LECTURE III: EXTREMAL $\mathcal{N}=2$ CFT'S

1. SUMMARY OF KEY POINTS FROM LECTURES I + II.

2. QUANTUM GRAVITY IN 2+1 DIMENSIONS.

3. DEFINING EXTREMAL $\mathcal{N}=2$ CFT

4. COUNTING POLAR POLYNOMIALS

5. COUNTING WEAK JACOBI FORMS

6. SEARCH FOR THE EXTREMAL E.G.

7. NEAR-EXTREMAL $\mathcal{N}=2$ CFT

8. CONCLUDE: TWO OPEN PROBLEMS

SECTIONS 3–7 ARE UNPUBLISHED RESULTS WITH

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1. SUMMARY OF KEY POINTS FROM LECTURES I & II.

A. A VECTOR-VALUED NEARLY HOLOMORPHIC MODULAR FORM OF WEIGHT $w<0$ IS DETERMINED BY ITS POLAR PART

B. THE ELLIPTIC GENUS OF AN $\mathcal{N}=2$ THEORY WITH INTEGRAL U(1) CHARGES AND $c=6m$ IS A WEAK JACOBI FORM $\chi(\tau,z;\mathcal{C}) \in \tilde{J}_{0,m}$

C. JACOBI FORMS ARE EQUIVALENT TO V-V-N-H MOD. FORMS

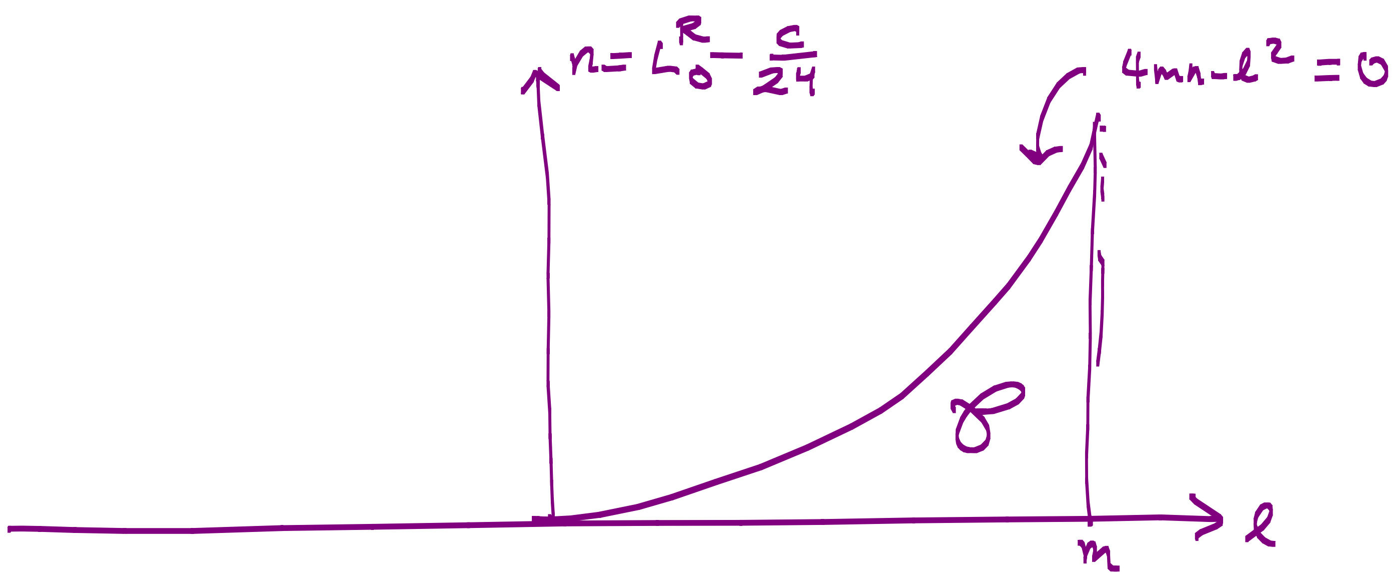
$$\chi(\tau,z) = \sum_{n,\ell} c(n,\ell)\, q^n y^\ell$$

$$= \sum_{\mu \bmod 2m} h_\mu(\tau)\, \Theta_{\mu,m}(\tau,z)$$

D. THE POLAR TERMS IN $h_\mu(\tau)$ ARE THE TERMS $c(n,\ell)$ WITH

$$p = 4mn - \ell^2 < 0$$

AFTER SPECTRAL FLOW & CHARGE CONJUGATION THE INDEPENDENT $c(n,\ell)$ HAVE $(\ell,n) \in \mathcal{P}$



E. IF $\mathcal{C}$ HAS AdS_3 DUAL, BTZ BLACK HOLES ONLY CONTRIBUTE TO THE NON-POLAR PART OF χ.

2. QUANTUM GRAVITY IN 2+1 DIM'S

. RECENTLY, WITTEN REVIVED AN OLD (1986) PROPOSAL THAT QUANTUM GRAVITY IN 2+1 DIM'S IS EXACTLY SOLUBLE.

FOCUS ON THE CASE $\Lambda < 0$

$$\boxed{\text{WHAT IS THE HOLOGRAPHIC DUAL OF 2+1 Q.G. ?}}$$

TO MOTIVATE WITTEN'S ANSWER

LET US RECALL WHY PEOPLE THINK THAT 2+1 Q.G. IS EXACTLY SOLUBLE:

THE ACTION

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) + \frac{k}{4\pi}\int \mathrm{Tr}\left(\omega\, d\omega + \frac{2}{3}\omega^3\right)$$

ℓ - AdS LENGTH

ω - SPIN CONNECTION

k - QUANTIZED

IS CLASSICALLY EQUIV. TO:

$$S = \frac{k_+}{4\pi}\int \mathrm{Tr}\left(A_+ dA_+ + \frac{2}{3}A_+^3\right) - \frac{k_-}{4\pi}\int \mathrm{Tr}\left(A_- dA_- + \frac{2}{3}A_-^3\right)$$

$$k_\pm = \frac{\ell}{16G} \pm \frac{k}{2}, \quad A_\pm = \omega \mp {*e}/\ell$$

WITTEN SUGGESTS THAT "THEREFORE" THE HOLOGRAPHICALLY DUAL PARTITION FUNCTION IS FACTORIZED:

LET'S JUST ACCEPT IT.

$$\boxed{Z(\tau,\bar\tau) = Z_{k_+}(\tau)\,\overline{Z_{k_-}(\tau)}}$$

ON A COMPACT SPACE C.S. HAS NO LOCAL DEGREES OF FREEDOM.

NEITHER DOES 2+1 GRAVITY: NO GRAV. WAVES. LOCALLY SOLUTIONS ARE JUST AdS

BROWN + HENNEAUX: EDGE STATES

= VIRASORO ALGEBRA DESCENDENTS

$$c_L + c_R = \frac{3\ell}{G} = 24(k_+ + k_-)$$

$$c_L - c_R = 24(k_+ - k_-) \quad \left[\begin{matrix}\text{LORENTZ}\\ \text{ANOMALY}\end{matrix}\right]$$

$$c_L = 24k_+ \,,\quad c_R = 24k_-$$

$$Z_k(\tau) \stackrel{?}{=} \chi_{VAC} = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}$$

(NOTE THAT $L_{-1}|0\rangle$ IS A NULL STATE)

$$\chi_{VAC}(\tau) = q^{-k+\frac{1}{12}}(1-q)\frac{1}{\eta(\tau)}$$

EVIDENTLY, THIS IS NOT MODULAR

BUT WE EXPECT MODULARITY IN A DIFF - INVT THEORY.

WHAT TO DO?

WITTEN PROPOSES THAT THE P.F. SHOULD BE AS CLOSE TO THE VIRASORO CHARACTER AS POSSIBLE:

$$Z_k(\tau) = \left[q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q^{\leq 0}} + O(q)$$

AS I EXPLAINED IN LECTURE 1 THIS MEANS:

$$Z_k(\tau) = a_k j^k + \cdots + a_0$$

THAT IS - BY ADDING NONPOLAR TERMS WE CAN RENDER $Z(\tau)$ MODULAR

WITTEN INTERPRETS THESE TERMS AS THE CONTRIBUTION OF BTZ BLACK HOLES.

THIS FITS IN PERFECTLY WITH THE FAREYTAIL STORY OF LECTURE II.

OBVIOUSLY THE REASONING IS FAR FROM AIR-TIGHT. WHETHER YOU LIKE IT OR NOT, WITTEN MAKES A SHARP PROPOSAL:

LET $Z_k(\tau)$ BE THE UNIQUE MOD. INVT. FUNCTION SUCH THAT:

$$Z_k(\tau) = \left[q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q^{\leq 0}} + O(q)$$

DEF AN EXTREMAL CFT OF LEVEL K IS A CFT WITH PARTITION FUNCTION $Z_k(\tau)$

WITTEN'S PROPOSAL: THE HOLOGRAPHIC DUAL OF PURE 2+1 GRAVITY IS $\mathcal{C}_k \otimes \widetilde{\mathcal{C}}_k$ WHERE $\mathcal{C}_k$ IS AN $ECFT_k$.

- BUT DO ECFT'S EXIST?
- YES FOR $k = 1$
- CONTROVERSIAL FOR $k > 1$

THIS LEADS TO A 4 QUESTION: MAYBE EXTREMAL SUPERCONFORMAL THEORIES ARE EASIER TO FIND....

THE STORY FOR $N=1$ IS DISCUSSED IN WITTEN'S PAPER AND IS SIMILAR TO $N=0$.

3. DEFINING EXTREMAL $\mathcal{N}=2$ CFT

I WILL NOW DESCRIBE SOME WORK IN PROGRESS WITH

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SOMETHING QUALITATIVELY NEW HAPPENS IN THE $\mathcal{N}=2$ CASE.

WE WILL USE MODULARITY OF THE ELLIPTIC GENUS TO PUT CONSTRAINTS ON THE SPECTRUM OF $\mathcal{N}=2$ PRIMARY FIELDS.

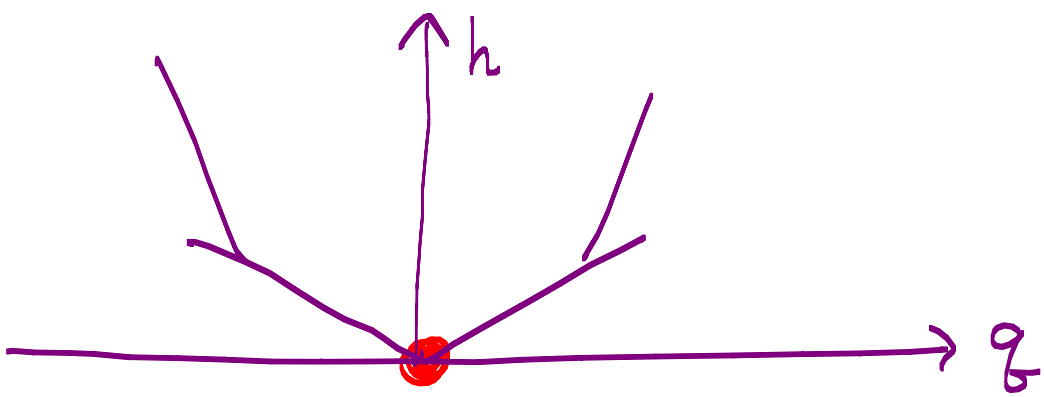
HOW SHOULD WE DEFINE AN "EXTREMAL $\mathcal{N}=2$ CFT" ?

IT'S UP TO US.

LET'S FOLLOW WITTEN'S LEAD

AND TAKE IT TO BE SUCH THAT THE PARTITION FUNCTION

$Z(z,\tau;\bar{z},\bar{\tau})$ DEFINED LAST TIME IS "AS CLOSE AS POSSIBLE" TO THE VACUUM CHARACTER.



VACUUM CHARACTER FOR $|h,q\rangle = |0,0\rangle$ AND $c = 6m$:

$$\chi_{vac}(\tau,z) = q^{-m/4} \frac{\prod_{j=1}^{\infty} \overbrace{\left(1+y\,q^{j+1/2}\right)}^{G^+_{-j-1/2}} \overbrace{\left(1+y^{-1}q^{j+1/2}\right)}^{G^-_{-j-\frac{1}{2}}}}{\prod_{j=1}^{\infty} \underbrace{\left(1-q^{j}\right)}_{J_{-j}} \prod_{j=2}^{\infty} \underbrace{\left(1-q^{j}\right)}_{L_{-j}}}$$

THIS IS NOT SPECTRAL FLOW INVT & DOES NOT HAVE GOOD MODULAR PROPERTIES – SO WE FORCE IT TO HAVE THESE PROP'S:

<u>DEF</u>: AN $\mathcal{N}=(2,2)$ EXTREMAL CFT IS A <u>HYPOTHETICAL</u> THEORY WITH

$$Z_{NSNS} = \Big| \sum_{\theta \in \mathbb{Z}} SF_\theta \chi_{vac} \Big|^2 +$$

$$+ \sum_{\substack{p>0 \\ \text{OR} \\ \tilde{p}>0}} c(n\,\ell;\tilde{n}\,\tilde{\ell})\, q^n y^\ell \bar{q}^{\tilde{n}} \bar{y}^{\tilde{\ell}}$$

WE COULD HAVE FORMULATED THIS IN THE RAMOND OR NS SECTOR, BUT THE DEF. SEEMS BEST MOTIVATED IN THE NS-SECTOR.

THIS DEFINITION IMPLIES THAT AN $\mathcal{N}=2$ ECFT MUST HAVE ELLIPTIC GENUS

$$\chi_{EXT}(\tau, z) = 2(-1)^m \sum_{\theta \in \mathbb{Z}+1/2} SF_\theta \, \chi_v + \text{NONPOLAR}$$

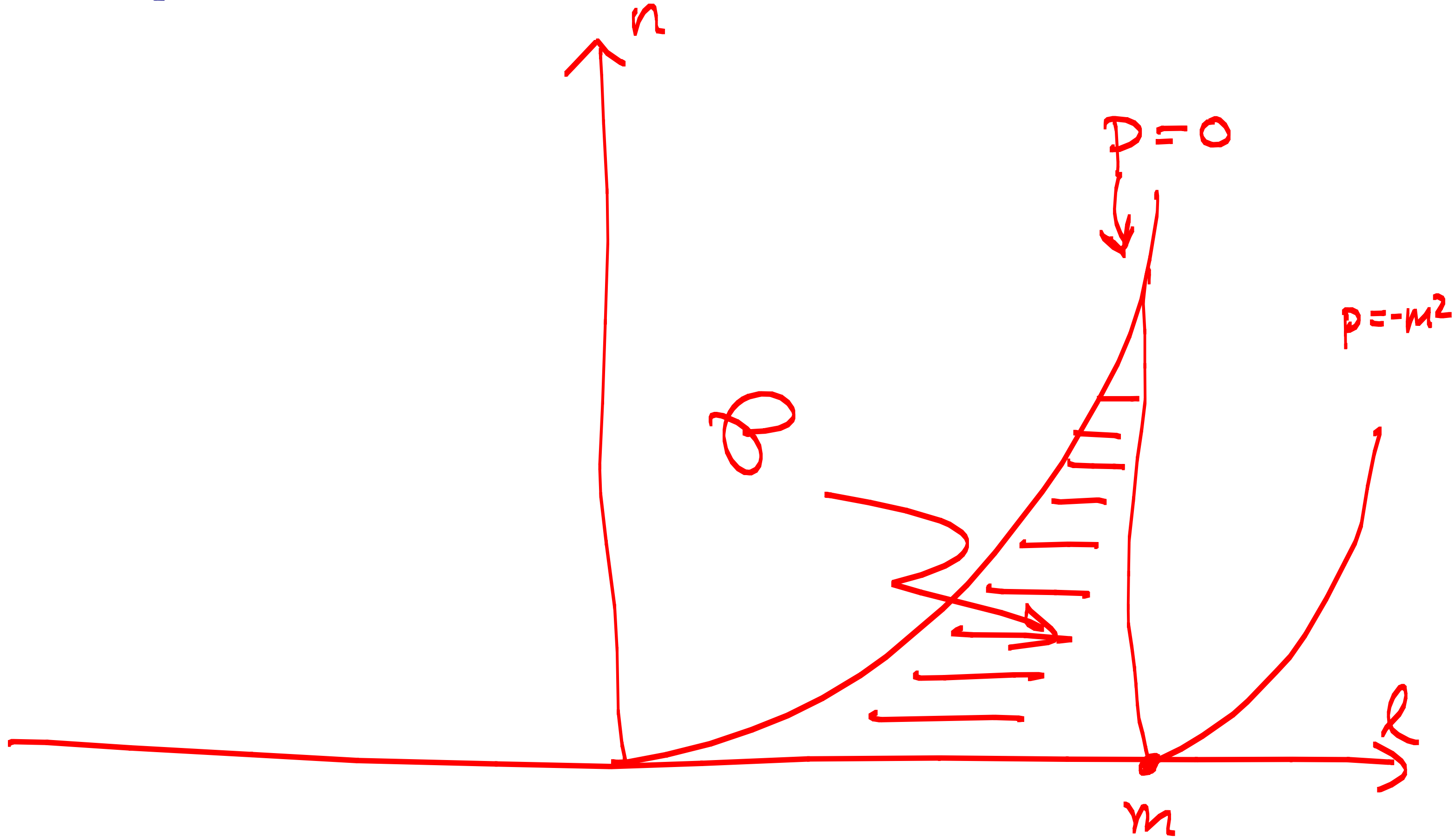
$$= 2(-1)^m \left\{ (1-q) y^m \prod_{m=1}^{\infty} \frac{(1-yq^{m+1})(1-y^{-1}q^m)}{(1-q^m)^2} + (y \to y^{-1}) + \text{NONPOLAR} \right\}$$

- IS THIS COMPATIBLE WITH MODULAR INVARIANCE ?
- DOES THERE EXIST SUCH A

$$\chi_{EXT} \in \tilde{J}_{0,m} \quad ?$$

4. COUNTING POLAR POLYNOMIALS

AS I EXPLAINED LAST TIME THE ELLIPTIC GENUS IS COMPLETELY DETERMINED BY ITS FOURIER COEFFICIENTS IN THE POLAR REGION $\mathcal{P}$



$$p = 4mn - \ell^2$$

Def: IF $\phi = \sum \hat{\phi}(n,\ell) q^n y^\ell$ IS ANY FOURIER SERIES IT'S POLAR POLYNOMIAL IS:

$$\mathrm{Pol}(\phi) := \sum_{(\ell,n)\in \mathcal{P}} \hat{\phi}(n,\ell) q^n y^\ell$$

LET US COUNT THE DIMENSION OF

$\mathcal{P}_m :=$ VECTOR SPACE OF POLAR POLY'S.

$$p(m) = \dim \mathcal{P}_m = \sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil$$

THIS IS THE NUMBER OF LATTICE POINTS IN THE SHADED REGION $\mathcal{P}$.

IT IS NOT ENTIRELY TRIVIAL TO EVALUATE, AND WE EXPLAIN IN THE NOTES THAT

$$P(m) = \underbrace{\frac{m^2}{12}}_{\text{AREA}} + \underbrace{\frac{5m}{8}}_{\text{LATTICE CORRECTION}} + A(m)$$

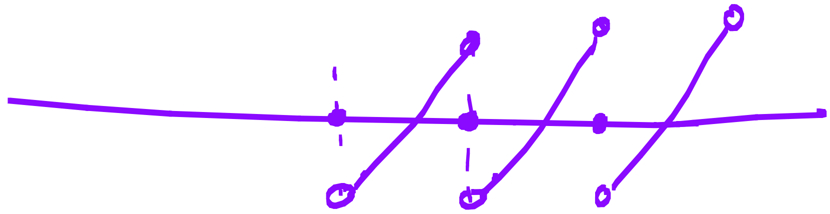
$A(m) =$ #-THEORETIC EXPRESSION INVOLVING CLASS NUMBERS,

MAIN POINT FOR US IS THAT FOR LARGE m IT GROWS <u>ROUGHLY</u> LIKE $\Theta(m^{1/2})$

Proof (time permitting)

Introduce sawtooth function

$$((x)) = x - \frac{1}{2}\left(\lfloor x \rfloor + \lceil x \rceil\right) = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha & x = n + \alpha, \quad 0 < \alpha < 1 \end{cases}$$



Write

$$\sum \left\lceil \frac{r^2}{4m} \right\rceil = \underbrace{\sum \frac{r^2}{4m}}_{A} - \underbrace{\sum \left(\left(\frac{r^2}{4m}\right)\right)}_{B} + \frac{1}{2}\underbrace{\sum \left(\left\lceil \frac{r^2}{4m} \right\rceil - \left\lfloor \frac{r^2}{4m} \right\rfloor \right)}_{C}$$

$$A = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24}$$

$C = \frac{m}{2}$ + correction when $\frac{r^2}{4m} \in \mathbb{Z}$ happens $O(m^{1/2})$

$= \frac{m}{2} - \left\lfloor \frac{b}{2} \right\rfloor$ $\quad b$ = largest integer $b^2 \mid m$

B Roughly, a random walk between $-1/2$ and $+1/2$. So $O(m^{1/2})$. Can express exactly in terms of class numbers.

5. COUNTING WEAK JACOBI FORMS

NOW THE ELLIPTIC GENUS MUST BE A WEAK JACOBI FORM OF WEIGHT ZERO AND INDEX m.

WHAT IS THE DIMENSION OF $\tilde{J}_{0,m}$?

THM [EICHLER-ZAGIER] THE BIGRADED RING $\tilde{J}_{*,*}$ OF WEAK JACOBI FORMS IS A POLYNOMIAL RING:

$$\tilde{J}_{*,*} = \mathbb{C}[E_4, E_6, \phi_{0,1}, \phi_{-2,1}]$$

NOTE THE SIMILARITY TO THE THEOREM WE PROVED ABOUT M_* IN LECTURE I. THE PROOF IS NOT HARD – BUT SKIP IT.

WE THEREFORE HAVE A BASIS FOR $\tilde{J}_{0,m}$

$$(\phi_{-2,1})^a (\phi_{0,1})^b E_4^c E_6^d$$

$$* \begin{cases} a+b=m \\ -2a+4c+6d=0 \end{cases} \qquad a,b,c,d \in \mathbb{Z}_+$$

$j(m) := \dim \widetilde{J}_{0,m}$ IS THE # OF SOLUTIONS TO $(*)$

A STRAIGHTFORWARD COMPUTATION SHOWS THAT

$$j(m) = \frac{m^2}{12} + \frac{m}{2} + \widetilde{A}(m)$$

$$\widetilde{A}(m) = \left(\delta_{s,0} + \frac{s}{2} - \frac{s^2}{12}\right) \; ; \quad m = s \bmod 6, \; 0 \le s \le 5$$

BUT $P(m) = \dfrac{m^2}{12} + \dfrac{5m}{8} + O(m^{1/2})$!

$P(m) > j(m)$ FOR $m \geq 5$

$$P(m) - j(m) = \frac{m}{8} + O(m^{1/2})$$

RECALL THAT WHEN WE DISCUSSED THE RECONSTRUCTION FORMULA IN LECTURE II

$$f(\tau) = \frac{1}{2} \sum_{\Gamma_\infty \backslash \Gamma} \left(j(\gamma,\tau)^{-w} f^{-}(\gamma\tau) + \text{REG} \right)$$

I SAID THAT IN GENERAL YOU CANNOT TAKE AN ARBITRARY f^{-} AND GET A MODULAR FORM.

THE OBSTRUCTION IS MEASURED BY A SPACE OF CUSP FORMS

$$0 \to \tilde{J}_{0,m} \xrightarrow{\text{Pol}} P_m \longrightarrow S_{5/2}(\Gamma, M)$$

AS DESCRIBED IN PAPER WITH MANSCHOT.

RECENTLY J. MANSCHOT COMPUTED THE DIMENSION OF THE OBSTRUCTION SPACE AND REPRODUCES THE FORMULA FOR $P(m) - j(m)$. [TO APPEAR.]

6. SEARCH FOR THE EXTREMAL ELLIPTIC GENUS

NOW RETURN TO OUR HYPOTHETICAL EXTREMAL $\mathcal{N}=2$ CFT WITH

$$\chi_{EXT} = SF_{1/2}\chi_V + SF_{-1/2}\chi_V + N.P.$$

THERE IS NO GUARANTEE THAT

$$p^m_{EXT} := Pol(\chi_{EXT}) \in I_m(\widetilde{J}_{0,m})$$

BUT MAYBE THERE IS MAGIC...

CHOOSE A BASIS ϕ_i FOR $\widetilde{J}_{0,m}$.

IF THE EXTREMAL $\mathcal{N}=2$ CFT EXISTS THEN

$$\exists x_i \qquad \sum_{i=1}^{j(m)} x_i \, Pol(\phi_i) = p^m_{EXT} \qquad (*)$$

EVEN IF WE FIND SOLUTIONS THERE IS A FURTHER TEST SINCE

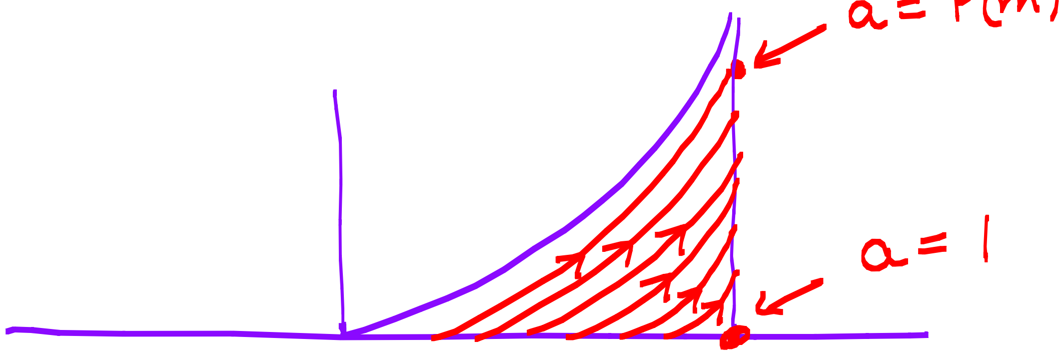
$$\sum_i x_i \phi_i = \sum c(n,\ell) q^n y^\ell$$

MUST HAVE $c(n,\ell) \in \mathbb{Z}$

OF COURSE, THESE ARE NECESSARY, NOT SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE EXTREMAL THEORY.

TO ANALYZE EQUATION $(*)$ INTRODUCE A POLARITY-ORDERED BASIS FOR P_m,

$$q^{n(a)} y^{\ell(a)} \qquad a = 1, \ldots, P(m)$$



POLARITY $\nearrow$ AS $a \nearrow$

FOR $a=1$ y^m HAS $p=-m^2$,

DEFINE:

$$\mathrm{Pol}\,\phi_i := \sum_{a=1}^{P(m)} N_{ia}\, q^{n(a)} y^{\ell(a)}$$

$$P^m_{EXT} := \sum_{a=1}^{P(m)} d_a\, q^{n(a)} y^{\ell(a)}$$

EQUATION $\circledast$ IS:

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \qquad a=1,\ldots,P(m)$$

RECALL: $P(m) > j(m)$ $m \geq 5$

COMPUTER: $1 \leq m \leq 36$

SOLN'S X_i EXIST FOR: $1 \leq m \leq 5, 7, 8, 11, 13$

$\nexists$ X_i FOR $m = 6, 9, 10, 12, 14 \leq m \leq 36$

MOREOVER: IN THE CASES WHERE SOLUTIONS EXIST THE $c(n,\ell)$ SEEM TO BE INTEGRAL.

(THIS IS NONTRIVIAL. E.G. $m=2$

$$\frac{1}{6}\phi_{0,1}^2 + \frac{5}{6}\phi_{-2,1}^2 E_4 \quad .)$$

RECENTLY, WE FOUND AN ANALYTIC ARGUMENT:

FOR m SUFF. LARGE, SOLUTIONS X_i DO NOT EXIST

THUS, EXTREMAL $\mathcal{N}=2$ THEORIES AT BEST EXIST FOR A FINITE "SPORADIC" SET OF m.

7. NEAR EXTREMAL $\mathcal{N}=2$ CFT

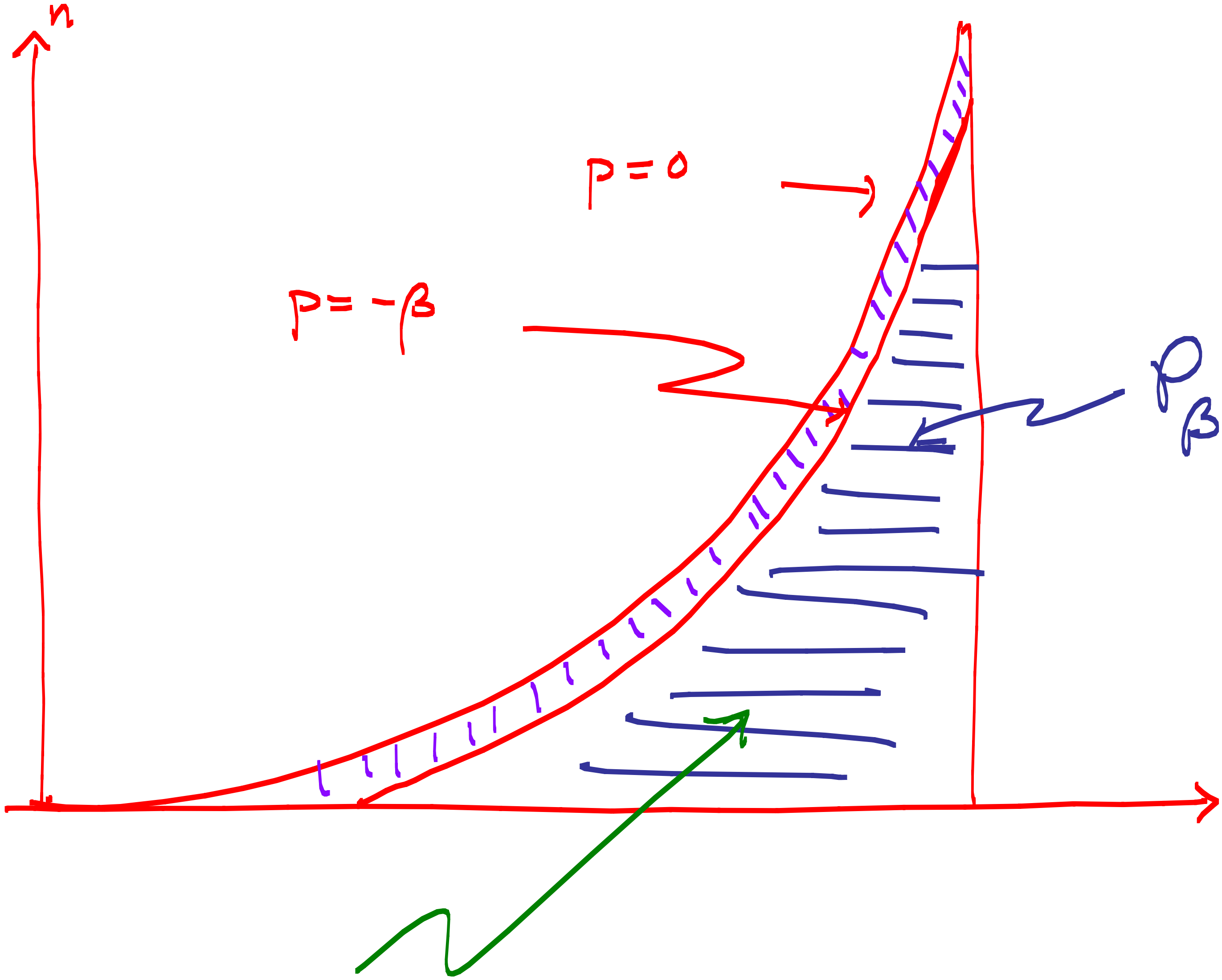
PERHAPS OUR DEFINITION WAS TOO RESTRICTIVE...

MAYBE THERE ARE QUANTUM CORRECTIONS TO THE COSMIC CENSORSHIP BOUND...

LET US DEFINE A

β- EXTREMAL $\mathcal{N}=2$ CFT

IF WE ONLY DEMAND AGREEMENT WITH THE VACUUM CHARACTER UP TO POLARITY $-\beta$



ONLY DESCENDENTS OF THE VACUUM

NOW WE ONLY TRY TO MATCH THE POLAR DEGENERACIES IN ρ_β.

SO WE INCREASE β UNTIL
WE ARE SOLVING

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \qquad a = 1, \cdots, j(m)$$

THIS HAPPENS FOR $\beta = \frac{m}{2} + O(m^{1/2})$

COMPUTER:

- SOLUTIONS x_i DO EXIST!
  $1 \leq m \leq 36$

- AND
  $$\sum_i x_i \phi_i = \sum c(n,\ell)\, q^n y^\ell$$

DO HAVE $c(n,\ell) \in \mathbb{Z}$ ...

EXCEPT FOR $m=17$!

QUI IN ITALIA IL DICIASSETTE PORTA SFIGA!

FOR $m=17$, WE MUST LOWER THE POLARITY CUTOFF STILL FURTHER....

ON THE OTHER HAND,

THERE IS SOME $\beta_*(m)$ SO THAT WE CAN MATCH χ_{VAC} FOR ALL STATES OF POLARITY LESS THAN $-\beta_*(m)$

THE COMPUTER EVIDENCE SUGGESTS

$$\beta_*(m) \geq \frac{m}{2} + O(m^{1/2})$$

FOR ALL m.

IF TRUE, THEN FOR ANY $\mathcal{N}=(2,2)$ CFT THERE MUST BE AN $\mathcal{N}=2$ PRIMARY WITH

$$4m\left(h^R - \frac{c}{24}\right) - \ell^2 > -\beta_*(m)$$

THUS WE FORMULATE THE CONJECTURE:

ANY $\mathcal{N}=(2,2)$ CFT WITH INTEGRAL $U(1)$ CHARGES MUST HAVE A STATE $\psi_L \otimes \widetilde{BRS}$ WHERE ψ_L IS AN $\mathcal{N}=2$ PRIMARY WITH:

$$h^{NS} > \frac{m}{4} + \frac{(J_0^{NS})^2}{4m} - \frac{1}{8} + O(m^{-1/2})$$

ON THE OTHER HAND - USING A DIFFERENT METHOD - IT IS POSSIBLE TO SHOW HOW TO CONSTRUCT A χ WITH ONLY CONTRIBUTIONS FROM DESCENDENTS WITH

$$h^{NS} \leq \frac{5m}{16}$$

NOW – IT IS ABOUT TIME I REVEAL MY REAL MOTIVATION FOR PURSUING THIS PROBLEM:

IN RECENT YEARS THERE HAS BEEN MUCH ACTIVITY IN FLUX COMPACTIFICATON AND MODULI STABILIZATION. (C.f. L. McAllister TALKS) SHAMIT KACHRU HAS BEEN A DRIVING FORCE IN THIS DEVELOPMENT.

WITH MANY CHOICES FOR FLUX THERE ARE, FAMOUSLY, MANY POSSIBLE COMPACTIFICATIONS

WE EXPECT THERE ARE SOME WITH

- $AdS_3 \times \mathcal{K}$

$\Rightarrow \exists$ HOLOGRAPHIC DUAL CFT_2

- $0 < -\frac{\Lambda}{M_{Pl}^2} = +\left(\frac{1}{\ell M_{Pl}}\right)^2 \ll 1$

$$\Rightarrow \quad c = \frac{3\ell}{2G} \gg 1$$

- KK LENGTHSCALE OF $\mathcal{K}$ IS ORDER c IN ADS UNITS

$\Rightarrow$ "MOST" PRIMARIES HAVE A LARGE GAP FROM THE VACUUM.

THE SPECTRUM RESEMBLES AN EXTREMAL CFT !

8. CONCLUSION

I WILL LEAVE YOU WITH TWO OBVIOUS OPEN PROBLEMS

- PROVE THE BOUND ON h^{NS}
- DOES THIS BOUND PUT ANY INTERESTING CONSTRAINTS ON AdS_3 FLUX COMPACTIFICATIONS?