LECTURE II: $\mathcal{N}=2$ & THE ELLIPTIC GENUS

Now we are going to combine the constraints of modularity with extended supersymmetry.

1. Review $\mathcal{N}=2$ algebra, SF, unitarity.
2. Modular invce of partition fns
3. Elliptic genus
4. Jacobi forms
5. Physical interpretation of $\oplus$-decomp.
6. Polar states for the elliptic genus
7. Explicit reconstruction formula.
8. Rademacher
9. AdS_3/CFT_2 + Fareytail

1. $\mathcal{N}=2$ Superconformal symmetry

$$T(z) = \sum L_n z^{-n-2}$$

$$J(z) = \sum J_n z^{-n-1} \qquad U(1) \text{ current}$$

$$G^{\pm}(z) = \sum_r G^{\pm}_r z^{-r-3/2}$$

Main relation:

$$\boxed{\{G^{\pm}_r, G^{\mp}_s\} = 2L_{r+s} \pm (r-s) J_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}}$$

$r \in \mathbb{Z} \pm a$ in a $\mathbb{Z}$ torsor

$a \in \mathbb{Z} + 1/2$: "NS algebra"

$a \in \mathbb{Z}$ "R algebra"

SPECTRAL FLOW ISOMORPHISM $\quad a \to a+\theta$

$$G^{\pm}_{n\pm a} \longrightarrow G^{\pm}_{n\pm(a+\theta)}$$

$$L_n \longrightarrow L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0}$$

$$J_n \longrightarrow J_n + \frac{c}{3}\theta\,\delta_{n,0}$$

Notation: $c = 3\hat{c} = 6m$

Susy σ-MODEL(X)

$\mathcal{N}=(2,2)$ $\quad \hat{c} = d \quad X$ Kähler, $\dim = d$

$=(4,4) \quad m = d/2 \quad X$ Hyperkähler

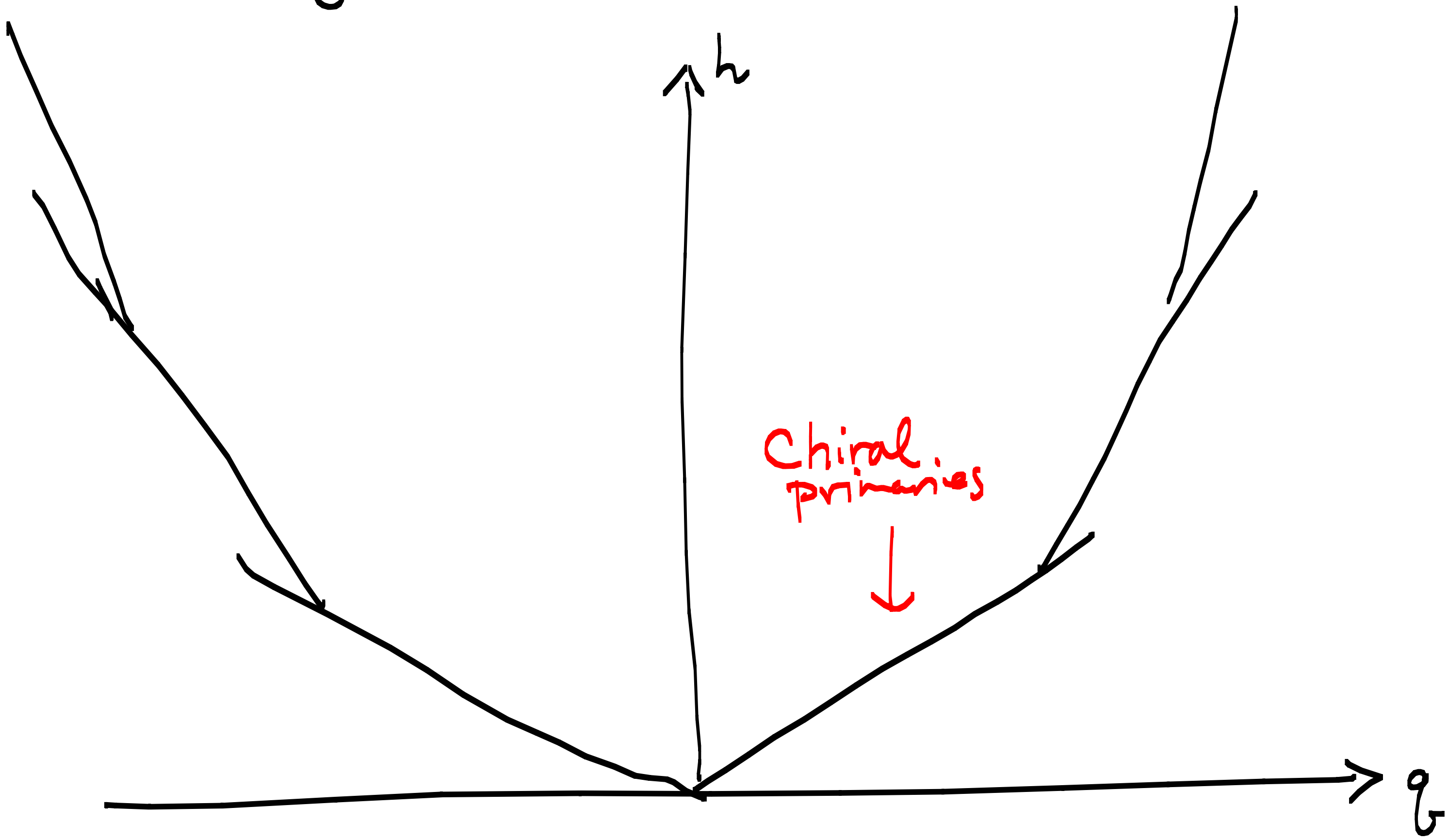
<u>Exercise:</u> Show That

$$4mL_0 - J_0^2 \qquad \text{spectral flow invt.}$$

$$\mathcal{H} = \bigoplus V_{h,q} \otimes \tilde{V}_{\tilde{h},\tilde{q}}$$

$$V_{h,q}: \begin{cases} (G^{\pm}, J, L)_{>0} \, |h,q\rangle = 0 \\ L_0 \, |h,q\rangle = h \, |h,q\rangle \\ J_0 \, |h,q\rangle = q \, |h,q\rangle \end{cases}$$

Constraints of unitarity were worked out by Boucher-Friedan-Kent.
We just summarize the main facts.



NS - SECTOR

$$0 \le \| G^{\pm}_{-1/2} |h,q\rangle \|^2 \Rightarrow 0 \le \langle h,q | \{ G^{\mp}_{\frac{1}{2}}, G^{\pm}_{-\frac{1}{2}} \} | h,q \rangle$$

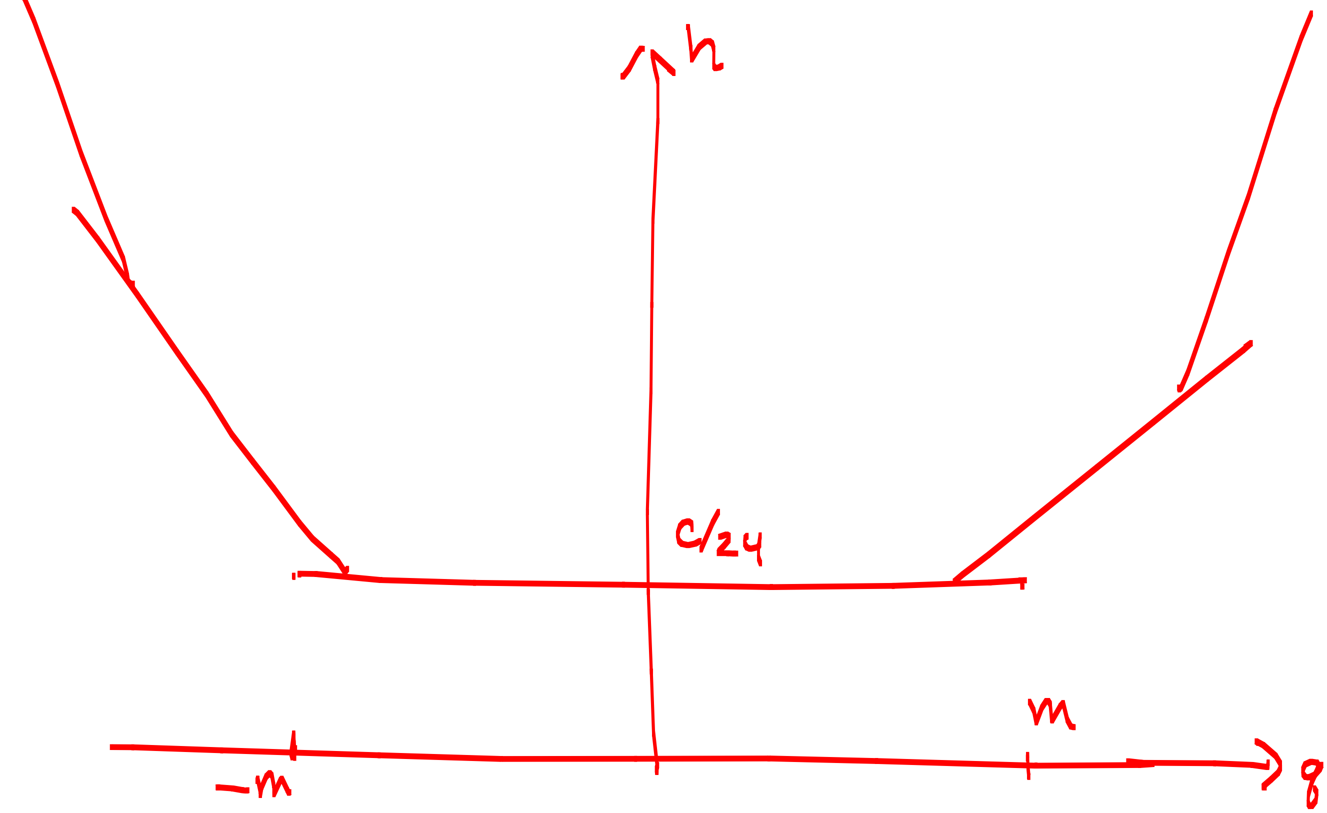
$$\Longrightarrow \quad h \ge \tfrac{1}{2}|q| \qquad \text{BPS BOUND}$$

$h = q/2 \qquad G^{+}_{-1/2} |h,q\rangle = 0$

$h = -q/2 \qquad G^{-}_{-1/2} |h,q\rangle = 0$

BPS STATE "CHIRAL PRIMARY"

R Sector



$$\| G_0^{\pm} |h,q\rangle \|^2 \geq 0$$

$$\{ G_0^+, G_0^- \} = 2\left(h - \frac{c}{24}\right)$$

SPECTRAL FLOW TAKES CHIRAL PRIMARIES TO RAMOND GROUNDSTATES, SO WE REFER TO THOSE AS BPS STATES.

2. PARTITION FUNCTIONS

NOW CONSIDER $\mathcal{C} = (2,2)$ CFT

WE DEFINE

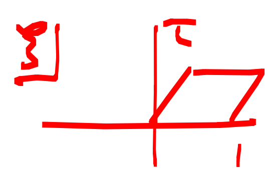
$$Z_{RR} = \mathrm{Tr}_{\mathcal{H}_{RR}}\, q^{L_0 - c/24} e^{2\pi i z J_0}\, \bar{q}^{\tilde{L}_0 - c/24} e^{2\pi i \bar{z} \tilde{J}_0} \cdot e^{i\pi(J_0 - \tilde{J}_0)}$$

THIS HAS THE INTERPRETATION OF A PATH INTEGRAL ON THE TORUS

$$Z_{RR} = \left\langle e^{2\pi i \int_{E_\tau} \left(A^{01} J + A^{10} \tilde{J} \right)} \right\rangle_{E_\tau}$$

$$A = \frac{i}{2\,\mathrm{Im}\,\tau} \left(\bar{z}\, d\xi - z\, d\bar{\xi} \right)$$

E_τ



$\Longrightarrow$ GOOD MODULAR PROPERTIES

NEED HOLOMORPHY

3. ELLIPTIC GENUS

PUT $\bar{z} = 0$

Then, on the R-movers we are computing the Witten index: $\mathrm{Tr}\ \bar{q}^{\bar{L}_0 - c/24}(-1)^F$.

On highest weight repn's:

$$\mathrm{Tr}_{V_{h,q}}\, q^{L_0 - c/24}\, e^{i\pi J_0} = \begin{cases} e^{i\pi q} & h = \frac{c}{24} \\ 0 & h > \frac{c}{24} \end{cases}$$

<u>Exercise</u>: Use $G_0^{\pm}$ to write states in B/F pairs.

DEF: THE (2,2) ELLIPTIC GENUS

$$\chi(\tau, z; \mathcal{C}) := Z_{RR}(\tau, z; \bar{\tau}, 0)$$

- HOLOMORPHIC
- COUNTS $(*, \mathrm{BPS})$

- MODULAR: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\gamma \cdot (\tau, z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} \right)$$

$$\chi\big(\gamma\cdot(\tau, z)\big) = e^{2\pi i m \frac{c z^2}{c\tau+d}} \chi(\tau, z)$$

(AS EXPLAINED IN THE NOTES)

FOR SIMPLICITY:

1. $m \in \mathbb{Z}$
2. $\mathrm{Spec}(J_0) \subset \mathbb{Z}$

THEN SPECTRAL FLOW $\Longrightarrow$

$$\chi(\tau, z+\theta\tau+\theta') = e^{-2\pi i m(\theta^2\tau + 2\theta z)} \chi(\tau, z)$$

$$\theta, \theta' \in \mathbb{Z}$$

4. JACOBI FORMS

$$\phi\left(\gamma\cdot(\tau,z)\right)=(c\tau+d)^{w}\, e^{2\pi i m \frac{cz^2}{c\tau+d}}\,\phi(\tau,z)$$

$$\phi\left(\tau,\ z+\theta\tau+\theta'\right)=e^{-2\pi i m(\theta^2\tau+2\theta z)}\,\phi(\tau,z)$$

$\theta,\theta'\in\mathbb{Z}$

Are the transformations of a Jacobi form of weight w and index m.

Note that the transformation laws imply that there is a Fourier expansion:

$$\phi(\tau,z)=\sum_{n,\ell\in\mathbb{Z}} c(n,\ell)\,q^n y^\ell$$

$$y:=e(z)\qquad q:=e(\tau).$$

As with modular forms, there is a growth condition.

WEAK JACOBI FORMS: $c(n,\ell)=0$ IF $n<0$: $\tilde{J}_{w,m}$

(Jacobi forms: $c(n,\ell)=0$ if $4mn-\ell^2<0$ and don't concern us.)

$n = L_0 - c/24$: SO UNITARITY $\Rightarrow$ WK JAC. FRM. CONDITION

$$\boxed{N=(2,2) \text{ ELLIPTIC GENUS } \in \tilde{J}_{0,m}}$$

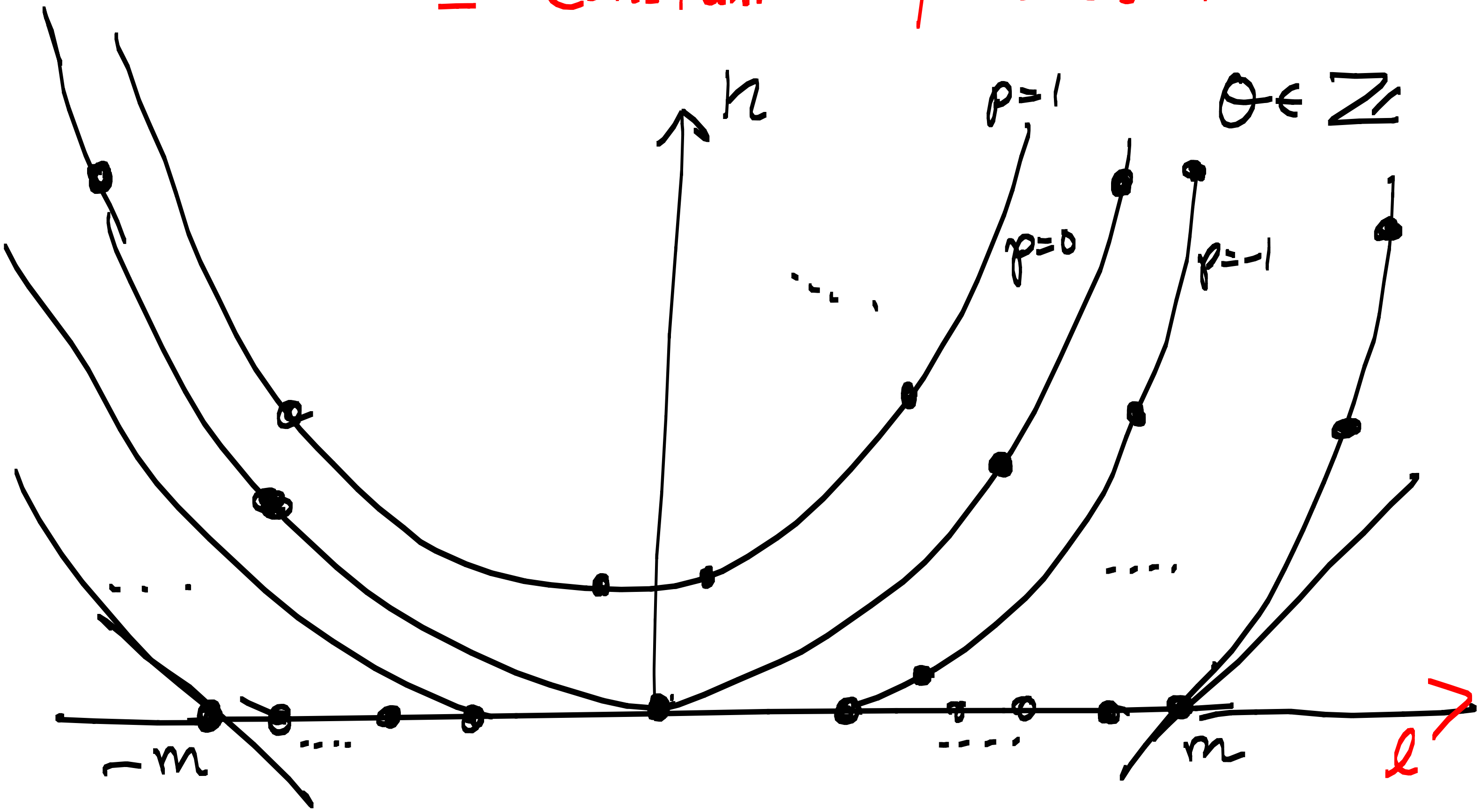
NOW WE EXPLAIN AN IMPORTANT FACT

WEAK JACOBI FORM $\Longleftrightarrow$ V.V. N.H. MOD. FORM

To see this one translates the spectral flow condition to one on the Fourier coefficients:

$$c(n,\ell) = c(n+\ell\theta+m\theta^2, \ell+2m\theta)$$

$=$ Constant on parabolas $4mn-\ell^2=\text{cnst.}$



ON THE OTHER HAND, WE CAN ALWAYS BRING ℓ INTO A FUNDAMENTAL DOMAIN FOR THE $\mathbb{Z}$-ACTION

$$\ell \longrightarrow \ell + 2m\theta \qquad \theta\in\mathbb{Z}$$

$$\Longrightarrow \qquad c(n,\ell) = c_\mu(p)$$

$$\mu = \ell \bmod 2m$$

$$p := 4mn - \ell^2 \quad \text{"polarity"}$$

SO NOW LOOK AT THE FOURIER EXP. OF ϕ. SUM OVER (n,ℓ) BY FIXING μ, THEN p, THEN SUM OVER POINTS ON THE PARABOLA — THAT SUM IS A Θ-FUNCTION:

$$\sum c(n,\ell) q^n y^\ell =$$

$$= \sum_{\mu \bmod 2m} \sum_{p} c_\mu(p) q^{\frac{p}{4m}} \underbrace{\sum_{\ell = \mu \bmod 2m} q^{\frac{\ell^2}{4m}} y^\ell}_{:= \Theta_{\mu,m}(\tau, z)}$$

$$\Rightarrow \quad \phi(\tau, z) = \sum_{\mu \bmod 2m} h_\mu(\tau) \Theta_{\mu,m}(\tau, z)$$

$$h_\mu(\tau) = \sum_{p = -\mu^2 \bmod 2m} c_\mu(p) q^{\frac{p}{4m}}$$

NOW WE CAN COMPLETE THE PROOF BECAUSE ONE CAN SHOW BY POISSON SUMMATION:

$$\Theta_{\mu,m}(\gamma\cdot(\tau,z)) = (c\tau+d)^{1/2}\, e\left(m\,\frac{cz^2}{c\tau+d}\right)\cdot$$

$$\cdot\, M_{\mu\nu}(\gamma)\,\Theta_{\nu,m}(\tau,z)$$

$\Rightarrow$ $h_\mu(\tau)$ is a nearly holomorphic vector-valued modular form of wt $w-1/2$

with mult. system $M^{tr,-1}$

5. PHYSICAL INTERPRETATION

THE THETAFUNCTION DECOMPOSITION OF THE ELLIPTIC GENUS HAS A NICE PHYSICAL INTERPRETATION

U(1) CURRENT $\Rightarrow$ CHIRAL BOSON

$$J = i\sqrt{2m}\,\partial\phi \quad , \quad R^2 = m$$

$$SF = \text{MULT. BY } e^{i\sqrt{2m}\,\phi}$$

WE ARE SEPARATING OUT THE CONTRIBUTIONS OF THIS FIELD

$$\chi(\tau, z) = \sum_{\mu} \tilde{h}_{\mu}(\tau) \frac{\Theta_{\mu,m}(z,\tau)}{\eta}$$

MOREOVER

IF $\mathcal{C}$ HAS A HOLOGRAPHIC DUAL...

$$AdS_3 \times K_7$$

THEN THE U(1) CURRENT COUPLES

TO A GAUGE FIELD A ON AdS_3

WITH A CHERN-SIMONS TERM

$$S = \int \frac{1}{e^2} F * F + m \int A dA + \cdots$$

AT LONG DISTANCES CS-TERM

LEADS TO

SINGLETON MODES/EDGE STATES

AND THESE ARE DESCRIBED BY

THE CHIRAL SCALAR ϕ

$$\chi = \sum_{\mu \bmod 2m} \tilde{h}_\mu(\tau) \frac{\Theta_{\mu,m}(z,\tau)}{\eta}$$

SINGLETON

BULK MODES IN

CHARGE SECTOR

$\mu \bmod 2m$

6. POLAR STATES FOR ELLIPTIC GENUS

LET US RETURN TO THE MAIN THEME OF LECTURE I: THE POLAR TERMS OF A V-V. N.H. MOD. FORM DETERMINE THE WHOLE FORM.

DEF: A POLAR STATE IN $\mathcal{C}$ IS AN EIGENSTATE OF $\left(L_0 - \frac{c}{24}, J_0\right) = (n, \ell)$ WITH

$$p = 4mn - \ell^2 < 0.$$

The states of negative polarity are precisely the states which contribute to the polar terms of h_μ.

Thus, we are interested in determining the polar degeneracies because we want to know nonpolar deg's.

INDEPENDENT POLAR DEG'S ?

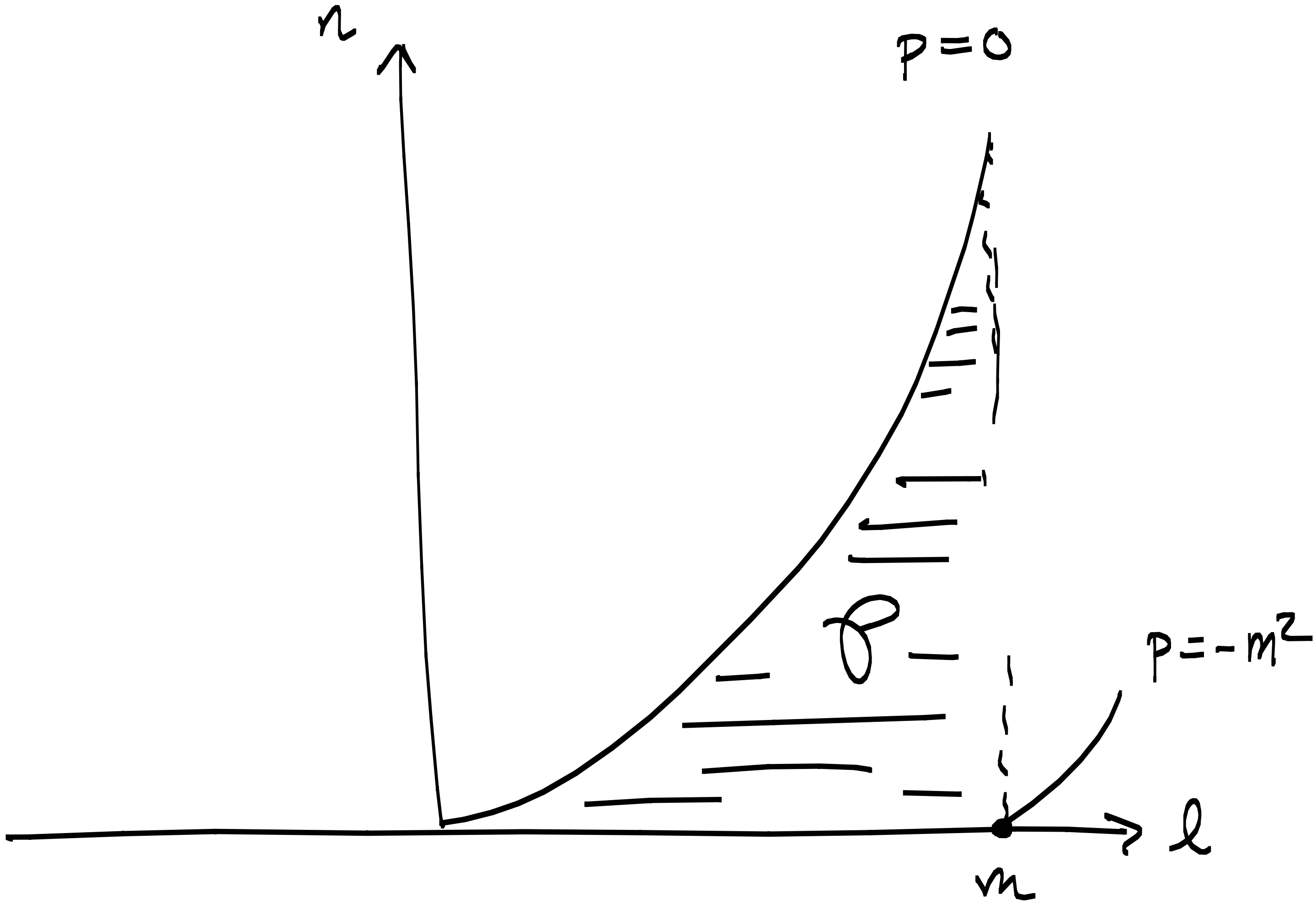
$$\mathcal{S} = \{(n,\ell) \mid -m^2 \underset{\substack{\uparrow \\ \text{UNITARITY}}}{\leq} p < 0\}$$

S.F. = $\mathbb{Z}$-ACTION PRESERVING P.

CHARGE CONJ. $(n,\ell) \longrightarrow (n,-\ell)$

$$\gamma = -1 \implies c(n,\ell) = c(n,-\ell)$$

SO $D_\infty = \mathbb{Z}_2 \ltimes \mathbb{Z}$ ACTS ON $\mathcal{S}$



LATTICE POINTS IN $\mathcal{P}$ = INDPT POLAR DEG'S

χ HAS wt = 0 $\Rightarrow$ h_μ HAS wt $= -\frac{1}{2}$

$\Rightarrow$ FINITE # OF POLAR DEG'S DETERMINE ALL $c(n,\ell)$

EXPLICIT FORMULA ?

7. EXPLICIT FORMULA.

FOR ANY V.V. N.H. MOD FORM WITH $w<0$ WE CAN ASK

RECONSTRUCT f_μ FROM f_μ^- ?

YES! FOR SIMPLICITY TAKE $M(\gamma)=1$

NAIVE GUESS

$$f(\tau) = \frac{1}{2}\sum_{\Gamma_\infty \backslash \Gamma} j(\gamma,\tau)^{-w} f^{-}(\gamma\tau)$$

- FORMALLY TRANSFORMS AS MODFORM
- HAS CORRECT POLAR PART.
- BUT $w<0 \Rightarrow$ CON~~V~~ERGE

$$f^{-}(\tau) = \sum_{n-\Delta<0} \hat{f}(n)\, e^{2\pi i(n-\Delta)\tau}$$

Typical term:

$$(c\tau+d)^{|w|} \underbrace{\exp\left(2\pi i (n-\Delta)\frac{a\tau+b}{c\tau+d}\right)}_{e^{2\pi i(n-\Delta)\frac{a}{c}}\, e^{-2\pi i(n-\Delta)\frac{1}{c(c\tau+d)}}}$$

SO MUST REGULATE:

$$\boxed{f(\tau) = \frac{1}{2} \sum_{\Gamma_\infty \backslash \Gamma} \left\{ j(\gamma,\tau)^{-w} f^-(\gamma\tau) + \text{REG.} \right\}} \quad (*)$$

(LEAVE ON THE BLACKBOARD!)

IF WE START WITH A RANDOM f^- THEN THE REGULATOR SPOILS MOD. INVCE.

BUT! IF f^- = POLAR PART OF A MODULAR FORM SOMETHING MAGICAL HAPPENS: $\exists$ REG. PRESERVING MOD. INVCE: J. MANSCHOT + G.M.

3 APPLICATIONS

1. RADEMACHER
2. AdS_3/CFT_2 PARTITION FN. "FAREYTAIL"
3. OSV CONJECTURE

8. RADEMACHER EXPANSION

$\hat{f}(n)$ FOR $n-\Delta > 0$ IN TERMS OF $\hat{f}(n)$ FOR $n-\Delta < 0$.

$$\hat{f}(n) = \int_0^1 d\tau \, e^{-2\pi i (n-\Delta)\tau} f(\tau)$$

SUBST. EQUATION (*) AND EXCHGE SUM + INTEGRAL.

DO THE INTEGRALS & GET BESSEL FUNCTIONS. RESULT:

FOR $n-\Delta > 0$:

$$\hat{f}(n) = 2\pi \sum_{m-\Delta<0} \hat{f}(m) \sum_{c=1}^{\infty} \frac{1}{c} K_c(m-\Delta, n-\Delta) \left(\frac{|m-\Delta|}{n-\Delta}\right)^{\frac{1-w}{2}} I_{1-w}\left(\frac{4\pi}{c}\sqrt{|m-\Delta|(n-\Delta)}\right)$$

$I_\nu(x)$ = Bessel function.

$$K_c(A,B) = i^{-w} \sum_{\substack{-c<d<0 \\ (c,d)=1}} e\left(A\cdot\frac{a}{c} + B\cdot\frac{d}{c}\right)$$

= SUM OF PHASES : KLOOSTERMAN SUM.

- CONVERGENT SUM FOR $w<0$
- GENERALIZES TO V-V FORMS. (PUT IN μ, ν)
- COROLLARY: HARDY-RAMANUJAN-CARDY:

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x, \quad \operatorname{Re} x \to +\infty$$

SO WE HAVE A "CONVERGENT SUM OF EXPONENTIALS"

DOMINANT TERM FROM MOST NEGATIVE $m - \Delta < 0$.

UNITARY THEORY $\boxed{\Delta = +c/24}$

$$\Rightarrow \quad \log \hat{f}(n) \sim 4\pi\sqrt{\Delta n} + O(\log n)$$

$$= 2\pi\sqrt{\frac{c}{6} n} + O(\log n)$$

WARNING: ONLY VALID FOR

$$n - \Delta \gg 1$$

AS DeBoer mentioned in his lecture there are physical situations – e.g. the "entropy enigma" configurations where we need

$$n - \Delta = \mathcal{O}(1) \quad , \quad n \longrightarrow \infty$$

For example, let

$$\eta^{-\chi}(\tau) = q^{-\chi/24} \sum_{n=0}^{\infty} P_\chi(n)\, q^n$$

Then it turns out that

$$P_\chi\left(\frac{\chi}{24} + \ell\right) \sim \chi^{-1/2} \exp\left(\frac{\chi}{2}\left(1 + \log\frac{\pi^2}{6}\right) + \frac{\pi^2}{3}\ell\right)$$

$\chi \to \infty$, ℓ fixed.

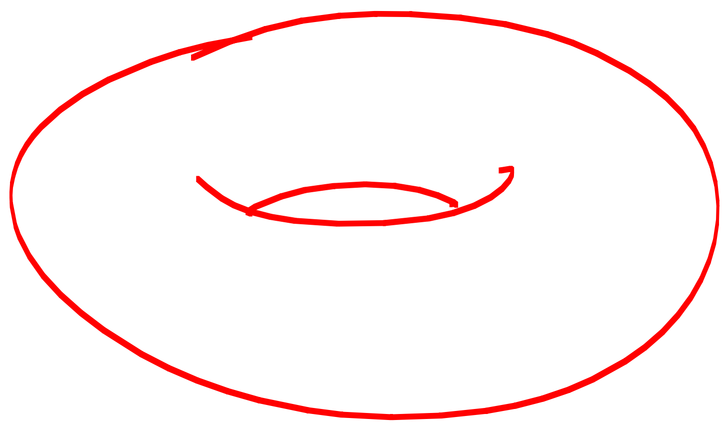
9. AdS_3/CFT_2 PARTITION FUNCTION

NOW LET US RETURN TO THE ELLIPTIC GENUS $\mathcal{X}$ OF OUR (2,2) THEORY $\mathcal{C}$

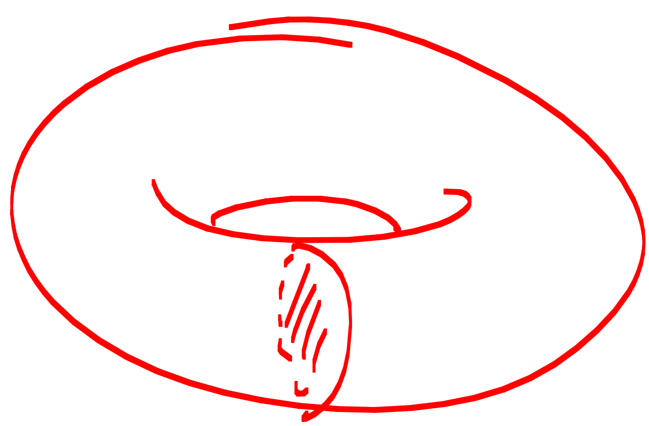
SUPPOSE IT HAS A DUAL

$$AdS_3 \times K$$

$\mathcal{X}(\tau, z)$ = P.F. ON

E_τ = 

WE MUST FILL THIS IN WITH A HYPERBOLIC GEOMETRY,

EUCL. BTZ  $= \mathbb{H}_3/\mathbb{Z}$

$AdS_3 / CFT_2 \Rightarrow \quad Z_{CFT} = Z_{STRING}$

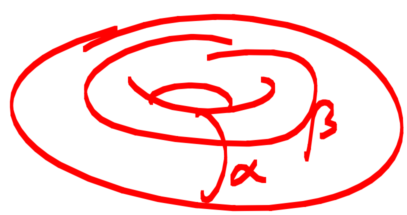
APPLYING OUR RECONSTRUCTION FORMULA TO $h_\mu(\tau)$ FOR χ WE GET A FORMULA LIKE

$$\chi(z,\tau) = \sum_{\mathcal{P}} c_\mu(\mathcal{P}) \sum_{\substack{\Gamma_\infty \backslash \Gamma \\ = \{(c,d)\}}} \left(q^{\frac{P}{4m}} \theta_{\mu,m} \right) \Big|_{\gamma_{c,d}}$$

INTERPRETATION:

1. TO WRITE ACTION OF A BH WE MUST CHOOSE CONTRACTIBLE CIRCLE OF EUCL. TIME

$= c\,\alpha + d\,\beta$



2. <u>SUM OVER $\mathcal{P}$?</u>

TRANSLATING FROM CFT TO GRAVITY QUANTITIES

$$4mn - \ell^2 = 4mM - J^2$$

COSMIC CENSORSHIP BOUND $4mM - J^2 \geq 0$!

NONPOLAR TERMS ~ BLACK HOLES

POLAR TERMS ~ PARTICLES IN ADS NOT SUFFICIENTLY MASSIVE TO COLLAPSE INTO A BLACK HOLE.

THIS RELATION BETWEEN NON-POLAR TERMS + BLACK HOLES HAS RECENTLY PLAYED A ROLE IN WITTEN'S DISCUSSION OF 2+1 QUANTUM GRAVITY.