A Few Remarks On The Interaction Of Theoretical Condensed Matter & Physical Mathematics

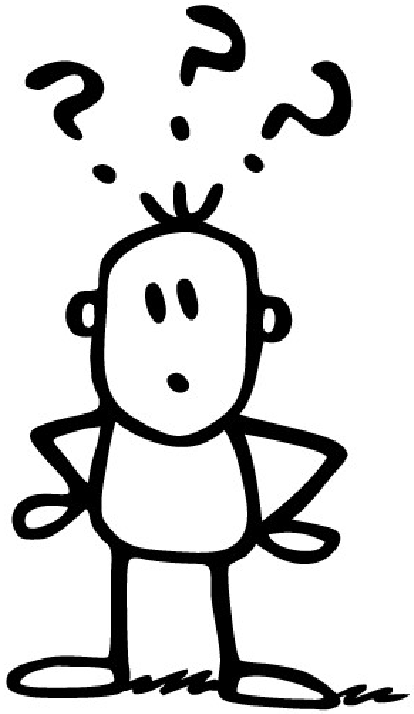
I will raise a few questions, suggest a few potentially interesting future directions.

Disclaimers

I haven't been working in this subject for a few years, even though I regard it as one of the most vibrant areas in Physical Mathematics.

So I'm probably pretty uninformed/out of the loop on some important points.

And some of the suggestions I make might be pretty silly.



So why did he agree to be on the panel?

What was he thinking ???

ACP(2015)Working group – expert opinions.

Birthday questions: Haldane, Freedman, Seiberg.

Two suggestions for interesting future directions.

Questions From An Aspen/Harvard Working Group

Dan Freed, Mike Freedman, Matt Hastings, Mike Hopkins, Anton Kapustin, Alexei Kitaev, Constantin Teleman

How to relate three distinct approaches to the mathematical classification of ``SRE'' or ``invertible'' topological phases.

Boundary conditions in topological field theory: When does a SRE phase only allow gapless boundary conditions?

Questions related to lattice realizations of topological phases.

First Approach: Lattice

Kitaev tries to define a space of lattice Hamiltonians on d-dimensional disk that are ``locally gapped''  such that the groundstate must be ``invertible'' in the sense that if Ψ is the groundstate then we can tensor with another system and groundstate Ψ' so $\Psi \otimes \Psi'$ can be brought to a product state using a bounded number of ``local quantum gates''. This gives a space $\mathcal{B}_d(\mathcal{D})$ of Hamiltonians that depends on details $\mathcal{D}$

Then we want to take some kind of limit

$$\mathcal{B}_d := \lim_{\mathcal{D}} \mathcal{B}_d(\mathcal{D})$$

Kitaev argues the set of spaces $\{\mathcal{B}_d\}_{d\geq 0}$ forms something known in topology as a "loop spectrum":

$$\mathcal{B}_d \to \Omega \mathcal{B}_{d+1}$$

Details can (more or less) be carried out for free fermions – leading to K and KO spectrum. The interacting case is less clear, and $\mathcal{B}_d$ is only clear for low values of d.

Second & Third Approaches: TFT

Kapustin et. al. ask what topological terms can be used in the low energy effective action and argue that these are bordism invariants. This leads to classification of G SPT -phases in $d+1$ spacetime dimesions by

$$Hom\left(\Omega_{d+1}^{structure}(BG), U(1)\right)$$

Freed-Hopkins PROVE a mathematical theorem classifying ``reflection positive extended invertible topological field theories'' and produce a very closely related answer, using somewhat more elaborate ideas from bordism theory.

20-20 hindsight: Agreement of Kapustin and Freed-Hopkins is not that surprising since an ``invertible topological field theory'' is essentially a formalization of what one ought to mean by a ``classical topological field theory''.

The Big Gap

Relating the lattice approach to the field theory approach faces some challenges:

Technically hard to define precisely the correct spaces of locally gapped lattice Hamiltonians with SRE ground state.

3d codes of Haah?

Field theory versions of nonabelian Dijkgraaf-Witten models?

Needed: Theory of Interfaces Between Topological Field Theories

When does a gapped phase (i.e. a TFT) only have gapless boundary conditions?

Examples

Gapless modes at boundaries of topological insulators and superconductors.

Witten has given a clear explanation of how this follows very naturally as an interpretation of index theorems with boundary: $Z(X) = (-1)^{Index(X)}$ but when X has a boundary the index is not topological – must be cured by gapless edge modes.

Example: Invertible 3+1 theory with domain category: Unoriented manifolds with SO(3) bundle $P \to X$. $Z(P) = (-1)^{w_2(P)^2[X]}$

Example (Hopkins): Fix a space M with a gerbe connection B and consider a 2-1-0 extended TFT with domain category: Manifold with map $f: X \to M$. Partition function on closed 2-surface: $Z(\mathrm{X}, f) = \exp\left(2\pi i \int f^*(B)\right)$

If B is not torsion can prove there is no TFT boundary condition.

Clarification Needed: Abelian Chern Simons

Theorem: (Belov and Moore) For G= $U(1)^r$ let Λ be the integral lattice corresponding to the classical theory. Then the *quantum theory* only depends on

a.) $\mathcal{D} = \Lambda^*/\Lambda$, the ``discriminant group''

b.) The quadratic function q: $\mathcal{D} \to \mathbb{R}/\mathbb{Z}$

c.) $\sigma(\Lambda)$ mod 24

Kapustin & Saulina: Boundary conditions in 1-1 correspondence with Lagrangian subspaces of $\mathcal{D}$

Freed-Hopkins-Lurie-Teleman: Claim to construct the entire 3-2-1-0 extended TFT but have MORE boundary conditions than Kapustin-Saulina (but they seem a little exotic).

Three Birthday Questions

Haldane 60: 2011

Freedman 60: 2011

Seiberg 60: 2016

From HaldaneFest talk 2011

A Question/Challenge to CMT

Thus, in topological band theory, a natural generalization of the 10 DAZ symmetry classes would be

$$[\tau] \in \left[\mathfrak{Twist}_{KR}(\mathbb{R}^3//\,\Gamma_0)\right]$$

And a natural generalization of the classification of topological phases for a given ``symmetry type'' $[\tau]$ would be

$$KR^{\tau}(\mathbb{R}^3//\,\Gamma_0)$$

Can such ``symmetry types'' and topological phases actually be realized by physical fermionic systems?

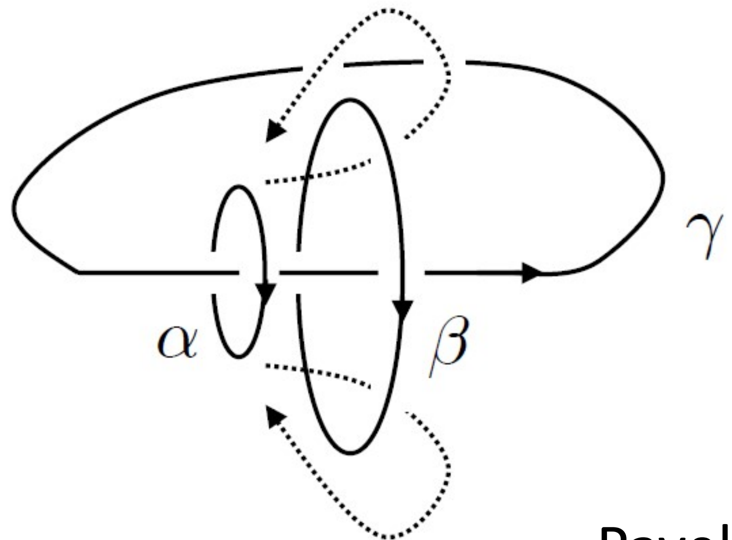
A Question From FreedmanFest 2011

Can Khovanov homology of knots and links be usefully applied to generalize Kitaev's idea for fault-tolerant quantum codes using topological phases of matter?

Q-bits: $\mathcal{H}(L)$

Quantum gates: $\Phi(\Sigma): \mathcal{H}(L_1) \to \mathcal{H}(L_2)$

Realization In Recent Literature?



Michael Levin and Chengie Wang
1403.7437

Chao-Ming Jian and Xiao-Liang Qi
1405.6688

Pavel Putrov, Juven Wang, Xiao-Gang Wen, and Shing-Tung Yau: 1602.05951, 1612.09298

Dima Galakhov computed the effect of the same bordism in Khovonov homology (using the Landau-Ginzburg approach suggested by Gaiotto-Witten) and found $\Phi(\Sigma)$ is given by signed permutations.

A Point From SeibergFest 2016:

Given a continuous family of Hamiltonians with a gap in the spectrum there is, in general, not one Berry connection, but rather a family of Berry connections.

This can have physical consequences:
I will illustrate that using examples
from topological band structure.

But the general remark should
have broad applications.

Two Possible New Directions -1/2

Modular tensor categories have had very little impact on string theory itself. (Despite the original motivation.)

Of course, they have played an important role in mathematics (VOA theory, knot theory, 3-manifold invariants) and in topological phases of matter.

(Rather surprisingly they played an important role in supersymmetric field theory just in the past few years.)

Can recent work on topological phases and ***generalizations*** of modular tensor categories be usefully applied to CFT ?

Example: There is interesting recent work on
``G-crossed braided tensor categories'' . [Barkeshli, et. a.l.]
Does it shed any useful insight into orbifold theories?

I realized only recently (with Jeff Harvey) that actually our knowledge of consistency conditions for asymmetric orbifolds is rather incomplete. Maybe the new ideas in the theory of topological phases can give a complete solution to this old problem.

We started to work this out with Nati Seiberg …..

Still, it seems a good project and there is plenty to do here.

Possible New Directions – 2/2

Will C* algebras and noncommutative geometry play a more dominant role in the theory of topological phases?

Haldane: Noncommutative coordinates in Laughlin ``wavefunctions''

Bellissard et. al. : Noncommutative geometries associated to homogeneous but aperiodic media.