Lecture 1: Modularity in 2D CFT

1. Introduction & Overview

Let me begin with a broad overview of these 4 lectures.

THIS IS A SERIES OF LECTURES ON

BPS STATECOUNTING

IN STRING THEORY (BPS STATES ARE SPECIAL STATES WHICH WE'LL DEFINE IN DUE COURSE)

An important role in BPS State counting is played by

Automorphic functions $\supset$ modular forms

So our first two lectures will emphasize that aspect.

Modular forms, and automorphic forms have played an important role in mathematics since the early 19^{th} century – and continue to be a very active area of research. E.g. the proof of FLT relies on the theory of modular forms.

The subject entered physics in the 1980's in the context of 2D CFT and string theory.

1980's: 2D CFT – Constrain spectrum

w.s. string – anomaly cancell.

- Constrains the spectrum of 2D CFT
- Consistency conditions on the worldsheet includes cancellation of diffeo anomalies = modular anomalies.

• But the current motivation comes largely from the program of accounting for the entropy of SUSY BH's using D-brane microstates.

$$\boxed{\text{BH MICROSTATES} \Longleftrightarrow \text{D-BRANES}}$$

This program began with the paradigmatic computation of Strominger + Vafa in 1995. Let us sketch in caricature what they did. The exact details are not crucial to what I will subsequently say.

S-V PARADIGM

S.V. considered type IIB string theory

$IIB /$	$\mathbb{R}_t \times$	$\mathbb{R}^4 \times$	$S^1 \times$	$K3$
Q1 D1	$\times$		$\times$	
Q5 D5	$\times$		$\times$	$\times$

For large radius S^1 they argue that the low energy dynamics is governed by a 1+1 dim'l CFT of maps

$$\mathbb{R}_t \times S^1 \longrightarrow (K3)^Q / S_Q := \mathrm{Sym}^Q K3$$

$$Q = Q_1 Q_5$$

COUNT BPS STATES WITH ELLIPTIC GENUS

$$\chi(\mathrm{Sym}^Q k3) \sim \sum c^{(Q)}(n) q^n$$

$n \sim$ third charge (momentum on S^1_R)

$c^{(N)}(n)$ "counts" BPS states of charge (Q_1, Q_5, n)

On the other hand, in 5D SUGRA of IIB/$S^1 \times K3$ $\exists$!

BH WITH CHARGES Q_1, Q_5, n LEAVING 8 SUPERSYMMETRIES UNBROKEN.

One computes the $\underline{\text{area}}$ of the horizon to be
$\qquad\qquad\quad 4$

$$S_{BH} = 2\pi \sqrt{Q_1 Q_5 n}$$

Called BH entropy because it behaves like an entropy: Major problem in GR is to acct for the entropy in terms of microstates.

But going back to the D-brane picture, χ is a modular function (more precisely, a weak Jacobi form) and modularity $\Longrightarrow$

For $n \gg Q$:

$$\log c^{(Q)}(n) \sim 2\pi\sqrt{Q\, n} = 2\pi\sqrt{Q_1 Q_5 n}$$

GREAT SUCCESS!

Lecture I: Basics of modular forms in 2D CFT

Lecture II Combine with extended SUSY:

Elliptic genus, Jacobi Forms,

Lecture IIIA: NEW RESULT

PROGRAM: Repeat this for more realistic BH's

a.) $d=4$

b.) Fewer susy

Current state of the art:

1. BACKGROUNDS WITH $d=4$ & 16 REAL SUSY's: GOOD CONTROL: SEE A. SEN'S LECTURES.

2. BCKGND'S WITH $d=4$ & 8 REAL SUSY'S: MUCH LESS IS KNOWN; WE ARE LEARNING.

$\exists$ charge regimes where we do not know how to compute microstate degeneracy, even at leading order;

and so the SV program is incomplete.

In terms of Jan-DeBoer's lectures

$$\Gamma \longrightarrow \lambda\Gamma \quad \lambda \to \infty \quad ?$$

$$\hat{q}_0 := q_0 - \frac{1}{2}(D_{ABC}P^C)^{-1} Q_A Q_B$$

$$-\hat{q}_0 \gg P^3 \qquad \checkmark$$

$$-\hat{q}_0 \sim P^3 \qquad ?$$

Lectures IIIB + IV will focus on BPS

Statecounting for boundstates of D-branes on a CY 3-fold –

For appropriate charges these can lead to $d=4$ BH's with 4 unbroken susy's.

The main theme of those lectures will be that even the index of BPS states depends on background fields via wall-crossing.

Nevertheless, Thanks to modularity we can make some interesting statements approximating the OSV conjecture.

One last preliminary remark: I can only say very little in 1 hour. Many more details are in the very preliminary lecture notes posted on the school web-page.

Now we will review some basic aspects of the theory of modular forms and how it is related to 2D conformal field theory.

We will take a point of view emphasizing the role of "polar states," and how they constrain the spectrum of the theory.

2. 2D CFT on a torus

2D CFT $\mathcal{C}$; HAS HILBERT SPACE

$$\mathcal{H} = \oplus N_{h\tilde{h}} V_h \otimes \tilde{V}_{\tilde{h}}$$

V_h: HW Rep: $L_0|h\rangle = h|h\rangle$, $L_n|h\rangle = 0$, $n>0$

$$Z(\tau,\bar{\tau}) := \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24}\, \bar{q}^{\tilde{L}_0 - \tilde{c}/24}$$

$$= \mathrm{Tr}\ e^{-2\pi\beta H + 2\pi i\theta P}$$

$q = e(\tau)$, $\tau = \theta + i\beta$, $e(x) := e^{2\pi i x}$

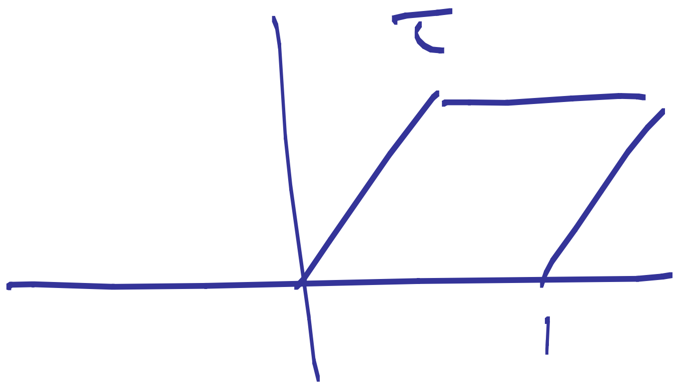
$$H = L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24}$$

$$P = L_0 - \tilde{L}_0 - \frac{c-\tilde{c}}{24}$$

Spectrum bounded below discrete $\Rightarrow$
Z has no singularities for $\tau \in \mathcal{H}$
Might for $\mathrm{Im}\,\tau \longrightarrow 0, \infty$

AS EXPLAINED IN THE NOTES THIS Z CAN BE INTERPRETED AS THE PARTITION FUNCTION OF $\mathcal{C}$ ON THE TORUS, MORE PRECISELY ON THE ELLIPTIC CURVE

$$E_\tau = \mathbb{C} / \mathbb{Z} \oplus \tau \mathbb{Z}$$



IF $\mathcal{C}$ IS DIFF-INVT THEN IN PARTICULAR Z IS INVT UNDER GLOBAL DIFF'S OF E_τ.

ORIENTATION-PRESERVING DIFFS

$$\Gamma = SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}),\ \det = 1 \right\}$$

ACTS BY $\quad \tau \longrightarrow \dfrac{a\tau+b}{c\tau+d}$

Facts about the Modular group $\Gamma := SL(2,\mathbb{Z})$

1. Generators and relations

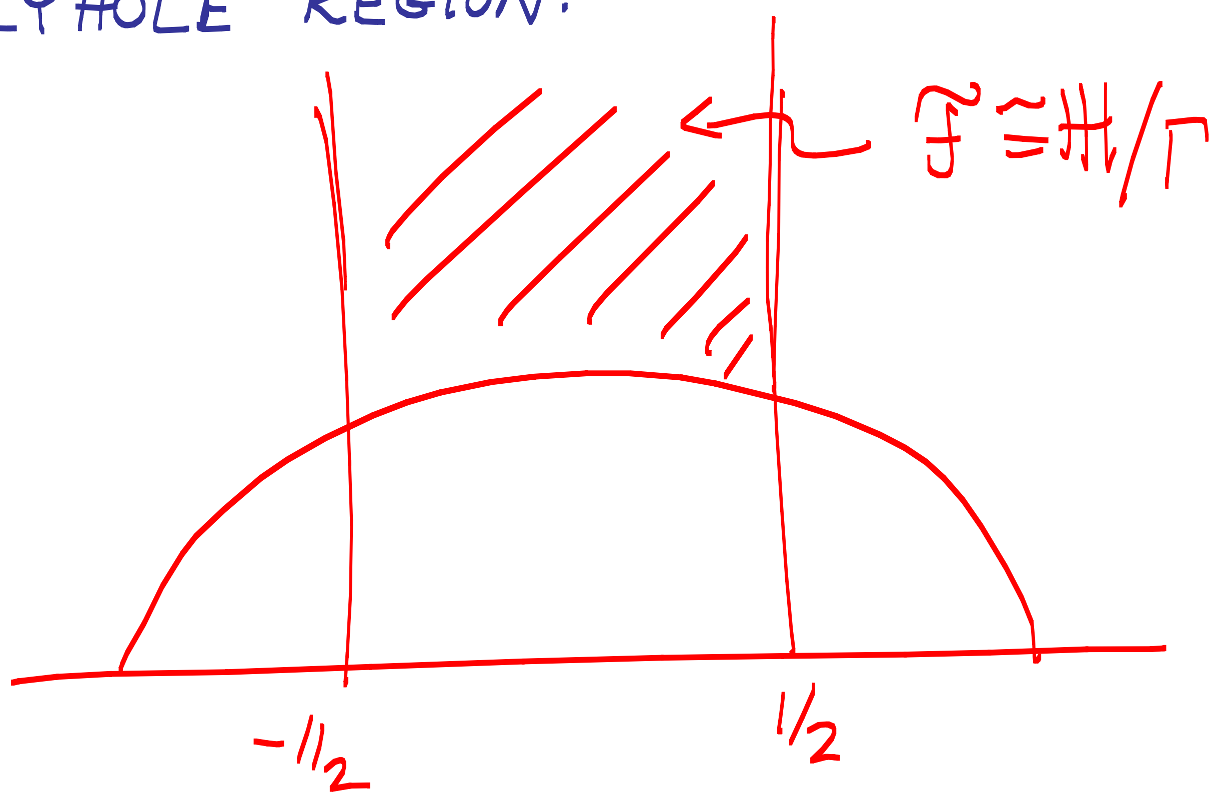
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S^2 = -1 \qquad (ST)^3 = (TS)^3 = -1$$

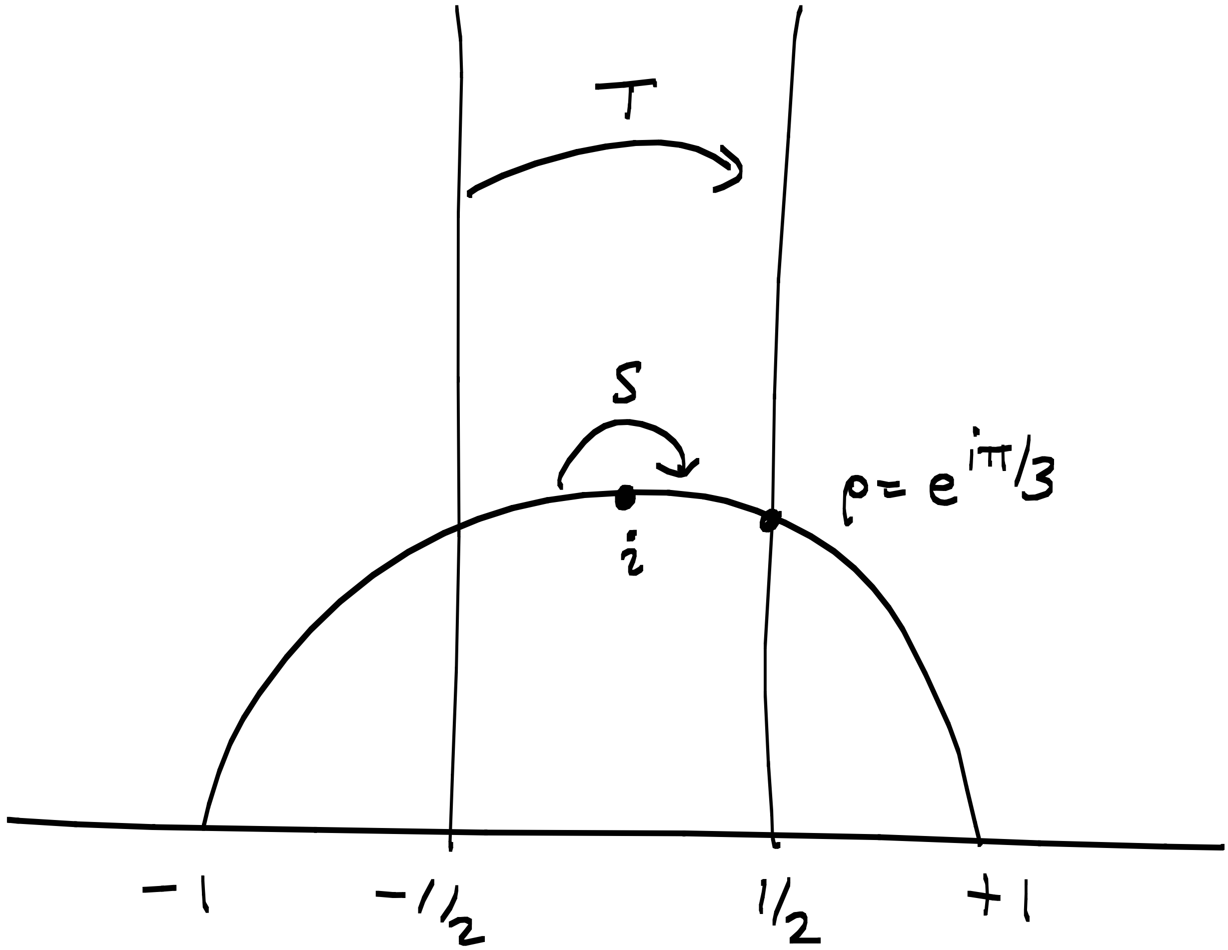
$\gamma = -1$ acts trivially on τ so

$S^2 = 1$ and $(ST)^3 = 1$ in $\bar{\Gamma} = PSL(2,\mathbb{Z})$

2. FROM THIS WE DERIVE THE STANDARD FUNDAMENTAL DOMAIN = KEYHOLE REGION:



3. 3 SPECIAL POINTS WITH NONTRIVIAL STABILIZER

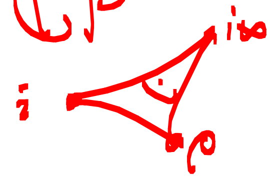


$$\tau = i \qquad \mathbb{Z}_2 = \langle S \rangle$$

$$\tau = \rho = e^{i\pi/3} \sim e^{2\pi i/3} \qquad \mathbb{Z}_3 = \langle ST \rangle$$

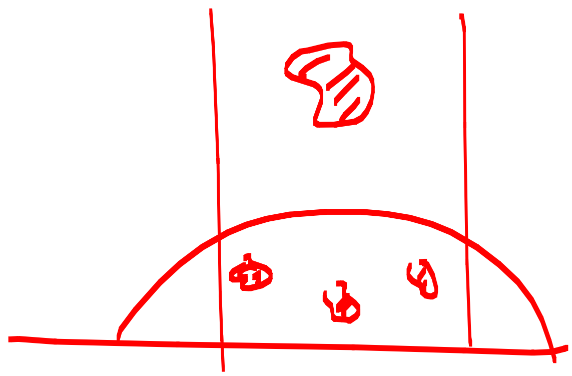
$$\tau = i\infty \quad \Gamma_\infty = \langle T \rangle = \left\{ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \middle| \ell \in \mathbb{Z} \right\}$$

4. $\hat{\mathbb{Q}} = \{i\infty\} \cup \mathbb{Q} \qquad (\mathbb{H} \cup \hat{\mathbb{Q}})/\Gamma \cong \mathbb{CP}^1$



3. Chiral Splitting and Holomorphy

By itself, modular invariance is not terribly strong: take *any* function with compact support in $\mathcal{F}$. By averaging over Γ we get a modular invt. function.



But holomorphy + modularity can impose strong constraints.

Already $Vir_L \oplus \widetilde{Vir}_R \Rightarrow$ A kind of holomorphic factorization of $Z(\tau, \bar\tau)$

$$Z(\tau, \bar\tau) = \sum_{h, \tilde h} N_{h\tilde h}\, \chi_h(\tau)\, \overline{\chi_{\tilde h}(\tau)}$$

THIS SPLITTING IS MORE POWERFUL THE FEWER THE TERMS IN THE SUM.

In general, when we have a <u>finite</u> decomposition

$$Z(\tau,\bar{\tau}) = \sum_i f_i(\tau)\,\tilde{f}_i(\bar{\tau})$$

THEN WE CONCLUDE

$$f_i(\gamma(\tau)) = M_{ij}(\gamma)\, f_j(\tau)$$

Where $M(\gamma)$ is a projective rep of Γ. $\tilde{f}_i$ transforms in the contragredient rep$^{\underline{n}}$.

To ILLUSTRATE THIS IDEA CONSIDER AN EXTREME CASE:

$$Z(\tau,\bar{\tau}) = f(\tau).$$

WHERE $f(\tau)$ is a <u>modular function</u>.

ONE WAY THIS CAN HAPPEN IS TO USE THE WITTEN INDEX AS WE'LL SEE NEXT TIME.

NOW WE CAN INVOKE

THM: FIELD OF MEROMORPHIC MODULAR FUNCTIONS $= \mathbb{C}(j)$

PF: $\exists \; j : (\mathbb{H} \cup \hat{\mathbb{Q}})/\Gamma \longrightarrow \mathbb{CP}^1$

$$\mathbb{C}(j) \cong \mathbb{C}(z)$$

WE'LL SHOW LATER:

$$j(\tau) = q^{-1} + 196884q + \cdots\cdots$$

NOW APPLY THIS: $Z(\tau)$ HAS NO SING'S ON $\mathbb{H}$ $\Longrightarrow$

$Z(\tau)$ = POLYNOMIAL IN j ORDER $\frac{c}{24}$.

THIS STRONGLY CONSTRAINS THE SPECT.
BUT WE CAN DO BETTER:

IN GENERAL, IF WE CAN WRITE

$$f(\tau) = \sum_{n \geq 0} \hat{f}(n)\, e^{2\pi i (n-\Delta)\tau}$$

- TERMS WITH $n-\Delta < 0$: POLAR
- TERMS WITH $n-\Delta > 0$: NONPOLAR

$$f(\tau) = f^{-}(\tau) + f^{+}(\tau)$$

- The constant term $n-\Delta = 0$ is polar or nonpolar depending on context. In the present case it is polar.

WE NOW USE THIS EXAMPLE TO ILLUSTRATE THE MAIN THEME OF THIS LECTURE:

$$\boxed{\text{THE FINITE SET OF POLAR DEG'S DETERMINES THE } \infty \text{ SET OF NONPOLAR DEG'S.}}$$

IN OUR EXAMPLE

$$Z(\tau) = Z^-(\tau) + Z^+(\tau)$$

$$= \sum_{n-\Delta \leq 0} d(n) e^{2\pi i (n-\Delta)\tau} + Z^+$$

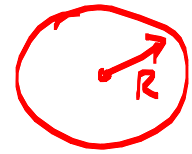
$$= a_\Delta j^\Delta + \cdots + a_0$$

No $+ \cdots$

PLUG IN q-EXPANSION FOR j TO GET A TRIANGULAR SYSTEM OF EQ'S FOR $a_0, \cdots a_\Delta$ IN TERMS OF $d(0), \cdots, d(\Delta)$.

4. SIMPLE EXAMPLE: "CHIRAL SCALARS"

GAUSSIAN: $\mathbb{R}\times S^1 \longrightarrow S^1_R$



$$\frac{(2\pi R)^2}{4\pi\alpha'}\int dX * dX \qquad X\sim X+1$$

A STANDARD EXERCISE IN CFT

$$Z(\tau,\bar\tau) = \frac{\#_{\Lambda_R}}{\eta\,\bar\eta}$$

$$\#_\Lambda = \sum_\Lambda q^{\frac{1}{2}p_+^2}\,\bar q^{\frac{1}{2}p_-^2}$$

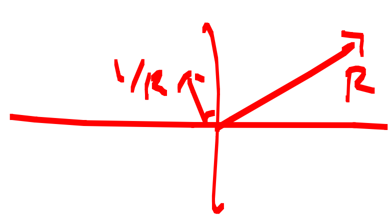
SIEGEL-NARAIN FOR $\Lambda\subset\mathbb{R}^{1,1}$

$$\Lambda_R = \{\, ne+mf \mid n,m\in\mathbb{Z}\,\}\subset\mathbb{R}^{1,1}$$

$$v=(v_+;v_-) \qquad v^2 = v_+^2 - v_-^2$$

$$e=\frac{1}{\sqrt{2}}\left(\frac{1}{R};\frac{1}{R}\right) \qquad f=\frac{1}{\sqrt{2}}(R;-R)$$

$$e^2=f^2=0 \qquad e\cdot f=1 \qquad \mathrm{II}^{1,1}$$



$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Poisson: $\Theta_\Lambda \longrightarrow (c\tau+d)^{1/2}(c\bar{\tau}+d)^{1/2}\,\Theta_\Lambda$

$$\Rightarrow \eta(\gamma\tau) = (c\tau+d)^{1/2} e^{i\phi(\gamma)}\eta(\tau)$$

$\uparrow$ 24th root of 1.

NOW THIS RESULT GENERALIZES TO (b_+, b_-) CHIRAL/ANTICHIRAL IN WHICH CASE

$$Z(\tau,\bar{\tau}) = \frac{\Theta_\Lambda}{\eta^{b_+}\bar{\eta}^{b_-}} \qquad \Lambda \subset \mathbb{R}^{b_+,b_-}$$

THE EMBEDDING OF Λ ENCODES GEOM. DATA OF THE TARGET SPACE.

THESE THEORIES TYPICALLY HAVE MODULAR ANOMALIES, ONLY FOR

- Λ EVEN UNIMODULAR
- $b_+ - b_- = 0 \mod 24$

$\dim \Lambda = 24k \;,\; b_- = 0 \;\Rightarrow Z(\tau)$

<u>DIGRESSION:</u> I'D LIKE TO MAKE A DIGRESSION FROM OUR MAIN THEME. WE MUST BE CAREFUL WHEN SPEAKING OF THE "THEORY OF A CHIRAL SCALAR"

WHAT WOULD ITS PARTITION FUNCTION BE? THE ZERO MODES OF CHIRAL/ANTICHIRAL ARE CORRELATED BY Λ.

HOWEVER IN SOME CASES A THEORY OF A CHIRAL SCALAR DOES EXIST – BUT IT IS VERY SUBTLE – AND NOT WELL APPRECIATED.

ILLUSTRATE THIS WITH GAUSSIAN MODEL

$$R^2 = p/q \quad \Longrightarrow$$

$$Z = \sum_{\mu,\nu \bmod 2m} N_{\mu\nu}\, f_\mu(\tau)\, \overline{f_\nu(\tau)}$$

$$f_\mu(\tau) = \frac{\Theta_{\mu,m}(0,\tau)}{\eta}$$

(WE'LL MEET THESE Θ-FUNCTIONS OF LEVEL m AGAIN IN THE NEXT LECT.)

HERE $m = 2pq$.

FINITE HOLOMORPHIC FACTORIZATION RESULTS FROM ENHANCED SYMMETRY

HEIS. EXT. OF $LS^1 = \check{H}^1(S^1)$

THESE HEIS. EXTS. HAVE A

LEVEL $= 2pq$.

SELF-DUAL SCALAR AT LEVEL ONE!

TO DEFINE IT WE HAVE TO TAKE $R^2 = 2$ (F.F. RADIUS; NOT THE SD RADIUS) AND TAKE A DOUBLE COVER of S^1.

THE RESULTING THEORY OF A CHIRAL SCALAR HAS P.F. IN TERMS OF LEVEL ½ -THETA FUNCTIONS

$$Z_\epsilon = \frac{\vartheta(\epsilon)}{\eta} \qquad \epsilon = \text{SPIN STR.}$$

THE REASON I STRESS THIS POINT IS THAT THIS IS JUST THE SIMPLEST EXAMPLE OF A SELF-DUAL THEORY, AND FURTHER EXAMPLES INCLUDE

- M5-BRANE
- TYPE II RR FIELDS

SIMILAR SUBLTETIES APPLY TO THESE IMPORANT CASES.

BUT THAT'S THE TOPIC OF ANOTHER LECTURE SERIES – ONE I ALMOST GAVE HERE, BUT THEN CHICKENED OUT AND DECIDED TO TALK ABOUT MORE MAINSTREAM TOPICS.

5. Modular Forms

The above examples motivate the study of more general functions than modular functions.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

define $j(\gamma, \tau) := c\tau + d$

<u>Def:</u> A vector-valued nearly-holomorphic modular form of weight w is a collection $f_\mu(\tau)$ holomorphic for $\tau \in \mathcal{H}$

$$f_\mu(\gamma\tau) = j(\gamma, \tau)^w M(\gamma)_{\mu\nu} f_\nu(\tau)$$

We need w nonintegral so

$$-\pi < \arg(z) \leq \pi$$

From the cocycle identity

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2\tau)\, j(\gamma_2, \tau)$$

Prove $M(\gamma)$ is a projective rep$^{\underline{n}}$ of Γ: "multiplier system."

Simplest case $M(\gamma) = 1$ 1×1.

Put $\gamma = -1 \Rightarrow w$ an even integer.

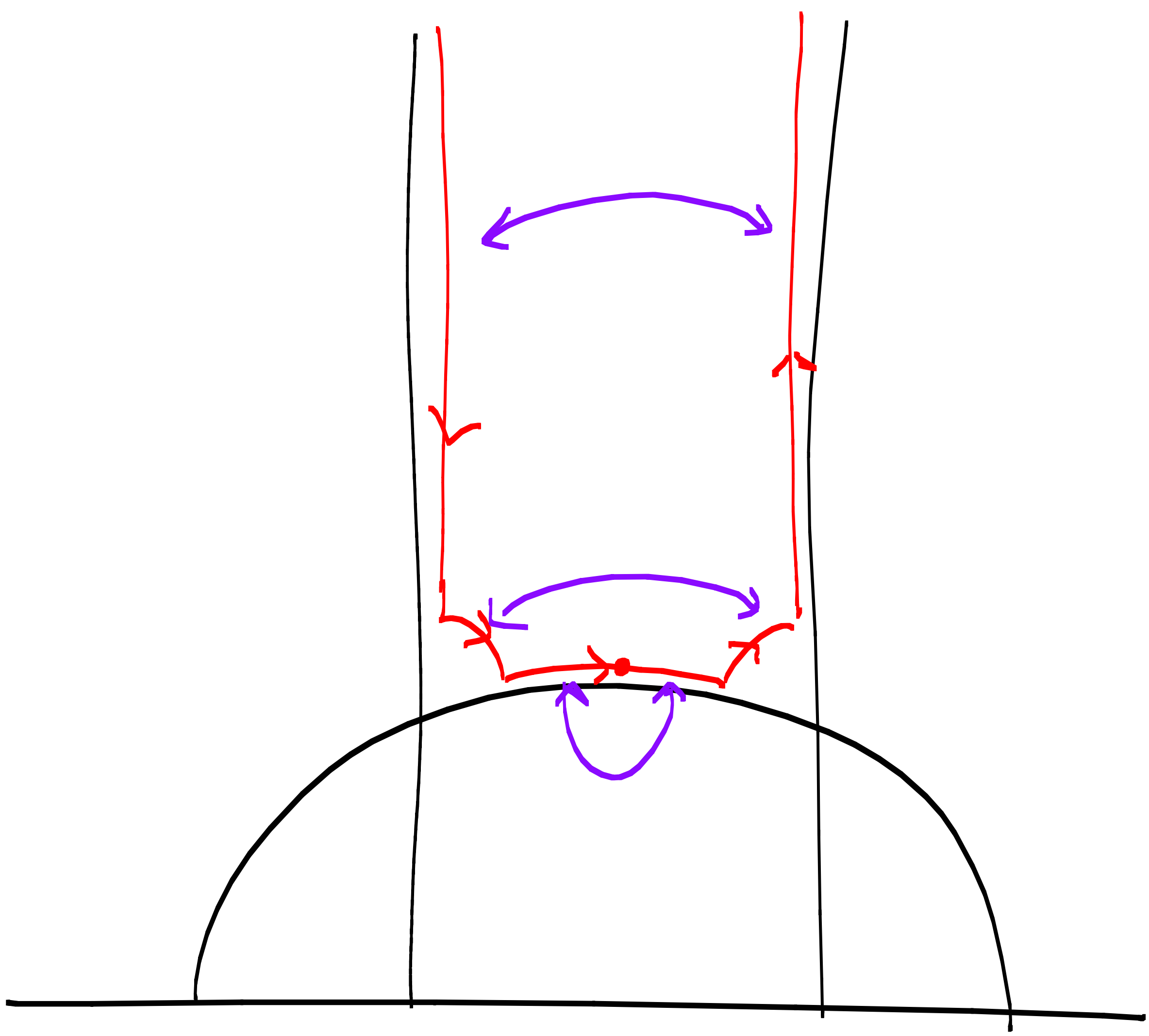
$$\gamma = T \Rightarrow f(\tau) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\, q^n$$

(Many interesting physical questions are related to asymptotics of $\hat{f}(n)$)

When $M(\gamma) = 1$ we can derive a useful constraint on any meromorphic function f s.t.

$$f(\gamma\tau) = (c\tau+d)^w f(\tau)$$

$V_p(f) =$ order of zero of f @ p.



For <u>any</u> meromorphic function transforming w/ wt = w

Integrate $\frac{1}{2\pi i}\frac{df}{f}$ around the red contour to get:

$$V_\infty(f) + \frac{1}{2}V_i(f) + \frac{1}{3}V_\rho(f) + \underbrace{\sum_{p\in H/\Gamma}^{*} V_p(f)}_{\text{only fin. many terms nonzero.}} = \frac{w}{12} \qquad (*)$$

Now, we have used "nearly holo" because in the math literature an important growth condition is imposed for the term "modular form".

IN MATH "MODULAR FORM" MEANS, $f(\tau)$ HAS SUB-EXP. GROWTH AT $\tau = i\infty$

$\Rightarrow$ $\hat{f}(n) = 0$ FOR $n < 0$.

WHAT WE GAIN:

$M_w(\Gamma)$ THE VECTOR SPACE OF MODULAR FORMS OF WEIGHT w IS EXPLICITLY KNOWN.

FIRST WE SHOW IT CAN BE NONEMPTY

$$G_w(\tau) := \sum_{\mathbb{Z}^2 - 0} \frac{1}{(m\tau+n)^w} \qquad w \in 2\mathbb{Z}, \quad w \geq 4 \text{ CONV.}$$

OBVIOUSLY MODULAR.

FOR LATER PURPOSES I WANT TO REWRITE THIS.

NOTE: $\Gamma_\infty \backslash \Gamma \cong \left\{ \begin{matrix} \text{Rel. prime} \\ (c,d) \end{matrix} \right\}$

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c\ell & b+d\ell \\ c & d \end{pmatrix}$$

Factor out common divisor $\Rightarrow$

$$G_w(\tau) = \zeta(w) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma,\tau)^{-w}$$

$$= 2\zeta(w) E_w(\tau) \qquad \text{"EISENSTEIN SERIES"}$$

NOTE $E(\tau) \longrightarrow 1$, $\tau \longrightarrow i\infty$

EXERCISE: VERIFY MODULARITY OF $\sum_{\Gamma_\infty \backslash \Gamma} j^{-w}$ USING COCYCLE IDENTITY.

(WITH A LITTLE MORE WORK CAN DERIVE q-EXP.)

Now the product of modular forms is modular and

$$M_*(\Gamma) = \bigoplus_w M_w(\Gamma) \quad \text{is a ring.}$$

THM: $M_*(\Gamma) = \mathbb{C}[E_4, E_6]$

This theorem is proven by systematically exploiting the identity $(*)$

$$v_p(f) \in \mathbb{Z}_+, \quad p \in \mathbb{H} \cup \hat{\mathbb{Q}} \quad \Longrightarrow$$

$$w < 0 \Rightarrow M_w = 0$$

$$w = 0 \Rightarrow M_0 = \mathbb{C} \cdot 1$$

$$w = 2 \Rightarrow M_2 = 0.$$

For weight $w=4$ (*) becomes

$$v_\infty + \frac{1}{2} v_i + \frac{1}{3} v_\rho + \sum^{*} v_p = \frac{1}{3}$$

with all v's ≥ 0 and integral.

The only solution is

$$v_\infty = v_i = v_p = 0 \; ; \quad v_\rho = 1$$

Thus: M_4 is one-dimensional and generated by E_4, which moreover has a simple zero at ρ and no others in $\mathcal{F}$.

EX: $M_6 = \mathbb{C} \cdot E_6$; FIND ZEROES OF E_6

SOMETHING NEW ~~HAPPENS~~ AT $W=12$

$$E_4^3 - E_6^2 := (12)^3 \Delta$$

$$\Delta = q + \cdots$$

Simple zero @ $q=0$; NO OTHER ZEROS IN $\mathcal{H}$.

If f is any modular form of weight 12 then

$$\frac{f - \hat{f}(0) \cdot E_{12}}{\Delta} \in M_0$$

hence a constant. So

$$M_{12} = \langle \Delta, E_{12} \rangle = \langle \Delta, E_4^3 \rangle = \langle \Delta, E_6^2 \rangle$$

The same argument also shows

$$M_w = \Delta \cdot M_{w-12} \oplus \langle E_w \rangle$$

So it follows that

$$\dim M_w = \begin{cases} \left[\frac{w}{12}\right] & w \equiv 2 \bmod 12 \\ \left[\frac{w}{12}\right] + 1 & w \not\equiv 2 \bmod 12. \end{cases}$$

This is dimension of polynomial ring.

Remarks:

1. A modular form with $\hat{f}(0)=0$, i.e. a form which vanishes for $q\to 0$ is called a "cusp form."

2. Given the transformation properties of η we see that

$$\Delta = \eta^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

manifestly showing that it does not vanish in $\mathbb{H}$.

3. Similarly, we can construct the j function:

$$J(\tau) = \frac{E_4^3}{\Delta} = \frac{E_6^2}{\Delta} + (12)^3$$

$$= q^{-1} + 744 + 196884q + \cdots$$

Now we come to a key point:
Given the examples of even the simplest partition functions in 2D CFT we should

1. Allow for negative weight
2. Allow for singularities at ∞

NOW WE CAN RETURN TO OUR MAIN THEME WHICH IS CRUCIAL TO THE PHYSICAL APPLICATIONS:

For negative weight nearly holomorphic forms the polar part uniquely determines the entire form.

In physical terms: the degeneracy of polar states completely determines the entire spectrum, including black hole states.

We can demonstrate this easily for the case $M(\gamma) = 1$ using our identity ⊛

By definition $v_p(f) \geq 0$ for $p \in \mathbb{H}$.

Therefore $w < 0$ forces $v_\infty(f) < 0$.

Moreover, if $f, \tilde{f}$ have the same polar piece then $f - \tilde{f}$ has $v_\infty = 0$ and hence must vanish.

Remark:

1. False if you drop holomorphy.
2. False if you drop modularity
3. Even false if you drop $w < 0$:

For nearly holomorphic positive weight forms you could always add a cusp form.

The above conclusion can be generalized to vector valued modular forms.