An Uncertainty Principle for Fluxes

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CUNY, June 28, 2006

Based on hep-th/0605198; hep-th/0605200

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Introduction

Today I'll be talking about some interesting subtleties in the quantization of Maxwell's theory and some of its generalizations.

Of course, there are many different generalizations of Maxwell's theory. I will discuss abelian gauge theories whose field-strengths are differential forms $F \in \Omega^{\ell}(M)$.

More technically - the result applies to the class of "generalized abelian gauge theories" – these are theories where the space of gauge-invariant field configurations "$\mathcal{A}/\mathcal{G}$" is a generalized differential cohomology group in the sense of Hopkins & Singer.

In particular "$\mathcal{A}/\mathcal{G}$" is an abelian group.

These kinds of theories arise naturally in supergravity and superstring theories, and indeed play a key role in the theory of D-branes and in recent claims of moduli stabilization in string theory.

Summary of the Results

1. Manifestly electric-magnetic dual formulation of the Hilbert space of generalized Maxwell theory.

2. The Hilbert space can be decomposed into electric and magnetic flux sectors, but the operators that measure electric and magnetic fluxes don't commute and cannot be simultaneously diagonalized.

This is surprising and nontrivial!

3. Group theoretic approach to the theory of a self-dual field.

4. In particular: the K-theory class of a RR field cannot be measured!

Generalized Maxwell Theory

Begin with generalized Maxwell theory on a spacetime M with $\dim M = n$.

It has a fieldstrength $F \in \Omega^{\ell}(M)$

Action $S = \pi R^2 \int_M F * F$

If $M = X \times \mathbb{R}$ we have a Hilbert space $\mathcal{H}$.

Grade the Hilbert space by (topological class of) magnetic flux:

$$\mathcal{H} = \oplus_m \mathcal{H}_m \qquad m \in H^{\ell}(X, \mathbb{Z}).$$

Electro-magnetic duality: An equivalent theory is based on a dual potential with $F_D \in \Omega^{n-\ell}(M)$ and

$$RR_D = \hbar$$

$\Rightarrow$ there must also be a grading by (topological class of) electric flux:

$$\mathcal{H} = \oplus_e \mathcal{H}_e \qquad e \in H^{n-\ell}(X, \mathbb{Z}),$$

Electric and Magnetic Flux Sectors

Can we simultaneously decompose $\mathcal{H}$ into electric and magnetic flux sectors?

$$\mathcal{H} \stackrel{?}{=} \oplus_{e,m} \mathcal{H}_{e,m}.$$

♮ response should be "yes!"

For the scalar field this is just decomposition into momentum and winding!

Measure magnetic flux: $\int_{\Sigma_1} F$ where $\Sigma_1 \in Z_\ell(X)$

Measure electric flux: $\int_{\Sigma_2} *F$ where $\Sigma_2 \in Z_{n-\ell}(X)$

Canonical momentum conjugate to A is $\Pi \sim (*F)_X$.

$$\begin{aligned} [\int_{\Sigma_1} F, \int_{\Sigma_2} *F] &\sim [\int_{\Sigma_1} F, \int_{\Sigma_2} \Pi] \\ &= [\int_X \omega_1 F, \int_X \omega_2 \Pi] \\ &= i\hbar \int_X \omega_1 d\omega_2 = 0 \end{aligned}$$

where ω_i are closed forms Poincaré dual to Σ_i.

Fallacy in the argument

But! the above period integrals only measure the flux modulo torsion.

The fluxes e, m are elements of abelian groups. These groups in general have nontrivial torsion subgroups.

The above discussion misses a very interesting uncertainty principle

Differential Cohomology

The space of gauge-inequivalent fields is described by *differential cohomology*, or *Deligne-Cheeger-Simons theory*.

We denote this space by $\check{H}^{\ell}(M)$.

Definition: By "generalized Maxwell theory" we mean a field theory such that the space of gauge inequivalent fields is $\check{H}^{\ell}(M)$ for some ℓ.

Our next goal is to get a clear picture of the space $\check{H}^{\ell}(M)$, based on two exact sequences.

Structure of the Differential Cohomology Group

Fieldstrength exact sequence:

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \xrightarrow{\text{fieldstrength}} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

Characteristic class exact sequence:

$$0 \to \underbrace{\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)}_{\text{Topologically trivial}} \to \check{H}^{\ell}(M) \xrightarrow{\text{char.class}} H^{\ell}(M;\mathbb{Z}) \to 0$$

The space of differential characters has the form:

$$\check{H}^{\ell} = T \times \Gamma \times V$$

T: Connected torus of topologically trivial flat fields:

$$\mathcal{W}^{\ell-1}(M) = H^{\ell-1}(M,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

Γ: Discrete (possibly infinite) abelian group of topological sectors: $H^{\ell}(M,\mathbb{Z})$.

V: Infinite-dimensional vector space of "oscillator modes." $V \cong \mathrm{Im} d^{\dagger}$.

Example: Loop Group of $U(1)$

Configuration space of a periodic scalar field on a circle:

$$\check{H}^1(S^1) = Map(S^1, U(1)) = LU(1)$$

• Topological components: Winding number $\in H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$

• Flat fields: The torus $\mathbf{T}$ of constant maps: $H^0(S^1, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$

• Vector space: $V = \Omega^0/\mathbb{R}$ are the loops admitting a logarithm:

$$\check{H}^1(S^1) = \mathbf{T} \times \mathbb{Z} \times V$$

This corresponds to the explicit decomposition,

$$\varphi(\sigma) = \exp\left[2\pi i \phi_0 + 2\pi i w \sigma + \sum_{n \neq 0} \frac{\phi_n}{n} e^{2\pi i n \sigma}\right]$$

The space of flat fields

$H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})$ is a compact abelian group...

- it is not necessarily connected!

Connected component of the identity

$$\mathcal{W}^{\ell-1}(M) = H^{\ell-1}(M,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

Group of components $= H^{\ell}_{T}(M;\mathbb{Z})$:

$$0 \to \mathcal{W}^{\ell-1}(M) \to H^{\ell-1}(M,\mathbb{R}/\mathbb{Z}) \to H^{\ell}_{T}(M;\mathbb{Z}) \to 0.$$

Example: $M = L_k$, $\ell = 2$, $\mathcal{W}^1 = 0$,

$$H^2(M;\mathbb{Z}) = H^2_T(M;\mathbb{Z}) = \mathbb{Z}_k$$
$$H^1(M;\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_k$$

These are *discrete Wilson lines*:

$$\chi_r(\gamma) = e^{2\pi i r/k} = \omega_k^r \qquad\qquad r \in \mathbb{Z}_k$$

defines the topologically nontrivial flat fields.

Poincaré-Pontryagin Duality

1. There is a very subtle product on differential characters inducing the structure of a graded ring:

$$\check{H}^{\ell_1}(M) \times \check{H}^{\ell_2}(M) \to \check{H}^{\ell_1+\ell_2}(M).$$

Denote it: $[\check{A}_1] * [\check{A}_2]$.

$= [A_1 dA_2]$ on topologically trivial fields. ★

2. If M is compact and oriented, and $\dim M = n$ then evaluation on the fundamental cycle $[M]$ – the "holonomy around M" - defines an integration map

$$\int_M^{\check{H}} : \check{H}^{n+1}(M) \to \mathbb{R}/\mathbb{Z}$$

3. Poincaré-Pontryagin duality: We have a perfect pairing

$$\check{H}^{\ell}(M) \times \check{H}^{n+1-\ell}(M) \to \mathbb{R}/\mathbb{Z}$$

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int_M^{\check{H}} [\check{A}_1] * [\check{A}_2]$$

Examples of the Pairing:

Example 1 : Chern-Simons terms

If $\dim M = 2p+1$, and $[\check{A}] \in \check{H}^{p+1}(M)$, then

$$\int_M^{\check{H}} [\check{A}] * [\check{A}] \in \mathbb{R}/\mathbb{Z}$$

For topologically trivial fields

$$\int_M^{\check{H}} [A] * [A] = \int_M^{H} AdA \mod \mathbb{Z}$$

Example 2: Cocycle on the Loop Group

Recall $\check{H}^1(S^1) = LU(1)$:

$\varphi = \exp(2\pi i \phi) \qquad \phi : \mathbb{R} \to \mathbb{R}$

$\phi(s+1) = \phi(s) + w \qquad w \in \mathbb{Z}$ is the winding number.

$$\langle \varphi^1, \varphi^2 \rangle = \int_0^1 \phi^1 \frac{d\phi^2}{ds} ds - w^1 \phi^2(0) \mod \mathbb{Z}$$

Note! This is (twice!) the cocycle which defines the basic central extension of $LU(1)$.

Hamiltonian Formulation of Generalized Maxwell Theory

Spacetime: $M = X \times \mathbb{R}$.

Generalized Maxwell fields: $[\check{A}] \in \check{H}^{\ell}(M)$.

$$S = \pi R^2 \int_M F * F$$

Hilbert space: $\mathcal{H} = L^2(\check{H}^{\ell}(X))$

This breaks manifest electric-magnetic duality.

There is a better way to characterize the Hilbert space.

Heisenberg Groups

Theorem A: Let G be a topological abelian group. Central extensions, $\tilde{G}$, of G by $U(1)$ are in one-one correspondence with continuous bimultiplicative maps $s : G \times G \to U(1)$ which are alternating (and hence skew).

1. s is *alternating*: $s(x,x) = 1$.
2. s is *skew* : $s(x,y) = s(y,x)^{-1}$.
3. s is *bimultiplicative*:

$$s(x_1+x_2, y) = s(x_1,y)s(x_2,y) \quad \& \quad s(x, y_1+y_2) = s(x,y_1)s(x,y_2)$$

If $x \in G$ lifts to $\tilde{x} \in \tilde{G}$

$$s(x,y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$$

Definition: If s is nondegenerate then $\tilde{G}$ is a *Heisenberg group.*

Theorem B: (Stone-von Neuman theorem). If $\tilde{G}$ is a Heisenberg group then the unitary irrep of $\tilde{G}$ where $U(1)$ acts canonically is unique up to isomorphism.

Heisenberg group for generalized Maxwell theory

Define:

$$\tilde{G} := \mathrm{Heis}\big(\check{H}^{\ell}(X) \times \check{H}^{n-\ell}(X)\big)$$

via the group commutator:

$$s\left(([\check{A}_1],[\check{A}_1^D]),([\check{A}_2],[\check{A}_2^D])\right) = \exp\left[2\pi i\big(\langle[\check{A}_2],[\check{A}_1^D]\rangle - \langle[\check{A}_1],[\check{A}_2^D]\rangle\big)\right].$$

Claim: The Hilbert space of the generalized Maxwell theory *is* the unique irrep of the Heisenberg group $\tilde{G}$

N.B! This formulation of the Hilbert space is *manifestly electric-magnetic dual.*

Explicit representation

For Heisenberg groups of the form

$$1 \to U(1) \to \mathrm{Heis}(S \times \hat{S}) \to S \times \hat{S} \to 1$$

where $\hat{S}$= Pontryagin dual, an explicit representation is given by $\mathcal{H} = L^2(S)$:

$\mathcal{H} = L^2(S)$ is a representation of S: $\forall s_0 \in S$

$$(T_{s_0}\psi)(s) := \psi(s + s_0).$$

$\mathcal{H} = L^2(S)$ is also a representation of $\hat{S}$: $\forall \chi \in \hat{S}$

$$(M_\chi \psi)(s) := \chi(s)\psi(s)$$

But!

$$T_{s_0} M_\chi = \chi(s_0) M_\chi T_{s_0}.$$

If $S = \check{H}^\ell(X)$, then PP duality $\Rightarrow$ $\hat{S} = \check{H}^{n-\ell}(X)$:

$\Rightarrow$ The unique irrep of $\tilde{G}$ is isomorphic to

$$\mathcal{H} \cong L^2(\check{H}^\ell(X))$$

Dual frame: $S = \check{H}^{n-\ell}(X)$ and $\hat{S} = \check{H}^\ell(X)$ $\Rightarrow$

$$\mathcal{H} \cong L^2(\check{H}^{n-\ell}(X))$$

Defining Electric Flux

Return to our question:

Can we simultaneously decompose $\mathcal{H}$ into electric and magnetic flux sectors?

$$\mathcal{H} \stackrel{?}{=} \oplus_{e,m} \mathcal{H}_{e,m}.$$

Need to understand grading by electric flux more deeply.

Diagonalizing $(*F)_X$ means diagonalizing Π, but Π is the generator of translations.

Definition: A state of definite topological class of electric flux is an eigenstate under translation by *flat fields*

$$\forall \phi_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}),$$

$$\psi(\check{A} + \check{\phi}_f) = \exp\left(2\pi i \int_X^H e\phi_f\right)\psi(\check{A})$$

The topological classes of electric flux are labelled by

$$e \in H^{n-\ell}(X, \mathbb{Z}).$$

Group theoretic approach to flux sectors

Electric flux sectors diagonalize the flat fields $H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$.

Magnetic flux sectors diagonalize dual flat fields $H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z})$.

These groups separately lift to commutative subgroups of $\tilde{G} := \mathrm{Heis}(\check{H}^{\ell} \times \check{H}^{n-\ell})$.

However they do not commute with each other!

$\mathcal{U}_E(\eta_e) :=$ translation operator by $\eta_e \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$

$\mathcal{U}_M(\eta_m) :=$ translation operator by $\eta_m \in H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z})$

Then we have the uncertainty relation:

$$[\mathcal{U}_e(\eta_e), \mathcal{U}_m(\eta_m)] = T(\eta_e, \eta_m) = \exp\left(2\pi i \int_X \eta_e \beta \eta_m\right)$$

T: torsion pairing, β= Bockstein: $\beta(\eta_m) \in H_T^{n-\ell}(X, \mathbb{Z})$.

Uncertainty Relation

$$[\mathcal{U}_e(\eta_e), \mathcal{U}_m(\eta_m)] = T(\eta_e, \eta_m) = \exp\left(2\pi i \int_X \eta_e \beta \eta_m\right)$$

T: torsion pairing, β: Bockstein.

$\Rightarrow$ Translations by $\mathcal{W}^{\ell-1}(X)$ and by $\mathcal{W}^{n-\ell-1}(X)$ commute

$\Rightarrow$ we can simultaneously diagonalize:

$$\mathcal{H} = \oplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}}.$$

However: The pairing does *not* commute on the subgroups of all flat fields.

It descends to the "torsion pairing" or "link pairing":

$$H^{\ell}_T(X) \times H^{n-\ell}_T(X) \to \mathbb{R}/\mathbb{Z}$$

This is a perfect pairing, so it is maximally noncommutative on torsion.

Example: Maxwell theory on $S^3/\mathbb{Z}_k \times \mathbb{R}$

$H^1(L_k; \mathbb{R}/\mathbb{Z}) \cong H^2(L_k; \mathbb{Z}) = \mathbb{Z}_k$ is all torsion

Acting on the Hilbert space the flat fields generate a Heisenberg group extension

$$0 \to \mathbb{Z}_k \to \mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \to \mathbb{Z}_k \times \mathbb{Z}_k \to 0$$

This has unique irrep P= clock operator, Q = shift operator

$$\boxed{PQ = e^{2\pi i/k} QP}$$

States of definite electric and magnetic flux

$$|e\rangle = \frac{1}{\sqrt{k}} \sum_m e^{2\pi i em/k} |m\rangle$$

This example already appeared in string theory in Gukov, Rangamani, and Witten, hep-th/9811048. They studied $AdS_5 \times S^5/\mathbb{Z}_3$, and in order to match nonperturbative states concluded that in the presence of a D3 brane one cannot simultaneously measure D1 and F1 number.

Remarks

• The pairing of topologically nontrivial flat fields has no tunable $\hbar$, and is always noncommuting, even in the large volume limit.

This is a quantum effect which does not disappear in the large volume, or $\hbar \to 0$ limit.

• Using an elaborate configuration of superconductors and Josephson junctions one can possibly test this uncertainty principle in the laboratory [work in progress with A. Kitaev and K. Walker]

Self-dual fields

Now suppose $\dim M = 4k+2$, and $\ell = 2k+1$.

We can impose a self-duality condition

$$F = *F$$

For the non-self-dual field we represent

$$\mathrm{Heis}(\check{H}^\ell(X) \times \check{H}^\ell(X))$$

Proposal: For the self-dual field we represent:

$$\mathrm{Heis}(\check{H}^\ell(X))$$

Attempt to define this Heisenberg group via

$$s_{\mathrm{trial}}([\check{A}_1],[\check{A}_2]) = \exp 2\pi i \langle [\check{A}_1],[\check{A}_2]\rangle.$$

It is skew and and nondegenerate, but *not* alternating!

$$s_{\mathrm{trial}}([\check{A}],[\check{A}]) = (-1)^{\int_X \nu_{2k} \; m}$$

(Gomi 2005).

$\mathbb{Z}_2$-graded Heisenberg groups

Theorem A'**:** Skew bimultiplicative maps classify $\mathbb{Z}_2$-graded Heisenberg groups.

$\mathbb{Z}_2$ grading in our case:

$$\epsilon([\check{A}]) = \begin{cases} 0 & \int \nu_{2k}\ m = 0 \bmod 2 \\ 1 & \int \nu_{2k}\ m = 1 \bmod 2 \end{cases}$$

Theorem B': A $\mathbb{Z}_2$-graded Heisenberg group has a unqiue $\mathbb{Z}_2$-graded irreducible representation.

This defines the Hilbert space of the self-dual field.

Remark: One can show that the nonself-dual field at a special radius, $R^2 = 2\hbar$, decomposes into

$$\mathcal{H}_{nsd} \cong \oplus_\alpha \mathcal{H}_{sd,\alpha} \otimes \mathcal{H}_{asd,\alpha}$$

The subscript α is a sum over "generalized spin structures" - a torsor for 2-torsion points in $H^{2k+1}(X;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$.

Example: Self-dual scalar: $k = 0$. By bosonization $\psi = e^{i\phi}$. The $\mathbb{Z}_2$-grading is just *fermion number*! α labels R and NS sectors.

$\Rightarrow$ SD theory generalizes VOA theory

Ramond-Ramond fields

• The (gauge equivalence class) of a RR field in type II theory is in *differential K-theory*, $\check{K}(X)$, an analog of $\check{H}(X)$. Many structures are formally the same:

$$0 \to \overbrace{K^{-1,B}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{K}^{0,\check{B}}(M) \to \Omega(M;R)^0_{d_H,\mathbb{Z}} \to 0$$

$$0 \to \underbrace{\Omega(M;R)^{-1}/\Omega(M;R)^{-1}_{d_H,\mathbb{Z}}}_{\text{Topologically trivial}} \to \check{K}^{0,\check{B}}(M) \to K^{0,B}(M) \to 0$$

$\exists$ a perfect pairing: $\langle [\check{C}_1], \overline{[\check{C}_2]} \rangle = \int^{\check{K}} [\check{C}_1] * \overline{[\check{C}_2]}$

The RR field is *self-dual* $\Rightarrow$

$\Rightarrow$ The Hilbert space is the unique $\mathbb{Z}_2$-graded irrep of the $\mathbb{Z}_2$-graded Heisenberg group

$$\mathrm{Heis}(\check{K}(X))$$

$\Rightarrow$ the full K-theory class is not measurable!

Rather, the Hilbert space is a representation of the "quantum K-theory"

$$0 \to U(1) \to QK(M;\mathbb{R}/\mathbb{Z}) \to K(M;\mathbb{R}/\mathbb{Z}) \to 0$$

Open Problems

1. If one cannot measure the complete K-theory class of RR flux what about the D-brane charge?

a.) If no, we need to make an important conceptual revision of the standard picture of a D-brane

b.) If yes, then flux-sectors and D-brane charges are classified by different groups $\Rightarrow$ tension with AdS/CFT and geometric transitions.

2. What happens when X is noncompact?

3. What is the physical meaning of the fermionic sectors in the RR Hilbert space?

4. How is this compatible with noncommutativity of 7-form Page charges in M-theory?