Three Points In A Talk

(To say nothing of the prologue!)

G.K.G.

Prologue

There were four of us – Nati, Jeffrey Samuel Harvey, and myself, and Maldacena. We were sitting in my office, skyping, and talking about how confused we were – confused from a conceptual point of view I mean, of course.

With me, it was my talk that was out of order. I knew it was my talk that was out of order, because I had just been reading a blog in which were detailed various mistakes by which a man could tell when his talk was out of order. I made them all.

Nati said: You know, you're on the wrong track altogether. You must not think of the results you can do with, but only of the results that you can't do without. Nati comes out really quite sensible at times. You'd be surprised. I call that downright wisdom.

Chapter One

AN ORBIFOLD BY T-DUALITY? –

- A ONE LOOP ANOMALY AND A PUZZLE -

- NATI IS SARCASTIC – AN OLD CONDITION REVISITED -

Can You Orbifold By T-Duality?

Comes up in some recent work on Mathieu Moonshine with Jeff …

Would be a nice way to construct ``monodrofolds'' and ``T-folds''

[Hellerman & Walcher 2005]

Example: Consider a periodic scalar at the self-dual radius:

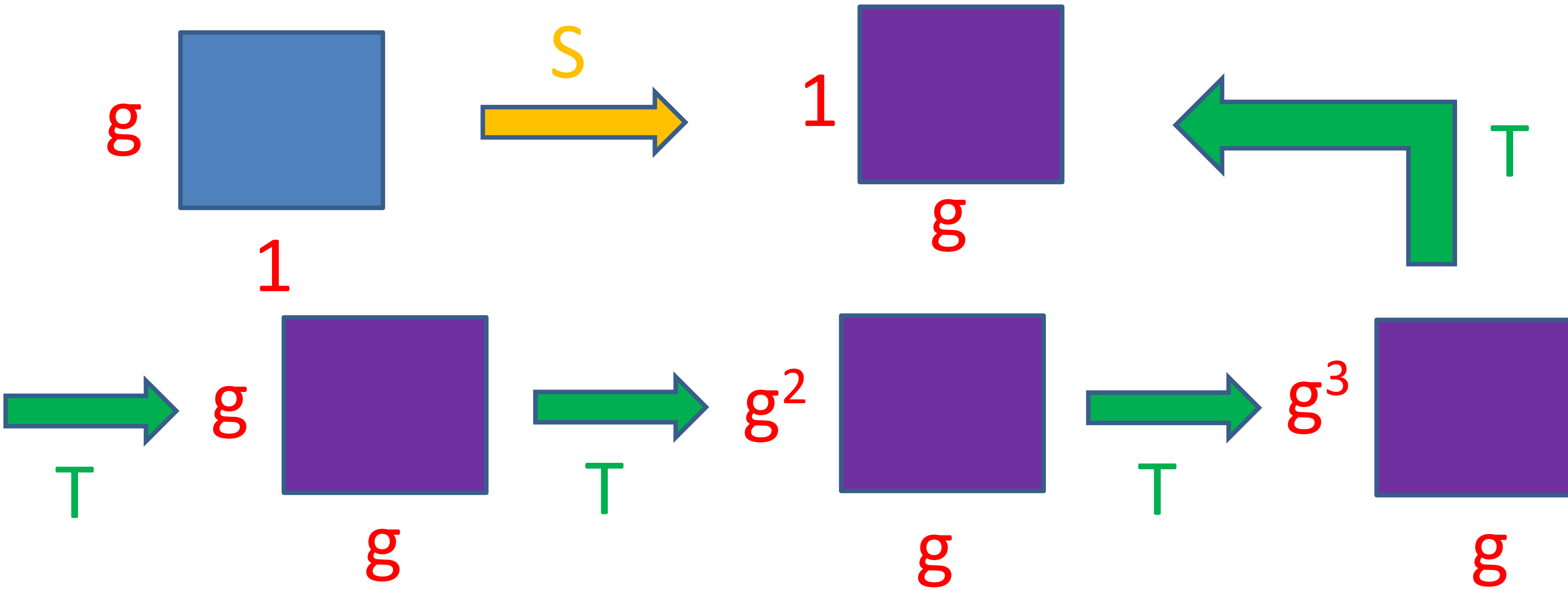
$$g : X_L \to X_L \qquad\qquad g : X_R \to -X_R$$

One loop modular anomaly in g-twisted sector:

$$E_L = \frac{1}{2} p_L^2 + \mathbb{Z} \in \frac{1}{4}\mathbb{Z} \qquad E_R = \frac{1}{16} + \frac{1}{2}\mathbb{Z}$$

But <u>*four*</u> copies of the Gaussian model has no one-loop anomaly.

Still, even with level matching satisfied, something peculiar is going on:



So, in the twisted sector, g is actually order four!

But computation of the partition function in the g^2-twisted sector gives negative half-integral degeneracies....

“Why don’t you think what you are doing? You go about things in such a slap-dash style. You’d get a scaffolding pole entangled you would!”

Gaussian model at self-dual point has $SU(2)_L \times SU(2)_R$ symmetry.

T-duality is a 180 degree rotation in $SO(3)_R$:

$$J^3 \to -J^3 \qquad J^{\pm} \to J^{\mp}$$

Therefore lifts to an order ***four*** element in $SU(2)_R$

-- even in the untwisted sector!

Now the method of orbits gives a sensible Z

A Condition For Asymmetric Orbifolds

These are examples of asymmetric orbifolds.

Narain, Sarmadi, Vafa (1987) is an important paper giving general consistency conditions for asymmetric orbifolds. But one condition was a little mysterious:

$p \cdot gp = 0 \mod 2$ for all Narain vectors p.

Violated by T-duality orbifold of four copies of the Gaussian model

Conjecture: Not a true consistency condition: It means the group acting on the CFT is really $\mathbb{Z}_4$

Chapter Two

PHILOSOPHY – ORIGINS & AFFINE SPACE –
- HILBERT BUNDLES –BERRY'S CONNECTION –
- A SIMPLE MAN - BAND STRUCTURE –
- A 3D QUANTUM HALL EFFECT

Philosophy

If a physical result is not mathematically natural, there might well be an underlying important physical issue.

I will illustrate this with continuous families of quantum systems

i.e. quantum systems parametrized by a space X of control parameters.

In this context one naturally encounters Berry connections – an enormously successful idea.

A Little Subtlety

Given a continuous family of Hamiltonians with a gap in the spectrum there is, in general, not one Berry connection, but rather a family of Berry connections.

This can have physical consequences:
I will illustrate that using examples
from topological band structure.

But the general remark should
have broad applications.

The origin of the problem is the problem of the origin.

Affine Space

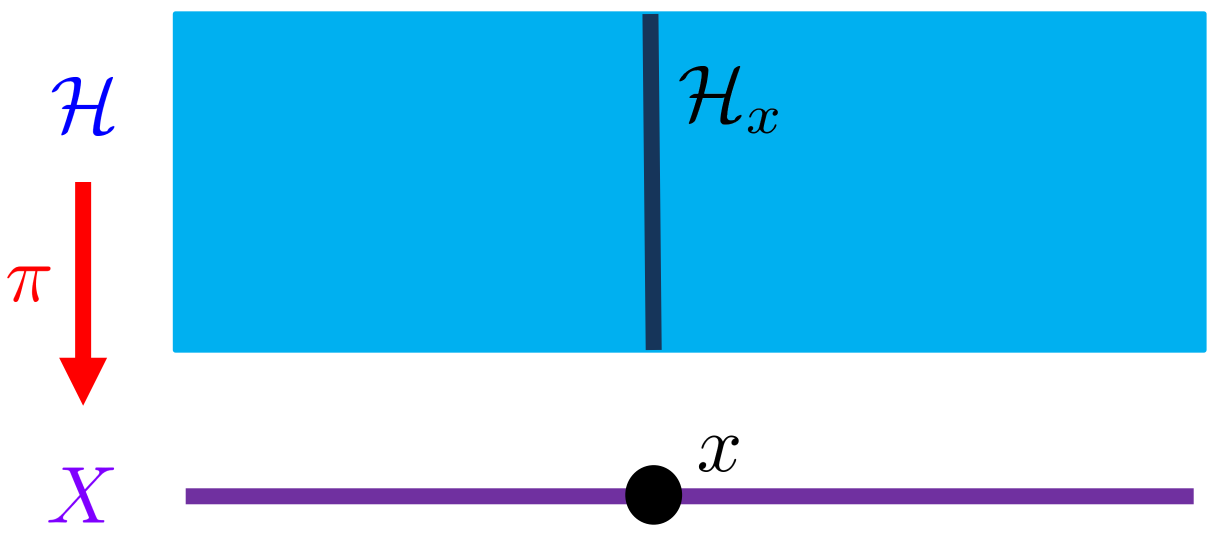
Like a vector space – but no natural choice of origin.

Official Definition: An affine space is space with a transitive and free action of a vector space.

Hilbert Bundles

Hilbert bundle over a space X of control parameters

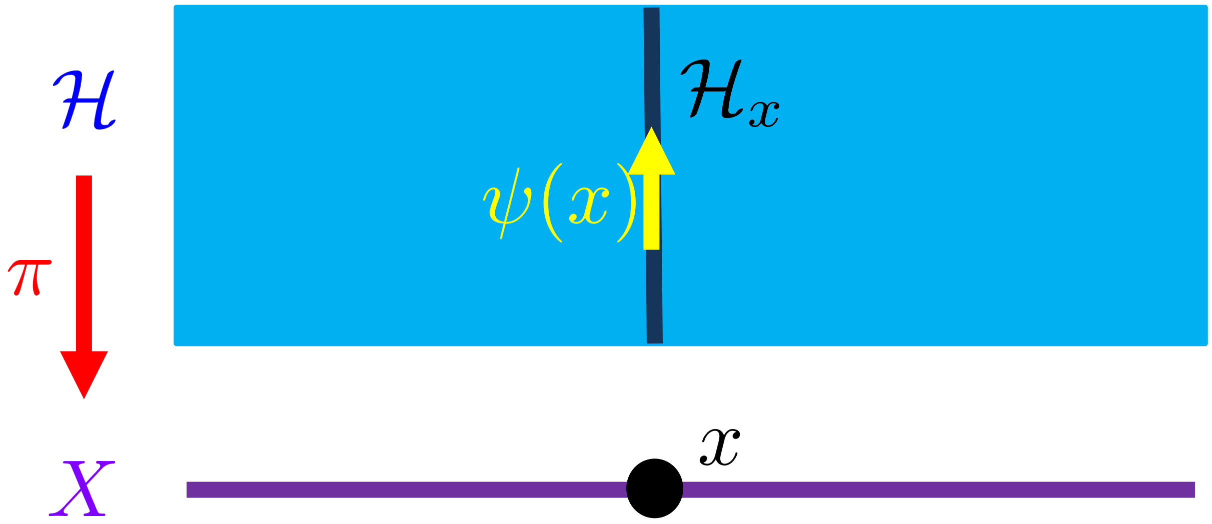
$$\begin{array}{ccc} \mathcal{H}_x & & \mathcal{H} \\ \downarrow & & \downarrow \pi \\ x & \hookrightarrow & X \end{array}$$



Sections Of A Hilbert Bundle

Space of sections: $\Gamma[\mathcal{H} \to X]$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$



Connection On A Vector Bundle $\mathcal{V}$

Connection:

$$\nabla : \Gamma(\mathcal{V}) \to \Omega^1(\mathcal{V})$$

$$\nabla(f\Psi) = df \otimes \Psi + f\nabla\Psi$$

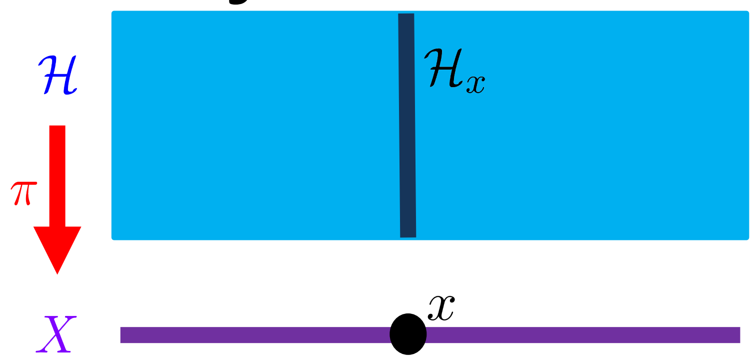
Remark: The space of connections on a vector bundle is an affine space modeled on the vector space: $\Omega^1(\mathrm{End}(\mathcal{V}))$

$$\nabla_1^{\mathcal{V}} - \nabla_2^{\mathcal{V}} = dx^\mu \otimes \alpha_\mu$$

Unless the bundle is trivial: $\mathcal{V} = X \times V_0$ for some fixed vector space V_0

there is no natural origin in the space of all connections.

Projected Bundles



Given a continuous family of projection operators: $$P(x) : \mathcal{H}_x \to \mathcal{H}_x$$

Projected bundle $\mathcal{V}$: Subbundle with sections:

$$\Gamma(\mathcal{V}) := \{\psi(x) | P(x)\psi(x) = \psi(x)\} \subset \Gamma(\mathcal{H})$$

$$\begin{array}{c} \mathcal{H} = S^2 \times \mathbb{C}^2 \\ \pi \downarrow \\ \hat{x} \in S^2 \end{array} \qquad P(\hat{x}) = \frac{1}{2}(1 + \hat{x} \cdot \vec{\sigma})$$

$\mathcal{V}$ is the Hopf line bundle

Berry Connection

Given a continuous family of Hamiltonians H_x on $\mathcal{H}_x$, if there is a gap:

$$E_{\text{gap}} \notin \cup_{x \in X} \text{Spec}(H_x)$$

we have a continuous family of projection operators: $$P(x) = \Theta(E_{\text{gap}} - H_x)$$

$$\nabla^B := P \circ \nabla^{\mathcal{H}} \circ \iota$$
$$\iota : \Gamma(\mathcal{V}) \hookrightarrow \Gamma(\mathcal{H})$$

[M. Berry (1983); B. Simon 1983)]

Note that it requires a CHOICE of $\nabla^{\mathcal{H}}$

Commonly assumed: $\mathcal{H}$ has been trivialized:

$$\mathcal{H} = X \times \mathcal{H}_0$$

Natural choice of $\nabla^{\mathcal{H}}$:

The trivial connection.

$$\nabla^{\mathcal{H}} \psi(x) = dx^\mu \frac{\partial}{\partial x^\mu} \psi(x)$$

$$\vec{A}^{\mathrm{Berry}} = \langle \psi | \vec{\nabla}_{\vec{R}} | \psi \rangle$$

But in general there is no natural trivialization of $\mathcal{H}$!

I’m a simple man….

Hilbert Bundle Over Brillouin Torus

Crystal in n-dimensional affine space: $C \subset \mathbb{A}^n$

Invariant under a lattice of translations: $L \subset \mathbb{R}^n$

Brillouin torus: = {unitary irreps of L}.

Reciprocal lattice: $L^\vee \subset \mathcal{K} \cong (\mathbb{R}^n)^\vee \cong \mathbb{R}^n$

$$\bar{k} \in T^\vee = \mathcal{K}/L^\vee \quad \chi_{\bar{k}}(R) = e^{2\pi \mathrm{i} k \cdot R} \quad R \in L$$

Bloch states define a Hilbert bundle $\mathcal{H}$ over the Brillouin torus:

$$\mathcal{H}_{\bar{k}} := \{\psi_{\bar{k}} | \psi_{\bar{k}}(x+R) = \chi_{\bar{k}}(R)\psi_{\bar{k}}(x)\}$$

Trivializations Of $\mathcal{H}$

$\mathcal{H}$ can be trivialized by choosing sets of Bloch functions:

$$\psi_{n,\bar{k}}(x+R) = e^{2\pi \mathrm{i} k\cdot R}\psi_{n,\bar{k}}(x) \quad n \in \mathbb{N}$$

For fixed n : smooth in $\bar{k}$

For fixed $\bar{k}$ $\{\psi_{n,\bar{k}}\}_{n=1}^{\infty}$ A ***basis*** for Hilbert space $\mathcal{H}_{\bar{k}}$

But in general there is no natural trivialization of $\mathcal{H}$!

A Family Of Connections on $\mathcal{H}$

So: There is no such thing as ``THE'' Berry connection in the context of band structure.

But, there ***is*** a natural family of connections on $\mathcal{H}$:

[Freed & Moore, 2012]

$$\nabla^{\mathcal{H},x_0}$$

They depend on a choice of origin x_0 modulo L:

$$\nabla^{\mathcal{H},x_0} - \nabla^{\mathcal{H},x_0'} = \alpha$$

$$\alpha = 2\pi \mathrm{i}\, dk \cdot (x_0 - x_0') \otimes 1_{\mathcal{H}}$$

Berry Connections For Insulators

Insulator: Projected bundle $\mathcal{F}$ of filled bands:

$$\mathcal{F}_{\bar{k}} = \Theta(E_f - H_{\bar{k}}) \cdot \mathcal{H}_{\bar{k}} \subset \mathcal{H}_{\bar{k}}$$

$$\nabla^{B,x_0} - \nabla^{B,x_0'} = \alpha$$

$$\alpha = 2\pi \mathrm{i}\, dk \cdot (x_0 - x_0') \otimes 1_{\mathcal{F}}$$

So what?

$$F(\nabla^{B,x_0}) = F(\nabla^{B,x_0'})$$

$\Longrightarrow$ All Chern numbers unchanged....

Electric polarization:

$$\langle K, P/e \rangle = \int_{T_K^\perp} \mathrm{Im} \log \det \mathrm{Hol}(\nabla^{B,x_0}, \gamma_K) \mod 2\pi$$

[King-Smith & Vanderbilt (1993); Resta (1994)]

Magnetoelectric Polarizability

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{Maxwell}} \supset \int_{\mathbb{R}^4} \alpha^{ij} E_i B_j$$

``Axion angle'' $\theta(x_0) = \frac{1}{3}\alpha^i{}_i = \int_{T^\vee} CS(\nabla^{B,x_0})$

[Qi, Hughes, Zhang; Essin, Joel Moore, Vanderbilt]

Dependence Of Axion Angle On x_0

$$CS(\nabla + \alpha) - CS(\nabla) = \mathrm{Tr}(2\alpha F + \alpha D_A \alpha + \tfrac{2}{3}\alpha^3)$$

$$\vec{c} := \int_{T^\vee} c_1(\mathcal{F}) \in L^\vee$$

$$\theta(x_0) - \theta(x_0') = 2\pi \vec{c} \cdot (x_0 - x_0')$$

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{Maxwell}} \supset \frac{1}{4\pi} \int_{\mathbb{R}^4} \langle \vec{c}, d\vec{x} \rangle \wedge CS(A^{\mathrm{Maxwell}})$$

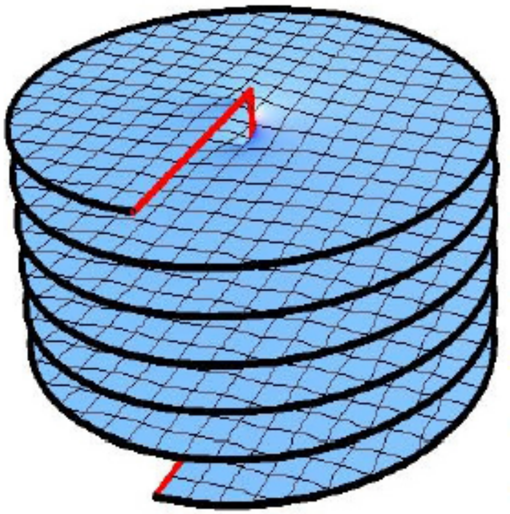
⇒ QHE in the *bulk* of the ``insulator'' in the planes orthogonal to $\vec{c}$

3D QHE

$$J^i = \frac{e^2}{h}\epsilon^{ijk}c_j E_k$$

B. Halperin 1987;
Kohmoto,Halperin, Wu 1992

Dislocations support chiral modes and give physical realizations of surface defects:



$\vec{D} \in L$

$\vec{c}_1 \cdot \vec{D}$ chiral bosons.

Closely related: Ran, Zhang, Vishwanath 2008 & Bulmash, Hosur, Zhang, Qi 2015

Chapter Three

QUANTUM SYSTEMS– CONTINUOUS FAMILIES –
- ALL IN THE FAMILY – NONCOMMUTATIVE GEOMETRY –
- HILBERT MODULES - A BORN RULE AT LAST -
- QUANTUM INFORMATION THEORY –
- HEXAGONS & PENTAGONS

Quantum Systems

Set of physical ``states'' $\mathcal{S}$

Set of physical ``observables'' $\mathcal{O}$

Born Rule: $BR : \mathcal{S} \times \mathcal{O} \to \mathcal{P}$

$\mathcal{P}$ Probability measures on $\mathbb{R}$.

$m \in \mathfrak{M}(\mathbb{R}) \Rightarrow 0 \leq \wp(m) \leq 1$

$m = [r_1, r_2] \subset \mathbb{R}$ $BR(\mathrm{s}, \mathrm{O})([r_1, r_2])$

is the probability that a measurement of the observable O in the state s has value between r_1 and r_2 .

Standard Dirac-von Neumann Axioms

$\mathcal{S}$ Density matrices ρ: Positive trace class operators on Hilbert space of trace =1

$\mathcal{O}$ Self-adjoint operators T on Hilbert space

Spectral Theorem: There is a one-one correspondence of self-adjoint operators T and projection valued measures:

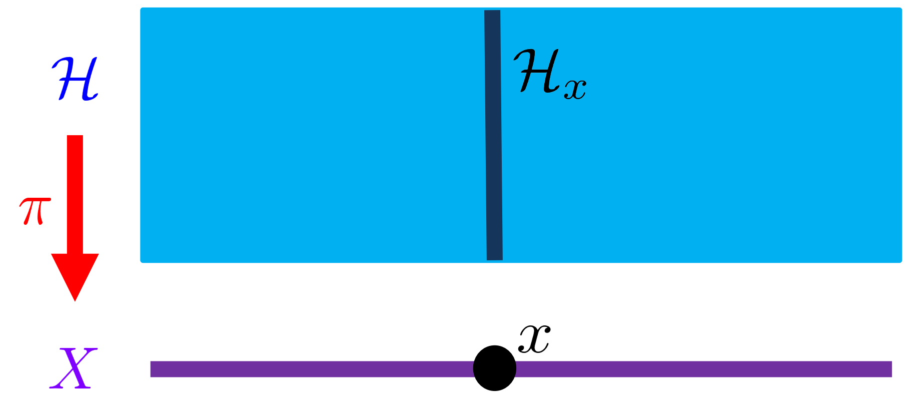
$$m \subset \mathbb{R} \to P_T(m)$$

Example: $T = \sum_\lambda \lambda P_\lambda \qquad P_T([r_1, r_2]) = \sum_{r_1 \leq \lambda \leq r_2} P_\lambda$

$$BR(\rho, T)(m) = \mathrm{Tr}_{\mathcal{H}} \left(\rho P_T(m)\right)$$

Continuous Families Of Quantum Systems

Hilbert bundle over space X of control parameters.



For each x get a probability measure $\wp_x$:

$$m \in \mathfrak{M}(\mathbb{R}) \mapsto \wp_x(m) := \mathrm{Tr}_{\mathcal{H}_x}(\rho_x P_{T_x}(m))$$

$$BR : \mathcal{S} \times \mathcal{O} \times X \to \mathcal{P}$$

$$BR(\rho, T, x) = \wp_x$$

All In The Family

Let’s replace the family X of control parameters by a

noncommutative space

Curiosity.

Indeed, formulating the Born rule proves to be an interesting challenge.

With irrational magnetic flux the Brillouin torus is replaced by a noncommutative manifold. (Bellisard, Connes, Gruber,…)

Noncommutative moduli of vacua of susy field theories. e.g. NC tt* geometry (S. Cecotti, D. Gaiotto, C. Vafa)

Boundaries of Narain moduli spaces of toroidal heterotic string compactifications are NC

(closely related to the most-cited paper of Seiberg & Witten)

The ``early universe'' might be NC

C* Algebras

A C* algebra is a (normed) algebra $\mathfrak{A}$ over the complex numbers with an involution:

$$a \in \mathfrak{A} \to a^* \in \mathfrak{A} \qquad (ab)^* = b^* a^*$$

such that ….

Example 1: $\mathfrak{A} = C(X) := \{f : X \to \mathbb{C}\}$

Example 2: $\mathfrak{A} = Mat_n(\mathbb{C})$

Self-adjoint: $a^* = a$

Positive: $a = b^* b$

Gelfand's Theorem

The topology of a (Hausdorff) space X is completely captured by the C*-algebra of continuous functions on X:

$$C(X) := \{f : X \to \mathbb{C}\}$$

Points ``are'' 1D representations:

$$\mathrm{ev}_{x_0} : f \in C(X) \mapsto f(x_0) \in \mathbb{C}$$

Commutative C* algebra: $\mathfrak{A} \Longrightarrow \mathrm{Irrep}(\mathfrak{A})$ A topological space

$$\mathfrak{A} \cong C(\mathrm{Irrep}(\mathfrak{A}))$$

Noncommutative Geometry

Statements about the topology/geometry of X are equivalent to algebraic statements about C(X)

Replace C(X) by a noncommutative C* algebra $\mathfrak{A}$

Interpret $\mathfrak{A}$ as the ``algebra of functions on a noncommutative space'' …

… even though there are no points.

``pointless geometry''

Example: Noncommutative torus:

$$U_i U_i^* = U_i^* U_i = 1 \quad U_i U_j = e^{2\pi \mathrm{i} \phi_{ij}} U_j U_i$$

Noncommutative Control Parameters

We would like to define a family of quantum systems parametrized by a NC manifold whose ``algebra of functions'' is a general C* algebra $\mathfrak{A}$

What are observables?

What are states?

What is the Born rule?

What replaces the Hilbert bundle?

Noncommutative Hilbert Bundles

Definition: A ***Hilbert-module*** $\mathcal{E}$ over C*-algebra $\mathfrak{A}$:

Complex vector space $\mathcal{E}$ with a right-action of $\mathfrak{A}$
and an ``inner product'' valued in $\mathfrak{A}$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

$$(\Psi_1, \Psi_2)^*_{\mathfrak{A}} = (\Psi_2, \Psi_1)_{\mathfrak{A}}$$

$$(\Psi, \Psi)_{\mathfrak{A}} \geq 0$$ (Positive element of the C* algebra.)

such that .....

Like a Hilbert space, but ``overlaps'' are valued in a (possibly) noncommutative algebra.

Quantum Mechanics With Noncommutative Amplitudes

Basic idea: Replace the Hilbert space by a *Hilbert-module*

$$\mathcal{H} \to \mathcal{E}$$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

So the Born rule is not obvious:

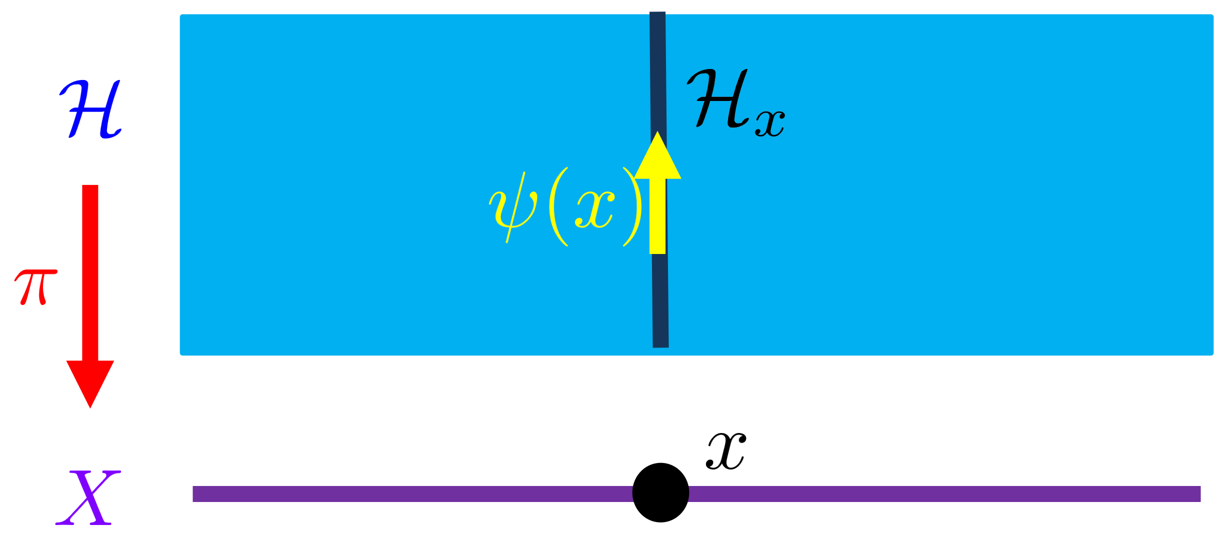
QM: $$0 \leq \ \wp(\lambda) = (\psi_\lambda, \psi)(\psi_\lambda, \psi)^* \ \leq 1$$

QMNA: $$(\Psi_\lambda, \Psi)(\Psi_\lambda, \Psi)^* \in \mathfrak{A}$$

Example 1: Hilbert Bundle Over A Commutative Manifold

$$\mathcal{E} = \Gamma[\mathcal{H} \to X] \qquad \mathfrak{A} = C(X)$$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$



$$(\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A} := C(X)$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}}(x) := (\psi_1(x), \psi_2(x))_{\mathcal{H}_x} \in \mathbb{C}$$

Example 2: Hilbert Bundle Over A Fuzzy Point

Def: ``fuzzy point'' has $\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C})$

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C})$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}} = \Psi_1^{\dagger}\Psi_2$$

Observables In QMNA

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^*\Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators
$\mathfrak{B}$ are another C* algebra.

Definition: *QMNA observables*
are self-adjoint elements of $\mathfrak{B}$

(Technical problem: There is no spectral theorem for self-adjoint elements of an abstract C* algebra.)

C* Algebra States

Definition: A *C*-algebra state* $\omega \in \mathcal{S}(\mathfrak{A})$ is a positive linear functional

$$\omega : \mathfrak{A} \to \mathbb{C} \qquad \omega(\mathbf{1}) = 1$$

$$\mathfrak{A} = C(X) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$$\omega(f) = \int_X f \, d\mu$$ dμ = a positive measure on X:

$$\mathfrak{A} \cong \mathrm{Mat}_{a \times a}(\mathbb{C}) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$$\omega(T) = \mathrm{Tr}_{\mathcal{H}}(\rho T)$$ ρ = a density matrix

QMNA States

Definition: A ***<u>QMNA state</u>*** is a *<u>completely</u>* *<u>positive</u>* unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

``Completely positive'' comes up naturally both in math and in quantum information theory.

A natural generalization of
the Schrodinger (actually, Lindblad)
equation exists.

QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of φ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

$$\mathrm{BR}(\varphi, T, \omega)(m) = \omega(\varphi(P_T(m)))$$

Family Of Quantum Systems Over A Fuzzy Point

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C}) = \mathbb{C}^b \otimes \mathbb{C}^a = \mathcal{H}_{\mathrm{Bob}} \otimes \mathcal{H}_{\mathrm{Alice}}$$

$$\mathfrak{A} = Mat_a(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Alice}})$$

$$\mathfrak{B} = Mat_b(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Bob}})$$

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

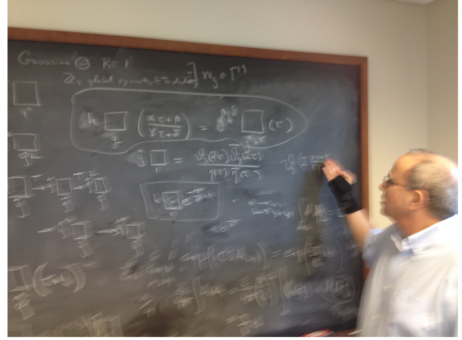
``A NC measure $\omega \in \mathcal{S}(\mathfrak{A})$'' is equivalent to a density matrix ρ_A on $\mathcal{H}_{\mathrm{A}}$

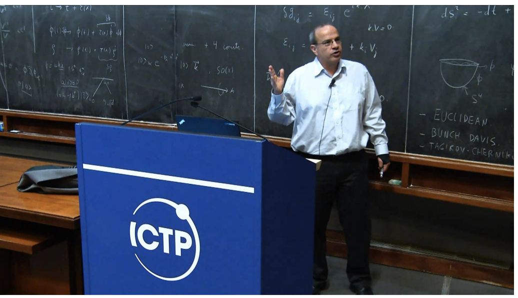
QMNA state: $\varphi(T) = \sum_\alpha E_\alpha^\dagger T E_\alpha \quad \sum_\alpha E_\alpha^\dagger E_\alpha = 1$

Quantum Information Theory & Noncommutative Geometry

$$
\begin{aligned}
BR(\varphi, T, \omega)(m) &= \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m)) \\
&= \textstyle\sum_\alpha \mathrm{Tr}_{\mathcal{H}_A} \rho_A E_\alpha^\dagger (P_T(m)) E_\alpha \\
&= \textstyle\sum_\alpha \mathrm{Tr}_{\mathcal{H}_B} E_\alpha \rho_A E_\alpha^\dagger P_T(m) \\
&= \mathrm{Tr}_{\mathcal{H}_B} \mathcal{E}(\rho_A) P_T(m)
\end{aligned}
$$

Last expression is the measurement by Bob of T in the state $\mathcal{E}(\rho_A)$ where Alice prepared ρ_A and sent it to Bob through quantum channel $\mathcal{E}$.

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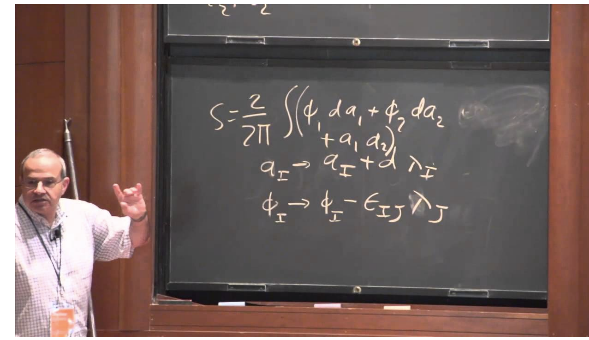

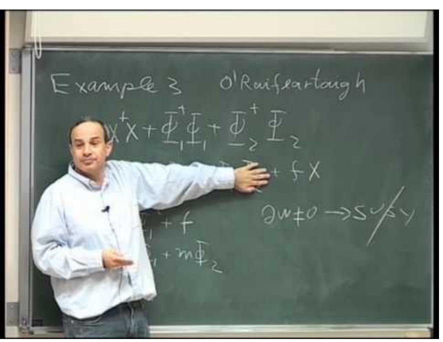
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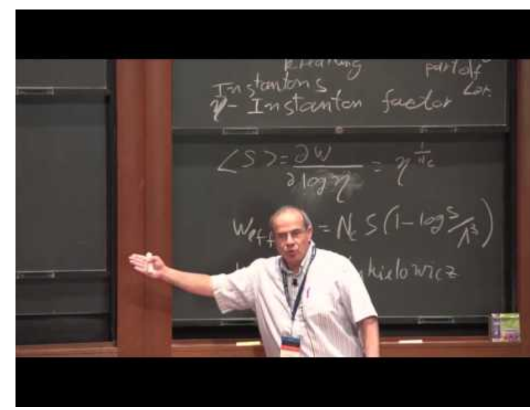
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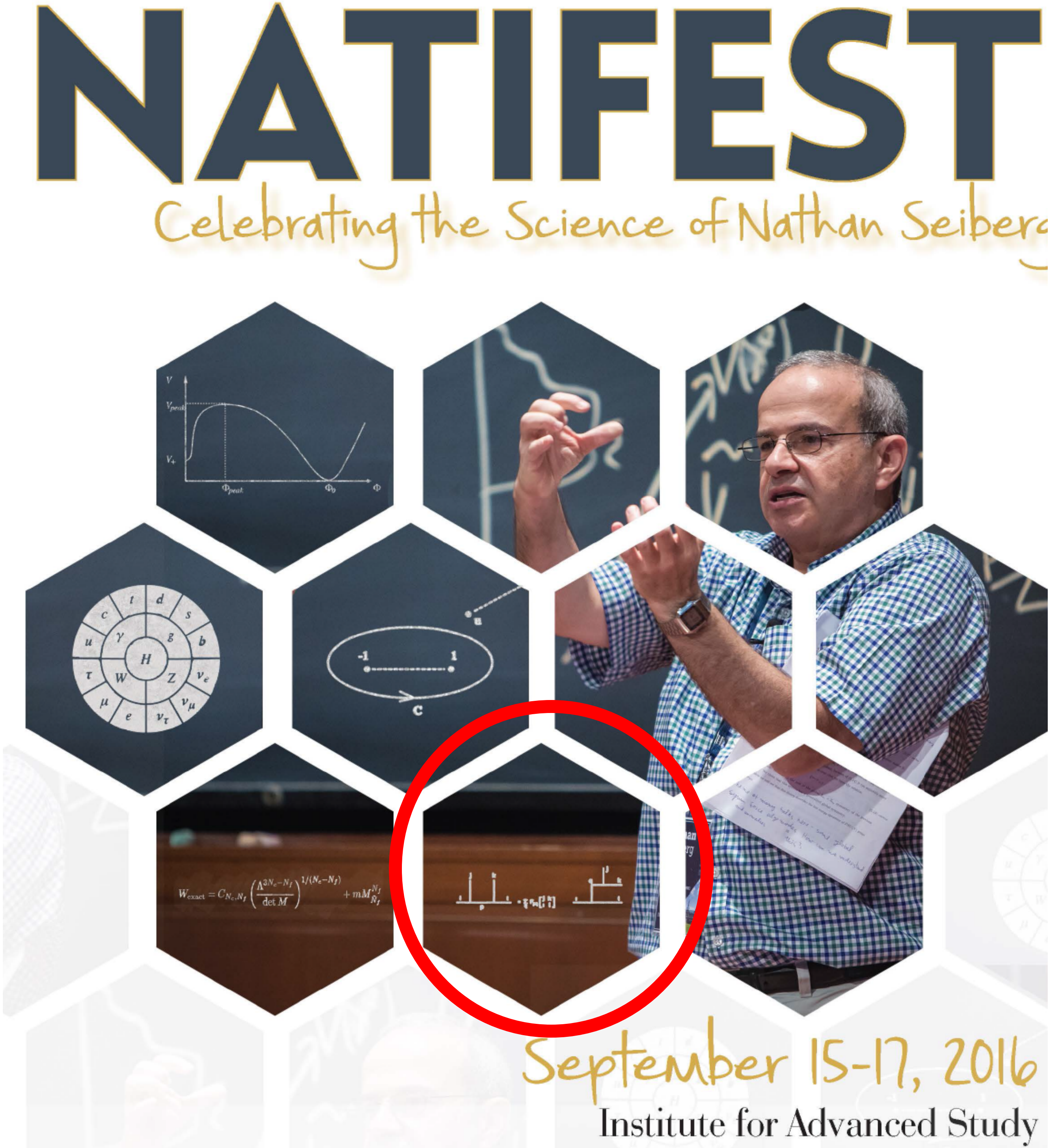
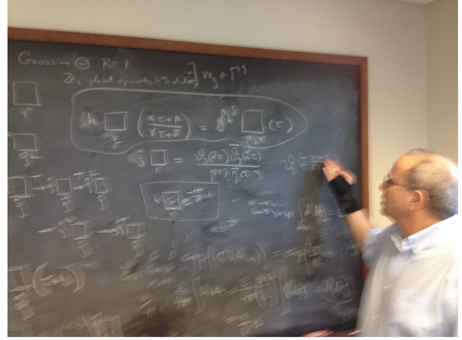
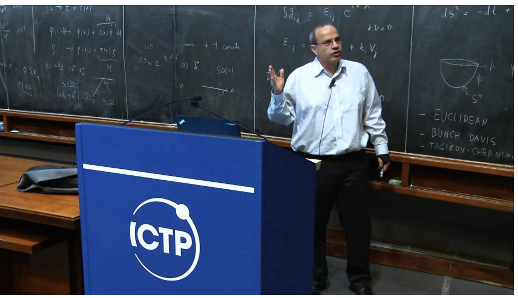
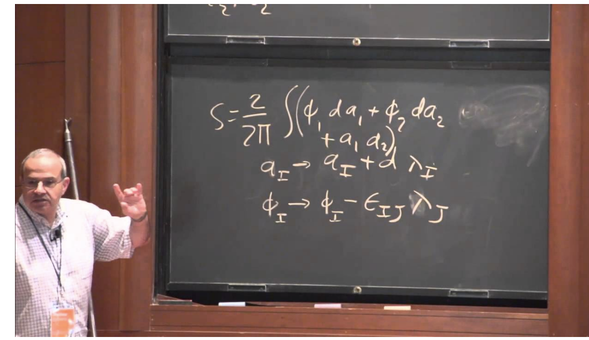

NATIFEST
Celebrating the Science of Nathan Seiber
September 15-17, 2016
Institute for Advanced Study







Congratulations
&
Happy Birthday !