N=2* SYM, Four Manifold Invariants, And Mock Modularity

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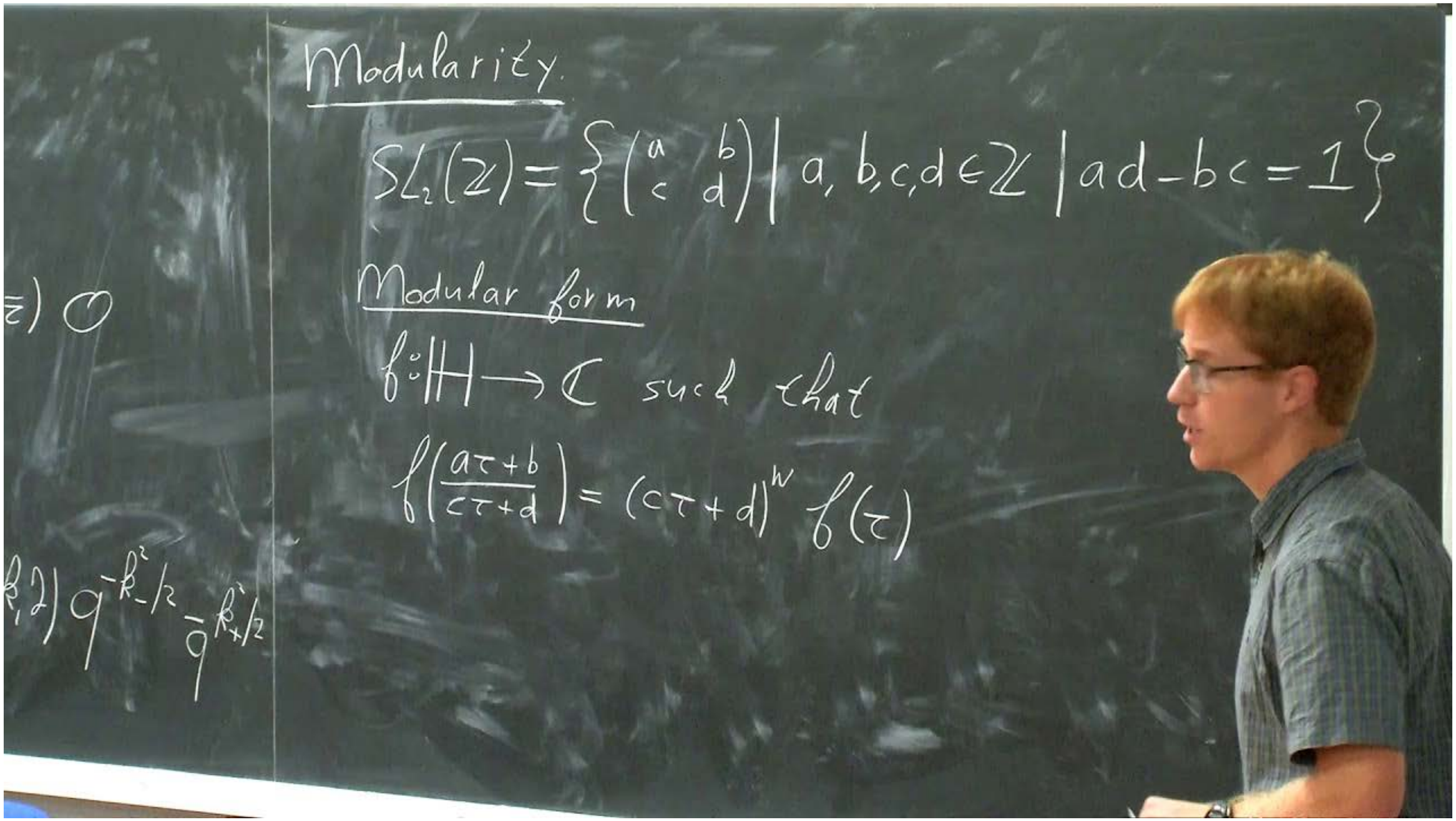
KITP - Modularity
December 3, 2020

Work with JAN MANSCHOT



Project has taken a few years. I first spoke about it at Simons Foundation Conference, Sept. 2017(run by Jeff and Shamit)

1 Introduction & Preliminaries

2 Summary Of Main Claims

3 The N=2* Theory: UV Meaning Of Invariants

4 Remarks On S-Duality Orbits Of Partition Functions

5 Coulomb Branch Integral: Measure & Evaluation

6 LEET Near Cusps & Explicit Results

Preliminaries

X: d=4, Smooth, compact, oriented, $\partial X = \emptyset$.

$b_2^+(X)$: Odd & positive

We study a TQFT on X

d=4 N=2* SYM. $G = SU(2), SO(3)$

The partition function generalizes both the Donaldson invariants and the Vafa-Witten invariants, and interpolates between them.

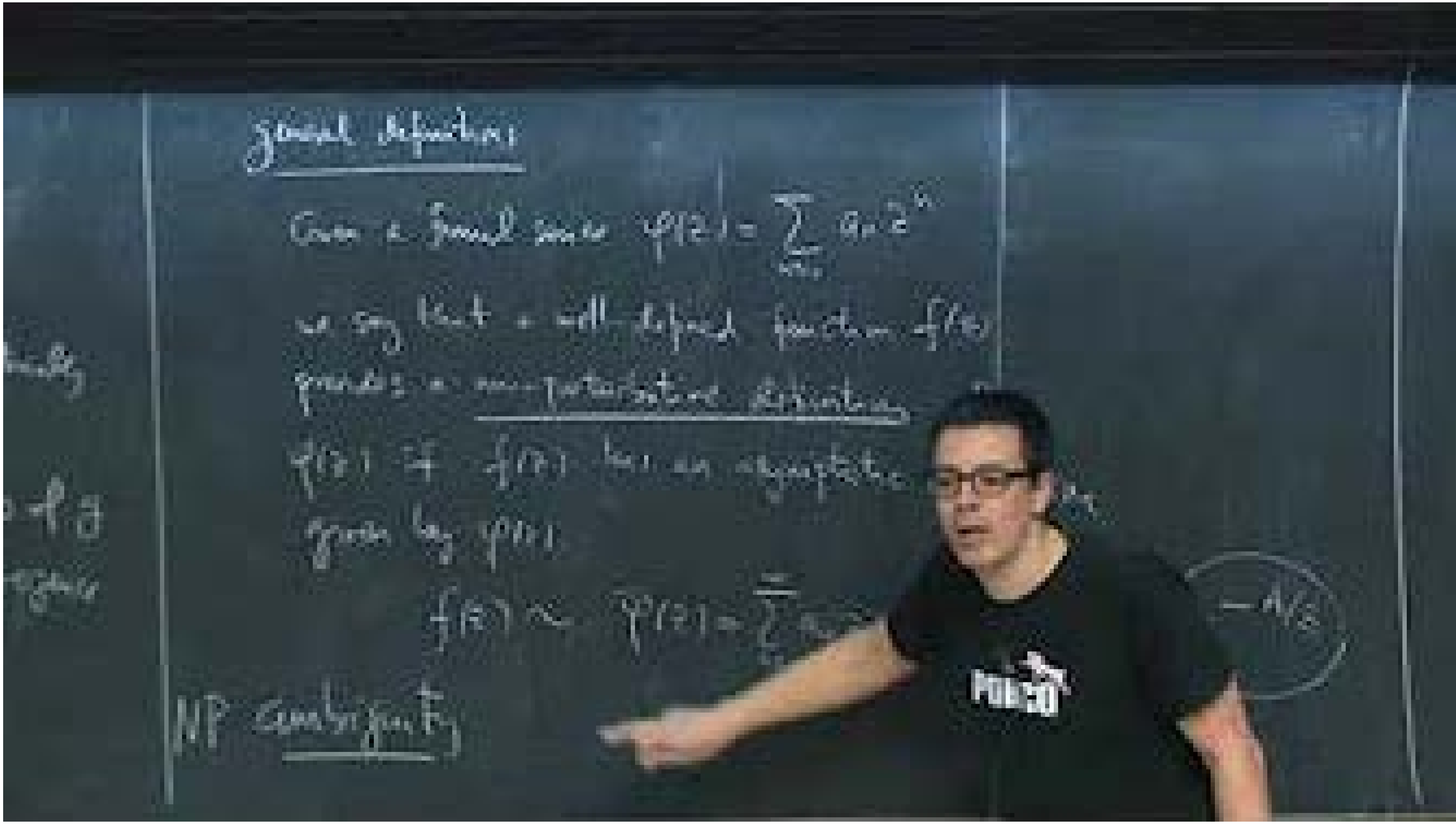
Preliminaries

The theory depends on a choice of background spin-c structure $\mathfrak{s}$.

Labastida-Marino [1995] noticed need to introduce $\mathfrak{s}$

The detailed dependence has not previously been discussed. Including it turns out to be nontrivial. We believe we have solved the problem completely.

Long, long, ago, at the ITP in 1998….


general definition
we say that a well-defined function
NP ambiguity

Preliminary: $Spin^c$-structure

$$Spin^c(4) := \{\ (u_1, u_2) | \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$$

$$1 \to U(1) \to Spin^c(4) \to SO(4) \to 1$$

Spin-c structure on X:

Reduction of structure group of TX to $Spin^c(4)$

``Spinors'': Associated rank 2 bundles $W^{\pm}$

$$c(\mathfrak{s}) := c_1(\det W^{\pm}) \in H^2(X;\mathbb{Z})$$

$$\ell = \frac{c(\mathfrak{s})^2 - 2\chi - 3\,\sigma}{8} \in \mathbb{Z}$$

Preliminary: $Spin^c$ & ACS

An ACS $\mathcal{J}$ defines a canonical spin-c structure $\mathfrak{s}(\mathcal{J})$:

Almost Complex Structure (ACS):
Reduction of structure group of TX to U(2)

$$Spin^c(4) := \{ (u_1, u_2) | \det(u_1) = \det(u_2) \} \subset U(2) \times U(2)$$

Use diagonal homomorphism $U(2) \rightarrow Spin^c(4)$.

For $c = c(\mathfrak{s})$ for $\mathfrak{s}$ an ACS we have

$$c^2 = 2\chi + 3\,\sigma \qquad \ell = 0$$

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Intro & Main Claims – 1/6

Data needed to formulate the partition function:

$$\tau_{uv} \in \mathcal{H} \, ; q_{uv} := e^{2\pi i \, \tau_{uv}}$$

$m \in \mathbb{C}$ Λ: UV scale $t := m/\Lambda$

(UV) Spin-c structure: $\mathfrak{s}$, $c_{uv} := c_1(\mathfrak{s}) \in H^2(X, \mathbb{Z})$

$$\nu \in H^2(X; \mathbb{Z}/2\mathbb{Z})$$

Intro & Main Claims – 2/6

Path integral defines a ``function''

$$Z_\nu(\tau_{uv}, c_{uv}, t) \colon H_*(X;\mathbb{Z}) \to \mathbb{C}$$

$$Z_\nu(x;\tau_{uv}, c_{uv}, t) = \sum_{k\geq 0} q_{uv}^k \int_{\mathcal{M}_{Q,k}} e^{\mu(x)}\; Eul(\mathcal{E}_{\mathfrak{s}}; t)$$

$\mathcal{M}_{Q,k}$: Moduli of nonabelian monopole connections on a principal $SO(3)$ bundle $P \to X$ with $\nu = w_2(P)$ and instanton no. $= k$

$$\mu \colon H_*(X,\mathbb{Z}) \to H^{4-*}\left(\mathcal{M}_{Q,k};\mathbb{Q}\right)$$

$\mathcal{E}_{\mathfrak{s}}$: U(1)-equivariant virtual bundle

Intro & Main Claims – 3/6

Special cases were studied in
[Moore & Witten 1997; Labastida & Lozano 1998]

Those studies were limited to spin manifolds
with trivial spin-c structure.

Related work: Vafa-Witten & Dijkgraaf, Park, Schroers
1998 N=1 deformation of N=4 SYM,
Kähler 4-folds with $b_2^+ \geq 3$ & no observables

Also related: Recent work of
Göttsche, Kool, Nakajima, and Williams

Physical Mass Limits

$$m \to 0$$

$$[N = 2^* \, SYM] \to [N = 4 \, SYM]$$

SW94:

$$m \to \infty \;\&\; q_{uv} \to 0$$

$$\Lambda_0^4 = 4\, m^4 q_{uv}$$

$$\Rightarrow pure\ SYM$$

Intro & Main Claims – 4/6

1A: For $\mathfrak{s} = \mathfrak{s}(\mathcal{J})$ and $t \to 0$

$$Z_\nu(x, \tau_{uv}, c_{uv}, t) \to Z_\nu^{VW}(\tau_{uv})$$

1B: For ANY spin-c structure,

$m \to \infty$ & $q_{uv} \to 0$ $with$ $\Lambda_0^4 := 4m^4 q_{uv}$ fixed:

$$Z_\nu^{renorm}(x, \tau_{uv}, c_{uv}, t) \to Z_\nu^{DW}(x)$$

What we mean by Z_ν^{renorm} is an interesting story best discussed later

Intro & Main Claims – 5/6

Central Claim:

Z_ν can be computed by studying an integral over Coulomb Branch = Base of Hitchin system =(this case: modular curve $\mathcal{H}/\Gamma(2) \cong \mathbb{C} - 3pt$)

2a: Writing a single-valued measure

$\Rightarrow$ implications for class S generalization

2b: ***Integrand*** is a total derivative of a mock Maass-Jacobi form.

2c: ***Value*** of the integral is a nonholomorphic completion of a mock modular form.

Intro & Main Claims – 6/6

For $b_2^+ > 1$ Z_ν is a linear combination of SW invariants with coefficients in a ring of modular forms for τ_{uv} and obeys the ``proper'' S-duality covariance

Today I will skip much of the physics background – See previous talks.

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``Equations Of Motion"

$A \in \mathcal{A}(P)$ $M \in \Gamma(W^{+} \otimes adP \otimes \mathbb{C})$

$W^{+} \to X$: Positive chirality rank two bundle associated to uv spin-c structure $\mathfrak{s}$

Q –fixed point equations (need Riemannian metric)

$F^{+} + [M, \bar{M}] = 0$ $\boldsymbol{D} M = 0$

``Nonabelian monopole/SW equations"

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

When $\mathfrak{s}$ is associated to an ACS these are equivalent to the Vafa-Witten equations.

Index Computations

$$v \dim \mathcal{M}_{Q,k} = \dim G \frac{c_{uv}^2 - (2\chi + 3\sigma)}{4}$$

N.B. Independent of instanton number k !

$$\dim \mathcal{M}_k = 8k - \frac{3}{2}(\chi + \sigma)$$

$$Index\ \boldsymbol{D} = -8k + \frac{3}{8}(c_{uv}^2 - \sigma)$$

$\Rightarrow$ Correlation functions on $H_*(X)$ infinite q_{uv} - series

Operators In The TQFT

$$\mathcal{O}: H_*(X, \mathbb{Z}) \to \quad Q - coho$$

$$p \in H_0(X;\mathbb{Z}) \qquad \mathcal{O}(p) = [\, Tr\, \phi^2(p)]$$

$$S \in H_2(X;\mathbb{Z}) \qquad \mathcal{O}(S) = [\int_S Tr(\phi F + \psi^2)]$$

What do these mean mathematically?

$U(1)_b$ Symmetry

$$F^+ + [M, \bar{M}] = 0 \qquad \boldsymbol{D} M = 0$$

$$U(1)_b \;:\; M \to e^{i\,\theta}\, M$$

$U(1)_b$ acts on the moduli space $\mathcal{M}_{Q,k}$ of these eqs.

Q- coho $\cong$ $H^*_{U(1)_b}(\mathcal{M}_{Q,k})$

$t = \frac{m}{\Lambda}$: $U(1)_b$ equivariant parameter

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

$$\mathcal{O}(x) \leftrightarrow \mu(x)$$

Generating Function Of Correlators

$$Z_\nu(x;\tau_{uv},c_{uv},t) := \langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*}$$

Q-symmetry: Path integral $\rightarrow \int_{\mathcal{M}_{Q,k}} \cdots$

$$\langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*} = \sum_{k\geq 0} q_{uv}^k \int_{\mathcal{M}_{Q,k}} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}};t)$$

$\mathcal{E}_{\mathfrak{s}}$: Obstruction bundle for elliptic complex

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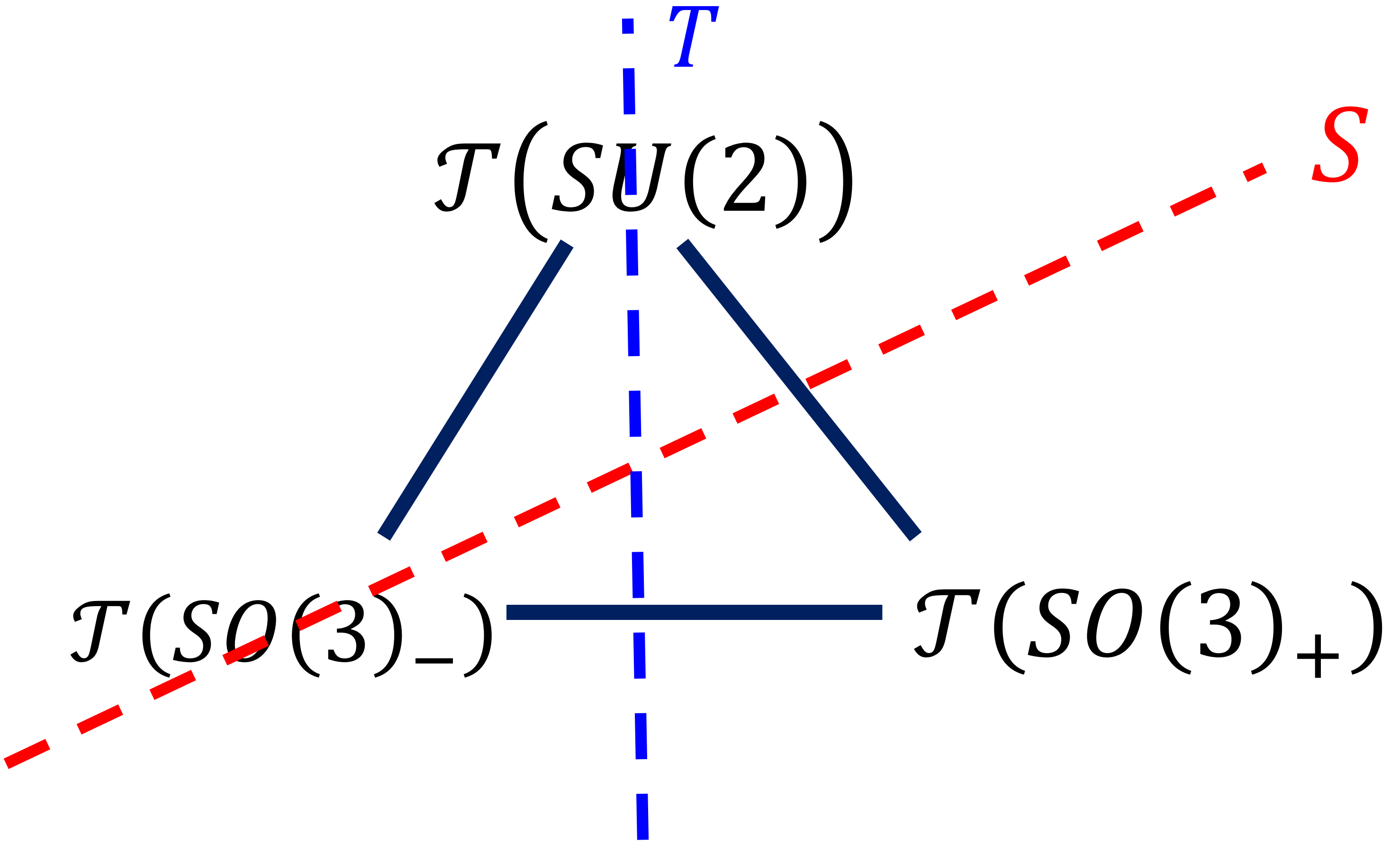
S-Duality

In the $SU(2)$ theory Z_ν is the partition function in the presence of 't Hooft flux $\nu \in H^2(X; \mathbb{Z}_2)$

The Z_ν span a vector space $\mathcal{V}$

But arbitrary linear combinations aren't physically meaningful

Three Distinct Theories



Gaiotto, Moore, Neitzke 2009;
Aharony, Seiberg, Tachikawa 2013

Partition Functions For The $SO(3)_\pm$ Theories

$$Z_\nu^{SO(3)_+} = \sum_\rho e^{i\,\pi\,\nu\cdot\rho} Z_\rho$$

$$Z_\nu^{SO(3)_-} = \sum_\rho e^{\frac{i\pi}{2}\rho^2 + i\pi\nu\cdot\rho} Z_\rho$$

$$\Delta S = \frac{i\pi}{2}\int P_2\big(w_2(P)\big)$$

Aharony, Seiberg, Tachikawa 2013

S-Duality Transformations

$$T: Z_\nu \to \xi_\nu \, Z_\nu$$

$$S: Z_\nu \to (-i\,\tau_0)^w \sum_\rho e^{i\,\pi\nu\cdot\rho} Z_\rho$$

$$w = -\frac{\chi}{2} - 4\ell \qquad \ell = \frac{c(\mathfrak{s})^2 - 2\chi - 3\,\sigma}{8}$$

$$\xi_\nu = \omega_{12}^{-\chi-2\ell}\,\omega_4^{-\nu^2}$$

Derivation from 6d ?

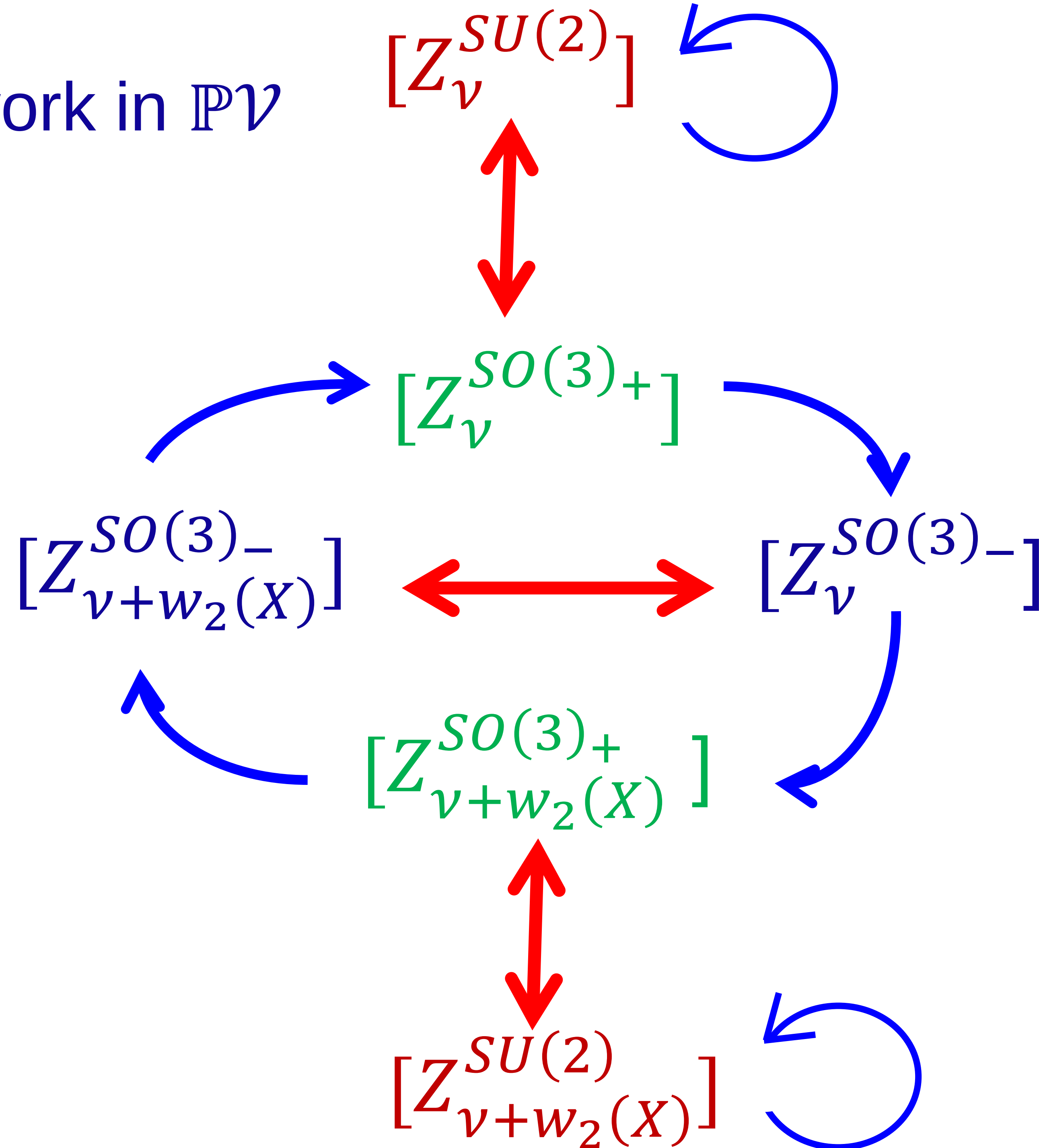
Orbit Of Partition Functions -1/2

The Z_ν span a vector space $\mathcal{V}$

The physical partition functions of the theories form an *orbit* in that vector space.

It is a finite covering of the triangle of theories.

For simplicity, work in $\mathbb{P}\mathcal{V}$



Full Modular Transformation Law

$$x = (p, S) \in H_0(X) \oplus H_2(X) \qquad \tau := \tau_{uv}$$

$$Z_\nu\left(\tilde{p}, \tilde{S}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w \sum_{\mu} B_{\mu,\nu}(\gamma) Z_\mu(p, S, \tau)$$

$$\tilde{S} = \frac{S}{(c\tau + d)^2}$$

$$\tilde{p} = \frac{1}{(c\tau + d)^2}\,(\,p - 2\pi i\, c\,(c\tau + d)S^2)$$

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Coulomb Branch Integral

In principle defined for general class S theory.

$$Z_{\nu}^{CB} = \int_{\mathcal{B}} du\, d\bar{u}\, \mathcal{H}\, \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: ***NOT holomorphic*** and ***metric- DEPENDENT***
``indefinite theta function''

Today: $u \in \mathbb{C} \cong \mathcal{B}$

5 Coulomb Branch Integral: Measure & Evaluation

5a Seiberg-Witten Review

5b Formulating The Measure And Integral

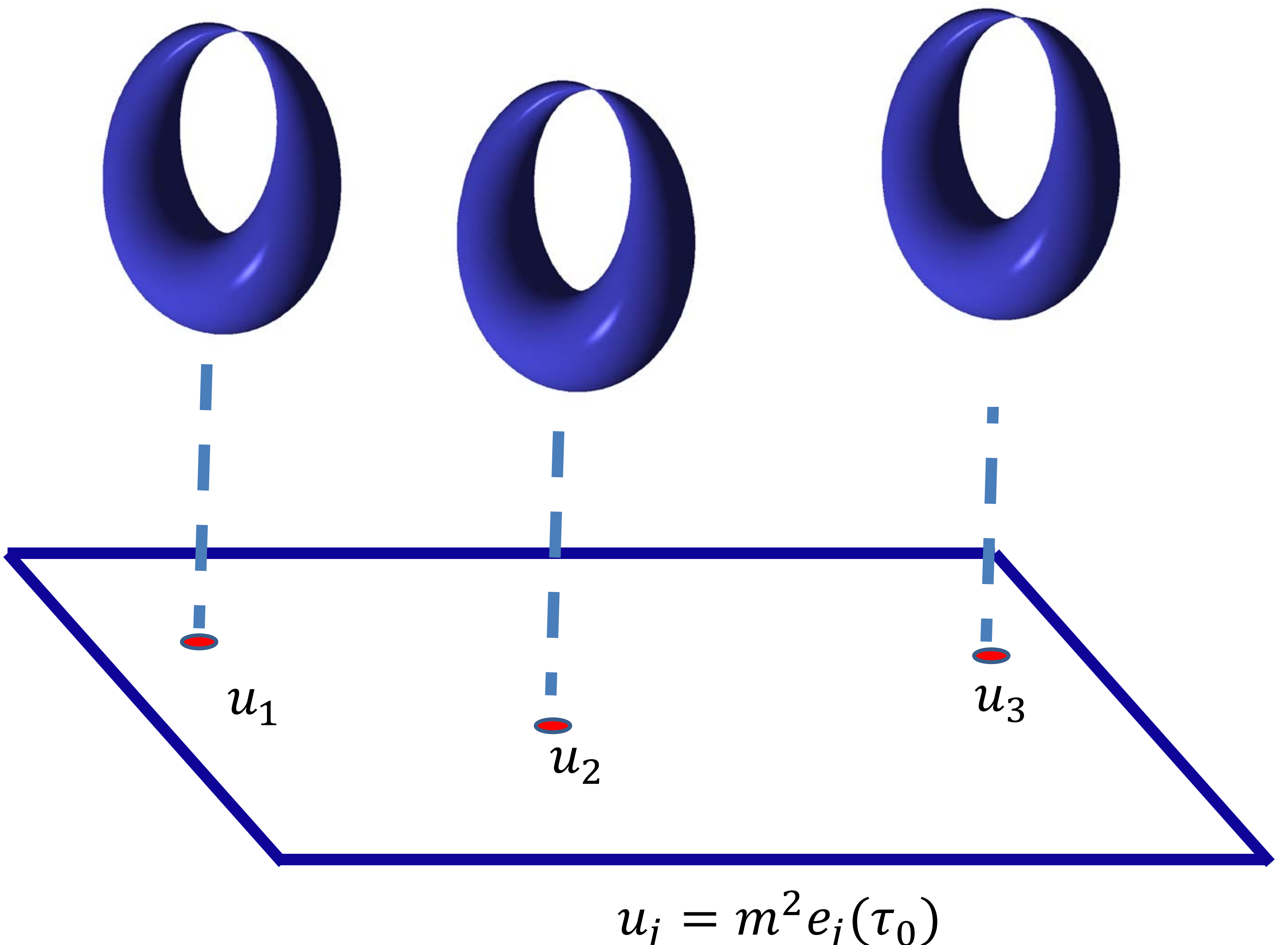
5c Evaluation Using Mock This & That

Seiberg-Witten Review – 1/6

$E_u \quad y^2 = \prod_{i=1}^{3} (x - \alpha_i) \qquad \alpha_i = u\, e_i(\tau_{uv}) + m^2 e_i(\tau_{uv})^2$

$e_i(\tau_{uv})$ half-periods of $E_{\tau_{uv}} = \mathbb{C}/(\mathbb{Z} + \tau_{uv}\mathbb{Z})$

Discriminant $\sim \eta^{24}(\tau_{uv}) \prod_{i=1}^{3} \left(u - m^2 e_i(\tau_{uv})\right)^2$


u_1
u_2
u_3
$u_j = m^2 e_j(\tau_0)$

Special Geometry

$H_1(E_u;\mathbb{Z})$: Fibers of a local system over $\mathcal{B}^*$

Definition: A ``duality frame'' is a local choice of A, B −cycles

Periods of λ define homomorphism $Z_u: H_1(E_u;\mathbb{Z}) \to \mathbb{C}$

$$a(u) := \oint_A \lambda \qquad a_D(u) := \oint_B \lambda$$

Fact: There is a locally holomorphic function $\mathcal{F}(a)$

$$a_D = \frac{d\mathcal{F}}{da}$$

$$\frac{da}{du} = \oint_A \frac{dx}{y} \qquad \frac{da_D}{du} = \oint_B \frac{dx}{y} \qquad \tau = \frac{da_D}{da} = \frac{d^2\mathcal{F}}{da^2}$$

N.B.

$\tau(u, m, \tau_{uv})$ should not be confused with τ_{uv}

$$\lim_{m \to 0} \tau(u, m, \tau_{uv}) = \tau_{uv} \qquad \lim_{u \to \infty} \tau(u, m, \tau_{uv}) = \tau_{uv}$$

Weak Coupling Prepotential

$u \to \infty$: $\exists$ Canonical duality frame (``weak coupling'') :

$$\mathcal{F}(a,m) = \frac{1}{2}\tau_{uv} a^2 + $$

$$+ m^2 \left(\log\left(\frac{2a}{m}\right) - \frac{3}{4} + \frac{3}{2}\log\left(\frac{m}{\Lambda}\right) \right)$$

$$+ a^2 \sum_{n=2}^{\infty} f_n(\tau_{uv}) \left(\frac{m}{a}\right)^{2n}$$

$f_n(\tau_{uv})$: polynomials:

E_2, E_4, E_6 wt $= 2n - 2$

[Minhahan, Nemeschansky, Warner; Dhoker, Phong]

Nekrasov: Instanton partition function ⇒

Λ, m dependence (also A,B couplings):

[Manschot, Moore, Xinyu Zhang 2019]

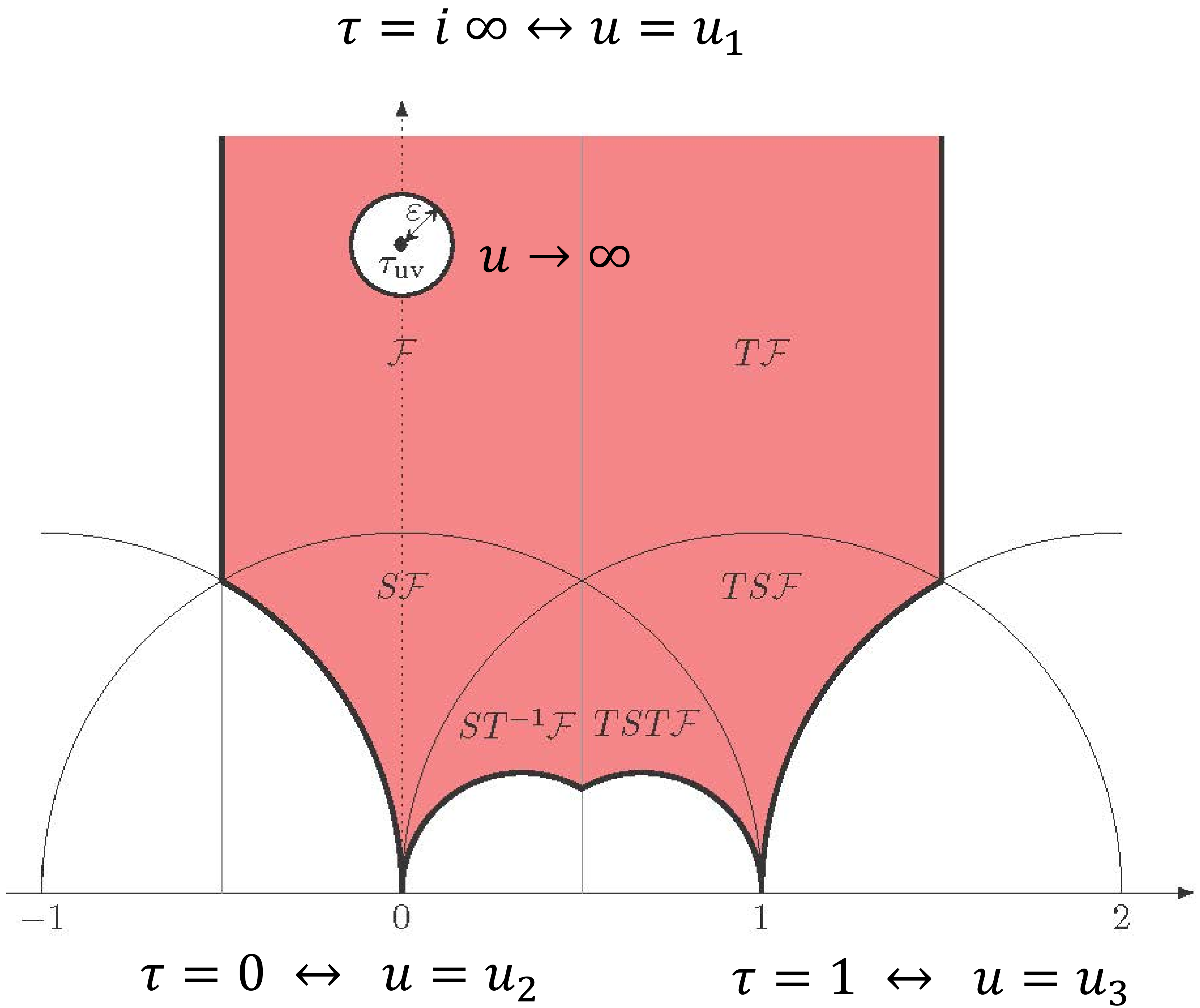
Modular Parametrization

Remarkably: One can invert these equations and express periods as bimodular forms in τ, τ_{uv}

$$m^2 \left(\frac{da}{du}\right)^2 = \frac{\vartheta_4^4(\tau)\vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau)\vartheta_4^4(\tau_{uv})}{\eta^6(\tau_{uv})}$$

$$m^{-2}\, u(\tau, \tau_{uv}) = \frac{e_1^2(\tau_{uv})\, e_{23}(\tau) + cycl.}{e_1(\tau_{uv}) e_{23}(\tau) + cycl}$$

$$\mathcal{B} \cong \mathcal{H}/\Gamma(2) \quad \cong \mathcal{F}\big(\Gamma(2)\big)$$


$\tau = i\infty \leftrightarrow u = u_1$
ε
$\tau_{\rm uv}$
$u \to \infty$
$\mathcal{F}$
$T\mathcal{F}$
$S\mathcal{F}$
$TS\mathcal{F}$
$ST^{-1}\mathcal{F}$
$TST\mathcal{F}$
-1
0
1
2
$\tau = 0 \leftrightarrow u = u_2$
$\tau = 1 \leftrightarrow u = u_3$

5 Coulomb Branch Integral: Measure & Evaluation

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Coulomb Branch Measure

$$Z_{\nu}^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega$$

$$\Omega = d\tau \wedge d\bar{\tau} \; \mathcal{H} \; \Psi_{\nu}^{J}$$

Begin with Maxwell partition function Ψ_{ν}^{J}

$$\Psi \sim \sum_{fluxes} e^{-S_{classical}}$$

Frame dependent.

Not holomorphic.

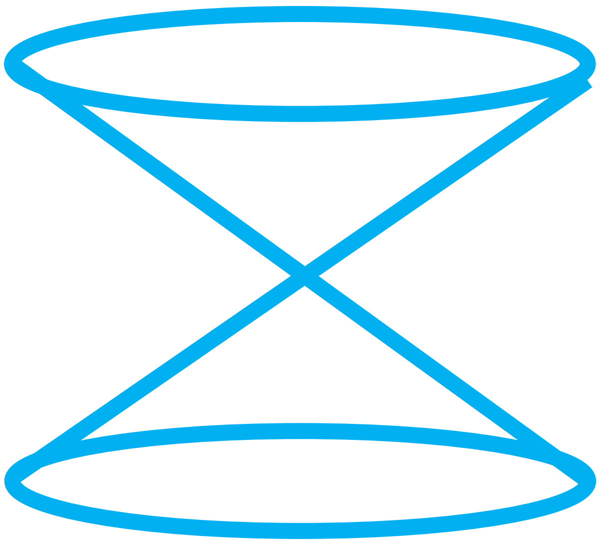
Metric dependent.

The ``Period Point'' J

$$b_2^+ > 1 \Rightarrow Z_\nu^{CB} = 0$$

$$b_2^+ = 1 \quad Z_\nu^{CB} \neq 0$$

$H^2(X; \mathbb{R})$



$*J = J$

$J^2 = 1$

$J \in$ Forward Light Cone

Maxwell Partition Function

$$\Psi_\nu \sim \sum_{fluxes} e^{-\int \bar{\tau}(u) f_+^2 + \tau(u)\, f_-^2}$$

Sum over the first Chern class

$$\lambda \in 2L + \bar{\nu}\ , \quad L = H^2(X;\mathbb{Z})$$

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\bar{\nu}} \partial_{\bar{\tau}} E_\lambda^J \;\; q^{-\frac{1}{4}\lambda^2}\, e^{\pi i\, \lambda \cdot z}$$

$$z = c_{uv}\, v(\tau, \tau_{uv}) + S\, \frac{du}{da}$$

Maxwell Partition Function

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \; e^{\pi i \, \lambda \cdot z}$$

$$z = c_{uv} \, v(\tau, \tau_{uv}) + S \, \frac{du}{da}$$

$$E_\lambda^J = Erf(x_\lambda) \qquad Erf(x) := \int_0^x e^{-\pi t^2} dt$$

$$x_\lambda = \sqrt{Im \, \tau}(\lambda + \frac{Im \, z}{Im \, \tau}) \cdot J$$

Maxwell Coupling To $\mathfrak{s}_{uv}$

$$\sim \exp\left(\int_X v\, F_b^+ f^+ + \bar{v} F_b^- f^-\right)$$

Folklore

$$v := \frac{d^2\mathcal{F}}{da\,da\,dm} = (a_D - a\tau)/m$$

Determines bimodular $v(\tau, \tau_{uv})$

$$\frac{\vartheta_2(\nu, 2\tau)}{\vartheta_3(\nu, 2\tau)} = \frac{\vartheta_2(0, 2\tau_{uv})}{\vartheta_3(0, 2\tau_{uv})}$$

Holomorphic Part Of Measure

$$\mathcal{H}_{bare} = A_1^{\sigma} A_2^{\chi} A_3^{c_{uv}^2}$$

Include observables:

$$\mathcal{H} = \mathcal{H}_{bare} A_4^{p} A_5^{c_{uv} \cdot S} A_6^{S^2}$$

Depend on duality frame –

- but the local system has nontrivial monodromy.

Local Topological Interactions

$$A_1 = \prod_i (u - u_i)^{\frac{1}{8}} =$$

$$(2m)^6 \frac{\eta(\tau_{uv})^{24}\eta(\tau)^{12}}{(\vartheta_4(\tau)^4\vartheta_3(\tau_{uv})^4 - \vartheta_3(\tau)^4\vartheta_4(\tau_{uv})^4)^3}$$

$$\mathrm{A}_2 = \left(\frac{da}{du}\right)^{-\frac{1}{2}}$$

$$A_3 \;\coloneqq \exp\left(-2\,\pi\,i\,\frac{d^2\mathcal{F}}{dm^2}\right) = \left(\frac{\Lambda}{m}\right)^{\frac{3}{2}} \frac{\vartheta_1(2\tau, 2v)}{\vartheta_2^2(\tau_{uv})\vartheta_4(2\tau)}$$

With all these ingredients we can now check that the CB measure is indeed monodromy invariant and hence well-defined. (Nontrivial!)

What about defining the integral of the measure?

$$u \to u_j$$

$$\mathcal{H} \to q_j^{-\frac{\ell}{2}} \, F(\tau_{uv}) \left(1 + \mathcal{O}(q_j)\right)$$

$u \to \infty$ i.e. $\tau \to \tau_{uv}$

$$(\tau - \tau_{uv})^{\ell - \frac{3}{2}} \sum_{\lambda} \cdots e^{-\frac{m}{\Lambda}(\tau - \tau_{uv})^{-\frac{1}{2}} S \cdot \lambda}$$

Do the phase integral first.

(as in string theory)

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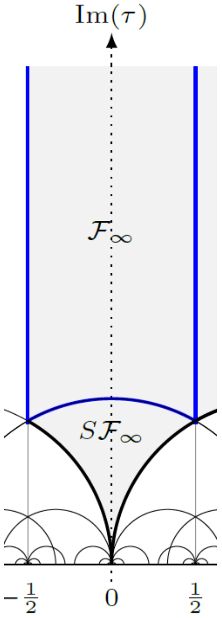
Relation To Mock Modular Forms -1.1

Z_{ν}^{CB} : A sum of integrals of the form

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} \, (Im\,\tau)^{-s} f(\tau,\bar{\tau})$$

Support of c is bounded below

$$f(\tau,\bar{\tau}) = \sum_{m-n\in\mathbb{Z}} c(m,n) q^m \, \bar{q}^n$$



Strategy: Find $\hat{h}(\tau,\bar{\tau})$ such that

$$\partial_{\bar{\tau}} \hat{h} = (Im\,\tau)^{-s} f(\tau,\bar{\tau})$$

$\hat{h}\,(\tau,\bar{\tau})$ is modular of weight (2,0)

Relation To Mock Modular Forms – 1.2

We choose an explicit solution

$$\partial_{\bar{\tau}} R = (Im\tau)^{-s} f(\tau, \bar{\tau})$$

vanishing exponentially fast at $Im\tau \to \infty$

R is not modular, but it's failure to be modular must be holomorphic.

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + R$$

$h(\tau)$: mock modular form

$$h(\tau) = \sum_{m \in \mathbb{Z}} d(m) q^m \qquad q = e^{2\pi i \tau}$$

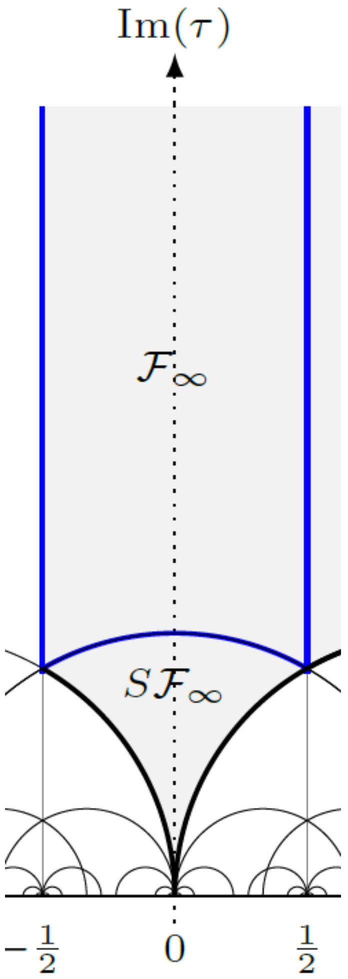
Doing The Integral

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, y^{-s}\, f(\tau,\bar{\tau})$$

$$\partial_{\bar{\tau}} \hat{h} = y^{-s}\, f(\tau,\bar{\tau})$$

$$I_f = d(0)$$

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m$$



Note: $d(0)$ undetermined by diffeq but fixed by the modular properties: Subtle!

Evaluation Of CB Integral ?

$$Z_{\nu}^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega \qquad \Omega = d\tau \wedge d\bar{\tau} \ \mathcal{H} \, \Psi_{\nu}^{J}$$

$$\Psi_{\nu}^{J} = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_{\lambda}^{J} \ q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \, \lambda \cdot z}$$

$$z = c_{uv} \, v(\tau, \tau_{uv}) + S \, \frac{du}{da}$$

$$\Omega = d \, \Lambda \qquad \Lambda = d\tau \, \mathcal{H} \ \hat{G} \qquad \Psi_{\nu}^{J} = \partial_{\bar{\tau}} \, \hat{G}$$

Evaluation Of CB Integral ?

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \, \lambda \cdot z}$$

$$\Psi_\nu^J = \partial_{\bar{\tau}} \hat{G}$$

$$\hat{G} = \sum_{\lambda \in 2L+\nu} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \lambda \cdot z}$$

??? NO!!! $\lim_{\lambda^2 \to +\infty} E_\lambda^J = \pm 1$

Evaluating Difference Of CB Integrals

$$\Psi^{J_1} - \Psi^{J_2} = \partial_{\bar{\tau}} \hat{G}^{J_1,J_2}$$

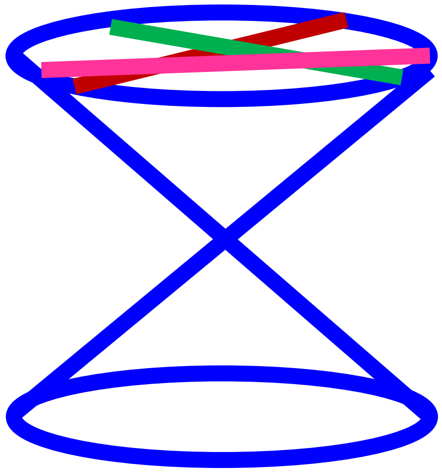
$$\widehat{G^{J_1,J_2}} = \sum_{\lambda \in 2L+\nu} E_\lambda^{J_1,J_2} q^{-\frac{1}{4}\lambda^2} e^{-2\pi i \lambda \cdot z}$$

$$E_\lambda^{J_1,J_2} = Erf\left(x_\lambda^{J_1}\right) - Erf\left(x_\lambda^{J_2}\right)$$

Converges nicely!

$\Rightarrow$ Can use this to evaluate the difference $Z_\nu^{CB,J_1} - Z_\nu^{CB,J_2}$ by a sum of residues.

Metric Dependence



Discontinuous jumps across walls:
Involves modular functions

For the boundary at $u \to \infty$ the modular parameter $\tau \to \tau_{uv}$. This leads to ***continuous*** metric dependence.

Closely related: Nonholomorphic in τ_{uv}

($\mathbb{CP}^2$ is a degenerate case.)

The Coulomb Branch Integral As Harmonic Maass Form

$$Z_{\nu}^{CB}(\tau_{uv}) = \int_{\mathcal{F}(\Gamma(2))} \Omega(\tau, \tau_{uv})$$

Z_{ν}^{CB} transforms under $SL(2,\mathbb{Z})$ as above

$$\frac{\partial}{\partial \bar{\tau}_{\mathrm{uv}}} \mathrm{Z}_{\nu}^{\mathrm{CB}} = \mathrm{y}^{-\frac{3}{2}} \eta^{-2\chi} \sum_{\lambda} K[\lambda_+, \lambda_-] \bar{q}^{\lambda_+^2} \mathrm{q}^{-\lambda_-^2}$$

The Special Period Point

For any manifold with $b_2^+ = 1$

$\exists$ special J_0 such that $\Psi_\nu^{J_0}$ factorizes:

$$\Psi_\nu^{J_0} = f_\nu \; \Theta_{L\_}(\tau, z)$$

$$f_\nu = \sum_{\lambda \in 2\mathbb{Z}+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \, \lambda \cdot z}$$

Measure As A Total Derivative

$$\Omega = d\,\Lambda \qquad \Lambda = d\tau\,\mathcal{H}\,\hat{G}$$

Where we can write $\hat{G}$ explicitly so that Λ is:

1. Well-defined
2. Nonsingular away from $\tau \in \{0, 1, i\infty, \tau_{uv}\}$
3. Good q_i expansion near cusps

Harmonic Jacobi-Maass Forms

These conditions determine $\widehat{G}$ uniquely.

Modular completion of an Appel-Lerche sum

$$F(\tau, z) \sim \frac{e^{-2\pi i\, z}}{\vartheta_4(2\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2-\frac{1}{4}}}{1+e^{4\pi i\, z} q^{2n-1}}$$

$$z = c_{uv}\, v(\tau, \tau_{uv}) + S\,\frac{du}{da}(\tau, \tau_{uv})$$

The Integral Is a Mock Modular Form

For $\mathfrak{s} = \mathfrak{s}(\mathcal{J})$ we find:

$$Z_{\nu}^{CB} = \hat{g}_{\nu}(\tau_{uv}, \bar{\tau}_{uv}) \ \Theta_{L\_}(\tau_{uv})/\eta^{2\chi}(\tau_{uv})$$

$$g_{\nu} = 3 \sum_{n \geq 0} H(4n - 2\mu) q_{uv}^{n - \frac{\nu}{2}}$$

… but other $\mathfrak{s}$ generalize …

For $\mathbb{CP}^2$ & $c_{uv} = 1$ (acs $\Rightarrow c_{uv} = 3$)

$$\frac{\partial}{\partial \bar{\tau}_{uv}} Z_\nu = y_{uv}^{-\frac{3}{2}} \, \eta^{-2} \, \widehat{E_2} \, \Theta_\nu(-\bar{\tau}_{uv})$$

Including Observables

n	Hol. part of $\eta(\tau_{\rm uv})^6\, \Phi^{\mathbb{P}^2}_{1/2}[u_{\rm D}^n]$
0	$q_{\rm uv}^{3/4} + 3\, q_{\rm uv}^{7/4} + 3\, q_{\rm uv}^{11/4} + 6\, q_{\rm uv}^{15/4} + \dots$
1	$-m^2 \left(\frac{3}{2}\, q_{\rm uv}^{7/4} + 12\, q_{\rm uv}^{11/4} + 35\, q_{\rm uv}^{15/4} + \dots \right)$
2	$m^4 \left(\frac{19}{16}\, q_{\rm uv}^{7/4} + \frac{31}{2}\, q_{\rm uv}^{11/4} + 89\, q_{\rm uv}^{15/4} + \dots \right)$
3	$-m^6 \left(\frac{15}{4}\, q_{\rm uv}^{11/4} + \frac{971}{16}\, q_{\rm uv}^{15/4} + \dots \right)$
4	$m^8 \left(\frac{85}{32}\, q_{\rm uv}^{\frac{11}{4}} + \frac{15151}{256}\, q_{\rm uv}^{\frac{15}{4}} + \dots \right)$

1 Introduction & Preliminaries

2 Summary Of Main Claims

3 The N=2* Theory: UV Meaning Of Invariants

4 Remarks On S-Duality Orbits Of Partition Functions

5 Coulomb Branch Integral: Measure & Evaluation

6 LEET Near Cusps & Explicit Results

Contributions Of The Cusps u_j

Physics ⇒ Near each cusp u_j , $j = 1,2,3$
the description of the vacuum changes:
We have a U(1) VM coupled to a charge 1 HM.
(In the appropriate duality frame) [Seiberg-Witten 94]

There is a separate contribution to the path integral coming from the path integral of these three LEET.

We add the contributions, because we sum over vacua:

$$Z_\nu = Z_\nu^{CB} + \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

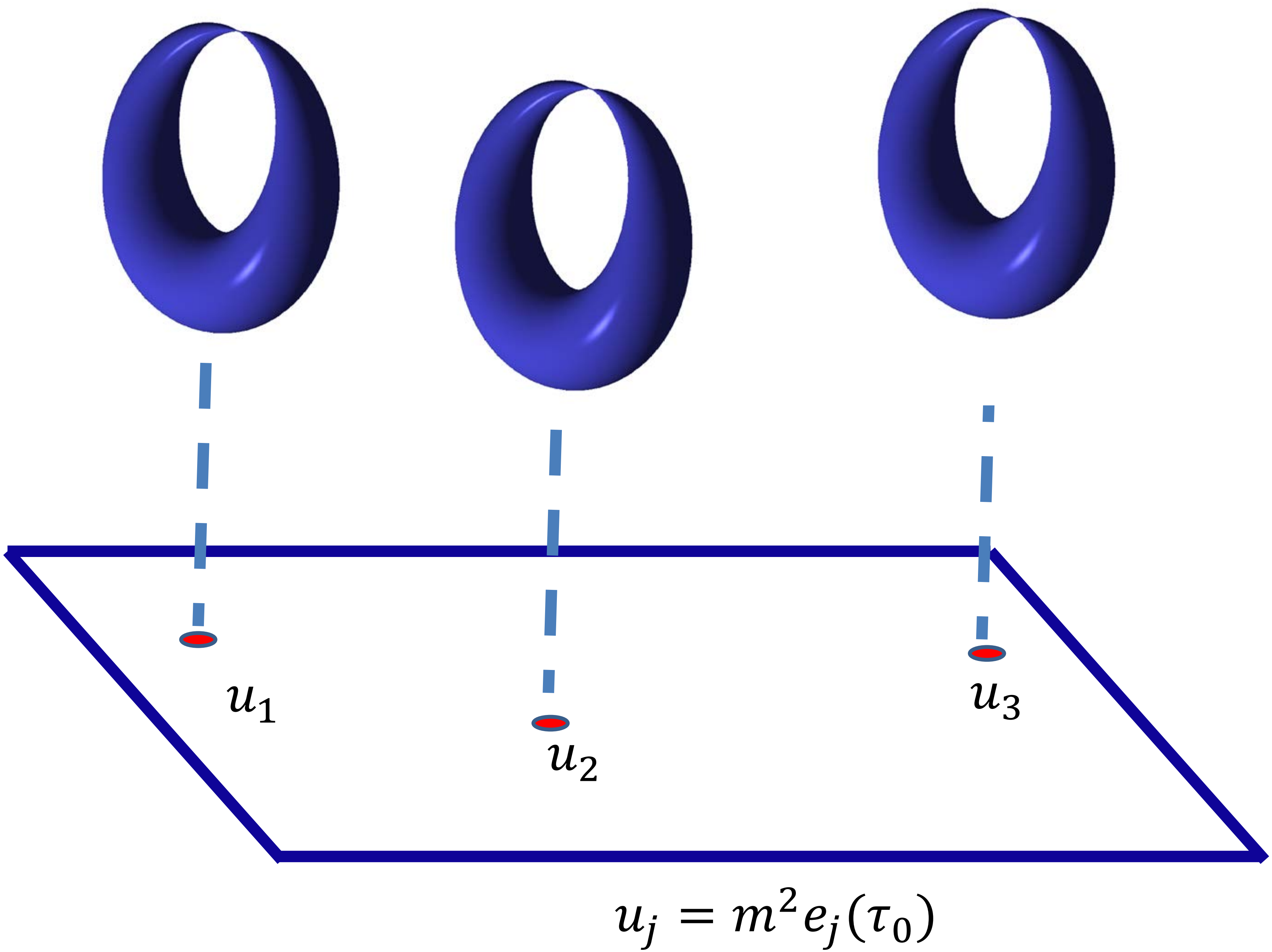
When $b_2^+ > 1$ $\mathrm{Z}_\nu^{\mathrm{CB}}$ vanishes –

- we get true topological invariants:

$$Z_\nu = \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

So it is quite interesting to determine

The three effective actions


u_1
u_2
u_3
$u_j = m^2 e_j(\tau_0)$

General Form Of Effective Action Near u_j

a: Local special coordinate vanishing at u_j

$$S_j^{SW} = \int \sum_n \kappa_n(a;\tau_{uv};t)\,\delta_n \quad + Q(*)$$

δ_n: Possible *local* topological couplings

$$e^{-S_j^{SW}}\Big|_{localize} = \prod_n F_{n,j}(\tau_{uv},t)^{\Delta_n}$$

Possible Topological Couplings Δ_n

$$X \Rightarrow \quad \chi \qquad \sigma$$

$$c_{uv}, \nu \;\Rightarrow\; c_{uv}^2 \qquad c_{uv} \cdot \nu \qquad \nu^2$$

$$c_{ir} \;\Rightarrow\; c_{ir}^2 \qquad c_{ir} \cdot c_{uv} \qquad c_{ir} \cdot \nu$$

$$S \;\Rightarrow\; S^2 \qquad S \cdot c_{ir} \qquad S \cdot c_{uv}$$

$$p$$

Determination Of Effective Action

$$Z_{v,j}^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv}; t)^{\Delta_n}$$

FINITE SUM!

MW97: The couplings κ_n at u_j can be determined from the wall-crossing behavior of Z_v^{CB} from u_j

→ Explicit formulae!

Comparison: Witten Conjecture

$$Z_\nu^{KMW}(p,S) = 2^{1+\kappa-\chi_h} e^{-\frac{i\pi}{2}\nu\cdot c_{uv}}[Z_{\nu,2}(p,S) + Z_{\nu,3}(p,S)]$$

$$\chi_h := \frac{\chi+\sigma}{4} \qquad \kappa = 2\chi + 3\,\sigma$$

$$Z_{\nu,2}(p,S) = e^{\frac{1}{2}S^2+p}\sum_{c_{ir}} SW(c_{ir})e^{c_{ir}\cdot S}e^{\frac{i\pi}{2}\nu\cdot c_{ir}}$$

$$Z_{\nu,3}(p,S) = e^{-\frac{1}{2}S^2-p}\sum_{c_{ir}} SW(c_{ir})e^{-ic_{ir}\cdot S}e^{\frac{i\pi}{2}\nu\cdot c_{ir}}$$

$$Z_{\nu} = \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

$$Z_{\nu,j}^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv};t)^{\Delta_n}$$

$$Z_{\nu,2}^{SW} = F_1^{\ell} F_2^{\chi_h} F_3^{\kappa} \sum_{c_{ir}} SW(c_{ir})\, F_4^{\left(\frac{c_{ir}+c_{uv}}{2}\right)^2}$$

$$F_1 = \; t^3 \left(\eta^4(\tau_{uv}) \vartheta_3(\tau_{uv}/2)\right)^{-1}$$

$$F_2 = \left(2\,\eta^{12}(\tau_{uv}/2)\right)^{-1}$$

$$F_4 = \vartheta_3(\tau_{uv}/2)/\vartheta_4(\tau_{uv}/2)$$

$$F_5^{p} \quad F_6^{S^2} \quad F_7^{S\cdot c_{uv}} \quad F_8^{S\cdot c_{ir}}$$

$$F_5^{p} = \exp\left(-\frac{t^2}{12}(\vartheta_2^4 + \vartheta_3^4)\, p\right)$$

$$F_8^{S\cdot c_{ir}} = \exp\left(-\frac{it}{4}(\vartheta_2\vartheta_3)^2\, S\cdot c_{ir}\right)$$

There are similar expressions for the other two cusps.

$$Z_{SW,1,\boldsymbol{\mu}}(\tau_{\rm uv}) = \left(-2\,\eta(2\tau_{\rm uv})^{12}\right)^{-\chi_{\rm h}} \left(\frac{(\Lambda/m)^3}{4\,\eta(\tau_{\rm uv})^4\,\vartheta_3(2\tau_{\rm uv})^4}\right)^{\ell} \left(\frac{\eta(\tau_{\rm uv})^2}{\vartheta_3(2\tau_{\rm uv})}\right)^{\lambda}$$

$$\times \sum_{\boldsymbol{x}=2\boldsymbol{\mu} \mod 2L} \mathrm{SW}(c_{\rm ir}) \left(\frac{\vartheta_3(2\tau_{\rm uv})}{\vartheta_2(2\tau_{\rm uv})}\right)^{\boldsymbol{x}^2}.$$

$$Z_{SW,3,\boldsymbol{\mu}}(\tau_{\rm uv}) = 2\,e^{2\pi i \boldsymbol{\mu}^2} \left(\frac{-(\Lambda/m)^3}{\eta(\tau_{\rm uv})^4\,\vartheta_3((\tau_{\rm uv}+1)/2)^4}\right)^{\ell}$$

$$\times \left(2\,\eta((\tau_{\rm uv}+1)/2)^{12}\right)^{-\chi_{\rm h}} \left(\frac{2\,\eta(\tau_{\rm uv})^2}{\vartheta_3((\tau_{\rm uv}+1)/2)}\right)^{\lambda}$$

$$\times \sum_{\boldsymbol{x}\in L} \mathrm{SW}(c_{\rm ir})\,(-1)^{2B(\boldsymbol{x},\boldsymbol{\mu})} \left(\frac{\vartheta_3((\tau_{\rm uv}+1)/2)}{\vartheta_4((\tau_{\rm uv}+1)/2)}\right)^{\boldsymbol{x}^2}.$$

Relation To Previous Results

For $\mathfrak{s}(\mathcal{I})$ and $m \to 0$ we recover and generalize formulae of [VW;DPS] for VW invariants.

For $c_{uv} = 0$ we recover formulae of Labastida-Lozano

For $m \to \infty$, $\mathrm{q}_{\mathrm{uv}} \to 0$ *after suitable renormalization* we recover the ``Witten conjecture'' for the Donaldson invariants in terms of the Seiberg-Witten invariants.

Recover and generalize explicit evaluation of u-plane integral for $\mathbb{CP}^2, S^2 \times S^2$ of Moore-Witten, Malmendier-Ono

A generalization and unification of the 1990's formulae:

VIRTUAL REFINEMENTS OF THE VAFA-WITTEN FORMULA

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with an appendix by Lothar Göttsche and Hiraku Nakajima

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Concept	This paper	GKNW
Geometry	Smooth, compact four-manifold X with $b_1 = 0$ of SW simple type	Projective complex surface S with $b_2^+ > 1$, $b_1 = 0$ of SW simple type
Mass/Scale	$m/\Lambda = t$	t
Modular parameter	$q_{\rm uv}$	x^4
UV Spin-c structure	$c_{\rm uv} \in \bar{w}_2(X) + H_2(X, 2\mathbb{Z})$	Canonical class K_S
IR Spin-c structure	$c_{\rm ir} \in \bar{w}_2(X) + H_2(X, 2\mathbb{Z})$	SW basic class $K_S - 2a_i$
't Hooft flux	$2\boldsymbol{\mu} \in H^2(X, \mathbb{Z})$	first Chern class c_1
0-observable	p	$-u$
2-observable	S	$i\alpha z$

Table 9: Dictionary between some of the concepts in this paper and in [13, 14, 85]

$U(1)_b$ Localization

$F^+ + [M, \bar{M}] = 0$ $\quad\quad$ $\boldsymbol{D} M = 0$

Fixed point set for $M \rightarrow e^{i\theta} M$ has TWO branches

Branch 1: $\mathcal{M}_{asd}$: $M = 0$ & $F^+ = 0$

Branch 2: $\mathcal{M}_{ab}$: $M \sim \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

$U(1)_b$: $\quad \int_{\mathcal{M}_{Q,k}} \cdots \rightarrow \int_{\mathcal{M}_{asd,k}} \cdots + \int_{\mathcal{M}_{ab}} \cdots$

$$\int_{\mathcal{M}_{asd}} \cdots \quad \sim \quad Z_{\nu,2}^{SW} + Z_{\nu,3}^{SW}$$

$$\int_{\mathcal{M}_{ab}} \cdots \quad \sim \quad Z_{\nu,1}^{SW}$$

Concluding Remarks

Twisted $N = 2^*$ on four-manifolds with a spin-c structure unifies and generalizes previous expressions for invariants of 4-manifolds derived from SYM.

Paper on the arXiv should appear ``soon.''

Hamiltonian formulation (Floer theory)?

Derivation from 6d (2,0) theory?

Generalization of these techniques to class S

X complex: Compute Refined Versions From Physics

REMARKS ON CLASS S: SLIDES FROM MY STRING MATH 2018 TALK IN SENDAI, JAPAN

Class S: General Remarks

$$\mathcal{H} = \alpha^{\chi}\, \beta^{\sigma} \det\left(\frac{da^i}{du_j}\right)^{1-\frac{\chi}{2}} \Delta_{phys}^{\frac{\sigma}{8}}$$

Δ_{phys} a holomorphic function on $\mathcal{B}$ with first-order zeros at the loci of massless BPS hypers

α, β will be automorphic forms on Teichmuller space of the UV curve C

α, β are related to correlation functions for fields in the (0,2) QFT gotten from reducing 6d (0,2)

Class S: General Remarks

$$\Psi \sim \sum_{\lambda} e^{i\pi\lambda\cdot\xi} e^{-i\pi\bar{\tau}(\lambda_+,\lambda_+)-i\pi\tau(\lambda_-\cdot\lambda_-)+\cdots}$$

$$\lambda \in \lambda_0 + \Gamma \otimes H^2(X;\mathbb{Z})$$

$$\xi \in \Gamma \otimes H^2(X;\mathbb{R})$$

$$\Gamma \subset H^1(\Sigma;\mathbb{Z})$$

Lagrangian sublattice

If $\xi = \rho \otimes w_2(X)\ mod\ 2$ ***then*** WC from interior of $\mathcal{B}$ will be cancelled by SW invariants

$\Rightarrow$ No new four-manifold invariants…

Ψ comes from a ``partition function" of a level 1 SD 3-form on $M_6 = \Sigma \times X$

Quantization: Choose a QRIF Ω on $H^3(M_6;\mathbb{Z})$

Natural choice: [Witten 96,99; Belov-Moore 2004]

$$\Omega(x) = \exp\left(i\pi \; WCS(\theta \cup x;\; S^1 \times M_6)\right)$$

Choice of weak-coupling duality frame + natural choice of $spin^c$ structure gives

$$\xi = \rho \otimes w_2(X)$$

HOWEVER!

Need For U(1)-valued QRIF

$e^{i\,\pi\lambda\cdot\xi}$ is a 6d generalization of the famous Witten phase: $(-1)^{w_2(X)\cdot\lambda}$

$$e^{\int \bar{v}\, F_b^+ F_{dyn}^+ + v\, F_b^- F_{dyn}^-} \rightarrow e^{i\,\pi \int w_2(X) \frac{F_{dyn}}{2\pi}}$$

So the $\mathbb{Z}_2$-phase generalizes to a U(1)-valued phase.

Important implications for the generalization of CB integral to class S theories: We do not want a $\mathbb{Z}_2$ –valued QRIF.

$$\mathcal{N} = 2^* \;\; SU(2)$$

$SL(2)$ Hitchin system on $E_{uv} = \mathbb{C}/(\mathbb{Z} + \tau_{uv}\mathbb{Z})$

Regular singularity at $z = 0$

Monodromy $\sim \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$

λ: Liouville form pulled back to $\Sigma \subset T^* E_{uv}$