Four Lectures on Web Formalism and Categorical Wall-Crossing

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collaboration with
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draft is ``nearly finished"…

Plan for four lectures

Lecture 1: Landau-Ginzburg models; Morse theory and SQM; Motivation from spectral networks; Motivation from knot homology

Lecture 2: Webology part 1: Plane webs. Definition of a Theory. Half-plane webs.

Lecture 3: Webology part 2: Vacuum and Brane A_∞ categories; Examples.

Lecture 4: Webology part 3: Domain walls and Interfaces; Composition of Interfaces; Parallel transport of Brane Categories; Categorified wall-crossing.

Three Motivations

1. IR sector of ***massive*** 1+1 QFT with N =(2,2) SUSY

2. Knot homology.

3. Spectral networks & categorification of 2d/4d wall-crossing formula [Gaiotto-Moore-Neitzke].

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

d=2, N=(2,2) SUSY

$$\{Q_+, \overline{Q_+}\} = H + P$$

$$\{Q_-, \overline{Q_-}\} = H - P$$

$$\{Q_+, Q_-\} = \bar{Z}$$

$$[F, Q_+] = Q_+ \qquad [F, \bar{Q}_-] = \bar{Q}_-$$

We will be interested in situations where two supersymmetries are unbroken:

$$U(\zeta) := Q_+ - \zeta^{-1}\overline{Q_-}$$

$$\{U(\zeta), \overline{U(\zeta)}\} = 2\left(H - \mathrm{Re}(\zeta^{-1}Z)\right)$$

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Overview of Results; Some Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- More about motivation from knot homology
- More about motivation from spectral networks

Example: LG Models - 1

$\phi, \psi_{\pm}, \bar{\psi}_{\pm}, \ldots$ Chiral superfield

$W(\phi)$ Holomorphic superpotential

$$S = \int d\phi * d\bar{\phi} - |\nabla W|^2 + \cdots$$

Massive vacua are Morse critical points:

$$dW(\phi_i) = 0 \qquad W''(\phi_i) \neq 0$$

Label set of vacua: $\phi_i \in \mathbb{V}$

Example: LG Models -2

More generally,….

(X,ω): Kähler manifold.

W: X $\longrightarrow \mathbb{C}$ Superpotential (A holomorphic Morse function)

$$\phi : D \times \mathbb{R} \to X$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

Boundary conditions for ϕ

Boundaries at infinity:

$$\phi \to \phi_i \qquad \phi \to \phi_j$$

$$x \to -\infty \qquad x \to +\infty$$

Boundaries at finite distance: Preserve ζ-susy:

$$\phi|_{x_\ell, x_r} \in \mathcal{L}_{\ell,r} \subset X$$

$$\iota^*_{\mathcal{L}}(\lambda) = dk$$

(Simplify: ω=dλ)

$$\pm \mathrm{Im}(\zeta^{-1} W) \geq \Lambda$$

Fields Preserving ζ-SUSY

U(ζ)[Fermi] =0 implies the ζ-*instanton* equation:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Time-independent: ζ-*soliton* equation:

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Projection to W-plane

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

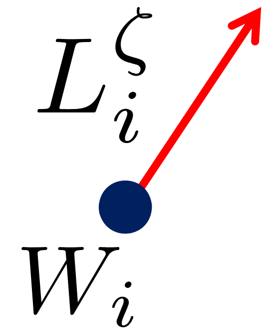
The projection of solutions to the complex W plane are contained in straight lines of slope ζ

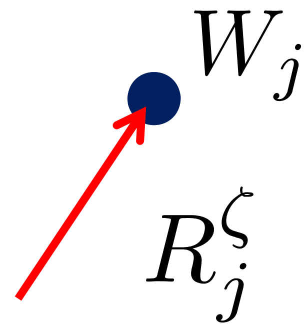
$$\frac{dW}{dx} = \frac{\partial W}{\partial \phi^I} \frac{\partial}{\partial x}\phi^I = \zeta \frac{\partial W}{\partial \phi^I} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$W(x) - W(x_0) = \zeta \int_{x_0}^{x} |\nabla W|^2 dx'$$

Lefshetz Thimbles

If D contains x $\longrightarrow$ -∞ $\phi \rightarrow \phi_i$





If D contains x $\longrightarrow$ +∞ $\phi \rightarrow \phi_j$

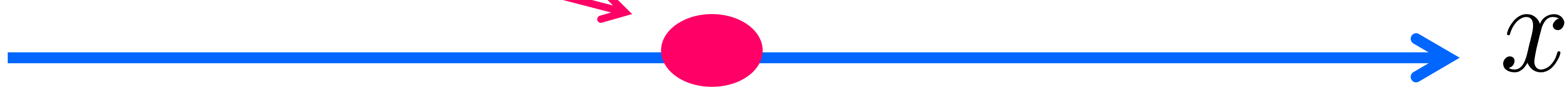
Inverse image in X of all solutions defines left and right Lefshetz thimbles

They are Lagrangian subvarieties of X

Solitons For D=$\mathbb{R}$

Scale set by W

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$



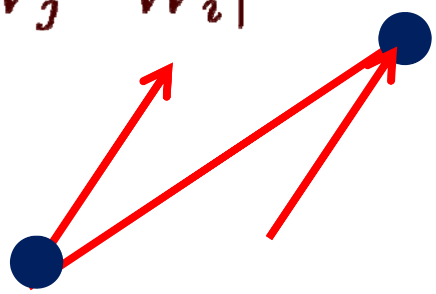
$\phi \cong \phi_i$ $\phi \cong \phi_j$

For general ζ there is no solution.

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

But for a suitable phase there is a solution

W_j



W_i

This is the classical soliton. There is one for each intersection (Cecotti & Vafa)

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

Near a critical point

$$W = W_i + \sum_I \frac{1}{2}\mu_I(\phi^I - \phi_i^I)^2$$

$$\phi^I = \phi_i^I + r^I\sqrt{\frac{\zeta\mu_I}{\kappa_I}}e^{\kappa_I x}$$

$$r^I \in \mathbb{R} \qquad |\kappa_I| = |\mu_I|$$

$$L_i^\zeta \quad \forall I \;\; \kappa_I > 0$$

$$R_i^\zeta \quad \forall I \;\; \kappa_I < 0$$

Witten Index

Some classical solitons are lifted by instanton effects, but the Witten index:

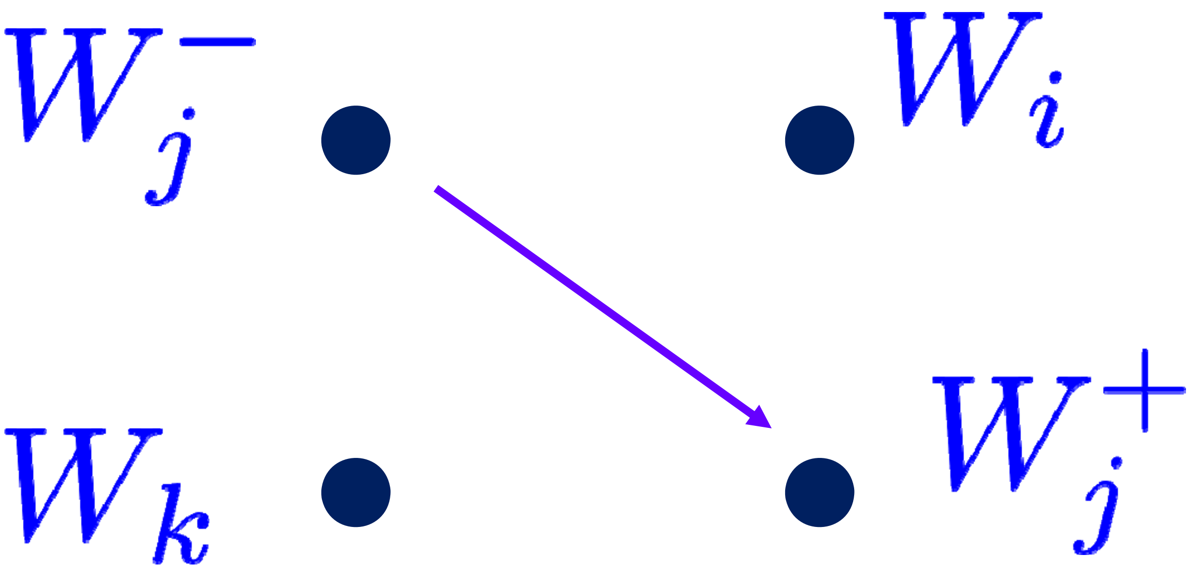
$$\mu_{ij} := \mathrm{Tr}_{\mathcal{H}_{ij}^{BPS}}(-1)^F$$

can be computed with a signed sum over classical solitons:

$$\mu_{ij} = \sum_{p \in L_i^\zeta \cap R_j^\zeta} (-1)^{\iota(p)}$$

These BPS indices were studied by [Cecotti, Fendley, Intriligator, Vafa and by Cecotti & Vafa]. They found the wall-crossing phenomena:

Given a one-parameter family of W's:



$$\mu_{ik}^- \to \mu_{ik}^+ = \mu_{ik}^- + \mu_{ij}\mu_{jk}$$

One of our goals will be to categorify this wall-crossing formula.

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Goals & Results - 1

Goal: Say everything we can about the theory in the far IR.

Since the theory is massive this would appear to be trivial.

Result: When we take into account the BPS states there is an extremely rich mathematical structure.

We develop a formalism – which we call the ``web-based formalism'' -- which shows that:

Goals & Results - 2

BPS states have ``interaction amplitudes'' governed by an L_∞ algebra

There is an A_∞ category of branes/boundary conditions, with amplitudes for emission of BPS particles from the boundary governed by an A_∞ algebra.

(A_∞ and L_∞ are mathematical structures which play an important role in open and closed string field theory, respectively. Strangely, they show up here.)

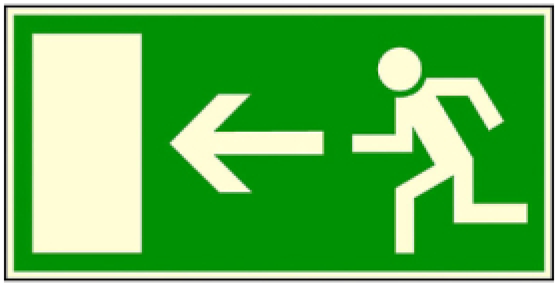
Goals & Results - 3

If we have continuous families of theories
(e.g. a continuous family of LG superpotentials)
then we can construct half-supersymmetric interfaces between the theories.

These interfaces can be used to ``implement'' wall-crossing.

Half-susy interfaces form an A_∞ 2-category, and to a continuous family of theories we associate a flat parallel transport of brane categories.

The flatness of this connection implies, and is a categorification of, the 2d wall-crossing formula.




EMERGENCY
EXIT ONLY

Some Old Questions

What are the BPS states on $\mathbb{R}$ in sector ij ? $\mathcal{H}_{ij}^{\mathrm{BPS}}$

Fendley & Intriligator; Cecotti, Fendley, Intriligator, Vafa; Cecotti & Vafa c. 1991

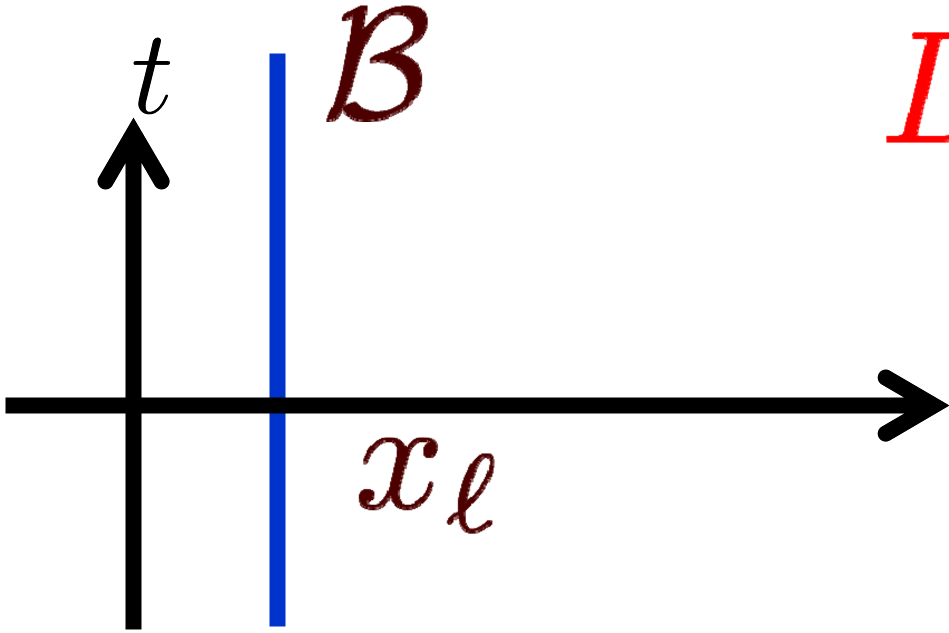
Some refinements. Main new point: L_∞ structure

What are the branes/half-BPS boundary conditions ? $\mathcal{B}$

Hori, Iqbal, Vafa c. 2000 & Much mathematical work on A-branes and Fukaya-Seidel categories.

We clarify the relation to the Fukaya-Seidel category & construct category of branes from IR.

Some New Questions -1



$$D = [x_\ell, \infty)$$

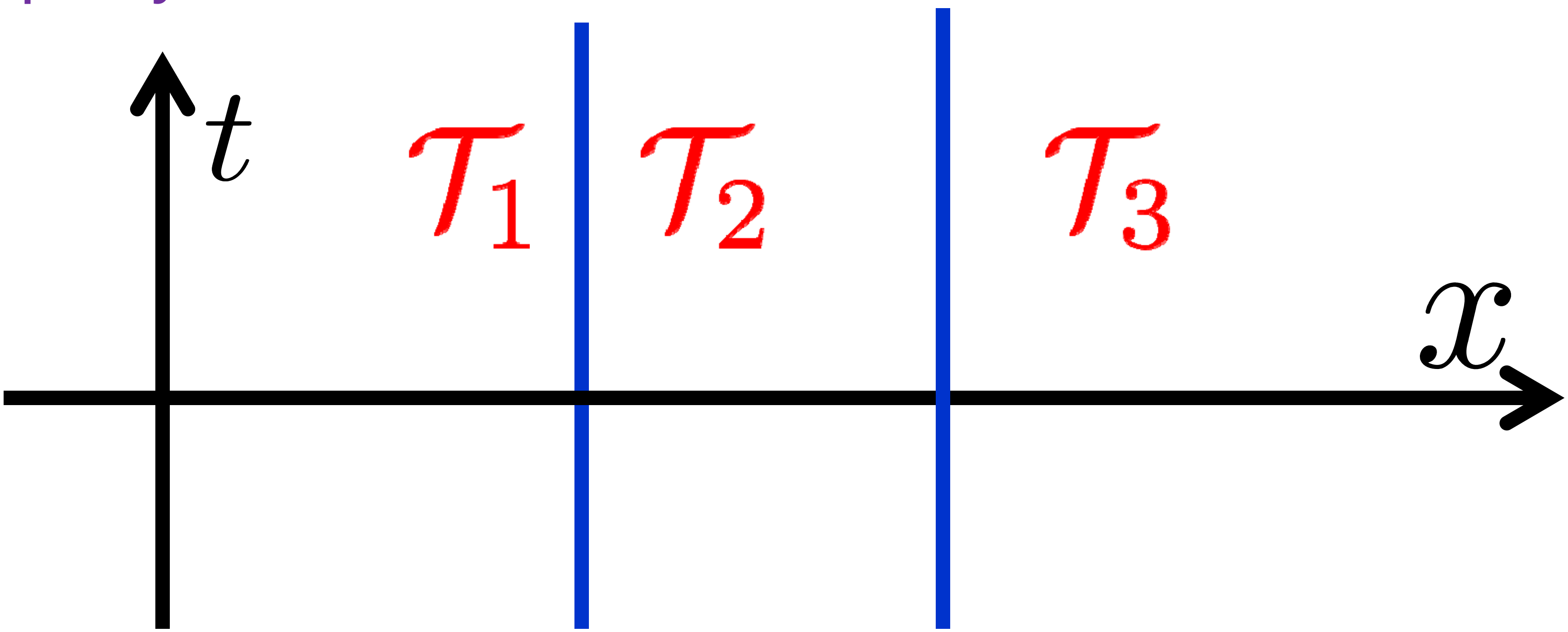
$$\phi \to \phi_j$$

What are the BPS states on the half-line ?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B},j}$$

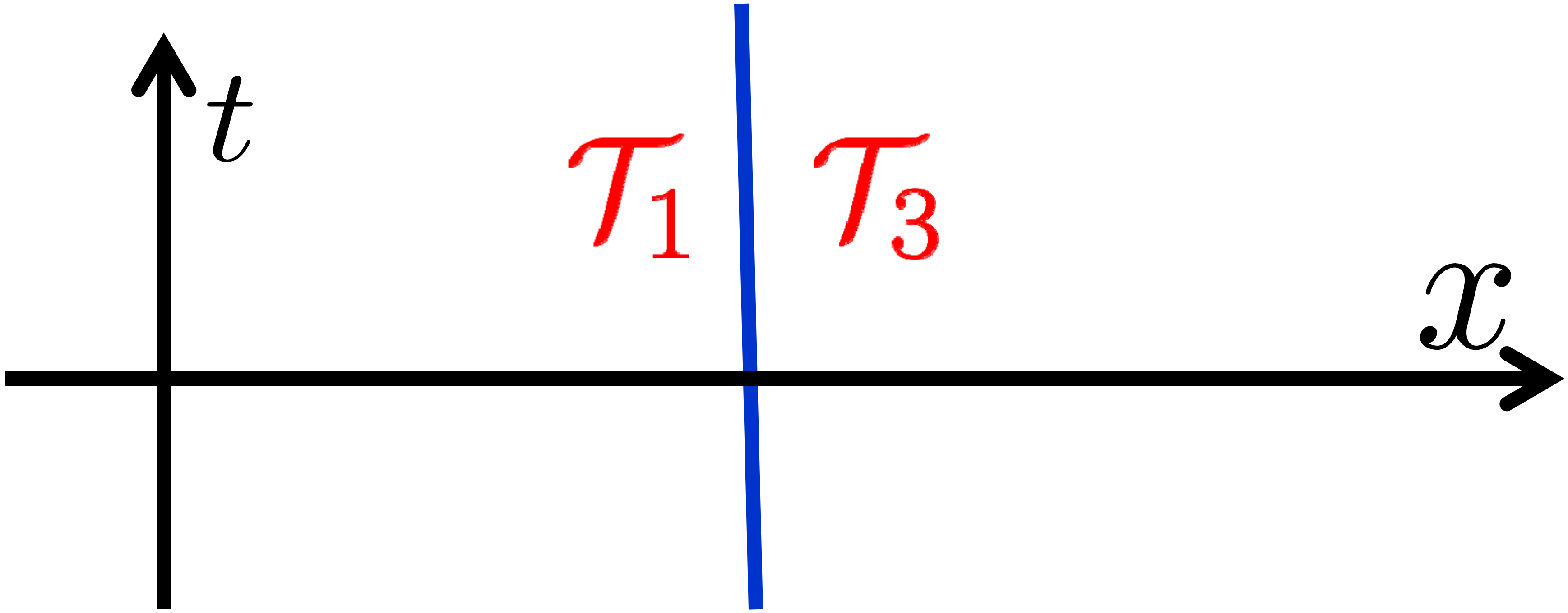
Some New Questions - 2

Given a pair of theories $\mathcal{T}_1, \mathcal{T}_2$ what are the supersymmetric interfaces?



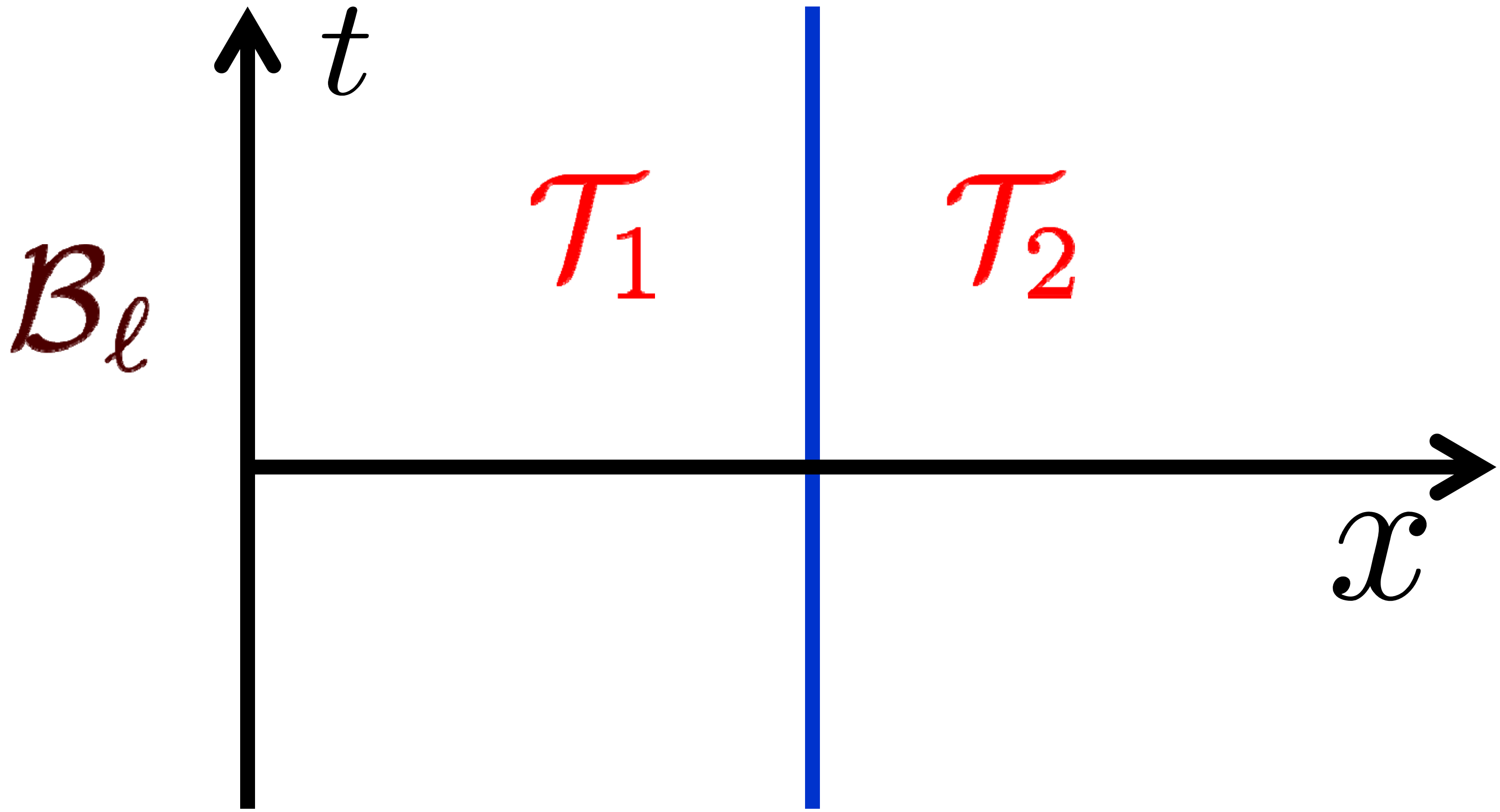
Is there an (associative) way of ``multiplying'' interfaces to produce new ones? And how do you compute it?

Some New Questions - 3



We give a method to compute the product. It can be considered associative, once one introduces a suitable notion of ``homotopy equivalence'' of interfaces.

Some New Questions - 4



Using interfaces we can ``map" branes in theory $\mathcal{T}_1$, to branes in theory $\mathcal{T}_2$.

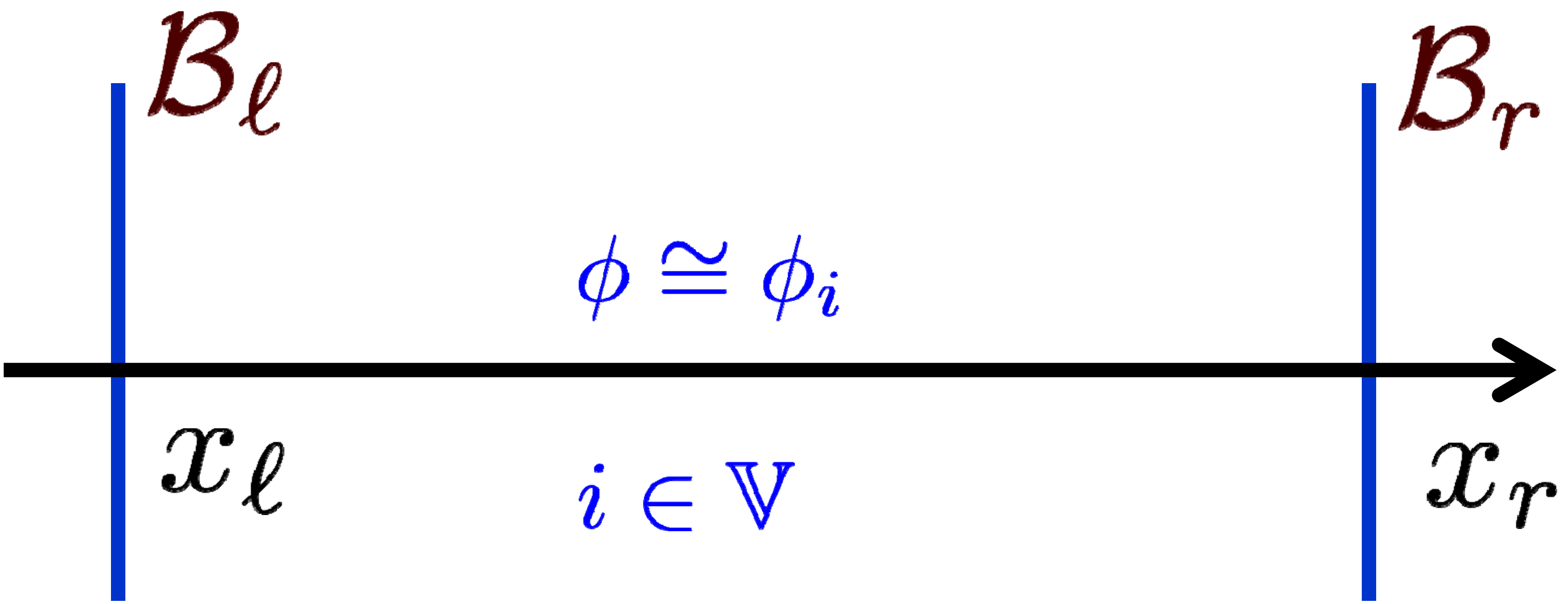
This will be the key idea in defining a ``parallel transport" of Brane categories.

Example of a surprise:

What is the space of BPS states on an interval ?

The theory is massive:

For a susy state, the field in the middle of a large interval is close to a vacuum:



Does the Problem Factorize?

For the Witten index: Yes

$$\mu_{\mathcal{B}_\ell,i} = \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i}}(-1)^F e^{-\beta H}$$

$$\boxed{\mu_{\mathcal{B}_\ell,\mathcal{B}_r} = \sum_{i\in\mathbb{V}} \mu_{\mathcal{B}_\ell,i} \cdot \mu_{i,\mathcal{B}_r}}$$

Naïve categorification?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,\mathcal{B}_r} \neq \sum_{i\in\mathbb{V}} \mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i} \otimes \mathcal{H}^{\mathrm{BPS}}_{i,\mathcal{B}_r}$$

No!

Enough with vague generalities!

Now I will start to be more systematic.

The key ideas behind everything we do come from Morse theory.

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SQM & Morse Theory (Witten: 1982)

M: Riemannian; h: *M* $\longrightarrow \mathbb{R}$, Morse function

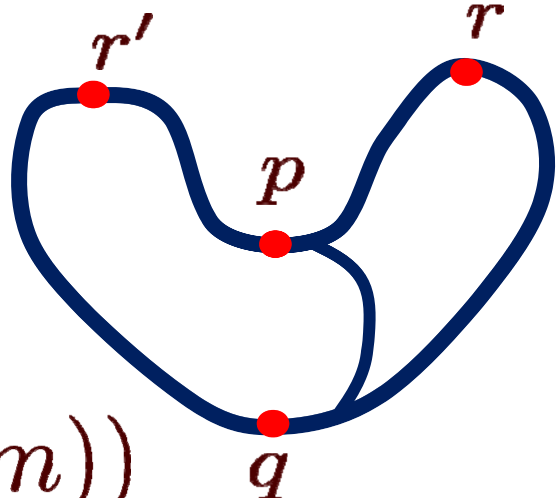
SQM: $q : \mathbb{R}_{\mathrm{time}} \to M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

$$L = g_{IJ}\dot{q}^I \dot{q}^J - g^{IJ}\partial_I h \partial_J h$$

$$+ g_{IJ}\bar{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \bar{\chi}^I \chi^J \; - R_{IJKL}\bar{\chi}^I \chi^J \bar{\chi}^K \chi^L$$

Perturbative vacua: $dh(m) = 0$

$\Longrightarrow \Psi(m)$



$$F(\Psi(m)) = \tfrac{1}{2}(d_{\uparrow}(m) - d_{\downarrow}(m))$$

Instantons & MSW Complex

Instanton equation: $$\frac{d\phi}{d\tau} = \pm g^{IJ} \frac{\partial h}{\partial \phi^J}$$

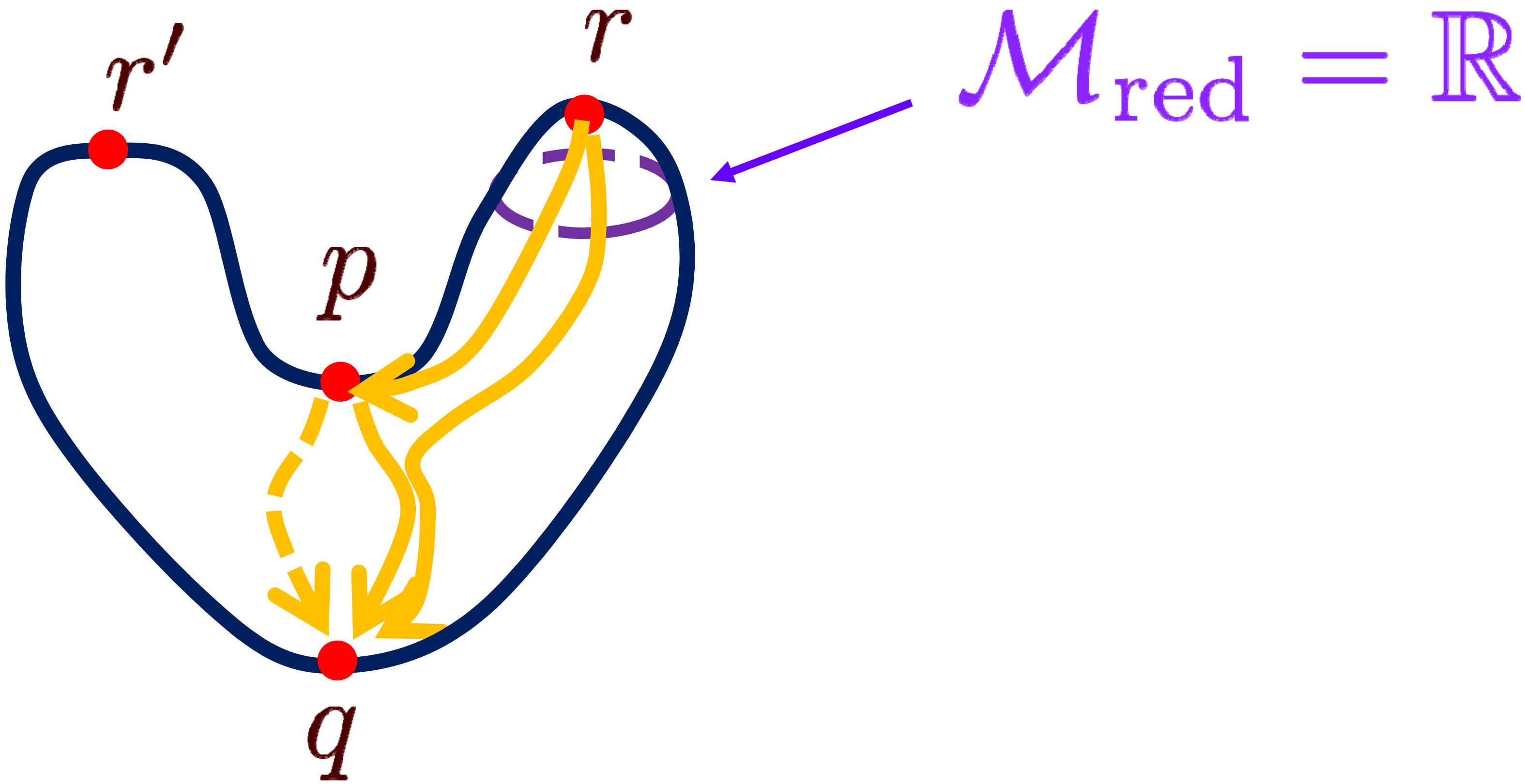
``Rigid instantons'' - with zero reduced moduli – will lift some perturbative vacua. To compute exact vacua:

MSW complex: $$\mathbb{M}^\bullet := \oplus_{p:dh(p)=0} \mathbb{Z} \cdot \Psi(p)$$

$$d(\Psi(p)) = \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

Space of groundstates (BPS states) is the *cohomology*.

Why $d^2 = 0$



Ends of the moduli space correspond to broken flows which cancel each other in computing $d^2 = 0$. A similar argument shows independence of the cohomology from h and g_{IJ}.

1+1 LG Model as SQM

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega \qquad \lambda = p dq$$

Recover the standard 1+1 LG model with superpotential: Two –dimensional ζ-susy algebra is manifest.

We now give two applications of this viewpoint.

Families of Theories

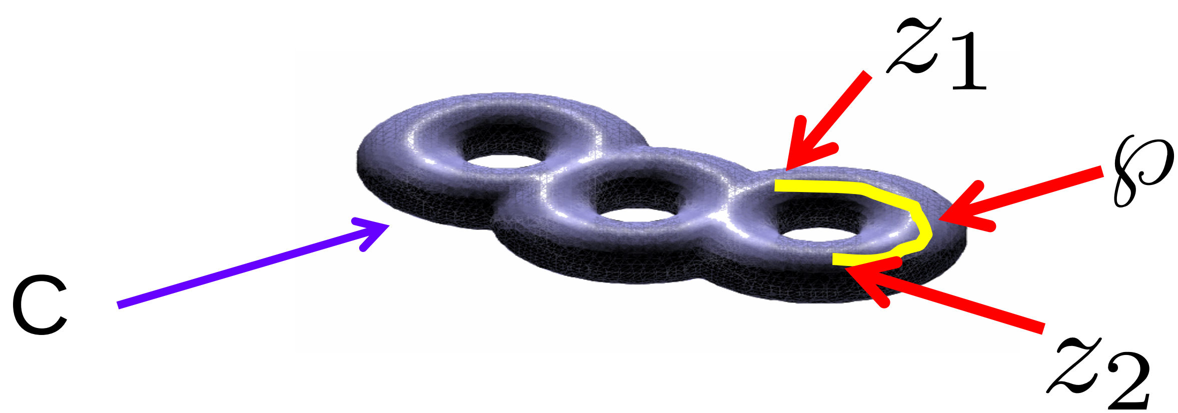
This presentation makes construction of half-susy interfaces easy:

Consider a *family* of Morse functions

$$W(\phi; z) \quad z \in C$$

Let $\wp$ be a path in C connecting z_1 to z_2.

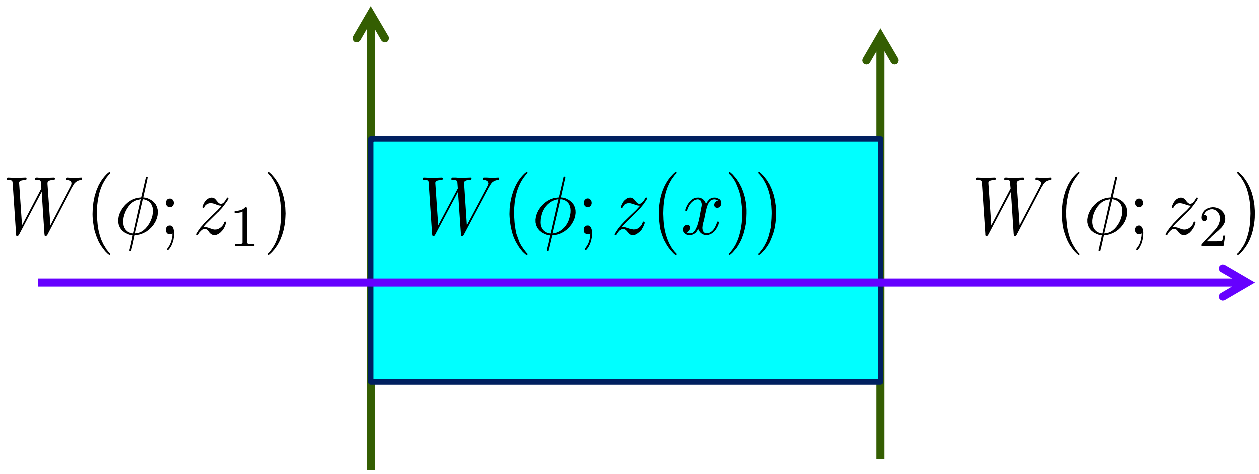
View it as a map z: $[x_l, x_r] \longrightarrow C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



Domain Wall/Interface

Using z(x) we can still formulate our SQM!

$$h = \int_D \phi^* (pdq) + \mathrm{Re}(\zeta^{-1} W(\phi; z(x)))dx$$



From this construction it manifestly preserves two supersymmetries.

MSW Complex

Now return to a single W. Another good thing about this presentation is that we can discuss ij solitons in the framework of Morse theory:

$$\frac{\delta h}{\delta \phi} = 0$$

Equivalent to the ζ-soliton equation

$$\mathbb{M}_{ij} = \oplus_{\mathrm{solitons}} \mathbb{Z} \cdot \Psi_{ij}$$

(Taking some shortcuts here….)

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

$$F = -\frac{1}{2} \eta (D - \epsilon)$$

Instantons

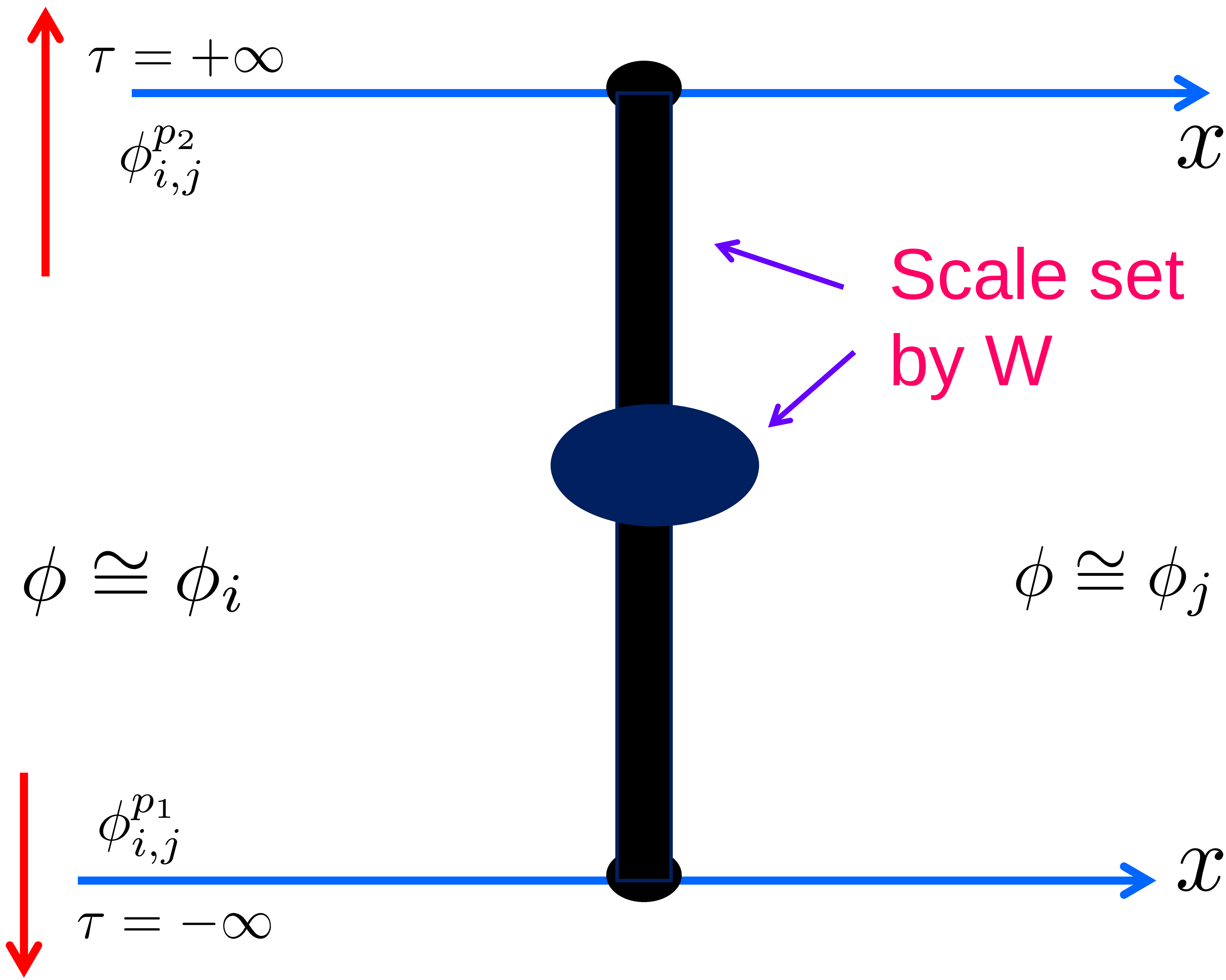
Instanton equation $$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

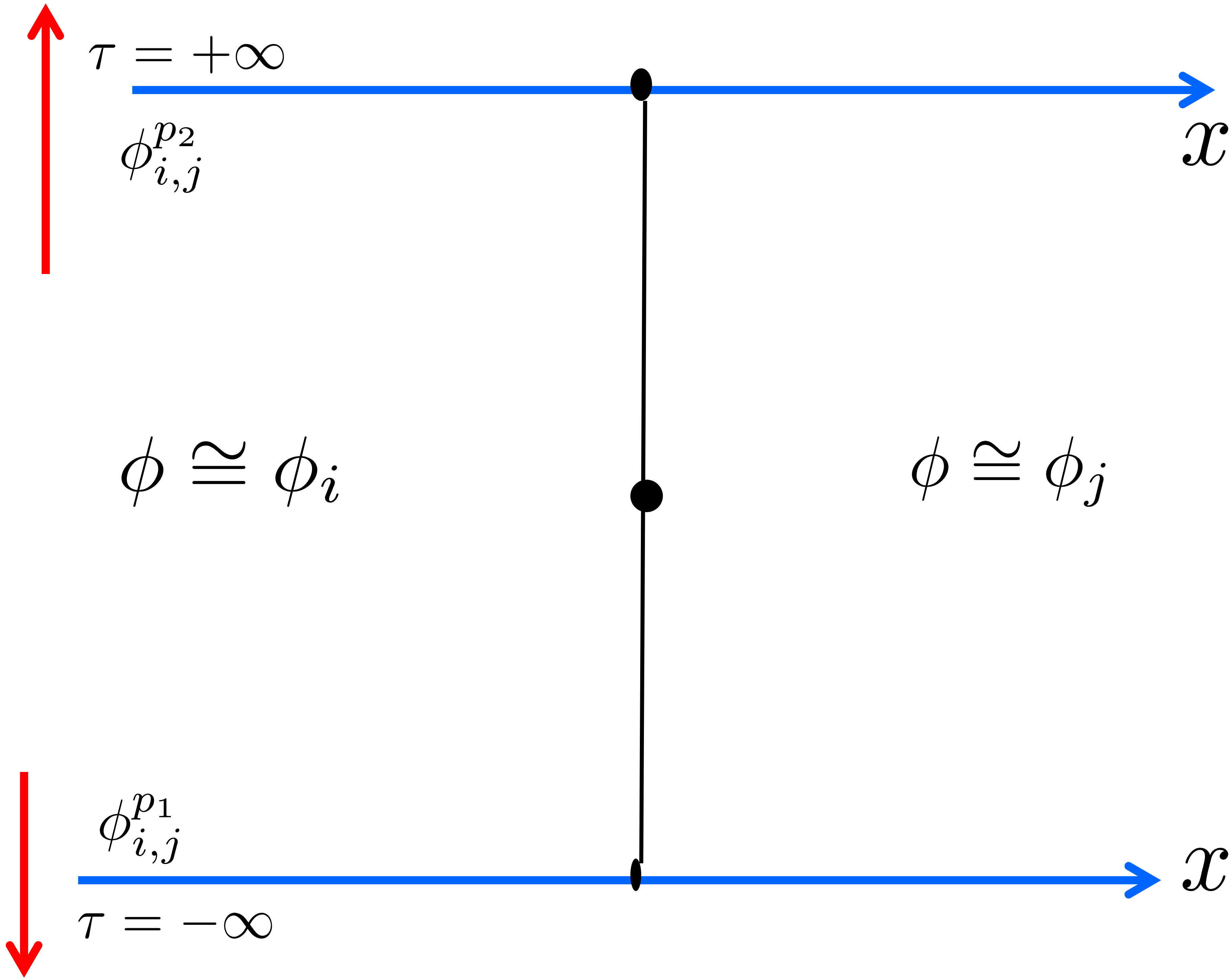
$$\bar{\partial}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

At short distance scales W is irrelevant and we have the usual holomorphic map equation.

At long distances the theory is almost trivial since it has a mass scale, and it is dominated by the vacua of W.



$\tau = +\infty$
$\phi^{p_2}_{i,j}$
x
Scale set
by W
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi^{p_1}_{i,j}$
x
$\tau = -\infty$



$\tau = +\infty$
$\phi_{i,j}^{p_2}$
x
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$

BPS Solitons on half-line D:

Semiclassically:

Q_ζ -preserving BPS states must be solutions of differential equation

$$\frac{\partial \phi^I}{\partial x} = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\phi|_{x_\ell} \in \mathcal{L}$$

$$\phi \to \phi_j$$
$$x \to \infty$$

Classical solitons on the positive half-line are labeled by: $p \in \mathcal{L} \cap R_j^\zeta$

Quantum Half-Line Solitons

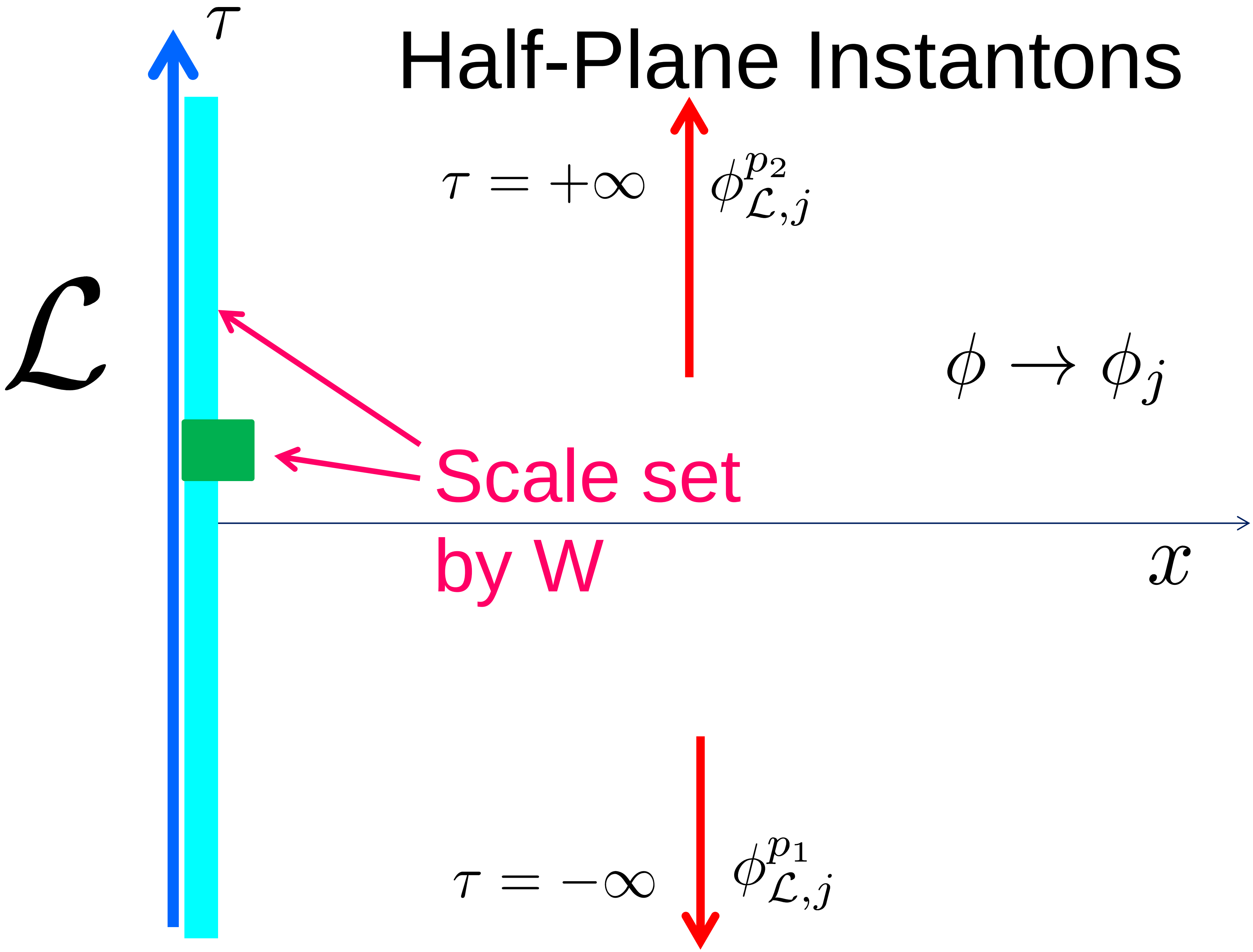
MSW complex: $$\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$$

Grading the complex: Assume X is CY and that we can find a logarithm:

$$w = \mathrm{Im} \log \frac{\iota^*(\Omega^{d,0})}{\mathrm{vol}(\mathcal{L})}$$

Then the grading is by $$f = \eta(D) - w$$

Half-Plane Instantons



Solitons On The Interval

Now return to the puzzle about the finite interval $[x_l, x_r]$ with boundary conditions $\mathcal{L}_l, \mathcal{L}_r$

When the interval is much longer than the scale set by W the MSW complex is

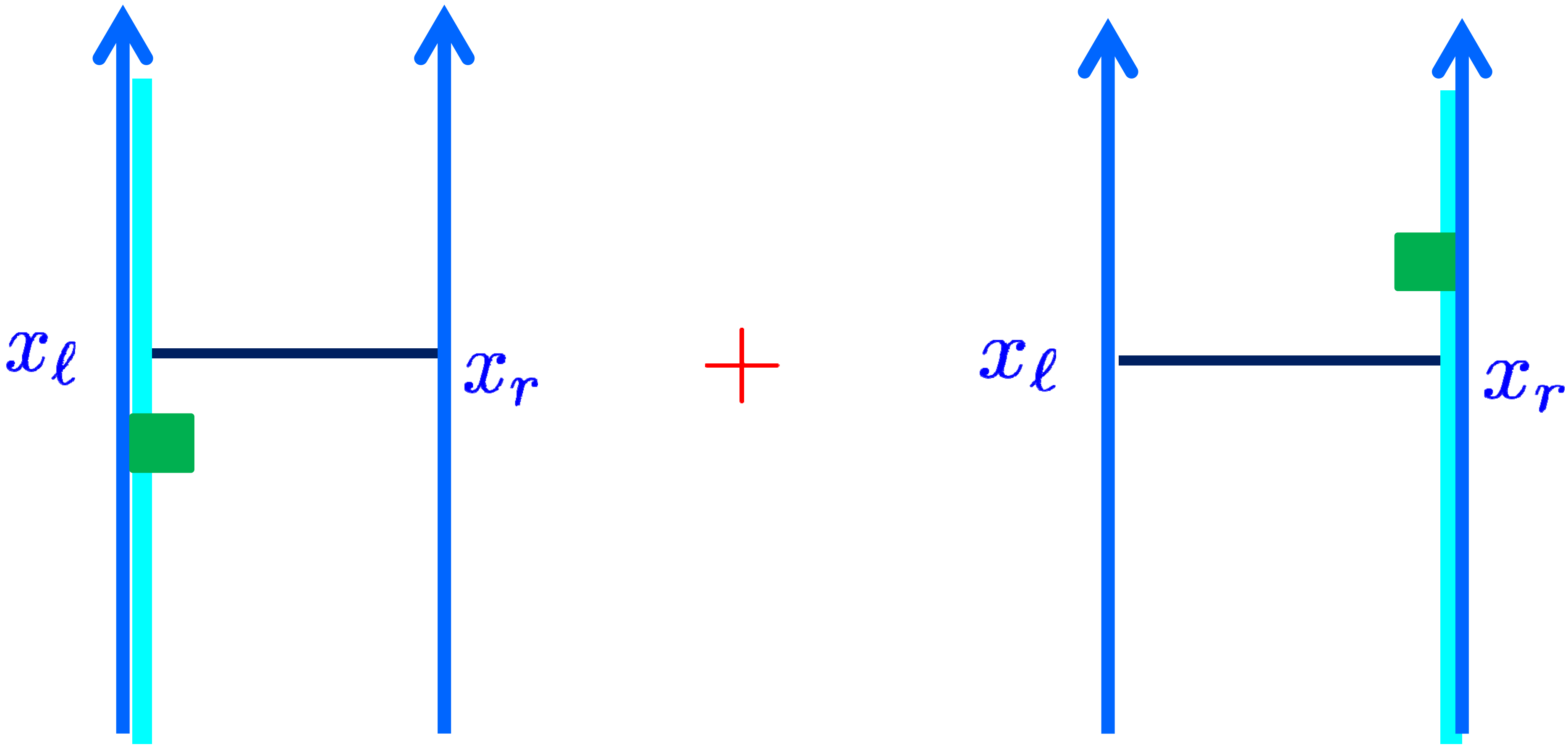
$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i} \otimes \mathbb{M}_{i,\mathcal{L}_r}$$

The Witten index factorizes nicely: $\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$

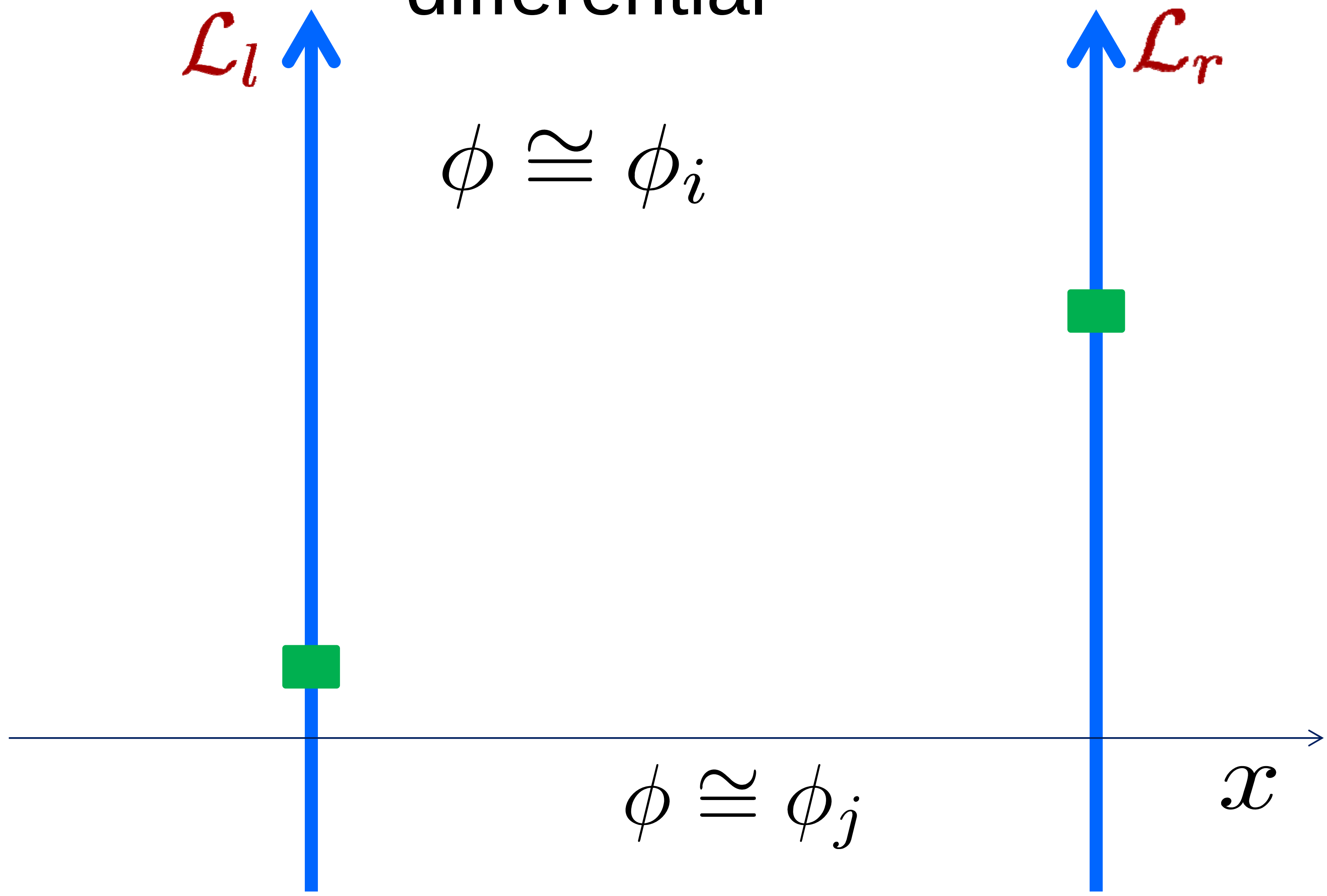
But the differential $d_{\mathcal{L}_\ell,i} \otimes 1 + 1 \otimes d_{i,\mathcal{L}_r}$

is too naïve !

$$\sum_i \left(d_{\mathcal{L}_\ell,i} \otimes 1 + 1 \otimes d_{i,\mathcal{L}_r}\right)$$



Instanton corrections to the naïve differential



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The Boosted Soliton - 1

We are interested in the ζ-instanton equation for a fixed generic ζ

We can still use the soliton to produce a solution for phase ζ

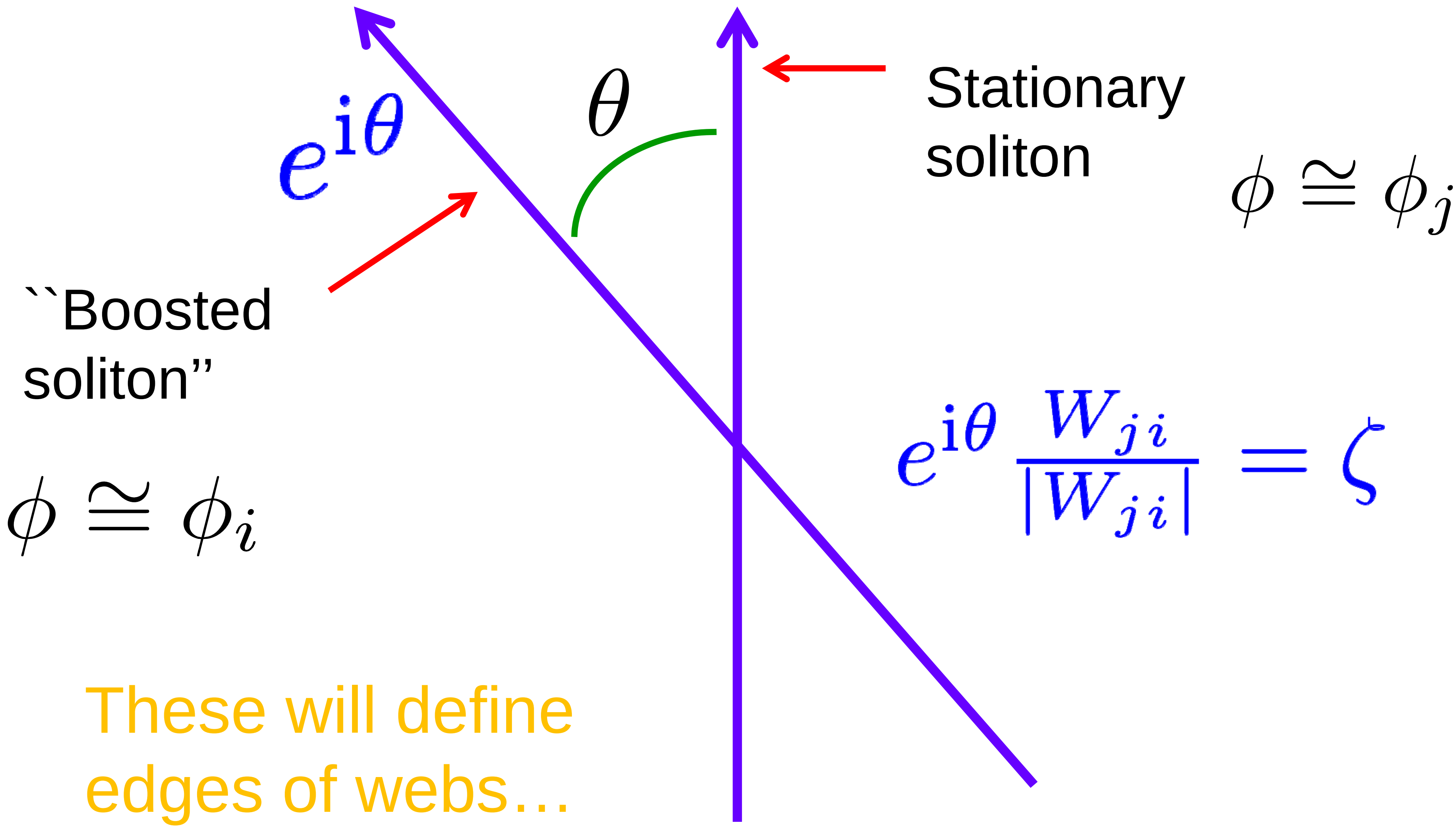
$$\phi_{ij}^{\rm inst}(x,\tau) := \phi_{ij}^{\rm sol}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + {\rm i}\frac{\partial}{\partial \tau}\right)\phi_{ij}^{\rm inst} = e^{{\rm i}\theta}\zeta_{ji}\frac{\partial \bar W}{\partial \bar\phi}$$

Therefore we produce a solution of the instanton equation with phase ζ if

$$\zeta = e^{{\rm i}\theta}\zeta_{ji} \qquad \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

The Boosted Soliton -2



These will define edges of webs…

The Boosted Soliton - 3

Put differently, the stationary soliton in ***Minkowski*** space preserves the supersymmetry:

$$Q_+ - \zeta_{ij}^{-1}\overline{Q_-}$$

So a boosted soliton preserves supersymmetry :

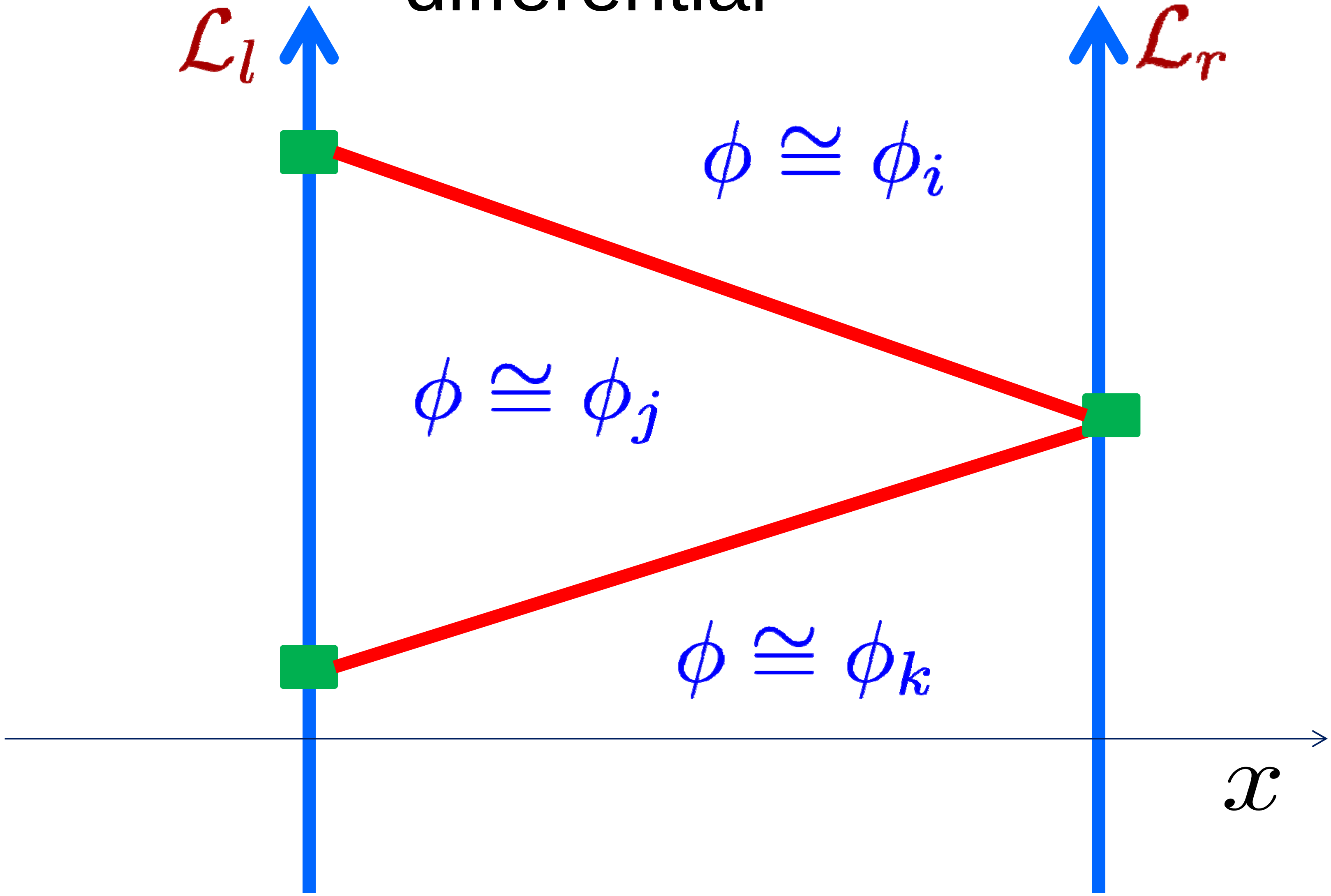
$$e^{\beta/2}Q_+ - \zeta_{ij}^{-1}e^{-\beta/2}\overline{Q_-}$$

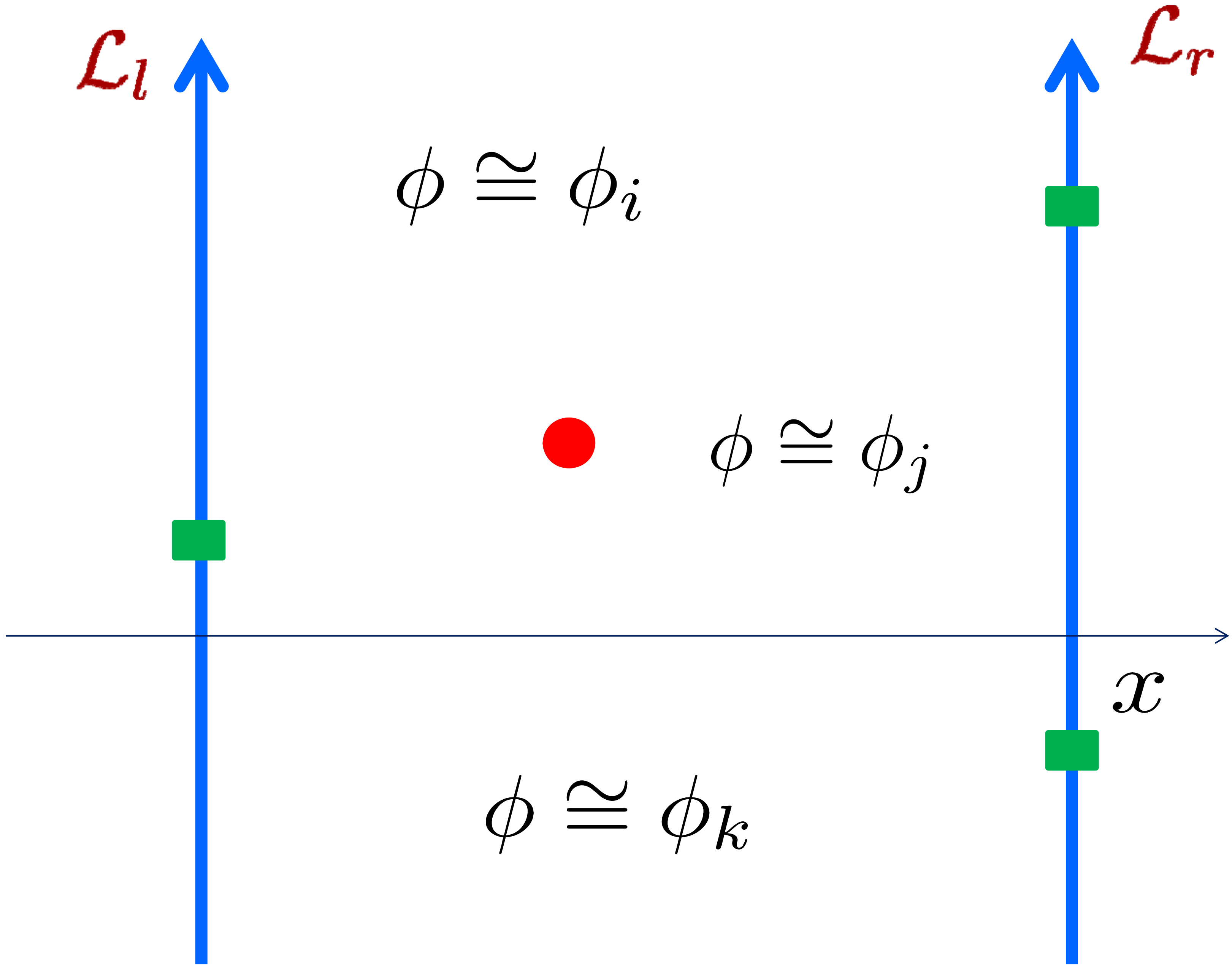
β is a real boost. In ***Euclidean*** space this becomes a rotation:

$$e^{i\theta/2}Q_+ - \zeta_{ij}^{-1}e^{-i\theta/2}\overline{Q_-}$$

And for suitable θ this will preserve ζ-susy

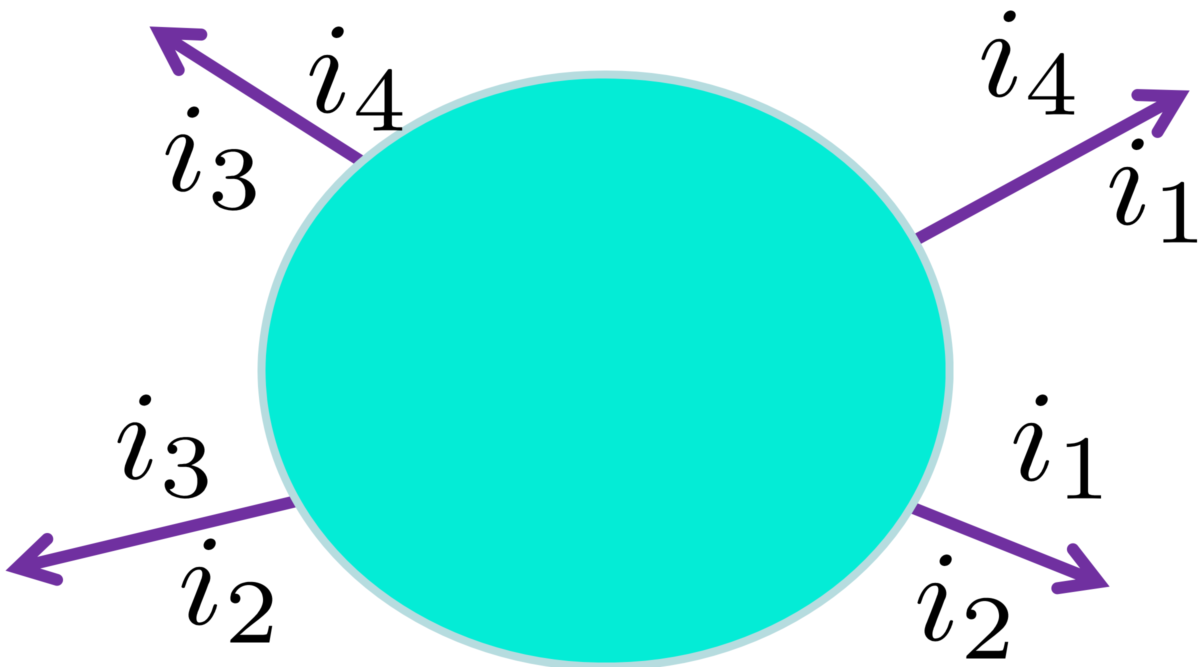
More corrections to the naïve differential





$\mathcal{L}_l$
$\mathcal{L}_r$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
x
$\phi \cong \phi_k$

Path integral on a large disk



Choose boundary conditions preserving ζ-supersymmetry:

Consider a cyclic ``fan of solitons''

$$\mathcal{F} = \{\phi^{\mathrm{inst}}_{i_1 i_2}, \cdots, \phi^{\mathrm{inst}}_{i_n i_1}\}$$

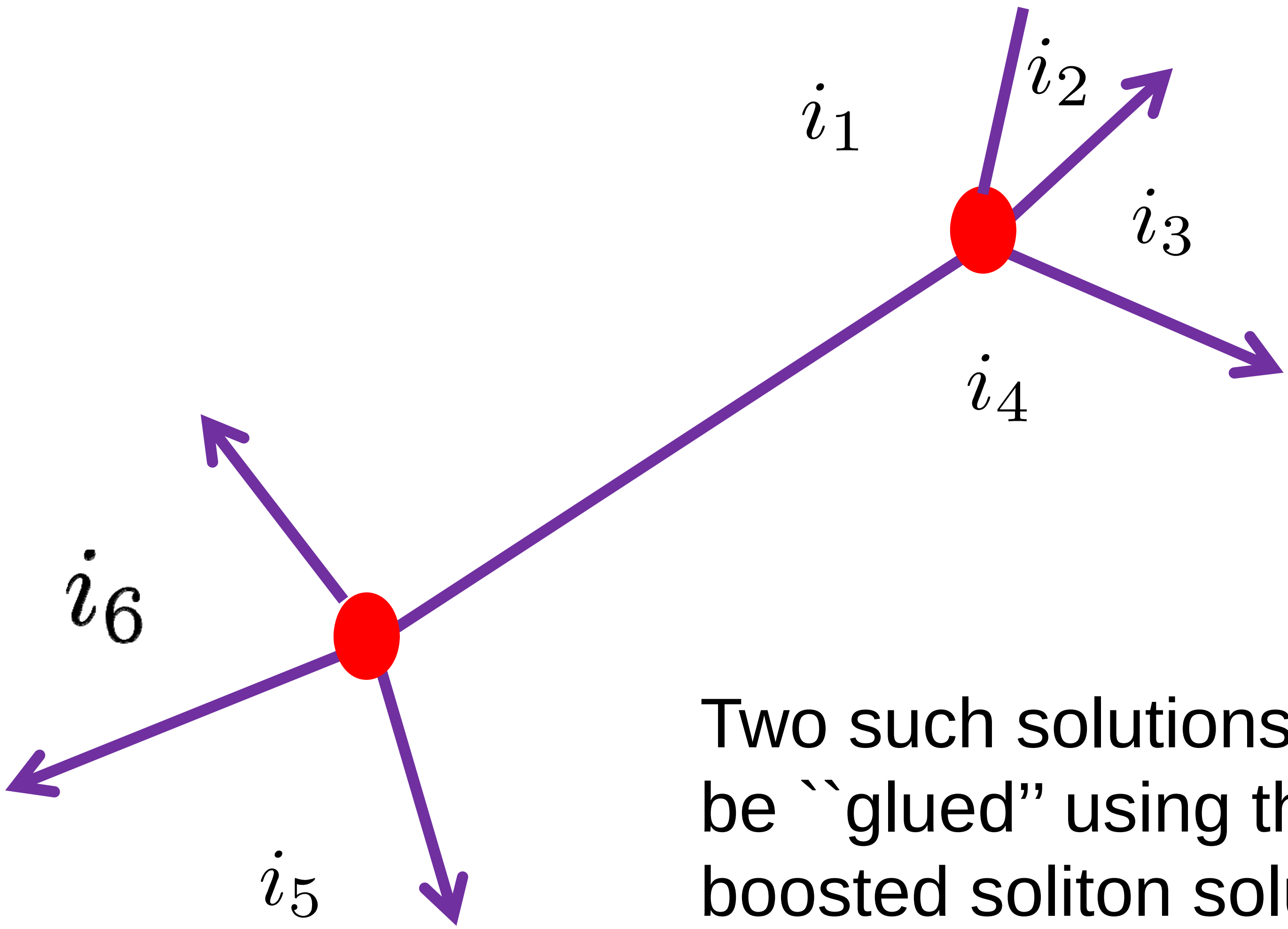
Localization

The path integral of the LG model with these boundary conditions (with A-twist) localizes on moduli space of ζ-instantons:

$$\mathcal{M}(\mathcal{F})$$

We assume the mathematically nontrivial statement that, when the index of the Dirac operator (linearization of the instanton equation) is positive then the moduli space is nonempty.

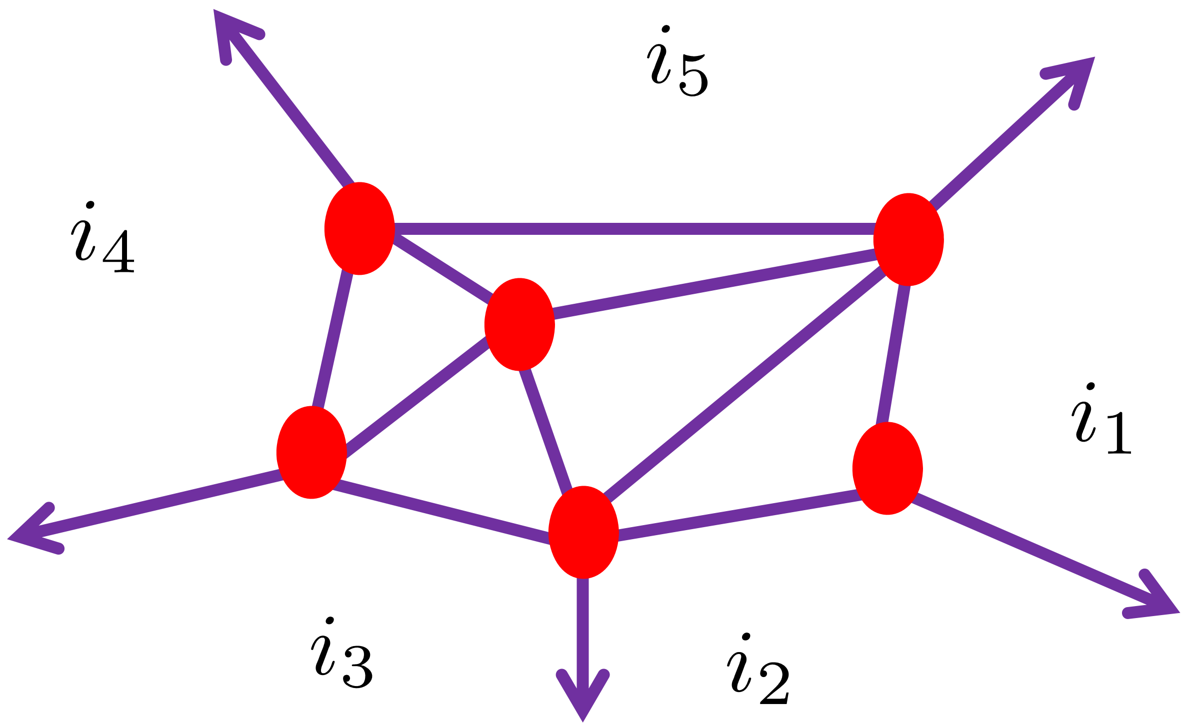
Gluing



Two such solutions can be ``glued'' using the boosted soliton solution -

Ends of moduli space

This moduli space has several "ends" where solutions of the ζ-instanton equation look like



We call this picture a ζ - *web*: $\mathfrak{w}$

ζ-Vertices

The red vertices represent solutions from the compact and connected components of

$$\mathcal{M}(\mathcal{F})$$

The contribution to the path integral from such components are called ``interior amplitudes.'' In the A-model for the zero-dimensional moduli spaces they count (with signs) the solutions to the ζ-instanton equation.

Path Integral With Fan Boundary Conditions

Just as in the Morse theory proof of $d^2=0$ using ends of moduli space corresponding to broken flows, here the broken flows correspond to webs $\mathfrak{w}$

Label the ends of $\mathcal{M}(\mathcal{F})$ by webs $\mathfrak{w}$. Each end contributes $\Psi(\mathfrak{w})$ to the path integral:

The total wavefunction is Q-invariant

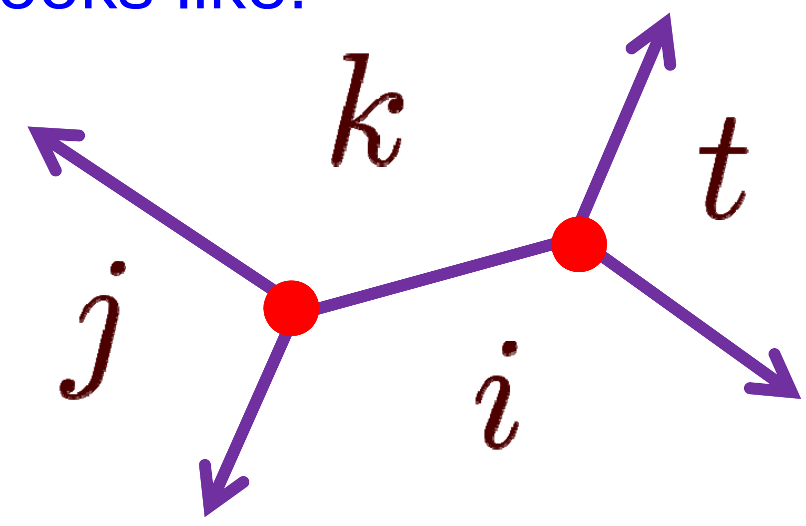
$$Q \sum_{\mathfrak{w}} \Psi(\mathfrak{w}) = 0$$

The wavefunctions $\Psi(\mathfrak{w})$ are themselves constructed by gluing together wavefunctions $\Psi(\mathfrak{r})$ associated with ζ-vertices $\mathfrak{r}$

⟹ L_∞ identities on the interior amplitudes

Example:

Consider a fan of vacua {i,j,k,t}. One end of the moduli space looks like:



The red vertices are path integrals with rigid webs. They have amplitudes β_{ikt} and β_{ijk}.

$$\mathcal{M} = \mathbb{R}^2_{transl} \times \mathbb{R}^+_{scale} \quad ?$$

Ends of Moduli Spaces in QFT

In LG theory (say, for X= $\mathbb{C}^n$) the moduli space cannot have an end like the finite bdy of $\mathbb{R}_+$

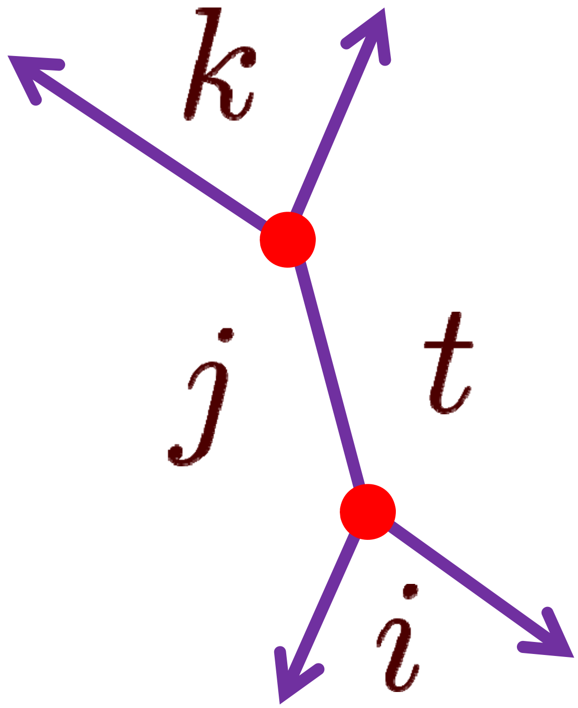
In QFT there can be three kinds of ends to moduli spaces of the relevant PDE's:

UV effect: Example: Instanton shrinks to zero size; bubbling in Gromov-Witten theory

Large field effect: Some field goes to ∞

Large distance effect: Something happens at large distances.

None of these three things can happen at the finite boundary of $\mathbb{R}_+$. So, there must be another end:



Amplitude: $\beta_{jkt}\beta_{ijt}$

The boundaries where the internal distance shrinks to zero must cancel leading to identities on the amplitudes like:

$$\beta_{ijk}\beta_{ikt} - \beta_{jkt}\beta_{ijt} = 0$$

This set of identities turns out to be the Maurer-Cartan equation for an L_∞ - algebra.

This is really a version of the argument for $d^2 = 0$ in SQM.

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Overview of Results; Some Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- More about motivation from knot homology
- More about motivation from spectral networks

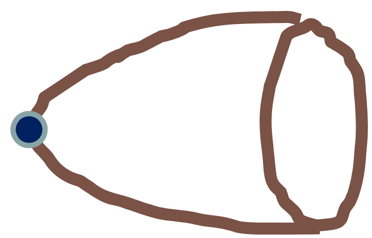
Knot Homology -1/5

(Approach of E. Witten, 2011)

Study (2,0) superconformal theory based on Lie algebra $\mathfrak{g}$

$$\mathbb{R} \times M_3 \times D$$

TIME

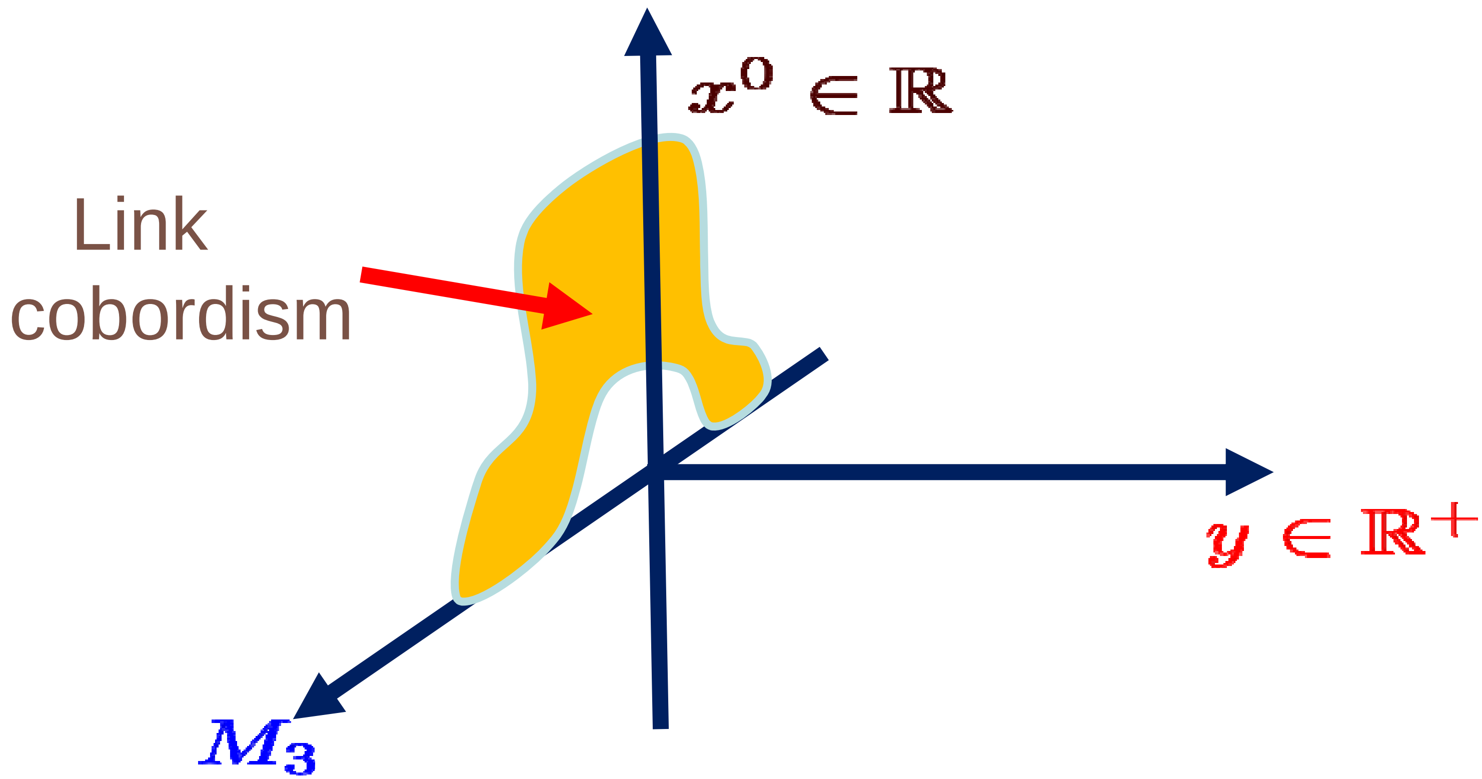
D: p 

M_3: 3-manifold containing a surface defect at $\mathbb{R}$ x L x {p}

More generally, the surface defect is supported on a link cobordism $L_1 \rightarrow L_2$:

Knot Homology – 2/5

Now, KK reduce by U(1) isometry of the cigar D with fixed point p to obtain 5D SYM on
$\mathbb{R} \times M_3 \times \mathbb{R}_+$



Knot Homology – 3/5

Hilbert space of states depends on M_3 and L:

$$\mathcal{H}_{\mathrm{BPS}}(M_3, L)$$

is identified with the knot homology of L in M_3.

This space is constructed from a chain complex using infinite-dimensional Morse theory on a space of gauge fields and adjoint-valued differential forms.

Knot Homology 4/5

Equations for the semiclassical states generating the MSW complex are the Kapustin-Witten equations for gauge field with group G and adjoint-valued one-form ϕ on the four-manifold $M_4 = M_3 \times \mathbb{R}^+$

$$F - \phi^2 + t(d_A\phi)^+ - t^{-1}(d_A\phi)^- = 0$$

$$d_A * \phi = 0$$

Boundary conditions at y=0 include Nahm pole and extra singularities at the link L involving a representation $R^\vee$ of the dual group.

Differential on the complex comes from counting ``instantons'' – solutions to a PDE in 5d written by Witten and independently by Haydys.

Knot Homology 5/5

In the case $M_3 = C \times \mathbb{R}$ with coordinates (z, x^1) these are precisely the equations of a ***gauged Landau-Ginzburg model*** defined on 1+1 dimensional spactime (x^0,x^1) with target space

$$\mathcal{X} : \mathcal{A} = A + i\phi \quad \tilde{M}_3 := C \times \mathbb{R}_+$$

$$\mathcal{G} = \mathrm{Map}(\tilde{M}_3, G^c)$$

$$W(\mathcal{A}) = \int_{\tilde{M}_3} \mathrm{Tr}(\mathcal{A} d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3)$$

Gaiotto-Witten showed that in some situations one can reduce this model to an ungauged LG model with finite-dimensional target space.

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Theories of Class S

(Slides 73-87 just a reminder for experts.)

Begin with the (2,0) superconformal theory based on Lie algebra $\mathfrak{g}$

Compactify (with partial topological twist) on a Riemann surface C with codimension two defects D inserted at punctures $\mathfrak{s}_n \in$ C.

Get a four-dimensional QFT with d=4 N=2 supersymmetry S[$\mathfrak{g}$,C,D]

Coulomb branch of these theories described by a Hitchin system on C.

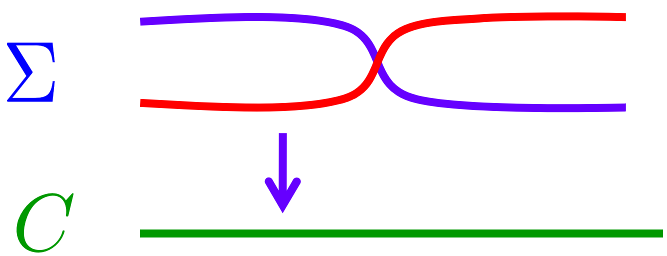
Seiberg-Witten Curve

UV Curve

$$\Sigma : \det(\lambda - \varphi(z,\bar{z})) = 0 \subset T^*C$$

$\lambda = pdq$ $\quad \lambda|_{\Sigma}$ SW differential

For $\mathfrak{g}$ =su(K) $\quad \pi : \Sigma \to C$



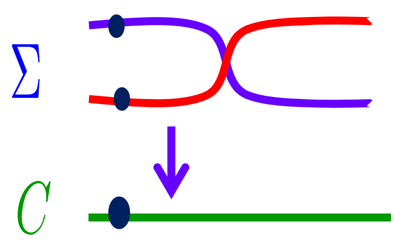
is a K-fold branched cover

$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

Canonical Surface Defect in $S[\mathfrak{g}, C, D]$

For $z \in C$ we have a ***canonical surface defect*** $\mathbb{S}_z$

It can be obtained from an M2-brane ending at $x^1 = x^2 = 0$ in $\mathbb{R}^4$ and z in C

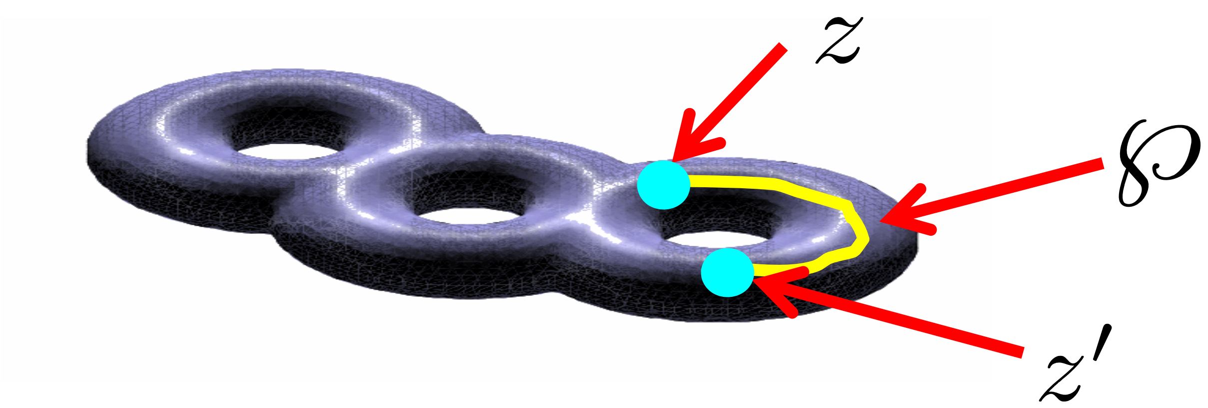


This is a 1+1 dimensional QFT localized at $(x^1, x^2) = (0,0)$ coupled to the ambient four-dimensional theory. In some regimes of parameters it is well-described by a Landau-Ginzburg model.

In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve over z.

Susy interfaces for S[$\mathfrak{g}$,C,D]

Interfaces between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$ are labeled by ***open paths*** $\wp$ on C

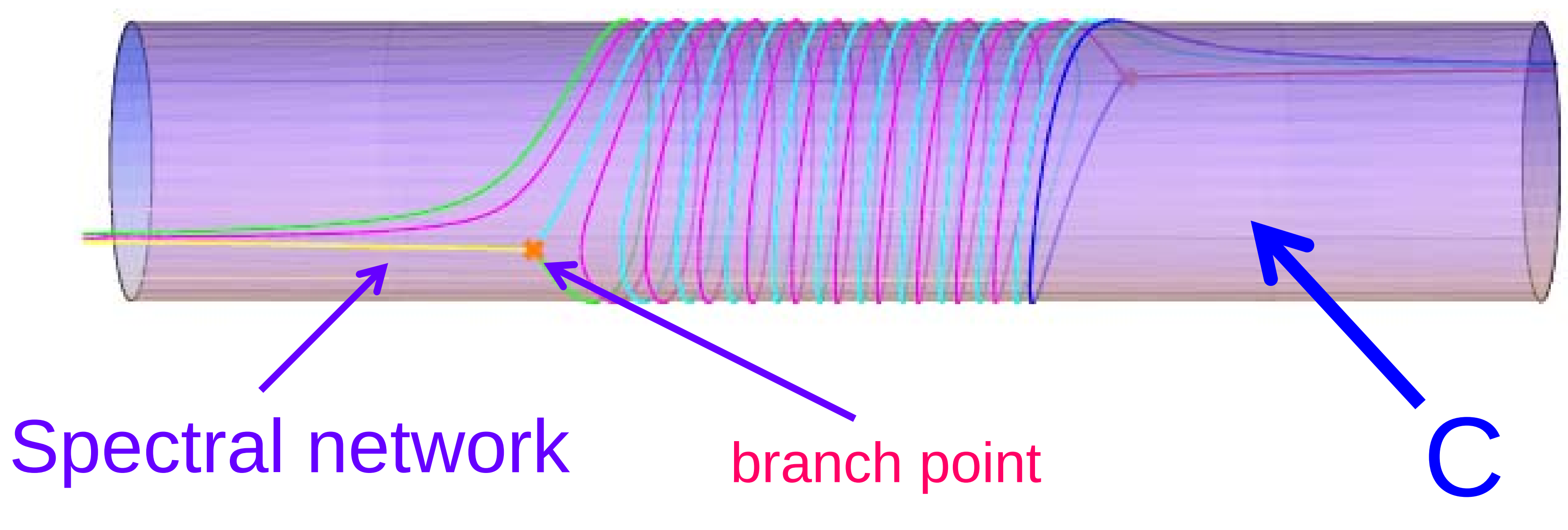


$L_{\wp,\vartheta}$ only depends on the homotopy class of $\wp$

Spectral networks

(D. Gaiotto, G. Moore, A. Neitzke)

Spectral networks are combinatorial objects associated to a branched covering of Riemann surfaces $\Sigma \longrightarrow C$



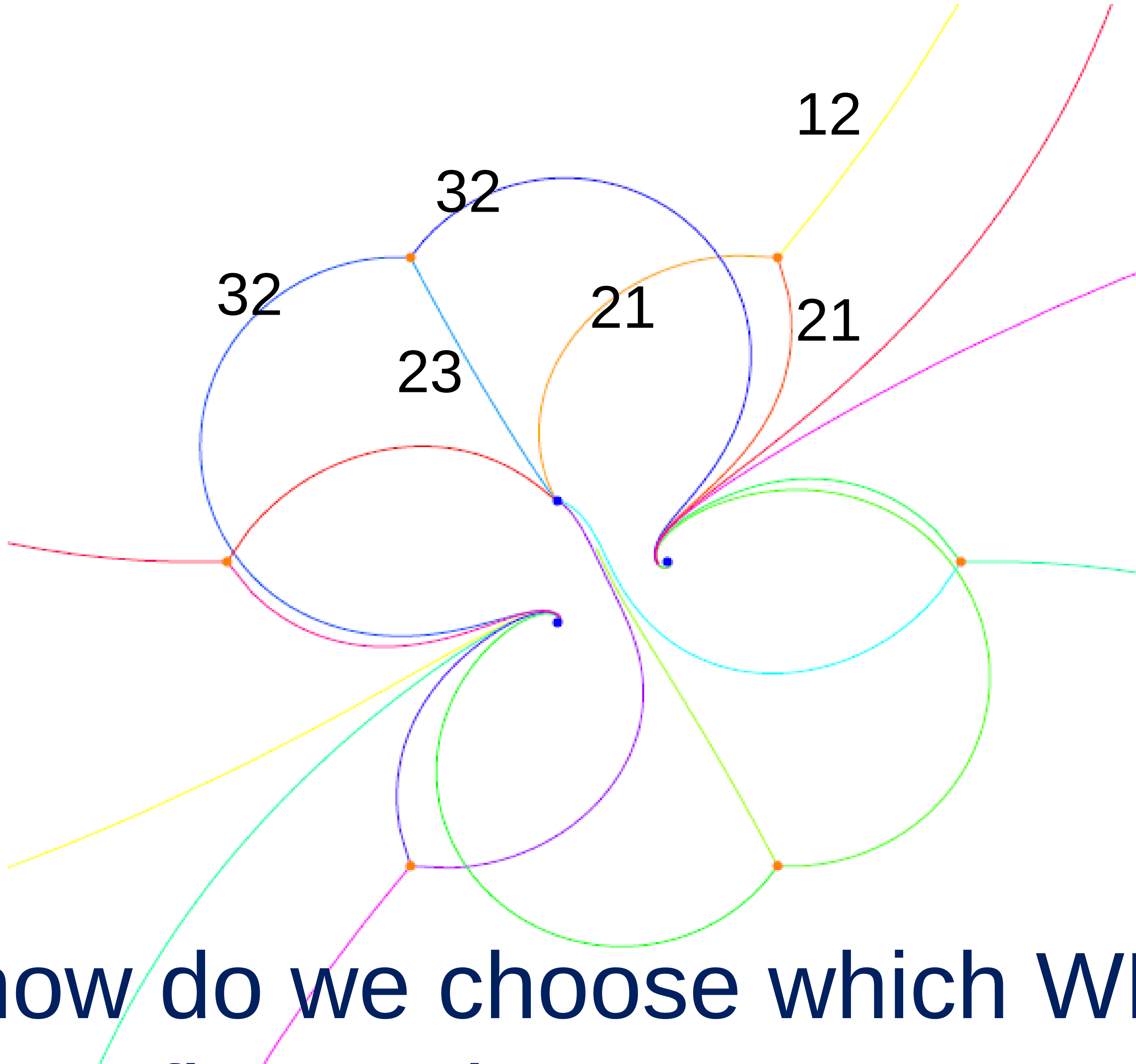
S-Walls

Spectral network $\mathcal{W}_\vartheta$ of phase ϑ is a graph in C.

Edges are made of WKB paths:

$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

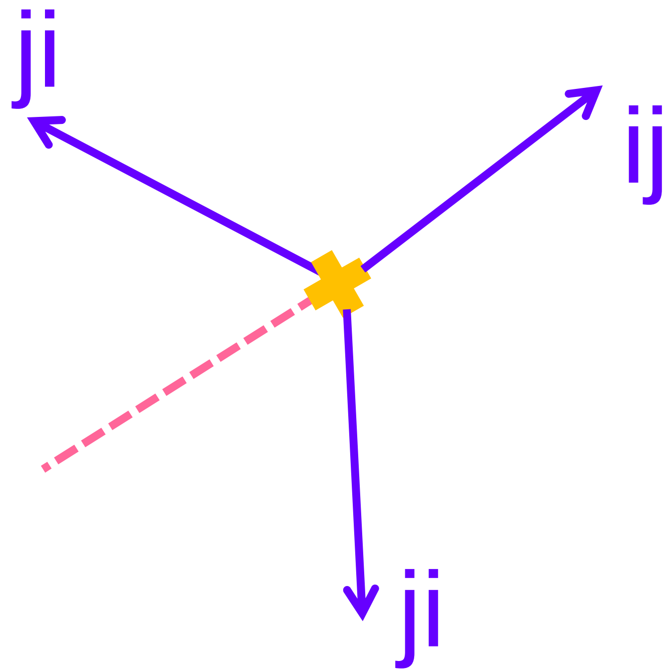
The path segments are ``***S-walls of type ij***''



But how do we choose which WKB paths to fit together?

Evolving the network -1/3

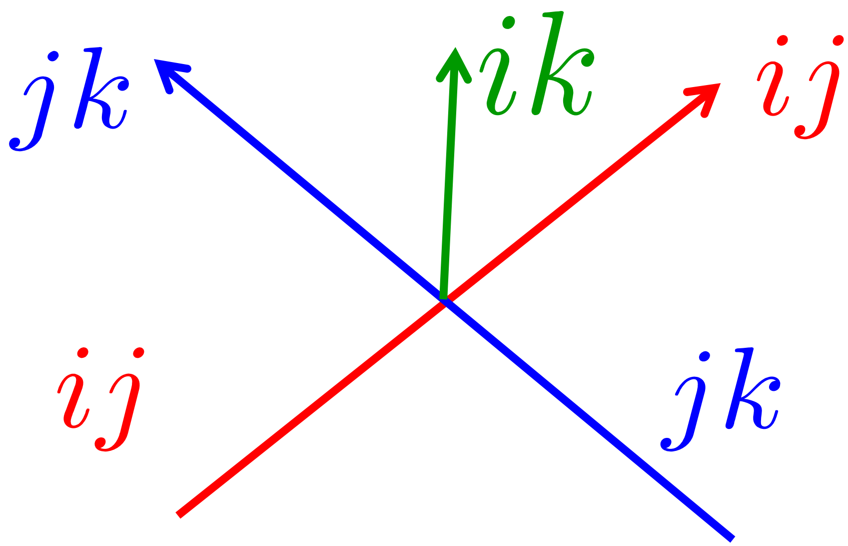
Near a (simple) branch point of type (ij):

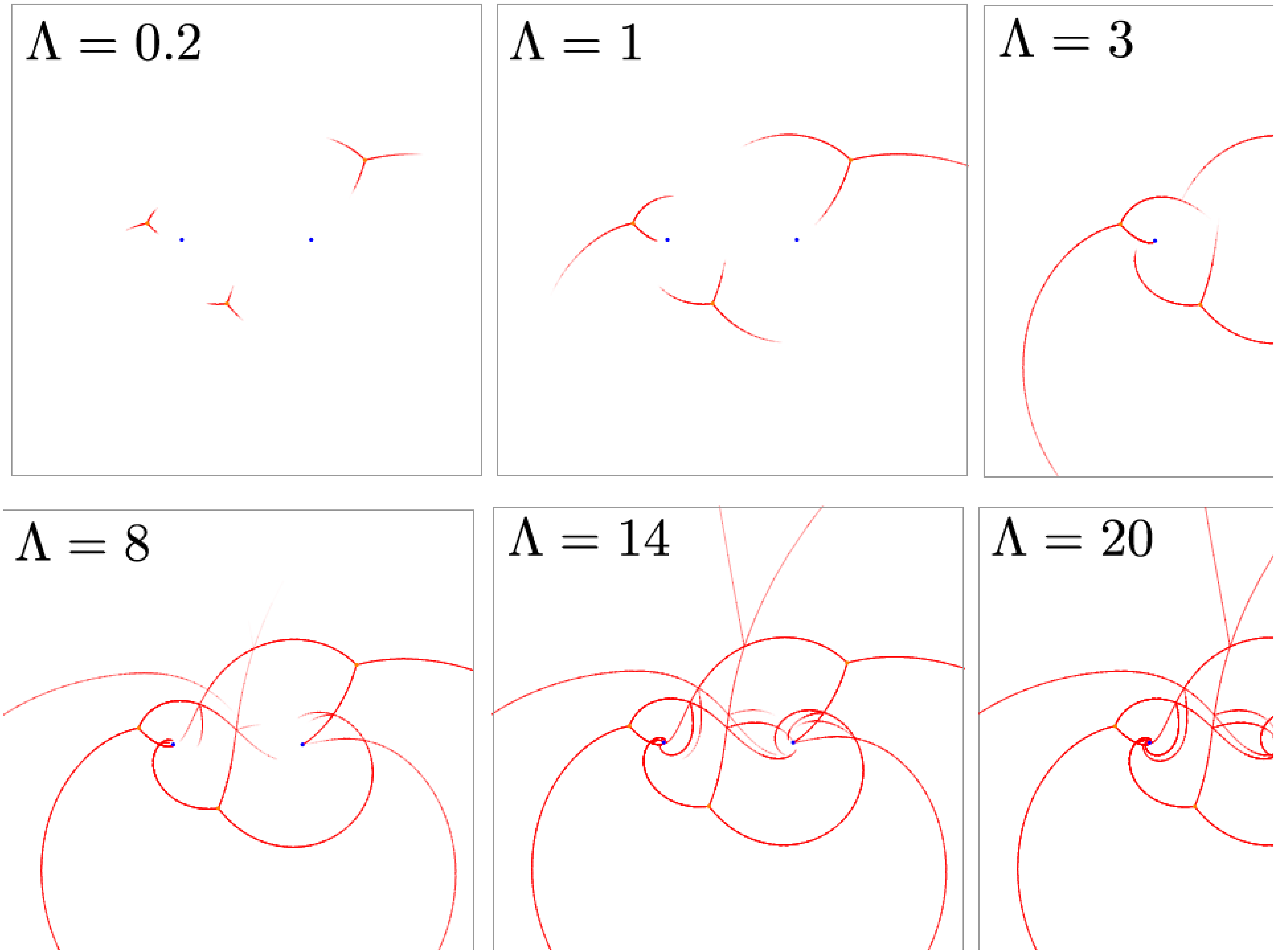


$$\int \lambda_i - \lambda_j \sim z^{3/2}$$

Evolving the network -2/3

Evolve the differential equation.
There are rules for how to continue when S-walls intersect. For example:

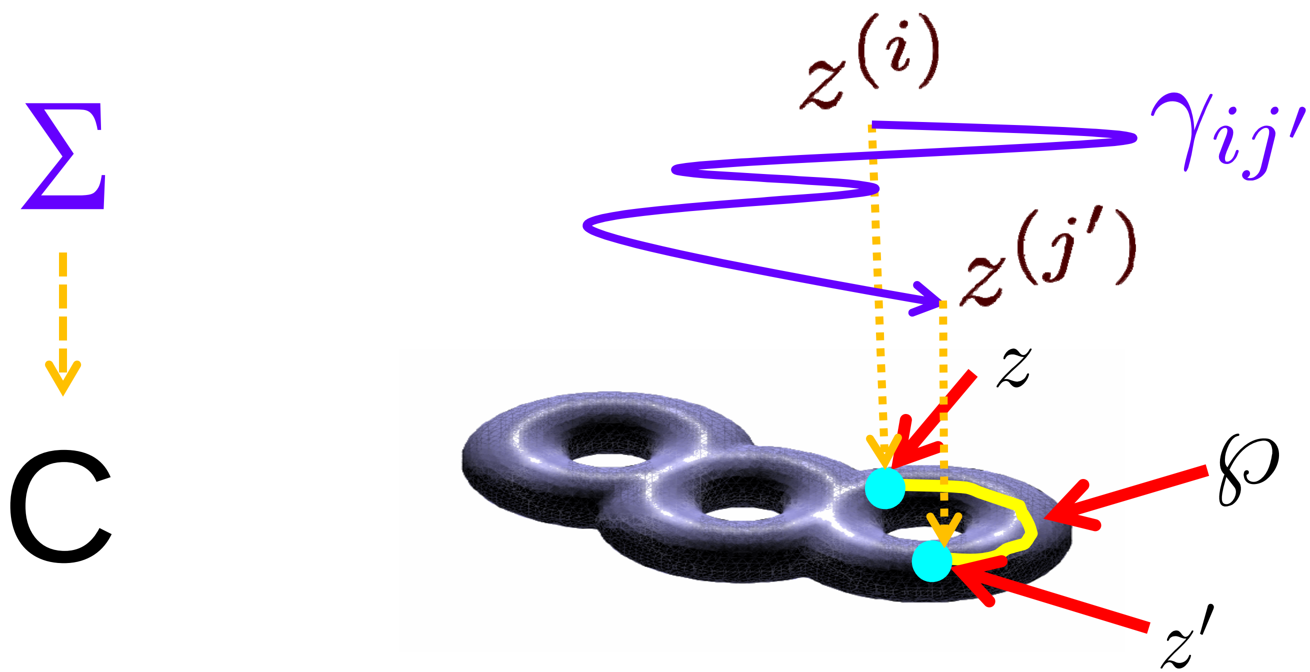



Λ = 0.2
Λ = 1
Λ = 3
Λ = 8
Λ = 14
Λ = 20

Formal Parallel Transport

Introduce the generating function of framed BPS degeneracies:

$$F(\wp,\vartheta) := \sum_{\Gamma_{ij'}} \overline{\underline{\Omega}}(L_{\wp,\vartheta},\gamma_{ij'}) X_{\gamma_{ij'}}$$



Homology Path Algebra

To any relative homology class

$a \in H_1(\Sigma,\{x_i, x_{j'}\}; \mathbb{Z})$ assign X_a

$$X_a X_b := \begin{cases} X_{a+b} & a, b \text{ composable} \\ 0 & \text{else} \end{cases}$$

X_a generate the "homology path algebra" of Σ

Four Defining Properties of F

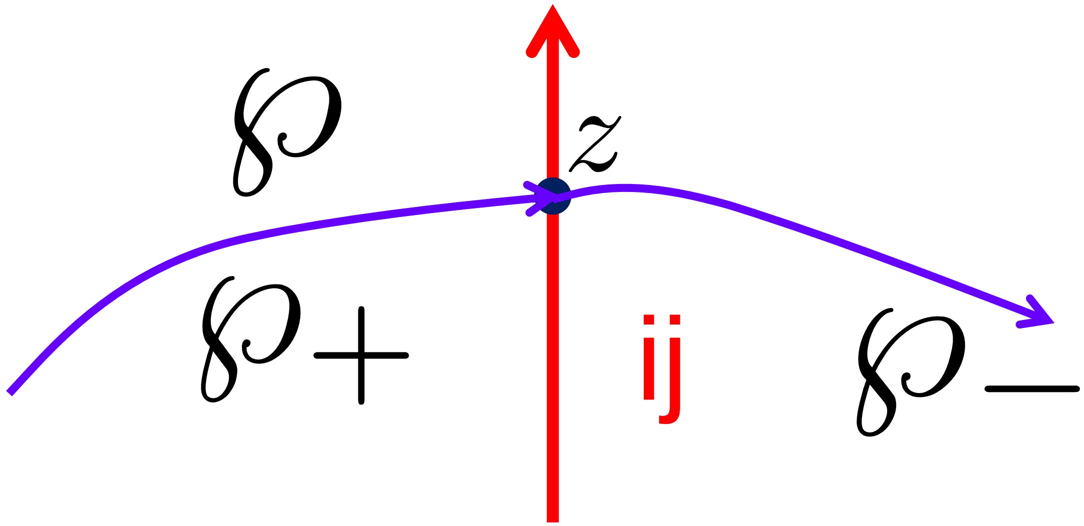
1 $$F(\wp,\vartheta)F(\wp',\vartheta) = F(\wp\wp',\vartheta)$$

2 Homotopy invariance $$F(\wp_1,\vartheta) = F(\wp_2,\vartheta)$$

3 If $\wp$ does NOT intersect $\mathcal{W}_\vartheta$: $$F(\wp,\vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}}$$

4 If $\wp$ DOES intersect $\mathcal{W}_\vartheta$: ``Wall crossing formula"

Wall Crossing for $\overline{\Omega}(L_{\wp,\vartheta}, a)$



$$F(\wp,\vartheta) = \sum_{s=1}^{K} X_{\wp^{(s)}}$$

$$+\sum_{\gamma_{ij}} \mu(\gamma_{ij}) X_{\wp_+^{(i)}} X_{\gamma_{ij}} X_{\wp_-^{(j)}}$$

Theorem: These four conditions completely determine both F($\wp$,ϑ) and μ

One can turn this formal transport into a rule for pushing forward a flat GL(1,$\mathbb{C}$) connection on Σ to a flat GL(K,$\mathbb{C}$) connection on C.

``Nonabelianization map''

We will want to categorify the parallel transport F($\wp$,ϑ) and the framed BPS degeneracies: $\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a)$

The next three lectures will be in a very different style:

On the blackboard.

Slower and more detailed.

The goal is to explain the mathematical ``web-based formalism" for addressing the physical problems outlined above.

No physics voodoo.