $K_G(X)$ at S.

Segal, and Atiyah-Segal consider a conjugacy class $\gamma \subset G$ and consider localizing at the prime ideal of representations whose character vanishes on γ. Then let

$$X^\gamma := \cup_{g \in \gamma} X^g \tag{4.107}$$

where for $g \in G$, X^g is the fixed point set of the action by g. There is a pullback map

$$\iota^* : K_G(X) \to K_G(X^\gamma) \tag{4.108}$$

and the key theorem states that this becomes an isomorphism when we invert the characters which do *not* vanish at γ. That is, if $S = \{\chi | \chi(\gamma) \neq 0\}$ then

$$\iota^* : S^{-1}K_G(X) \to S^{-1}K_G(X^\gamma) \tag{4.109}$$

is an isomorphism.

The main idea of the proof is to break X up into a union of orbits G/H. Whenever H is a proper subgroup of G there an argument in [10] shows there is an element $\chi_* \in S$ such that $\chi_* K_H(pt) = 0$, but $K_G(G/H) \cong K_H(pt)$ (the fiber of a G-equivariant bundle on G/H at the coset $1 \cdot H$ is an H-representation, and that fiber determines the entire bundle). Therefore, by remark 1 above, the "contribution" of any orbit is zero.

For our application we will not use exactly Segal's theorem, since we wish to consider twisted K-theory and since we will invert a slightly different set.

As a simple example suppose that $\phi = 1$ so we do not have time-reversing symmetries and moreover that the point group has the form $\mathbb{Z}_2 \times P'(C)$ where $\mathbb{Z}_2 = \{1, I\}$ and I is the inversion transformation $I : x \to -x$. A nontrivial example of such a point group is that of the diamond structure, discussed below. Let ϵ be the one-dimensional representation of $P(C)$ which is trivial on $P'(C)$ and is the sign representation of $\mathbb{Z}_2$. We will localize by inverting the multiplicative set $S = \{(1-\epsilon)^n\}_{n \geq 0}$.

If I takes $\bar{k}$ to $\bar{k}'$ with $\bar{k} \neq \bar{k}'$ then the restriction of $\iota^*\mathcal{E}$ to the set of two points $\{\bar{k}, \bar{k}'\}$ has an isomorphism $\iota^*\mathcal{E} \cong \epsilon\iota^*\mathcal{E}$ and hence the localization gives zero for all but the fixed-points.

Therefore, if $\phi = 1$ and $\mathcal{E}^+$ is infinite-dimensional we have

$$S^{-1}K^{\nu^{\rm can}}_{P(C)}(T^\vee) \cong S^{-1}K^{\nu^{\rm can}}_{P(C)}((T^\vee)^I) \tag{4.110}$$

Example: If $P(C) = \{1, I\} \cong \mathbb{Z}_2$ then the fixed point locus is the set of order two elements of $T^\vee$. If we think of $T^\vee \cong \mathbb{R}^d/\mathbb{Z}^d$ then these can be represented by the points $\bar{k} = (\bar{k}_1, \dots, \bar{k}_d)$ with $\bar{k}_i \in \{0, \frac{1}{2}\}$. (In the CM literature they are called the TRIM points.) The fixed points are isolated and at each fixed point $\bar{k}_i$, $i = 1, \dots, 2^d$, we have an element of the representation ring of $\mathbb{Z}_2$, $\mathcal{E}^-|_{\bar{k}_i} \cong n_i \oplus m_i\epsilon$. The localization just identifies this with an element of $\mathbb{Z}[\frac{1}{2}]$ as discussed above. If we know the total number of bands $n_i + m_i = n_B$ then from $n_i - m_i$ we can recover the number of parity-odd states at each TRIM point. These are the invariants discussed in [50].

4.6.3 Example: Some twisted equivariant K-theory invariants for diamond structure

As a somewhat more nontrivial example let us consider the three-dimensional diamond structure. We consider the case with $\phi = 1$ so the point group is a 48-element group isomorphic to $\mathbb{Z}_2 \times S_4$, where S_4 is realized as the proper cubic group and the $\mathbb{Z}_2$ factor

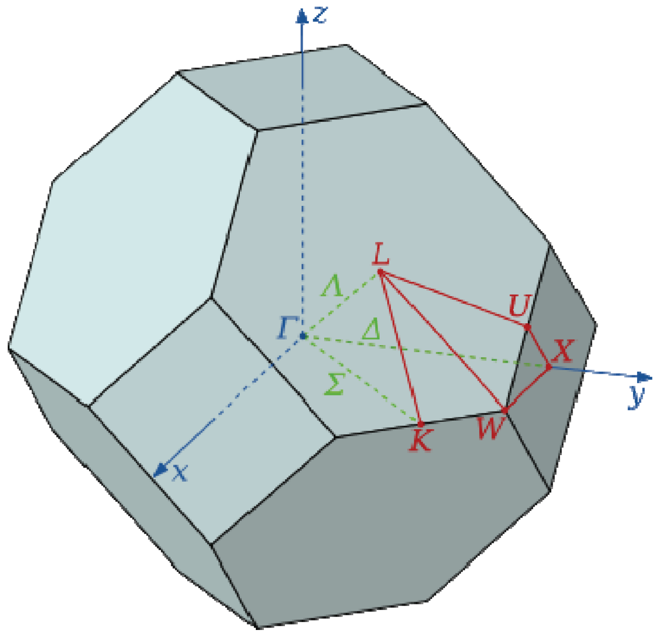

Figure 16: The Wigner-Seitz fundamental domain for the Brillouin torus of the diamond structure.

is generated by inversion. The Wigner-Seitz cell is shown in Figure 16. Identifying sides appropriately produces a copy of $T^\vee$.

♣Explain some more of the background details. ♣

We take again the multiplicative set $S = \{(1-\epsilon)^n\}$ as above. There are now 8 fixed points of I. These consist of the origin, labeled Γ in 16 together with two orbits under the point group of four points labeled L and three points labeled X, respectively.

At Γ the stabilizer is all of $O_h = \mathbb{Z}_2 \times S_4$. The contribution to the localized K-theory from this point is just a representation of this group - it is the representation provided by the valence bands.

There are four L-points. One of them is $k = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and has stabilizer $\cong \mathbb{Z}_2 \times S_3$. The canonical twist does not lead to a central extension and hence, again the K theory invariants correspond to a representation of this group.

There is an orbit of three X points $(\frac{1}{2}, 0, 0)$, $(0, \frac{1}{2}, 0)$, and $(0, 0, \frac{1}{2})$ each with isotropy group $\mathbb{Z}_2 \times D_4$. For example $(\frac{1}{2}, 0, 0)$ is fixed by a group of order 16 generated by ϵ_i and (23). To identify this with $\mathbb{Z}_2 \times D_4$ we observe that $I = \epsilon_1\epsilon_2\epsilon_3$ is in the center. Then $\epsilon_2(23)$ generates a $\pi/2$ rotation in the 23 plane and together with ϵ_2 generates D_4 acting in the 23 plane. The lifts to $G(C)$ of I and ϵ_2 are $\{I|s\}$ and $\{\epsilon_2|s\}$ and the group commutator is $\{1|(-1,1,1)\}$, and $\exp[2\pi i k \cdot v^{ii'}] = \exp[2\pi i (\frac{1}{2}, 0, 0)\cdot(-1,1,1)] = -1$ is nontrivial. All other group commutators are trivial. Therefore, the canonical twisting describes a nontrivial extension ♣CHECK!!! ♣

$$1 \to \mathbb{Z}_2 \to \widetilde{\mathbb{Z}_2 \times D_4} \to \mathbb{Z}_2 \times D_4 \to 1 \tag{4.111}$$

at the three X-points. Now the twisting makes the localization a bit more interesting. We are localizing in the representation ring of $\widetilde{\mathbb{Z}_2 \times D_4}$. Since $\{1, I\}$ is a quotient of this group ϵ is in the representation ring and we are inverting powers of $(1-\epsilon)$.

DISCUSS SOME MORE ABOUT THE LOCALIZATION IN THIS CASE

To summarize: By localization we have identified several nontrivial K-theory invariants:

1. A representation of O_h

2. A representations of $\mathbb{Z}_2 \times S_3$ given by the representation of the valence bands at the L-points.

3. A of $\widetilde{\mathbb{Z}_2 \times D_4}$ given by the representation of the valence bands at the X-points.

♣There be 2-torsion information lost upon localization... ♣

It would be very interesting to explore the physical implications in terms of edge modes and electromagnetic response of these "extra" K-theory invariants that go beyond those normally described in the literature on topological insulators and superconductors.

A. Background material: Cartan's symmetric spaces

Definition: A *symmetric space* is a (pseudo) Riemannian manifold (M, g) such that every point p is an isolated fixed point of an involutive isometry τ_p.

Near any point p, the involutive isometry τ_p can be expressed as the inversion of the geodesics through p. That is, if $(x^1, \dots, x^n)$ are normal coordinates in a neighborhood of p with $\vec{x} = 0$ the coordinate of p then $\tau_p(\vec{x}) = -\vec{x}$. Importantly, τ_p extends to an involutive isometry of the full Riemannian space (M, g)

One can show that the Riemannian curvature is covariantly constant, and hence there are three families of examples where the scalar curvature (which is must be constant) is positive, zero, or negative.

Cartan classified the symmetric spaces and found that they are always homogeneous spaces of Lie groups. The positive curvature examples are of the form G/K where G is a compact Lie group and K is a Lie subgroup.

Let us first examine G/K at the Lie algebra level. The tangent space of G at 1 is the Lie algebra $\mathfrak{g}$ and the tangent space of K at 1 is the Lie subalgebra $\mathfrak{k}$. If we write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{A.1}$$

then there is a natural identification of $\mathfrak{p}$ with $T_K(G/K)$. The involutive isometry τ_p where $p = 1 \cdot K$ has a differential $\theta = d\tau_p : \mathfrak{p} \to \mathfrak{p}$ which in fact can be shown to be the restriction of an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$. That is, θ is a Lie algebra homomorphism $\theta([X, Y]) = [\theta(X), \theta(Y)]$ which is an isomorphism of vector spaces and $\theta^2 = Id$. The $+1$ eigenspace is $\mathfrak{k}$ and the -1 eigenspace is $\mathfrak{p}$. The property that it is a Lie algebra automorphism implies that

♣subsets not proper? ♣

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &\subset \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{k} \end{aligned} \tag{A.2}$$

The decomposition (A.1) associated with θ is called a *Cartan decomposition.* [31]

Now let us consider G/K at the global level. The reduction of the τ_p to a single involutive automorphism θ of $\mathfrak{g}$ has a global analog: There is an involutive automorphism τ of the group. (That is, τ is a group automorphism and $\tau^2 = Id$) such that $d\tau = \theta$ at the identity. Given such an involutive automorphism τ we can define a subgroup K to be the fixed points of τ:

$$K = \{g \in G | \tau(g) = g\} \tag{A.3}$$

Given such an involution we have a *Cartan embedding* by the "anti-fixed points":

$$G/K \hookrightarrow \mathcal{O} := \{g \in G | \tau(g) = g^{-1}\} \tag{A.4}$$

$$gK \mapsto \tau(g)g^{-1}. \tag{A.5}$$

Note that this is well-defined and indeed $\tau(\tau(g)g^{-1}) = (\tau(g)g^{-1})^{-1}$ because τ is an involution. One checks it is an embedding by looking at the neighborhood of $g = 1$. Then we identify $d\tau_1 = \theta$. To see it is surjective note that $\mathcal{O}$ admits a left G-action by twisted adjoint action: If $g_0 \in G$ and $g \in \mathcal{O}$ then $\tau(g_0)gg_0^{-1} \in \mathcal{O}$, and the isotropy group of this action at $g = 1$ is precisely K. The metric tensor is just the pullback of the usual left-right-invariant metric $-\mathrm{Tr}(g^{-1}dg) \otimes (g^{-1}dg)$. The inversion τ_{g_*} through $g_* \in \mathcal{O}$ required by the definition is $\tau_{g_*} : g \mapsto g_* g^{-1} g_*$. One easily checks that this takes $\mathcal{O} \to \mathcal{O}$ and is an isometry of the metric. To see that g_* is an isolated fixed point of τ_{g_*} use the left G-action to translate to $g_* = 1$ and use the involution θ on $\mathfrak{g}$ above. We see that infinitesimally it is the exponential of elements of $\mathfrak{p}$ which lie in $\mathcal{O}$ in the neighborhood of 1.

It is also worth noting that the Cartan embedding $\mathcal{O}$ of G/K is a totally geodesic submanifold, as follows from the same reasoning used at the end of 1.2.3

Now that we have these definitions we give the 10 classes of compact classical symmetric spaces:

Whenever G is a compact simple Lie group the homogeneous space $(G \times G)/G_{\mathrm{diag}}$ is a symmetric space. Suppose the action of the diagonal subgroup is on the right, then we have an isomorphism of manifolds:

$$(G \times G)/G_{\mathrm{diag}} \cong G \tag{A.6}$$

where we take $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Warning! This is *not* a group homomorphism. The involution τ is just $\tau : (g_1, g_2) \mapsto (g_2, g_1)$. In particular, if we take $G = U(n, \mathbb{R}) = O(n)$, $G = U(n, \mathbb{C}) = U(n)$, or $G = U(n, \mathbb{H}) = Sp(n)$ then we get a series of 3 classical symmetric spaces:

$$(O(n) \times O(n))/O(n) \tag{A.7}$$

$$(U(n) \times U(n))/U(n) \tag{A.8}$$

$$(Sp(n) \times Sp(n))/Sp(n) \tag{A.9}$$

[31] A *Cartan involution* of a Lie algebra is an involutive Lie algebra automorphism s such that $B(X, sY)$ is positive definite. θ is related to a Cartan involution. ♣ Clarify some confusing terminology. See Helgason III.7 for the straight story.♣

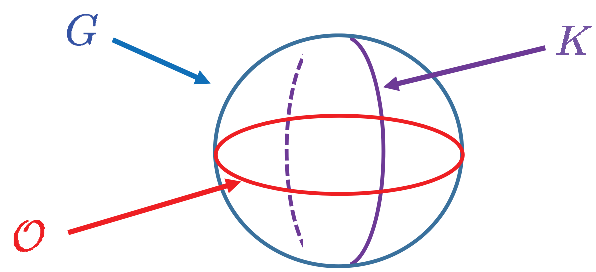

Figure 17: K and G/K locally divide the group into a product.

Another natural series of classical symmetric spaces are the Grassmannians. These arise from the involutive automorphism coming from conjugation

$$\tau(g) = g_0 g g_0^{-1} \qquad g_0 = \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix} \tag{A.10}$$

We can consider Grassmannians in real, complex, and quaternionic vector spaces to get

$$\mathrm{Gr}_k(\mathbb{R}^n) \cong O(n)/(O(k) \times O(n-k)) \tag{A.11}$$

$$\mathrm{Gr}_k(\mathbb{C}^n) \cong U(n)/(U(k) \times U(n-k)) \tag{A.12}$$

$$\mathrm{Gr}_k(\mathbb{H}^n) \cong Sp(n)/(Sp(k) \times Sp(n-k)) \tag{A.13}$$

With a little charity (regarding cases with $k \neq n-k$ as nonzero index analogs of the cases with $k = n-k$) we can consider this to be three more series of classical symmetric spaces.

Finally, as discussed in Section §1.6, we can put real, complex, or quaternionic structures on real, complex, or quaternionic spaces (when this makes sense). When these structures are made compatible with standard Euclidean metrics we obtain moduli spaces of structures. This gives us:

Real structures on complex vector spaces: $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$. Moduli space

$$U(n)/O(n) \tag{A.14}$$

This comes from $\tau(u) = u^*$.

Complex structures on real vector spaces: $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Moduli space:

$$O(2n)/U(n) \tag{A.15}$$

This comes from $\tau(g) = I_0 g I_0^{-1}$ where I_0 is (1.76).

Complex structures on quaternionic vector spaces: $\mathbb{C}^n \hookrightarrow \mathbb{H}^n$. Moduli space:

$$Sp(n)/U(n) \tag{A.16}$$

Viewing $Sp(n)$ as unitary $n \times n$ matrices over the quaternions the involution is $\tau(g) = -\mathfrak{i} g \mathfrak{i}$, i.e. conjugation by the unit matrix times $\mathfrak{i}$.

Quaternionic structures on complex vector spaces: $\mathbb{C}^{2n} \cong \mathbb{H}^n$. Moduli space:

$$U(2n)/Sp(n) \tag{A.17}$$

Viewing $Sp(n)$ as $USp(2n) := U(2n) \cap Sp(2n; \mathbb{C})$ we can use the involutive automorphism $\tau(g) = I_0^{-1} g^* I_0$ on $U(2n)$. The fixed points in $U(2n)$ are the group elements with $g I_0 g^{tr} = I_0$, but this is the defining equation of $Sp(2n, \mathbb{C})$.

When Cartan classified compact symmetric spaces he found the 10 series above (A.7) - (A.17) together with a finite set of exceptional cases. [32]

B. Group cohomology and central extensions

B.1 Central extensions

Now we study an important class of extensions. We change the notation from the previous section to emphasize this.

Let A be an abelian group and G any group.

Definition A *central extension* of G by A, [33] is a group $\tilde{G}$ such that

$$1 \to A \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \to 1 \tag{B.1}$$

such that $\iota(A) \subset Z(\tilde{G})$.

We stress again that what we called G in the previous section is here called $\tilde{G}$, and what we called Q in the previous section is here called G.

Example . An example familiar from the quantum mechanical theory of angular momentum, and which we will discuss later is:

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1 \tag{B.2}$$

Here the $\mathbb{Z}_2 \cong \{\pm 1\}$ is the center of $SU(2)$.

Remarks:

1. Central extensions are important in the theory of projective representations and occur quite frequently in quantum mechanics. A simple example are the spin representations of the rotation group. We will explain this relation in more detail later, but for

[32] The exceptional cases are presumably all related to the nonassociative real division algebra $\mathbb{O}$, but explaining how this comes about appears to be nontrivial [22].

[33] Some authors say an extension of A by G.

the moment the reader might find it useful to think about G as a group of classical symmetries of a physical system and $\tilde{G}$ as a group of corresponding operators in the quantum mechanical description of that physical system. The fiber of the map π can be thought of as possible c-number phases which can multiply the operator on Hilbert space representing a symmetry operation g. For a more detailed account of this see Chapter *** below.

2. Central extensions appear naturally in quantization of bosons and fermions. The Heisenberg group is an extension of a translation group. The symplectic group of linear canonical transformations gets quantum mechanically modified by a central extension to define something called the metaplectic group.
3. Central extensions are important in the theory of anomalies in quantum field theory.
4. Central extensions are very important in conformal field theory. The Virasoro group, and the Kac-Moody groups are both nontrivial central extensions of simpler objects.

Exercise *Another form of splitting*

Show that an equivalent definition of a split exact sequence for a central extension is that there is a homomorphism $t : \tilde{G} \to A$ which is a left-inverse to ι, $t(\iota(a)) = a$.

(Hint: Define $s(\pi(\tilde{g})) = \iota t(\tilde{g}^{-1}))\tilde{g}$.)

There is an interesting way to classify central extensions of G by A.

As before let $s : G \to \tilde{G}$ be a "section" of π. That is, a map such that

$$\pi(s(g)) = g \qquad \forall g \in G \tag{B.3}$$

As we have stressed, in general s is not a homomorphism. As in the general discussion above, when s is in fact a homomorphism we say that *the short exact sequence* (1.39) *splits*. In this case $\tilde{G}$ is isomorphic to a direct product $\tilde{G} \cong A \times G$ via

$$\iota(a)s(g) \to (a, g) \tag{B.4}$$

When the sequence splits the semidirect product of the previous section is a direct product because A is central, so $\omega_g(a) = a$.

Now, let us allow that (B.1) does not necessarily split. Let us choose any section s and measure by how much s differs from being a homomorphism by considering the combination:

$$s(g_1)s(g_2)\left(s(g_1g_2)\right)^{-1}. \tag{B.5}$$

Now the quantity (B.5) is in the kernel of π and hence in the image of ι. Since ι is injective we can *define* a function $f_s : G \times G \to A$ by the equation

$$\iota(f_s(g_1, g_2)) := s(g_1)s(g_2)\left(s(g_1g_2)\right)^{-1}. \tag{B.6}$$

That is, we can write:

$$s(g_1)s(g_2) = \iota(f_s(g_1, g_2))s(g_1g_2) \tag{B.7}$$

The function f_s satisfies the important *cocycle identity*

$$f(g_1, g_2g_3)f(g_2, g_3) = f(g_1, g_2)f(g_1g_2, g_3) \tag{B.8}$$

Exercise

Verify (B.8) by using (B.6) to compute $s(g_1g_2g_3)$ in two different ways.

(Note that simply substituting (B.6) into (B.8) is not obviously going to work because $\tilde{G}$ need not be abelian.)

Exercise *Simple consequences of the cocycle identity*

a.) By putting $g_1 = 1$ and then $g_3 = 1$ show that

$$f(g, 1) = f(1, g) = f(1, 1) \qquad \forall g \in G \tag{B.9}$$

b.) Show that

$$f(g, g^{-1}) = f(g^{-1}, g). \tag{B.10}$$

Now we introduce some fancy terminology:

Definition: In general

1. A *2-cochain on G with values in A*, $C^2(G, A)$ is a function

$$f : G \times G \to A \tag{B.11}$$

Note that $C^2(G, A)$ is naturally an abelian group.

2. A *2-cocycle* $f \in Z^2(G, A)$ is a 2-cochain $f : G \times G \to A$ satisfying (B.8). $Z^2(G, A)$ inherits an abelian group structure from $C^2(G, A)$.

So, in this language, given a central extension of G by A and a section s we naturally obtain a two-cocycle $f_s \in Z^2(G, A)$ via (B.6).

Now, if we choose a different section $\hat{s}$ then

$$\hat{s}(g) = s(g)\iota(t(g)) \tag{B.12}$$

for some function $t : G \to A$. It is easy to check that

$$f_{\hat{s}}(g_1, g_2) = f_s(g_1, g_2)t(g_1)t(g_2)t(g_1 g_2)^{-1} \tag{B.13}$$

where we have used that $\iota(A)$ is central in $\tilde{G}$.

Definition: In general two 2-cochains f and $\hat{f}$ are said to differ by a coboundary if they satisfy

$$\hat{f}(g_1, g_2) = f(g_1, g_2)t(g_1)t(g_2)t(g_1 g_2)^{-1} \tag{B.14}$$

for some function $t : G \to A$.

One can readily check that if f is a cocycle then any other $\hat{f}$ differing by a coboundary is also a cocycle. Moreover, being related by a cocycle defines an equivalence relation on the set of cocycles $f \sim \hat{f}$. Thus, we may define:

Definition: The *group cohomology* $H^2(G, A)$ is the set of equivalence classes of 2-cocycles modulo equivalence by coboundaries. Moreover, this set carries a natural structure of an abelian group.

The group multiplication making $H^2(G, A)$ into an abelian group is simply defined by

$$(f_1 \cdot f_2)(g, g') = f_1(g, g') \cdot f_2(g, g') \tag{B.15}$$

This descends to a well-defined muiltiplication on cohomology classes: $[f_1] \cdot [f_2] := [f_1 \cdot f_2]$.

Now, the beautiful theorem states that group cohomology classifies central extensions:

Theorem: Isomorphism classes of central extensions of G by an abelian group A are in 1-1 correspondence with the second cohomology set $H^2(G, A)$. Moreover, with its abelian group structure identity corresponds to the split extension $A \times G$.

Proof: From (B.6)(B.8)(B.13) we learn that given a central extension we can unambiguously form a group cohomology class which is independent of the choice of section. Moreover, if $\tilde{G} \cong \tilde{G}'$ are isomorphic central extensions and $\psi : \tilde{G} \to \tilde{G}'$ is an isomorphism, then ψ can be used to map sections of $\tilde{G} \to G$ to sections of $\tilde{G}' \to G$: $s'(g) = \psi(s(g))$. Then

$$\begin{aligned} s'(g_1)s'(g_2) &= \psi(s(g_1))\psi(s(g_2)) \\ &= \psi(s(g_1)s(g_2)) \\ &= \psi(\iota(f_s(g_1, g_2))s(g_1 g_2)) \\ &= \psi(\iota(f_s(g_1, g_2)))\psi(s(g_1 g_2)) \\ &= \iota'(f_s(g_1, g_2))s'(g_1 g_2) \end{aligned} \tag{B.16}$$

and hence we assign precisely the same 2-cocycle $f(g_1, g_2)$ to the section s'. Hence the isomorphism class of a central extension maps unambiguously to a cohomology class $[f]$.

Conversely, given a cohomology class $[f]$ we may construct a corresponding central extension as follows. Choose a representative 2-cocycle f. With such an f we may define $\tilde{G} = A \times G$ as a set with multiplication law:

$$(a_1, g_1)(a_2, g_2) = (a_1 a_2 f(g_1, g_2), g_1 g_2) \tag{B.17}$$

Now suppose that we use two 2-cocycles f and f' which are related by a coboundary as in (B.14) above. Then we claim that the map $\psi : \tilde{G} \to \tilde{G}'$ defined by

$$\psi : (a, g) \to (a t(g)^{-1}, g) \tag{B.18}$$

is an isomorphism of groups. (Check this!) On the other hand, we just showed above that if $[f] \neq [f']$ then $\tilde{G}$ cannot be isomorphic to $\tilde{G}'$. ♠

Remark: Using a coboundary one can usefully simplify cocycles. For example, by setting $t(1) = f(1,1)^{-1}$ we may assume $f(1,1) = 1$. Then, by (B.9) we have $f(g,1) = f(1,g) = 1$ for all g. Similarly, if $g \neq g^{-1}$ we may put $f(g, g^{-1}) = f(g^{-1}, g) = 1$. If $g = g^{-1}$ then we might not be able to set $f(g,g) = 1$. We can "preserve this gauge" with further coboundary transformations that satisfy $t(1) = 1$ and $t(g^{-1}) = t(g)^{-1}$.

Example 1 . Extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. WLOG we can take $f(1,1) = f(1,\sigma) = f(\sigma,1) = 1$. Then we have two choices: $f(\sigma,\sigma) = 1$ or $f(\sigma,\sigma) = \sigma$. Both of these choices satisfies the cocycle identity. In other words $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. For the choice $f = 1$ we obtain $\tilde{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$. For the nontrivial choice $f(\sigma,\sigma) = \sigma$ we obtain $\tilde{G} = \mathbb{Z}_4$. Let us see this in detail. We'll let $\sigma_1 \in A \cong \mathbb{Z}_2$ and $\sigma_2 \in G \cong \mathbb{Z}_2$ be the nontrivial elements so we should write $f(\sigma_2, \sigma_2) = \sigma_1$. Note that $(\sigma_1, 1)$ has order 2, but then

$$(1, \sigma_2) \cdot (1, \sigma_2) = (f(\sigma_2, \sigma_2), 1) = (\sigma_1, 1) \tag{B.19}$$

shows that $(1, \sigma_2)$ has order 4. Moreover $(\sigma_1, \sigma_2) = (\sigma_1, 1)(1, \sigma_2) = (1, \sigma_2)(\sigma_1, 1)$. Thus,

$$\begin{aligned} \Psi &: (\sigma_1, 1) \to \omega^2 = -1 \\ \Psi &: (1, \sigma_2) \to \omega \end{aligned} \tag{B.20}$$

where ω is a primitive 4^{th} root of 1 defines an isomorphism. In conclusion, the nontrivial central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ is:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1 \tag{B.21}$$

Recall that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 2. The generalization of the previous example to an odd prime p is extremely instructive. These are $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. So, let us study in detail the extensions

$$1 \to \mathbb{Z}_p \to G \to \mathbb{Z}_p \to 1 \tag{B.22}$$

where we will write our groups multiplicatively. Now, using methods of topology one can show that [34]

$$H^2(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p. \tag{B.23}$$

On the other hand, we know from the class equation and Sylow's theorems that there are exactly two groups of order p^2, up to isomorphism! How is that compatible with (B.23)? The answer is that there can be nonisomorphic extensions involving the same central group. To see this, let us examine in detail the standard extension:

$$1\to\mathbb{Z}_p\to\mathbb{Z}_{p^2}\to\mathbb{Z}_p\to 1 \tag{B.24}$$

We write the first, second and third groups in this sequence as

$$\begin{aligned}\mathbb{Z}_p&=\langle\sigma_1|\sigma_1^p=1\rangle\\ \mathbb{Z}_{p^2}&=\langle\alpha|\alpha^{p^2}=1\rangle\\ \mathbb{Z}_p&=\langle\sigma_2|\sigma_2^p=1\rangle\end{aligned} \tag{B.25}$$

respectively. Then the first injection must take

$$\iota(\sigma_1)=\alpha^p \tag{B.26}$$

since it must be an injection and it must take an element of order p to an element of order p. The standard sequence then takes the second arrow to be reduction modulo p, so

$$\pi(\alpha)=\sigma_2 \tag{B.27}$$

Now, we try to choose a section of π. Let us try to make it a homomorphism. Therefore we must take $s(1)=1$. What about $s(\sigma_2)$? Since $\pi(s(\sigma_2))=\sigma_2$ we have a choice: $s(\sigma_2)$ could be any of

$$\alpha,\alpha^{p+1},\alpha^{2p+1},\dots,\alpha^{(p-1)p+1} \tag{B.28}$$

Here we will make the simplest choice $s(\sigma_2)=\alpha$. The reader can check that the discussion is not essentially changed if we make one of the other choices. (After all, this will just change our cocycle by a coboundary!)

Now that we have chosen $s(\sigma_2)=\alpha$, if s were a homomorphism then we would be forced to take:

$$\begin{aligned}s(1)&=1\\ s(\sigma_2)&=\alpha\\ s(\sigma_2^2)&=\alpha^2\\ \vdots\quad&\quad\vdots\\ s(\sigma_2^{p-1})&=\alpha^{p-1}\end{aligned} \tag{B.29}$$

But now we are stuck! The property that s is a homomorphism requires two contradictory things. On the one hand, we must have $s(1)=1$ for any homomorphism. On the other

[34] You can also show it by examining the cocycle equation directly.

hand, from the above equations we also must have $s(\sigma_2^p) = \alpha^p$. But because $\sigma_2^p = 1$ these requirements are incompatible. Therefore, with this choice of section we find a nontrivial cocycle as follows:

$$s(\sigma_2^k)s(\sigma_2^\ell)s(\sigma_2^{k+\ell})^{-1} = \begin{cases} 1 & k+\ell < p \\ \alpha^p & p \le k+\ell \end{cases} \tag{B.30}$$

and therefore,

$$f(\sigma_2^k, \sigma_2^\ell) = \begin{cases} 1 & k+\ell < p \\ \sigma_1 & p \le k+\ell \end{cases} \tag{B.31}$$

In these equations we assume $1 \le k, \ell \le p-1$. We know the cocycle is nontrivial because $\mathbb{Z}_p \times \mathbb{Z}_p$ is not isomorphic to $\mathbb{Z}_{p^2}$.

But now let us use the group structure on the group cohomology. $[f]^r$ is the cohomology class represented by

$$f^r(\sigma_2^k, \sigma_2^\ell) = \begin{cases} 1 & k+\ell < p \\ \sigma_1^r & p \le k+\ell \end{cases} \tag{B.32}$$

This corresponds to replacing (B.27) by

$$\pi_r(\alpha) = (\sigma_2)^r \tag{B.33}$$

and the sequence will still be exact, i.e. $\ker(\pi_r) = \mathrm{Im}\,(\iota)$, if $(r,p) = 1$, that is, if we compose the standard projection with an automorphism of $\mathbb{Z}_p$. Thus π_r also defines an extension of the form (B.24). But we claim that it is not isomorphic to the standard extension! To see this let us try to construct ψ so that

$$\begin{tikzcd} & & \langle\alpha\rangle \arrow[dd, "\psi"] \arrow[dr, "\pi_r"] & & \\ 1 \arrow[r] & \langle\sigma_1\rangle \arrow[ur, "\iota"] \arrow[dr, "\iota"'] & & \langle\sigma_2\rangle \arrow[r] & 1 \\ & & \langle\alpha\rangle \arrow[ur, "\pi"'] & & \end{tikzcd} \tag{B.34}$$

In order for the right triangle to commute we must have $\psi(\alpha) = \alpha^r$, but then the left triangle will not commute. Thus the extensions $\pi_1, \dots, \pi_{p-1}$, all related by outer automorphisms of the quotient group $\mathbb{Z}_p = \langle\sigma_2\rangle$, define inequivalent extensions with the same group $\mathbb{Z}_{p^2}$ in the middle.

In conclusion, we describe the *group* of isomorphism classes of central extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$ as follows: The identity element is the trivial extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \to 1 \tag{B.35}$$

and then there is an orbit of $(p-1)$ nontrivial extensions of the form

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 1 \tag{B.36}$$

acted on by $\mathrm{Aut}(\mathbb{Z}_p)$.

B.2 Group cohomology in other degrees

Motivations:

a.) The word "cohomology" suggests some underlying chain complexes, so we will show that there is such a formulation.

b.) There has been some discussion of higher degree group cohomology in physics in

1. The theory of anomalies (Faddeev-Shatashvili; Segal; Carey et. al.; Mathai et. al.; ...)
2. Classification of rational conformal field theories (Moore-Seiberg; Dijkgraaf-Vafa-Verlinde-Verlinde; Dijkgraaf-Witten; Kapustin-Saulina)
3. Condensed matter/topological phases of matter (Kitaev; Wen; ...)
4. Three-dimensional Chern-Simons theory and three dimensional supersymmetric gauge theory.

So here we'll just give a few definitions.

B.2.1 Definition

Suppose we are given any group G and an abelian group A (written additively in this section) and a homomorphism

$$\alpha : G \to \mathrm{Aut}(A) \tag{B.37}$$

Definition: An n-cochain is a function $\phi : G^{\times n} \to A$. The space of n-cochains is denoted $C^n(G, A)$.

Note that $C^n(G, A)$ is an abelian group using the abelian group structure of A on the values of ϕ, that is: $(\phi_1 + \phi_2)(\vec{g}) := \phi_1(\vec{g}) + \phi_2(\vec{g})$.

Define a group homomorphism: $d : C^n(G, A) \to C^{n+1}(G, A)$

$$\begin{aligned}(d\phi)(g_1, \dots, g_{n+1}) := \alpha_{g_1}\left(\phi(g_2, \dots, g_{n+1})\right) & \\ - \phi(g_1 g_2, g_3, \dots, g_{n+1}) + \phi(g_1, g_2 g_3, \dots, g_{n+1}) \pm \cdots + (-1)^n \phi(g_1, \dots, g_{n-1}, g_n g_{n+1}) & \\ + (-1)^{n+1} \phi(g_1, \dots, g_n) &\end{aligned} \tag{B.38}$$

We interpret a 0-cochain ϕ_0 to be some element $\phi_0 = a \in A$. Then we have, for $n = 0$:

$$(d\phi_0)(g) = \alpha_g(a) - a \tag{B.39}$$

For $n = 1$ $\phi_1 : G \to A$ and the differential acts as:

$$(d\phi_1)(g_1, g_2) = \alpha_{g_1}\left(\phi_1(g_2)\right) - \phi_1(g_1 g_2) + \phi_1(g_1) \tag{B.40}$$

$$(d\phi_2)(g_1, g_2, g_3) = \alpha_{g_1}\left(\phi_2(g_2, g_3)\right) - \phi_2(g_1 g_2, g_3) + \phi_2(g_1, g_2 g_3) - \phi_2(g_2, g_3) \tag{B.41}$$

$$(d\phi_3)(g_1, g_2, g_3, g_4) = \alpha_{g_1}\left(\phi_3(g_2, g_3, g_4)\right) - \phi_3(g_1 g_2, g_3, g_4) + \phi_3(g_1, g_2 g_3, g_4) - \phi_3(g_1, g_2, g_3 g_4) + \phi_3(g_1, g_2, g_3) \tag{B.42}$$

The set of n-cocycles is defined to be the subgroup $Z^n(G,A) \subset C^n(G,A)$ that satisfy $d\phi_n = 0$.

Next, one can check that for any ϕ, $d(d\phi) = 0$. (We give a simple proof below.) Therefore, we can define a subgroup $B^n(G,A) \subset Z^n(G,A)$ by

$$B^n(G,A) := \{\phi_n | \exists \phi_{n-1} \qquad s.t. \qquad d\phi_{n-1} = \phi_n\} \tag{B.43}$$

Then the group cohomology is defined to be the quotient

$$H^n(G,A) = Z^n(G,A)/B^n(G,A) \tag{B.44}$$

Remarks:

1. Remembering that we are now writing our abelian group A additively, we see that the equation $(d\phi_2) = 0$ is just the twisted 2-cocycle conditions, and $\phi_2' = \phi_2 + d\phi_1$ are two different twisted cocycles related by a coboundary. See equations **** above.

2. *Homogeneous cocycles*: A nice way to prove that $d^2 = 0$ is the following. We define *homogeneous* n-cochains to be maps $\varphi : G^{n+1} \to A$ which satisfy

$$\varphi(hg_0, hg_1, \dots, hg_n) = \alpha_h\left(\varphi(g_0, g_1, \dots, g_n)\right) \tag{B.45}$$

Let $\mathcal{C}^n(G,A)$ denote the abelian group of such homogeneous group cochains. Define

$$\delta : \mathcal{C}^n(G,A) \to \mathcal{C}^{n+1}(G,A) \tag{B.46}$$

by

$$\delta\phi(g_0, \dots, g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i \varphi(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}) \tag{B.47}$$

where $\widehat{g_i}$ means the argument is omitted. It is then very straightforward to prove that $\delta^2 = 0$. Indeed, if $\varphi \in \mathcal{C}^{n-1}(G,A)$ we compute:

$$\begin{aligned}
\delta^2\varphi(g_0, \dots, g_{n+1}) &= \sum_{i=0}^{n+1} (-1)^i \Big(\sum_{j=0}^{i-1} (-1)^j \varphi(g_0, \dots, \widehat{g_j}, \dots, \widehat{g_i}, \dots, g_{n+1}) \\
&\qquad - \sum_{j=i+1}^{n+1} (-1)^j \varphi(g_0, \dots, \widehat{g_i}, \dots, \widehat{g_j}, \dots, g_{n+1}) \Big) \\
&= \sum_{0 \le j < i \le n+1} (-1)^{i+j} \varphi(g_0, \dots, \widehat{g_j}, \dots, \widehat{g_i}, \dots, g_{n+1}) \\
&\qquad - \sum_{0 \le i < j \le n+1} (-1)^{i+j} \varphi(g_0, \dots, \widehat{g_i}, \dots, \widehat{g_j}, \dots, g_{n+1}) \\
&= 0
\end{aligned} \tag{B.48}$$

Now, we can define an isomorphism $\psi : \mathcal{C}^n(G,A) \to C^n(G,A)$ by defining

$$\phi_n(g_1, \dots, g_n) := \varphi_n(1, g_1, g_1 g_2, \dots, g_1 \cdots g_n) \tag{B.49}$$

That is, when ϕ_n and φ_n are related this way we say $\phi_n = \psi(\varphi_n)$. Now one can check that the simple formula (B.47) becomes the more complicated formula (B.38). Put more formally: there is a unique d so that $d\psi = \psi\delta$.

3. Where do all these crazy formulae come from? The answer is in topology. We will indicate it briefly in our discussion of categories and groupoids below.

B.3 The topology behind group cohomology

Now, let us show that this point of view on the definition of a group can lead to a very nontrivial and beautiful structure associated with a group.

An interesting construction that applies to any category is its associated simplicial space $|\mathcal{C}|$.

This is a simplicial space whose simplices are:

0. 0-simplices = objects
1. 1-simplices = $\Delta_1(f)$ associated to each morphism $f : x_0 \to x_1 \in X_1$.
2. 2-simplices: $\Delta(f_1, f_2)$ associated composable morphisms $(f_1, f_2) \in X_2$.
3. 3-simplices: $\Delta(f_1, f_2, f_3)$ associated to 3 composable morphisms, i.e. elements of:

$$X_3 = \{(f_1, f_2, f_3) \in X_1 \times X_1 \times X_1 | p_0(f_i) = p_1(f_{i+1})\} \tag{B.50}$$

And so on. See Figures 14 and 15. The figures make clear how these simplices are glued together:

$$\partial\Delta_1(f) = x_1 - x_0 \tag{B.51}$$

$$\partial\Delta_2(f_1, f_2) = \Delta_1(f_1) - \Delta_1(f_1 f_2) + \Delta_1(f_2) \tag{B.52}$$

and for Figure 15 view this as looking down on a tetrahedron. Give the 2-simplices of Figure 14 the counterclockwise orientation and the boundary of the simplex the induced orientation from the outwards normal. Then we have

$$\partial\Delta(f_1, f_2, f_3) = \Delta(f_2, f_3) - \Delta(f_1 f_2, f_3) + \Delta(f_1, f_2 f_3) - \Delta(f_1, f_2) \tag{B.53}$$

Note that on the three upper faces of Figure 15 the induced orientation is the ccw orientation for $\Delta(f_1, f_2 f_3)$ and $\Delta(f_2, f_3)$, but with the cw orientation for $\Delta(f_1 f_2, f_3)$. On the bottom fact the inward orientation is ccw and hence the outward orientation is $-\Delta(f_1, f_2)$.

Clearly, we can keep composing morphisms so the space $|\mathcal{C}|$ has simplices of arbitrarily high dimension, that is, it is an infinite-dimensional space.

Let look more closely at this space for the case of a group, regarded as a category with one object. Then in the above pictures we identify all the vertices with a single vertex.

For each group element g we have a one-simplex $\Delta_1(g)$ beginning and ending at this vertex.

For each ordered pair (g_1, g_2) we have an oriented 2-simplex $\Delta(g_1, g_2)$, etc. We simply replace $f_i \to g_i$ in the above formulae, with g_i now interpreted as elements of G:

$$\partial\Delta(g) = 0 \tag{B.54}$$

$$\partial\Delta(g_1, g_2) = \Delta_1(g_1) + \Delta_1(g_2) - \Delta_1(g_1 g_2) \tag{B.55}$$

$$\partial\Delta(g_1, g_2, g_3) = \Delta(g_2, g_3) - \Delta(g_1 g_2, g_3) + \Delta(g_1, g_2 g_3) - \Delta(g_1, g_2) \tag{B.56}$$

See Figure 15.

And so on.

To put this more formally: We have $n+1$ maps from $G^n \to G^{n-1}$ for $n \geq 1$ given by

$$\begin{aligned}
d^0(g_1, \dots, g_n) &= (g_2, \dots, g_n) \\
d^1(g_1, \dots, g_n) &= (g_1 g_2, g_3, \dots, g_n) \\
d^2(g_1, \dots, g_n) &= (g_1, g_2 g_3, g_4, \dots, g_n) \\
&\cdots\cdots \\
&\cdots\cdots \\
d^{n-1}(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1} g_n) \\
d^n(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1})
\end{aligned} \tag{B.57}$$

On the other hand, we can view an n-simplex Δ_n as

$$\Delta_n := \{(t_0, t_1, \dots, t_n) | t_i \geq 0 \qquad \& \qquad \sum_{i=0}^{n} t_i = 1\} \tag{B.58}$$

Now, there are also $(n+1)$ *face maps* which map an $(n-1)$-simplex Δ_{n-1} into one of the $(n+1)$ faces of the n-simplex Δ_n:

$$\begin{aligned}
d_0(t_0, \dots, t_{n-1}) &= (0, t_0, \dots, t_{n-1}) \\
d_1(t_0, \dots, t_{n-1}) &= (t_0, 0, t_1, \dots, t_{n-1}) \\
&\cdots\cdots \\
&\cdots\cdots \\
d_n(t_0, \dots, t_{n-1}) &= (t_0, \dots, t_{n-1}, 0)
\end{aligned} \tag{B.59}$$

d_i embeds the $(n-1)$ simplex into the face $t_i = 0$ which is opposite the i^{th} vertex $t_i = 1$ of Δ_n.

Now we identify

$$\left(\amalg_{n=0}^{\infty} \Delta_n \times G^n\right) / \sim$$

via

$$(d_i(\vec{t}), \vec{g}) \sim (\vec{t}, d^i(\vec{g})). \tag{B.60}$$

The space we have constructed this way has a homotopy type denoted BG. Even for the simplest nontrivial group $G = \mathbb{Z}/2\mathbb{Z}$ the construction is quite nontrivial and BG has the homotopy type of $\mathbb{R}P^{\infty}$.

Now, an n-cochain in $C^n(G, \mathbb{Z})$ (here we take $A = \mathbb{Z}$ for simplicity) is simply an assignment of an integer for each n-simplex in BG. Then the coboundary and boundary maps are related by

$$\langle d\phi_n, \Delta \rangle = \langle \phi_n, \partial \Delta \rangle \tag{B.61}$$

and from the above formulae we recover, rather beautifully, the formula for the coboundary in group cohomology.

Remark: When we defined group cohomology we also used homogeneous cochains. This is based on defining G as a groupoid from its left action and considering the mapping of groupoids $G//G \to pt//G$.

♣Explain more here? ♣

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