d=4 $\mathcal{N}$=2 Field Theory
and
Physical Mathematics

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Johns Hopkins, April 11, 2016

Phys-i-cal Math-e-ma-tics, n.

Pronunciation: Brit. /ˈfɪzɨkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

Frequency (in current use):

1. Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, but equally central, goals of elucidating the laws of nature at their most fundamental level, together with discovering deep mathematical truths.

2014 G. Moore *Physical Mathematics and the Future,* http://www.physics.rutgers.edu/~gmoore

…….

1573 *Life Virgil* in T. Phaer & T. Twyne tr. Virgil *Whole .xii. Bks. Æneidos* sig. Aiv^{v}, Amonge other studies ….. he cheefly applied himself to Physick and Mathematickes.

1 What can d=4, $\mathcal{N}$=2 do for you?

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Defects in Quantum Field Theory

5 Theories of Class S & Spectral Networks

6 Conclusion

7 Wall Crossing 201

Some Physical Questions

1. Given a QFT what is the spectrum of the Hamiltonian?

and how do we compute forces, scattering amplitudes? More generally, how can we compute expectation values of operators ?

2. Find solutions of Einstein's equations,

and how can we solve Yang-Mills equations on Einstein manifolds?

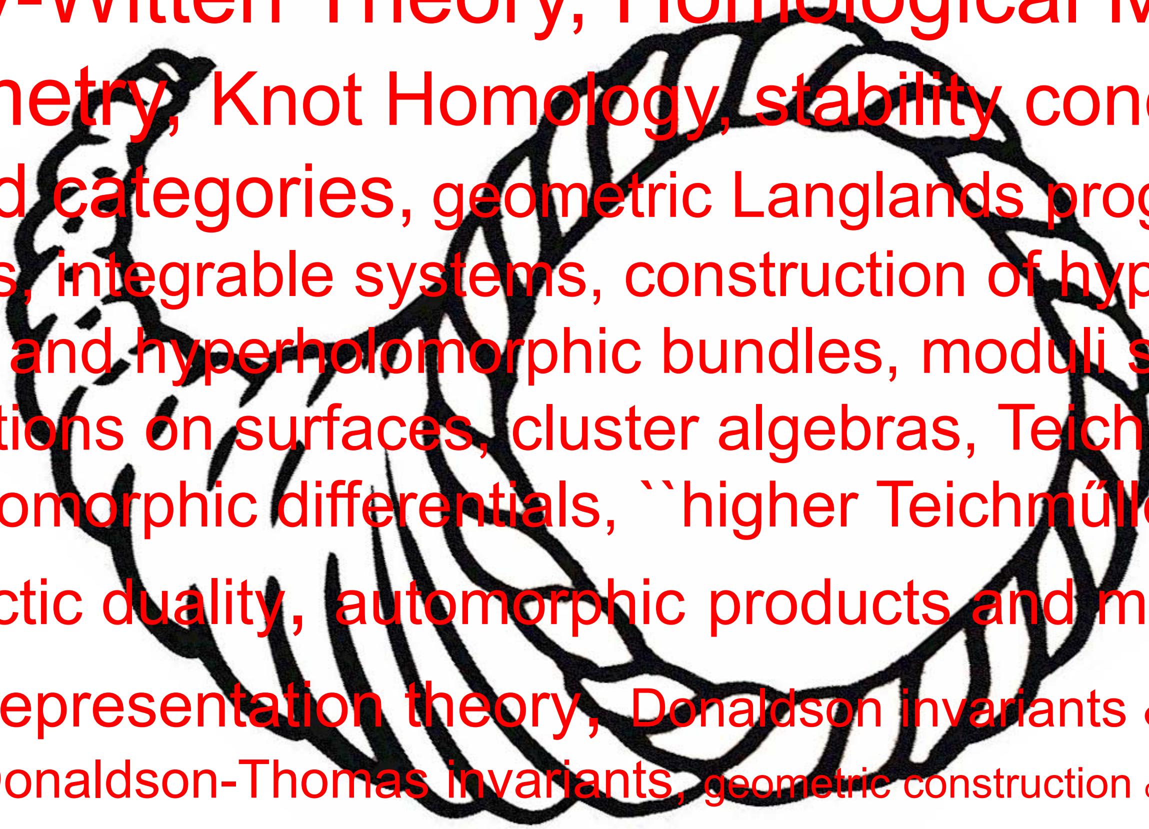
Exact results are hard to come by
in nontrivial situations …

But theories with ``extended supersymmetry'' in spacetime dimensions ≤ 4 have led to a wealth of results answering these kinds of questions.

(These developments are also related to explaining the statistical origin of black hole entropy –
but that is another topic for another time ….)

Cornucopia For Mathematicians

Provides a rich and deep mathematical structure.



Gromov-Witten Theory, Homological Mirror Symmetry, Knot Homology, stability conditions on derived categories, geometric Langlands program, Hitchin systems, integrable systems, construction of hyperkähler metrics and hyperholomorphic bundles, moduli spaces of flat connections on surfaces, cluster algebras, Teichműller theory and holomorphic differentials, ``higher Teichműller theory,'' symplectic duality, automorphic products and modular forms, quiver representation theory, Donaldson invariants & four-manifolds, motivic Donaldson-Thomas invariants, geometric construction of affine Lie algebras, McKay correspondence, ……….

The Importance Of BPS States

Much progress has been driven by trying to understand a portion of the spectrum of the Hamiltonian – the ``BPS spectrum'' –

BPS states are special quantum states in a supersymmetric theory for which we can compute the energy exactly.

So today we will just focus on the BPS spectrum in d=4, $\mathcal{N}$=2 field theory.

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2B *The Vacuum And Spontaneous Symmetry Breaking*

2C *BPS States: Monopoles & Dyons*

2D *Seiberg-Witten Theory*

2E *Unfinished Business*

Definition Of d=4, $\mathcal{N}$=2 Field Theory

There are many examples of d=4, $\mathcal{N}$=2 field theories

A d=4, $\mathcal{N}$=2 field theory is a field theory such that the Hilbert space of quantum states is a representation of the d=4, $\mathcal{N}$=2 super-Poincare algebra.

OK.....

..... So what is the
d=4, $\mathcal{N}$=2 super-Poincare algebra??

d=4,$\mathcal{N}$=2 Poincaré Superalgebra

(For mathematicians)

Super Lie algebra $\quad \mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1$

$$\mathfrak{s}^0 = \mathrm{poin}(1,3) \oplus \mathfrak{u}(2)_R \oplus \mathbb{R}^2_{\mathrm{central}}$$

$$\mathbb{R}^2_{\mathrm{central}} \cong \mathbb{C}$$

Generator Z = ``$\mathcal{N}$=2 central charge''

$$\mathfrak{s}^1 = [(2;2)_{+1} \oplus (2^*;2)_{-1}]_{\mathbb{R}}$$

$$\mathrm{Sym}^2\mathfrak{s}^1 \to \mathrm{transl} \oplus \mathbb{R}^2_{\mathrm{central}} \subset \mathfrak{s}^0$$

d=4, $\mathcal{N}$=2 Poincaré Superalgebra

(For physicists)

$\mathcal{N}$=1 Supersymmetry:

There is an operator Q on the Hilbert space $\mathcal{H}$

$$\{Q, Q^{\dagger}\} = 2H$$

$\mathcal{N}$=2 Supersymmetry:

There are ***two*** operators Q_1, Q_2 on the Hilbert space

$$\{Q_i, Q_j^{\dagger}\} = 2\delta_{i,j} H$$

$$\{Q_1, Q_2\} = 2Z$$

Constraints on the Theory

Representation theory:

Field and particle multiplets

Hamiltonians:

Typically depend on very few parameters for a given field content.

BPS Spectrum:

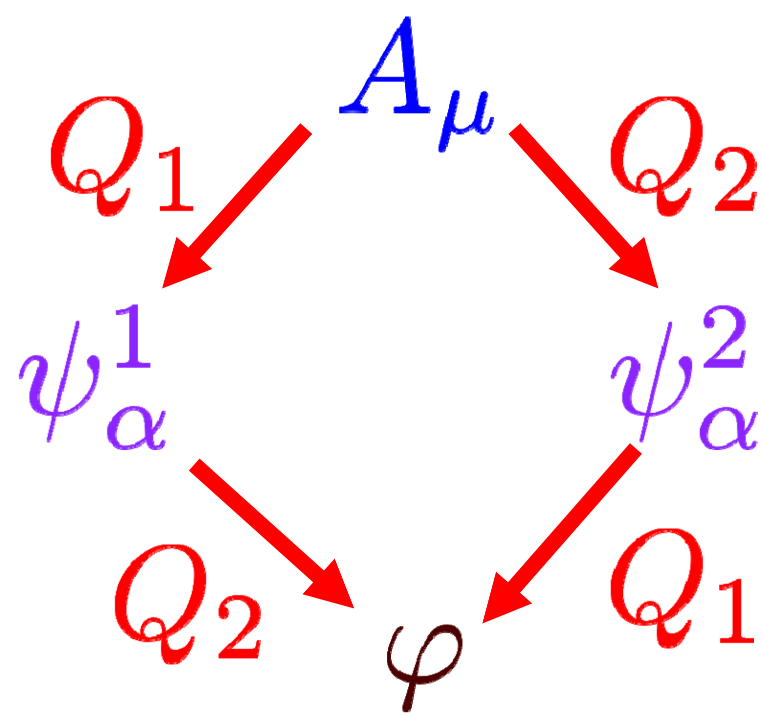
Special subspace in the Hilbert space of states

Example: $\mathcal{N}$=2 Super-Yang-Mills For U(K)

Gauge fields:

Doublet of gluinos:

Complex scalars
(Higgs fields):



All are K x K anti-Hermitian matrices (i.e. in $\mathfrak{u}$(K))

Gauge transformations: $\varphi \rightarrow g^{-1} \varphi g$

Hamiltonian Of $\mathcal{N}$=2 U(K) SYM

The Hamiltonian is completely determined, up to a choice of Yang-Mills coupling e_0^2

$$H = e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr}\left(\vec{E}^2 + \vec{B}^2 + |\vec{D}\varphi|^2 \right)$$
$$+ e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr}\left([\varphi, \varphi^\dagger]^2 \right)$$

Energy is a sum of squares.

Energy bounded below by zero.

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Classical Vacua

Classical Vacua: Zero energy field configurations.

$$H = e_0^{-2}\int_{\mathbb{R}^3} \mathrm{Tr}\left(\vec{E}^2 + \vec{B}^2 + |\vec{D}\varphi|^2\right) + e_0^{-2}\int_{\mathbb{R}^3} \mathrm{Tr}\left([\varphi,\varphi^\dagger]^2\right)$$

$$\vec{E} = \vec{B} = 0 \qquad \varphi = cnst.$$

$$[\varphi,\varphi^\dagger] = 0 \Rightarrow$$

$$\varphi = \mathrm{Diag}\{a_1,\dots,a_K\}$$

Any choice of $a_1,\dots a_K \in \mathbb{C}$ is a vacuum!

Quantum Moduli Space of Vacua

The continuous vacuum degeneracy is
an exact property of the quantum theory:

$$\langle \mathrm{Vac}|\varphi|\mathrm{Vac}\rangle = \mathrm{Diag}\{a_1, \ldots, a_K\}$$

The quantum vacuum is not unique!

Manifold of quantum vacua $\mathcal{B}$

Parametrized by the complex numbers a_1, …., a_K

Gauge Invariant Vacuum Parameters

$$u_s := \langle \mathrm{Vac}(u) | \mathrm{Tr}(\varphi^s) | \mathrm{Vac}(u) \rangle$$

$$\mathcal{B} := \{ u := (u_1, \dots, u_K) \}$$

Physical properties depend on the choice of vacuum u $\in \mathcal{B}$.

We will illustrate this by studying the properties of ``dyonic particles'' as a function of u.

Spontaneous Symmetry Breaking

$$\langle \mathrm{Vac}(u)|\varphi|\mathrm{Vac}(u)\rangle = \mathrm{Diag}\{a_1, \dots, a_K\}$$

broken to:

$$U(K) \longrightarrow U(1)^K$$

(For mathematicians)

φ is in the adjoint of U(K): stabilizer of a generic $\varphi \in \mathfrak{u}(K)$ is a Cartan torus

Physics At Low Energy Scales: LEET

Only one kind of light comes out of the flashlights from the hardware store….

Most physics experiments are described very accurately by using (quantum) Maxwell theory (QED). The gauge group is U(1).

The true gauge group of electroweak forces is SU(2) x U(1)

The Higgs vev sets a scale: $\langle \varphi \rangle = 246 \mathrm{GeV}$

The stabilizer subgroup of $<\varphi>$ is U(1) of E&M.

At energies << 246 GeV we can describe physics using Maxwell's equations + small corrections:

$\mathcal{N}$=2 Low Energy U(1)^K Gauge Theory

Low energy effective theory (LEET) is described by an $\mathcal{N}$=2 extension of Maxwell's theory with gauge group U(1)^K

K different ``electric'' and K different ``magnetic'' fields:

$$\vec{E}^I \quad \vec{B}^I \qquad I = 1, \ldots, K$$

& their $\mathcal{N}$=2 superpartners

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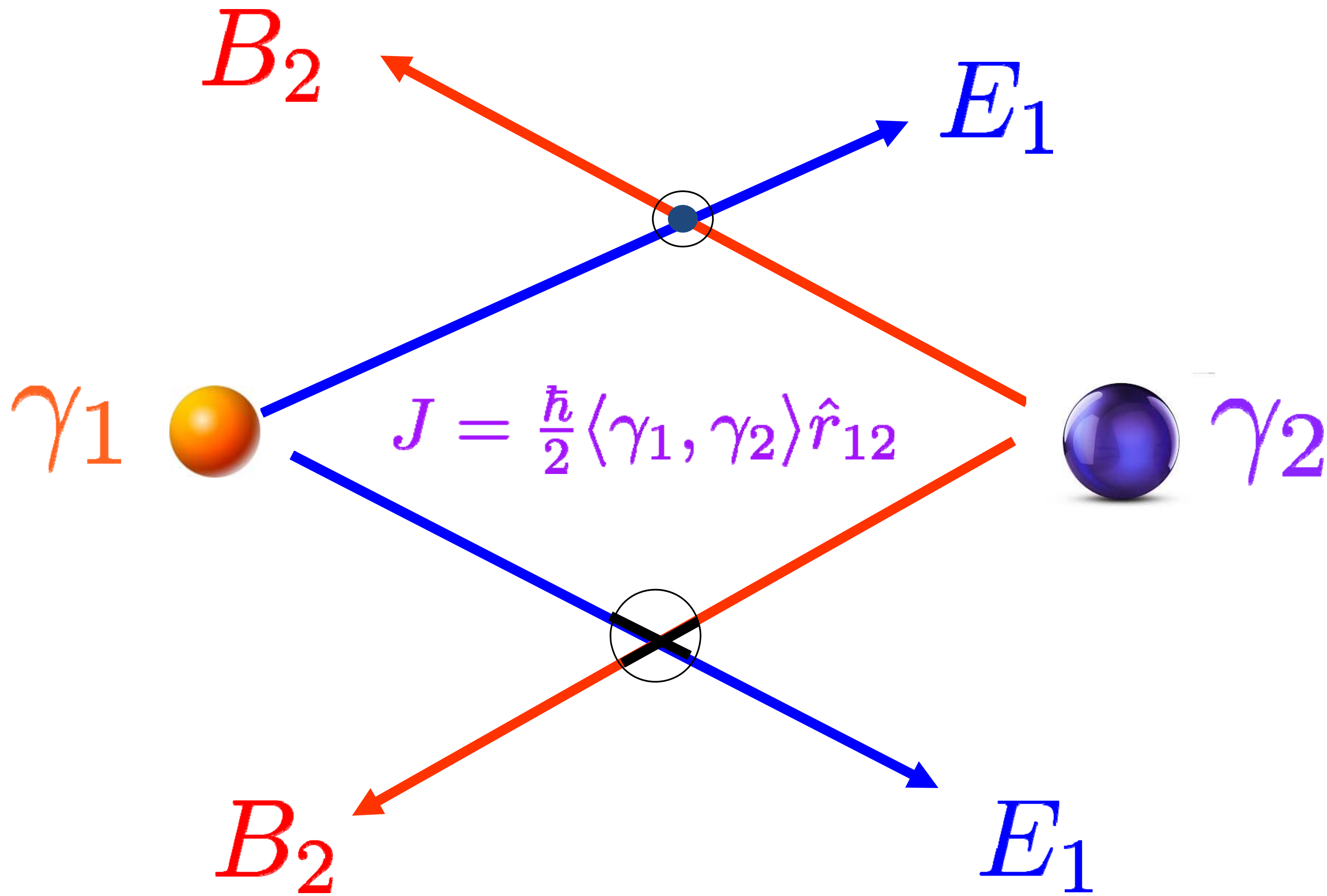
Electro-magnetic Charges

The theory will also contain ``dyonic particles'' – particles with electric and magnetic charges for the fields $\vec{E}^I \quad \vec{B}^I \quad I = 1, \dots, K$

(Magnetic, Electric) Charges:

$$\gamma = (p^I, q_I)$$

Dirac quantization: On general principles, the vectors γ are in a <u>symplectic</u> lattice Γ.



$$\langle \gamma_1, \gamma_2 \rangle = p_1^I q_{2,I} - p_2^I q_{1,I} \in \mathbb{Z}$$

BPS States: The Definition

Superselection sectors: $\mathcal{H} = \oplus_{\gamma\in\Gamma}\mathcal{H}_\gamma$

In the sector $\mathcal{H}_\gamma$ the central charge ***operator***

Z is just a c-number $Z_\gamma \in \mathbb{C}$

Bogomolny bound: In sector $\mathcal{H}_\gamma$

$$E \geq |Z_\gamma|$$

$$\mathcal{H}_\gamma^{\rm BPS} := \{\psi : E\psi = |Z_\gamma|\psi\}$$

The Central Charge Function

As a function of γ the $\mathcal{N}$=2 central charge is linear

$$Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2}$$

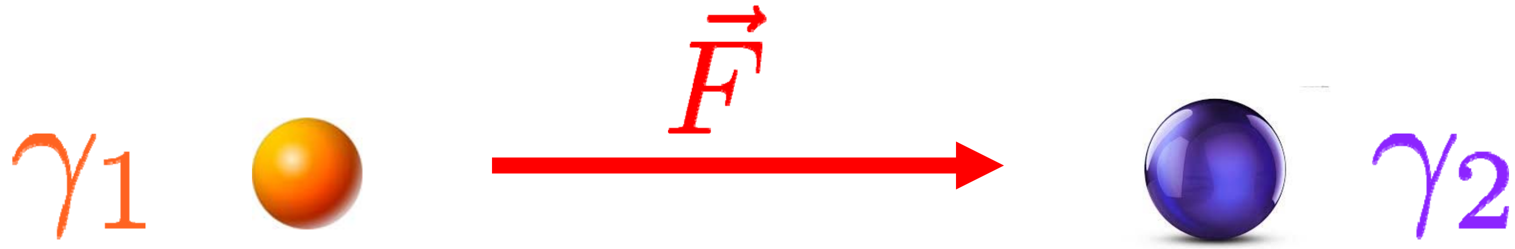
This linear function is ***also*** a function of u $\in \mathcal{B}$:

On $\mathcal{H}_{\gamma}^{\mathrm{BPS}}$ $\quad E = |Z_{\gamma}(u)|$

(In fact, it is a ***holomorphic*** function of u $\in \mathcal{B}$.)

So the mass of BPS particles depends on u $\in \mathcal{B}$.

Coulomb Force Between Dyons



$$\vec{F} = \left(q_I e^2_{IJ} q_J + p^I e^{-2}_{IJ} p^J\right) \frac{\hat{r}}{r^2}$$

$e^2_{IJ}(u)$ A nontrivial function of u $\in \mathcal{B}$

<u>It can be computed from $Z_\gamma(u)$</u>

Physical properties depend on the choice of vacuum u $\in \mathcal{B}$.

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So far, everything I’ve said follows fairly straightforwardly from general principles.

General d=4, $\mathcal{N}$=2 Theories

1. A moduli space $\mathcal{B}$ of quantum vacua,
   (a.k.a. the ``Coulomb branch'').

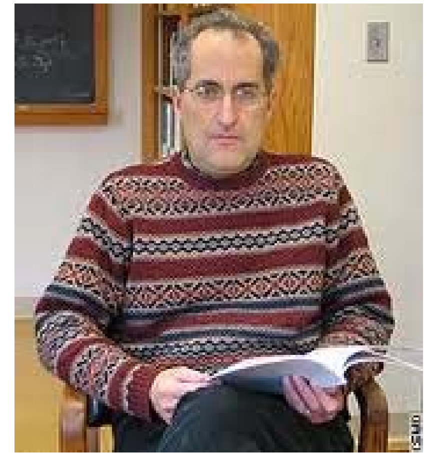
The low energy dynamics are described by an effective $\mathcal{N}$=2 abelian gauge theory.

2. The Hilbert space is graded by an integral lattice of charges, Γ, with integral

anti-symmetric form. There is a BPS subsector with masses given exactly by $|Z_\gamma(u)|$.

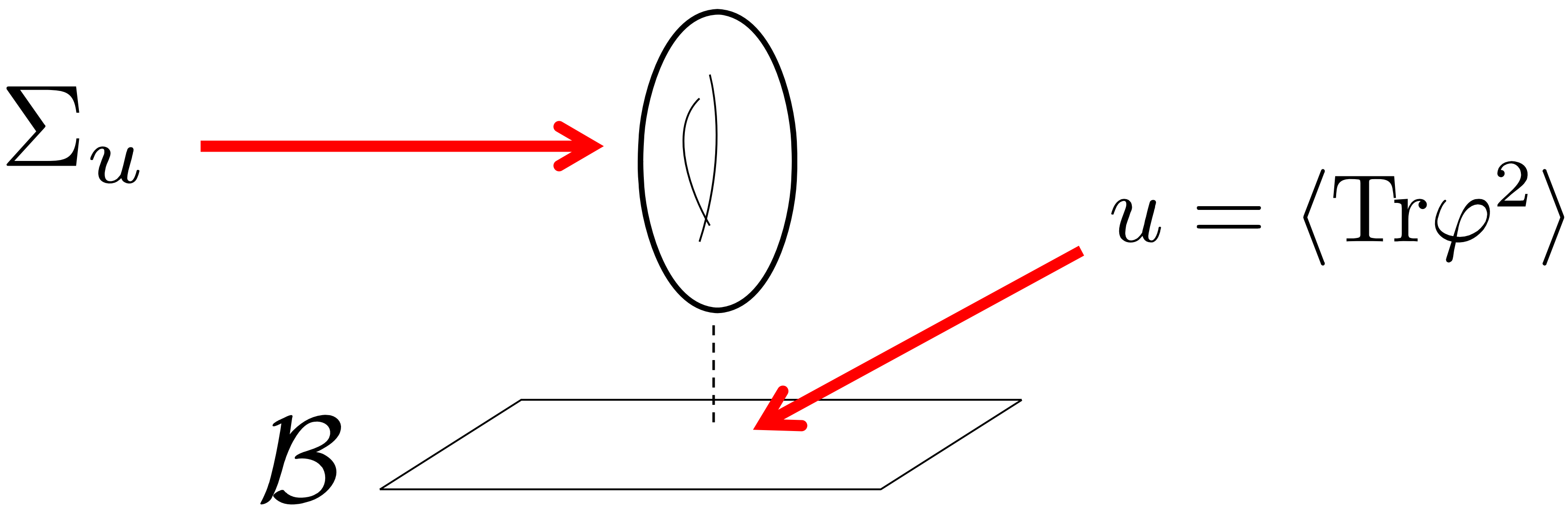
But how do we compute $Z_{\gamma}(u)$ as a function of u ?

Seiberg-Witten Paper



Seiberg & Witten (1994) found a way for the case of SU(2) SYM.

$Z_\gamma(u)$ can be computed in terms of the ***periods*** of a meromorphic differential form λ on a Riemann surface Σ both of which depend on u.



In more concrete terms: there is an integral formula like:

$$Z_\gamma(u) = \oint_\gamma \sqrt{\frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z}}dz$$

γ is a closed curve…

Because of the square-root there are different branches –
So the integral can be nonzero, and different choices of γ lead to different answers…

And, as realized in the 19th Century by Abel, Gauss, and Riemann, such functions (and line integrals) with branch points are properly understood in terms of surfaces with holes - Riemann surfaces.

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The Promise of Seiberg-Witten Theory: 1/2

So Seiberg & Witten showed how to determine the LEET exactly as a function of u, at least for G=SU(2) SYM.

They also gave cogent arguments for the exact BPS spectrum of this *particular* theory: d=4, $\mathcal{N}$=2 SYM with gauge group G=SU(2).

Their breakthrough raised the hope that in general d=4 $\mathcal{N}$=2 theories we could find many kinds of exact results.

The Promise of Seiberg-Witten Theory: 2/2

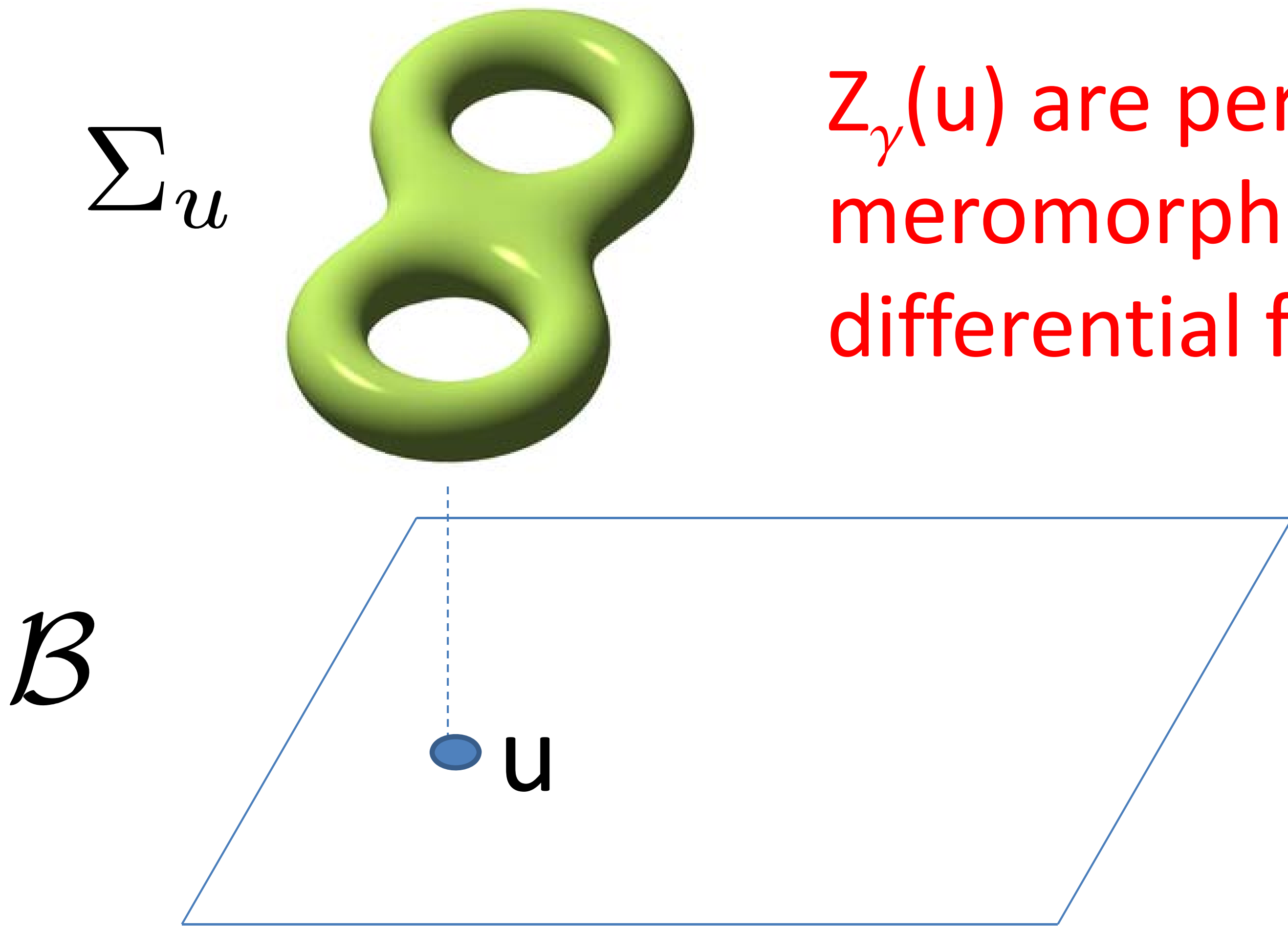
Promise 1: The LEET: Compute $Z_\gamma(u)$.

Promise 2: Exact spectrum of the Hamiltonian on a subspace of Hilbert space: the space of BPS states.

Promise 3: Exact results for path integrals – including insertions of ``defects’’ such as ``line operators,’’ ``surface operators’’, domain walls, Promise 1: The LEET: Compute $Z_{\gamma}(u)$.

Extensive subsequent work showed that the SW picture indeed generalizes to all known d=4 , $\mathcal{N}$=2 field theories:

$Z_\gamma(u)$ are periods of a meromorphic differential form on Σ_u



But, to this day, there is no general algorithm for computing Σ_u for a given d=4, $\mathcal{N}$=2 field theory.

But what about Promise 2: Find the BPS spectrum?

In the 1990's the BPS spectrum was *only* determined in a handful of cases…

(SU(2) with ($\mathcal{N}$=2 supersymmetric) quarks flavors: N_f = 1,2,3,4, for special masses: Bilal & Ferrari)

Knowing the value of Z_γ (u) in the sector $\mathcal{H}_\gamma$ does not tell us whether there are, or are not, BPS particles of charge γ. It does not tell us if $\mathcal{H}_\gamma^{BPS}$ is zero or not.

In the past 8 years there has been a great deal of progress in understanding the BPS spectra in a large class of other $\mathcal{N}=2$ theories.

One key step in this progress has been a much-improved understanding of the ``*wall-crossing phenomenon*.''

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Recall the space of BPS states is:

$$\mathcal{H}_{\gamma}^{\mathrm{BPS}} = \{\psi : E\psi = |Z_{\gamma}(u)|\psi\}$$

It is finite dimensional.

It depends on u, since $Z_{\gamma}(u)$ depends on u.

More surprising:

Even the dimension can depend on u !

BPS Index

As in the index theory of Atiyah & Singer, $\mathcal{H}^{\text{BPS}}$ is $\mathbb{Z}_2$ graded by $(-1)^F$ so there is an ***index***, in this case a Witten index, which behaves much better:

$$\Omega(\gamma) := \mathrm{Tr}_{\mathfrak{h}_\gamma^{\mathrm{BPS}}} (-1)^{2J_3}$$

J_3 is any generator of so(3)

Formal arguments prove: $\Omega(\gamma)$ is ***invariant*** under change of parameters such as the choice of u …

Index Of An Operator: 1/4

(For physicists)

Suppose T is a linear operator depending on parameters u $\in \mathcal{B}$

$$T_u : V \to W$$

If V and W are ***finite-dimensional*** Hilbert spaces then:

$$\dim(\ker T_u) - \dim(\ker T_u^\dagger) = \dim V - \dim W$$

independent of the parameter u!

Index Of An Operator: 2/4

Example: Suppose V=W is one-dimensional.

$$T_u(\psi) = u\psi \quad u \in \mathbb{C} \qquad \psi \in V$$

$$u \neq 0 \qquad \dim(\ker T_u) = \dim(\ker T_u^\dagger) = 0$$

$$u = 0 \qquad \dim(\ker T_u) = \dim(\ker T_u^\dagger) = 1$$

$$T_u = \begin{pmatrix} u & u & u^2 \\ \sin(u) & \sin(u) & \sin(u) \end{pmatrix} \quad \mathrm{Ind}(T_u) = 3 - 2 = 1$$

Index Of An Operator: 3/4

Now suppose T_u is a family of linear operators between two *infinite-dimensional* Hilbert spaces

$$\dim(\ker T_u) - \dim(\ker T_u^\dagger) = \dim \mathcal{H}_1 - \dim \mathcal{H}_2$$
$$= \infty - \infty$$

Still the LHS makes sense for suitable (Fredholm) operators and is ***invariant*** under *continuous* deformations of those operators.

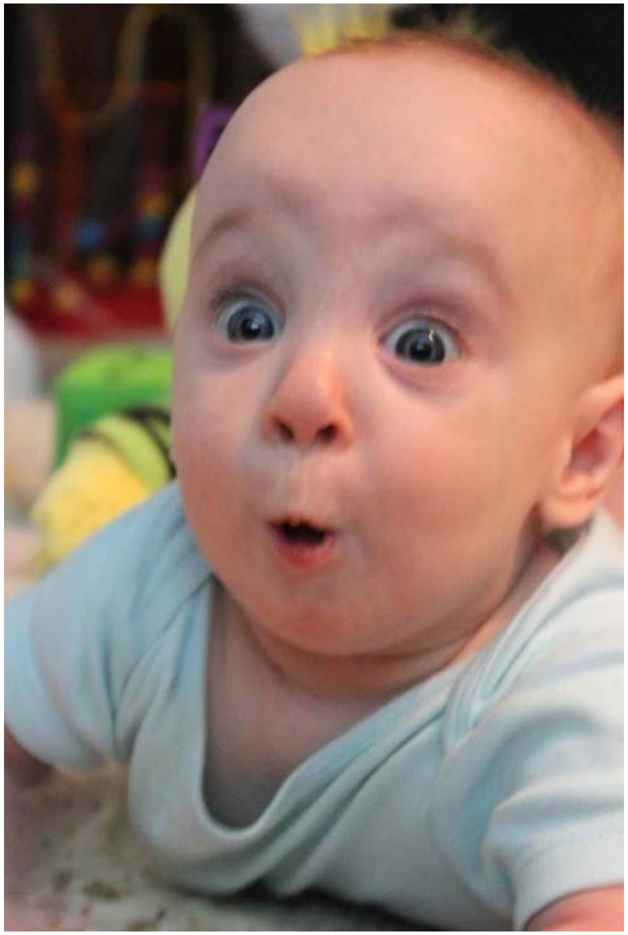
Index Of An Operator: 4/4

The BPS index *is* the index of the supersymmetry operator Q on Hilbert space.

(In the weak-coupling limit it is also the index of a Dirac operator on moduli spaces of magnetic monopoles.)

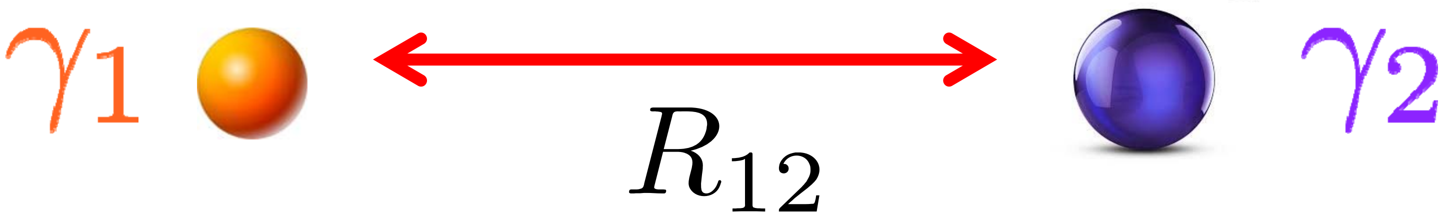
The Wall-Crossing Phenomenon

But even the ***index*** can depend on u !!



How can that be ?

BPS particles can form bound states which are themselves BPS!



Denef's Boundstate Radius Formula

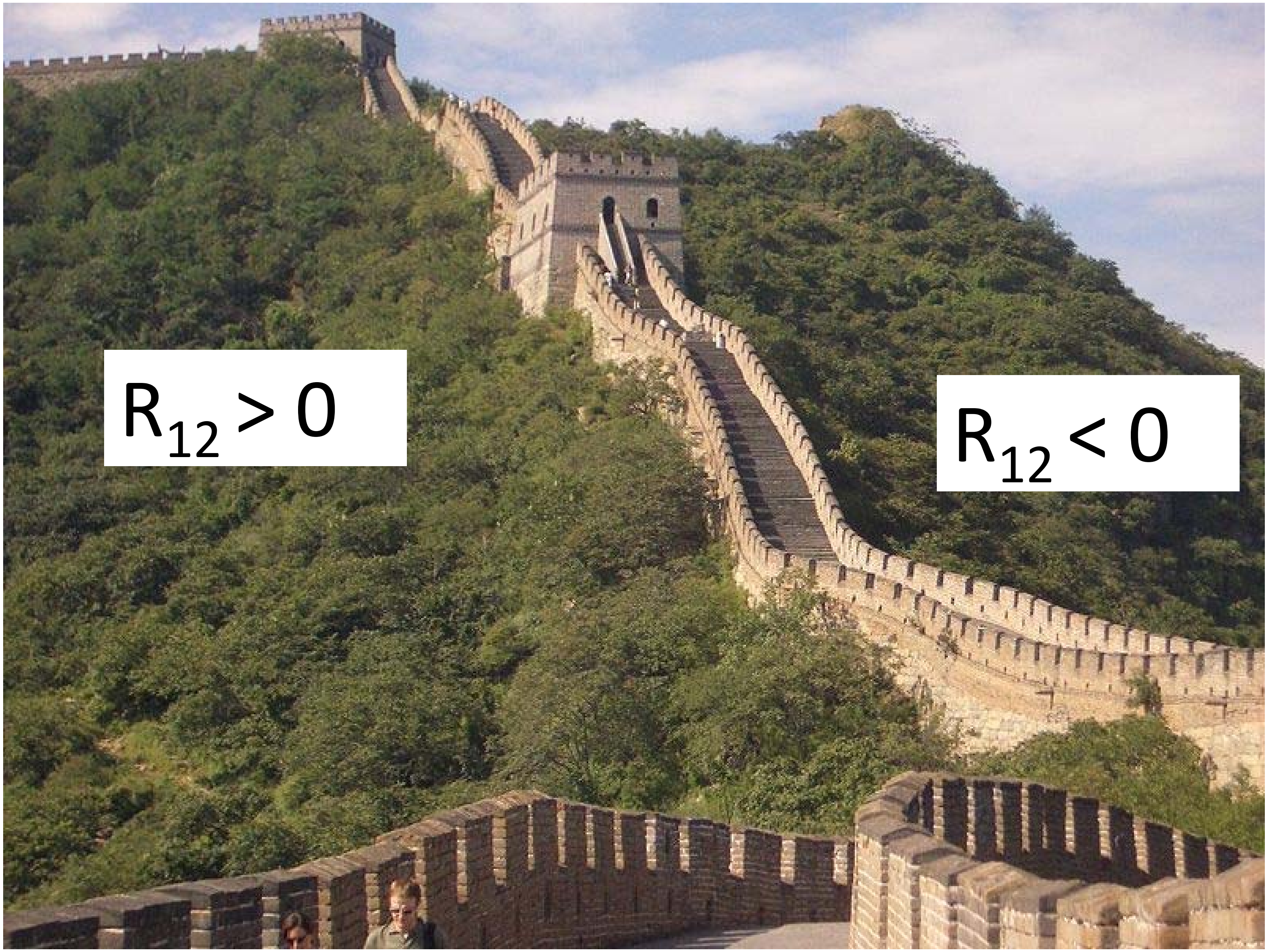
$$R_{12}(u) = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{2\mathrm{Im}\left(Z_{\gamma_1}(u) Z_{\gamma_2}(u)^*\right)}$$

The Z's are functions of the moduli $u \in \mathcal{B}$

So the moduli space of vacua $\mathcal{B}$

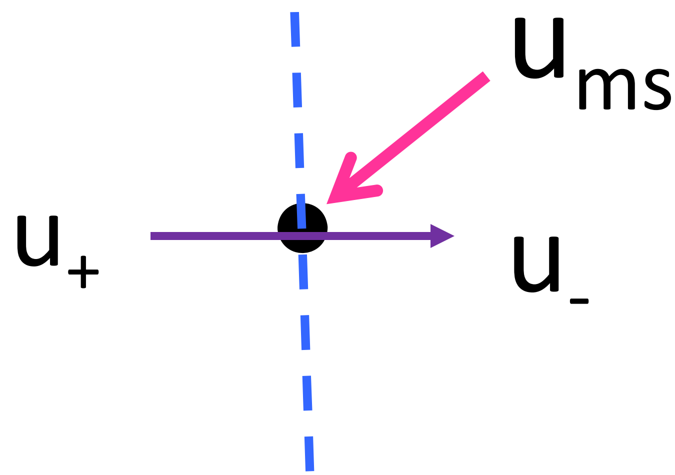
is divided into two regions:

$\mathrm{Im}(Z_1 Z_2^*) > 0$ **OR** $\mathrm{Im}(Z_1 Z_2^*) < 0$


$R_{12} > 0$
$R_{12} < 0$

Wall of Marginal Stability

Consider a path of vacua crossing the wall:



Exact binding energy:

$$|Z_{\gamma_1+\gamma_2}(u)| - (|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|) \leq 0$$

$$MS(\gamma_1, \gamma_2) := \{u | Z_{\gamma_1}(u) \parallel Z_{\gamma_2}(u)\}$$

The Primitive Wall-Crossing Formula

(Denef & Moore, 2007; several precursors)

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2 \mathrm{Im}(Z_1 Z_2^*)}$$

Crossing the wall: $\mathrm{Im}(Z_1 Z_2^*) \to 0$

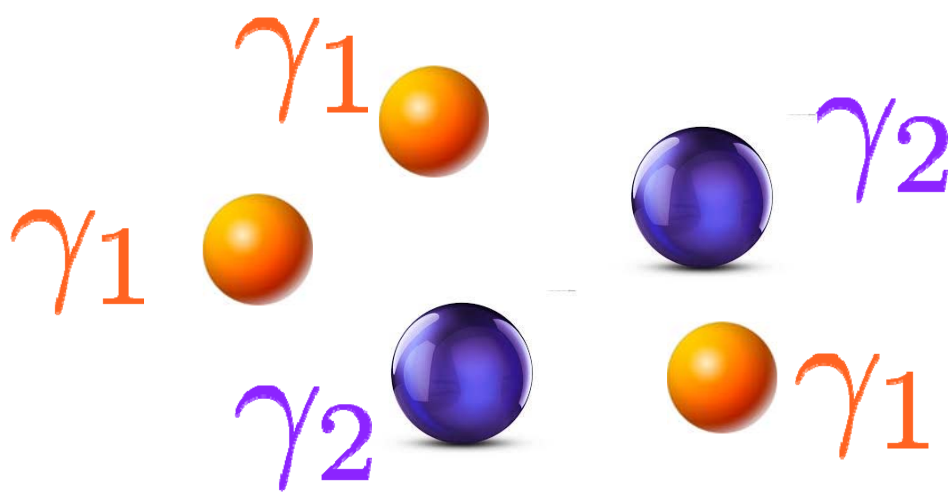
$$\Delta \mathcal{H} = \mathcal{H}^{\mathrm{spin}}_{J_{12}} \otimes \mathcal{H}^{\mathrm{BPS}}_{\gamma_1} \otimes \mathcal{H}^{\mathrm{BPS}}_{\gamma_2}$$

$$2J_{12} + 1 = |\langle \gamma_1, \gamma_2 \rangle|$$

Non-Primitive Bound States

But this is not the full story, since the *same* marginal stability wall holds for charges $N_1 \gamma_1$ and $N_2 \gamma_2$ for $N_1, N_2 > 0$

The primitive wall-crossing formula assumes the charge vectors γ_1 and γ_2 are *primitive vectors*.



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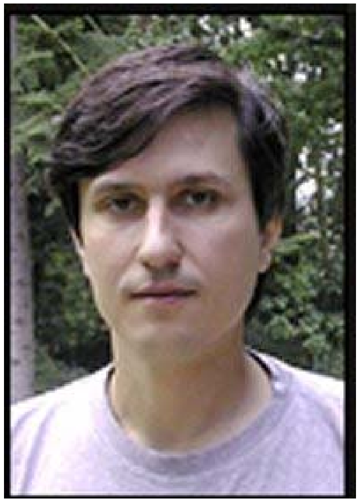

Kontsevich-Soibelman WCF

In 2008 K & S wrote a wall-crossing formula for Donaldson-Thomas invariants of Calabi-Yau manifolds…. But stated in a way that could apply to ``BPS indices'' in more general situations.

We needed a physics argument for why their formula should apply to d=4, $\mathcal{N}$=2 field theories.

There are now several physical derivations explaining that the KSWCF is indeed the appropriate formula for general boundstates.

In my view -- the best derivation uses ``line operators'' – or more properly - ``line defects.''

Gaiotto, Moore, Neitzke 2010; Andriyash, Denef, Jafferis, Moore 2010

Political Advertisement

tive WCF
or d=4 was
Moore in
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There are other physical derivations of the KSWCF due to Cecotti & Vafa and Manschot, Pioline, & Sen.

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Interlude: Defects in Local QFT

The very notion of ``what is a quantum field theory'' is evolving…

It no longer suffices just to know the correlators of all local operators.

Extended ``operators'' or ``defects'' have been playing an increasingly important role in recent years in quantum field theory.

Defects are local disturbances supported on submanifolds of spacetime.

Examples of Defects

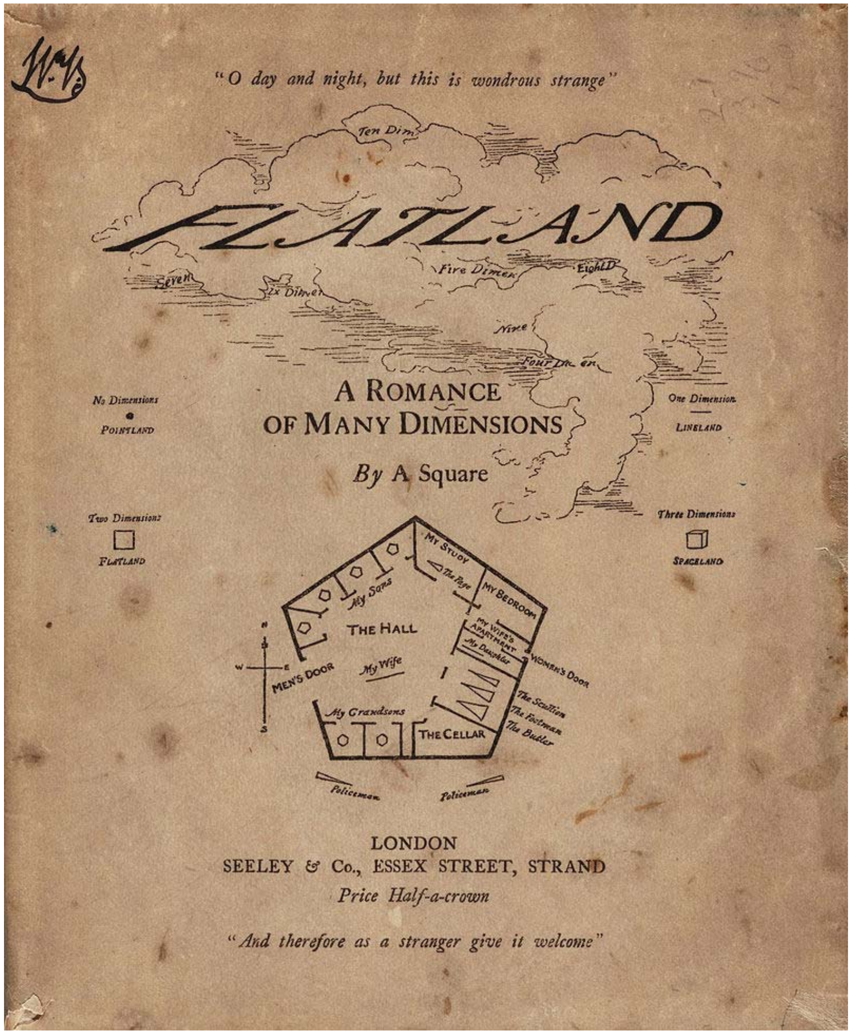
Example 1: d=0: Local Operators

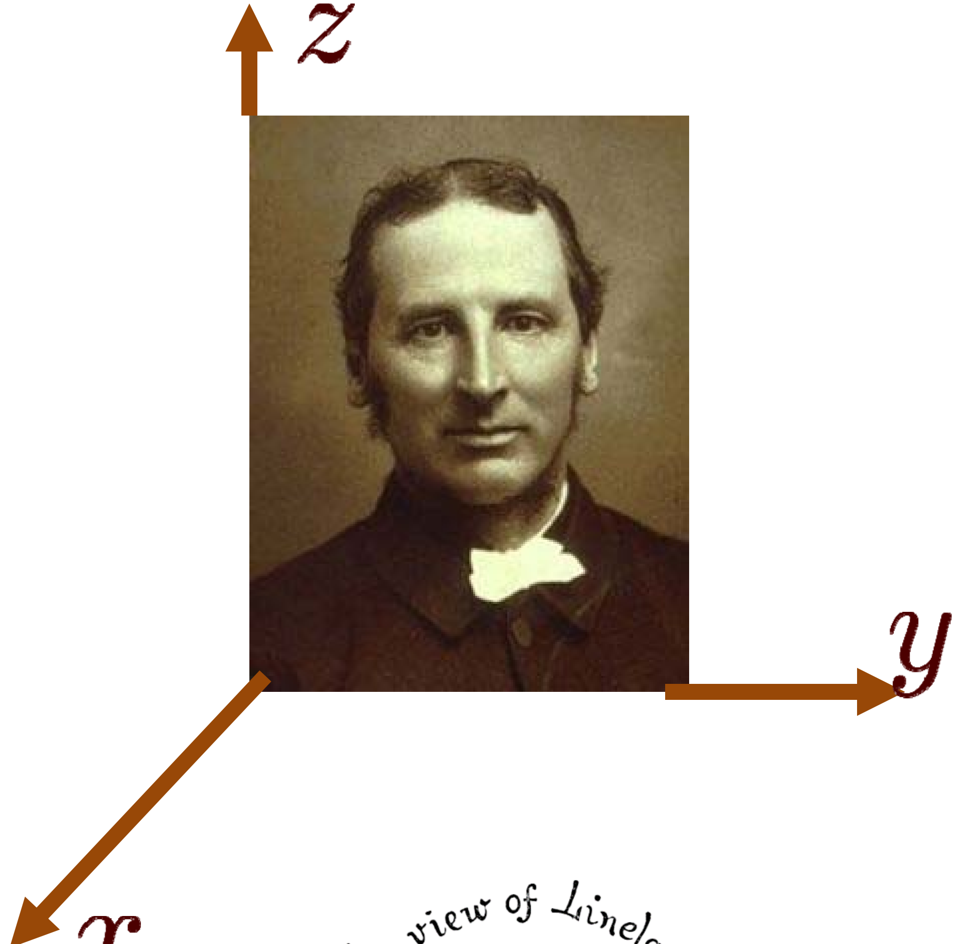
Example 2: d=1: ``Line operators''

Gauge theory
Wilson line:

$$W_R(\ell) = \mathrm{Tr}_R \mathrm{Pexp} \oint_\ell A$$

Example 3: Surface defects: Couple a 2-dimensional field theory to an ambient 4-dimensional theory.


"O day and night, but this is wondrous strange"
FLATLAND
A ROMANCE
OF MANY DIMENSIONS
By A Square
No Dimensions
POINTLAND
One Dimension
LINELAND
Two Dimensions
FLATLAND
Three Dimensions
SPACELAND
THE HALL
THE CELLAR
MEN'S DOOR
WOMEN'S DOOR
LONDON
SEELEY & Co., ESSEX STREET, STRAND
Price Half-a-crown
"And therefore as a stranger give it welcome"


z
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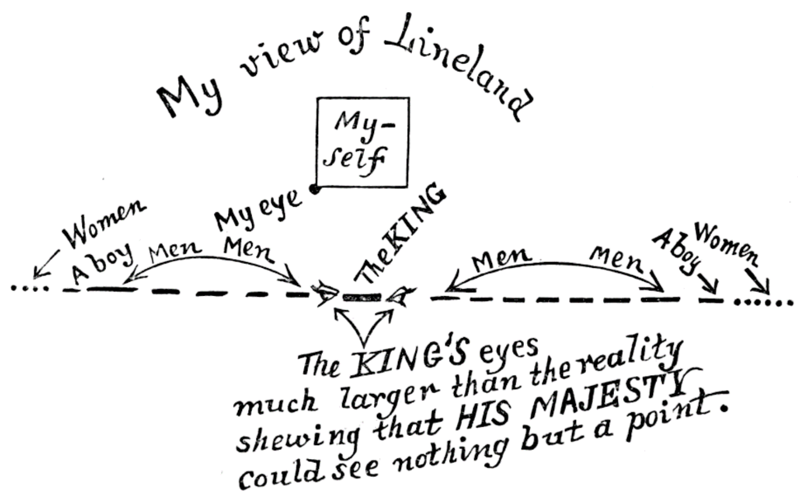
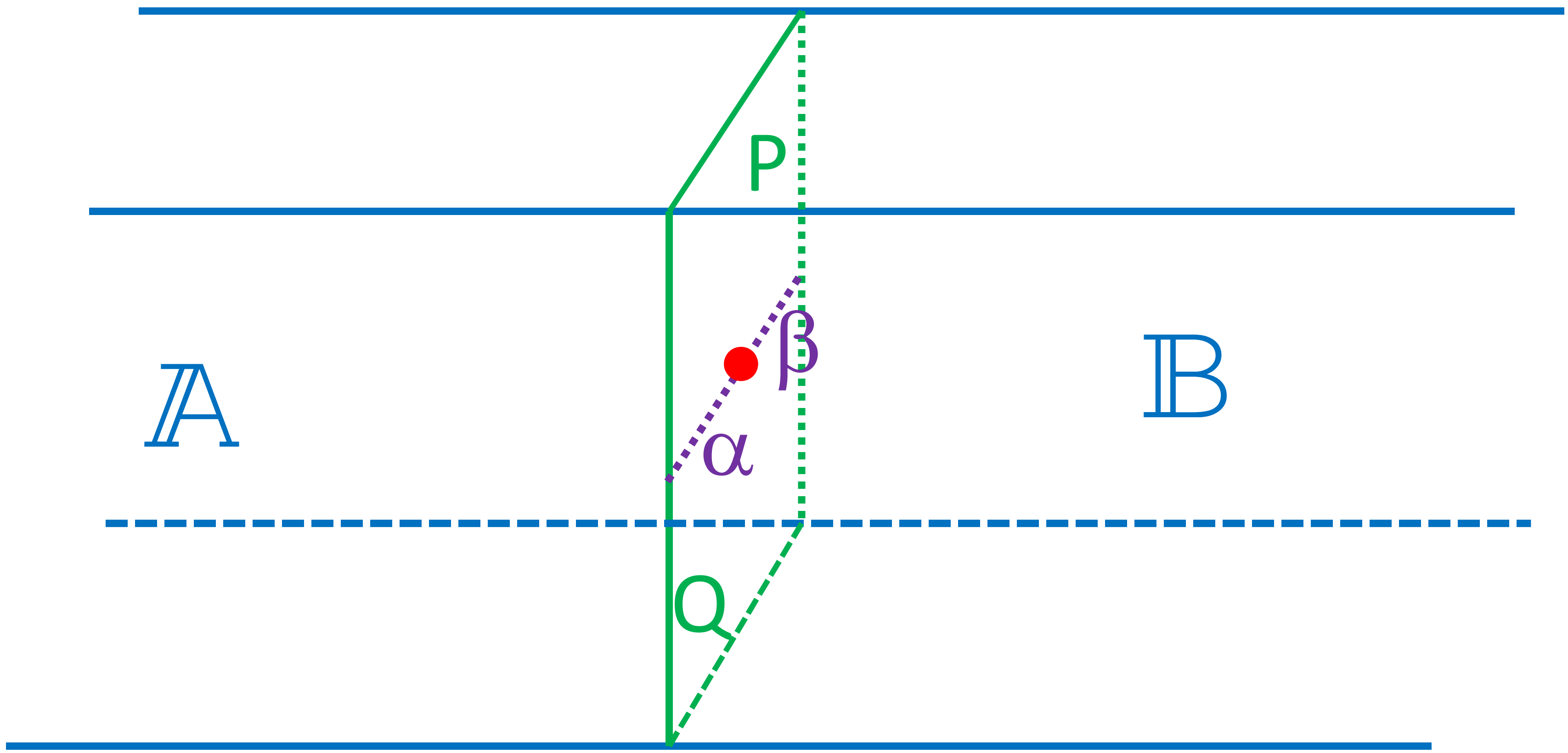
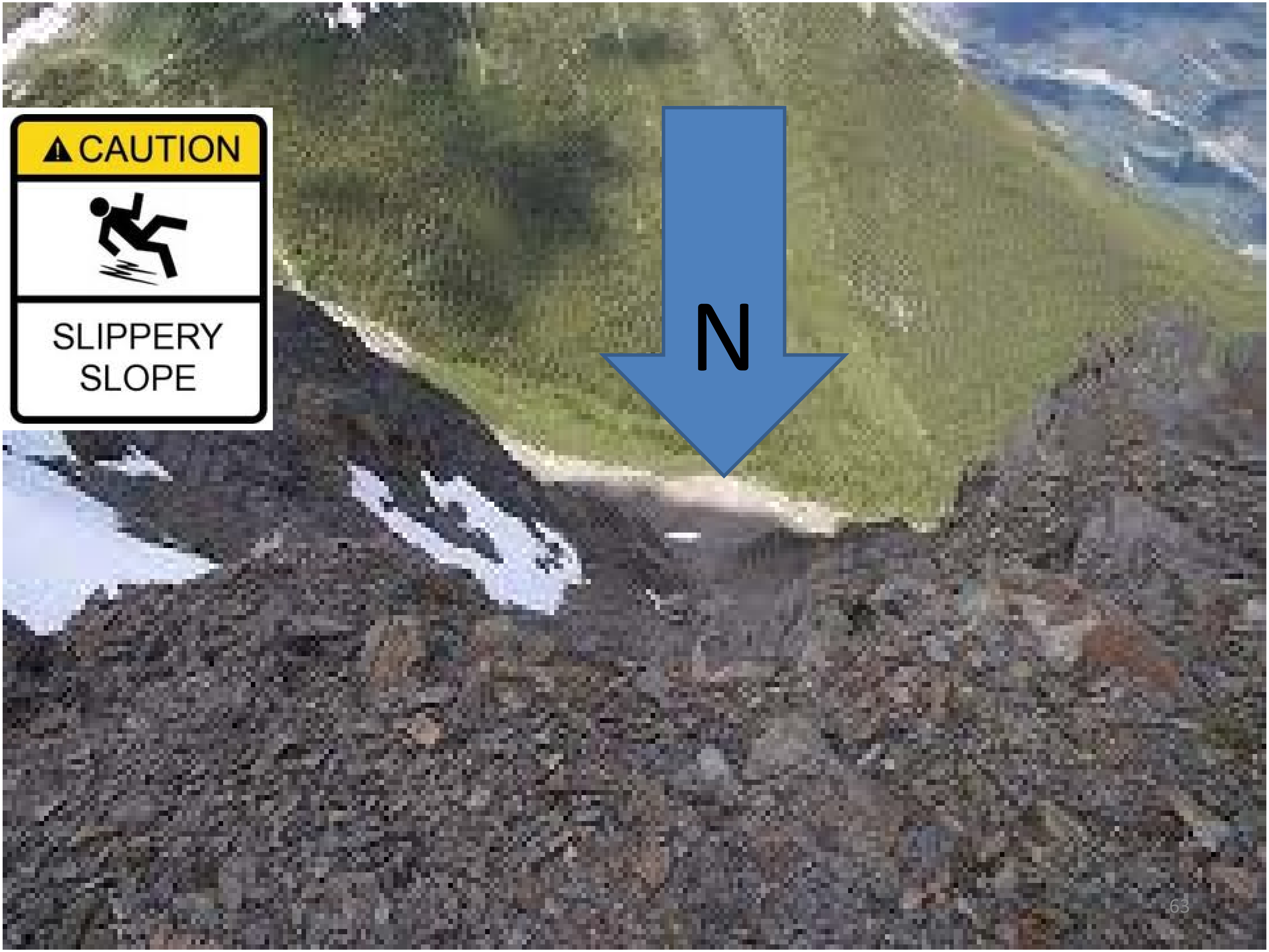

My view of Lineland
My-self
My eye
Women
A boy
Men
Men
The KING
Men
Men
A boy
Women
The KING'S eyes much larger than the reality shewing that HIS MAJESTY could see nothing but a point.

Defects Within Defects



Mathematically – related to N-categories….


CAUTION
SLIPPERY
SLOPE
N

1 What can d=4, $\mathcal{N}$=2 do for you?

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Defects in Quantum Field Theory

5 Theories of Class S & Spectral Networks

6 Conclusion

7 Wall Crossing 201

Wall-Crossing: Only half the battle…

The wall crossing formula only describes the CHANGE of the BPS spectrum across a wall of marginal stability.

It does NOT determine the BPS spectrum!

This problem has been solved for a large class of d=4 $\mathcal{N}$=2 theories known as

``theories of class S''

An important part of the GMN project focused on a special class of d=4, $\mathcal{N}$=2 theories,

the theories of class S.

("S" is for six)

The six-dimensional theories

Claim, based on string theory constructions:

There is a family of stable interacting field theories, S[$\mathfrak{g}$] , with six-dimensional (2,0) superconformal symmetry. (Witten; Strominger; Seiberg).

These theories have not been constructed – even by physical standards - but some characteristic properties of these hypothetical theories can be deduced from their relation to string theory and M-theory.

These properties will be treated as axiomatic. (c.f. Felix Klein lectures in Bonn). Later - theorems.

Theories Of Class S

d=6 superconformal theory → d=4 $\mathcal{N}$ =2 theory

Most ``natural'' theories are of class S:

For example, SU(K) $\mathcal{N}$=2 SYM coupled to ``quark flavors''.

But there are also (infinitely many) theories of class S with no (known) Lagrangian (Gaiotto, 2009).

In these theories many physical quantities have elegant descriptions in terms of Riemann surfaces and flat connections.

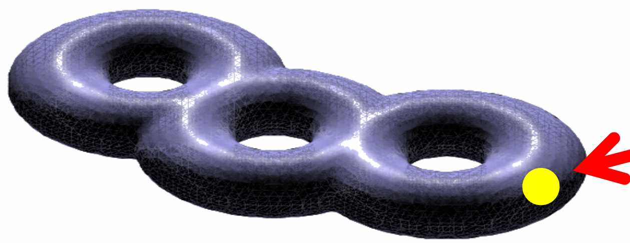
→

Relations to many interesting mathematical topics:

Moduli spaces of flat connections, character varieties, Teichmüller theory, Hitchin systems, integrable systems, Hyperkähler geometry …

Surface Defects In Theories Of Class S

C

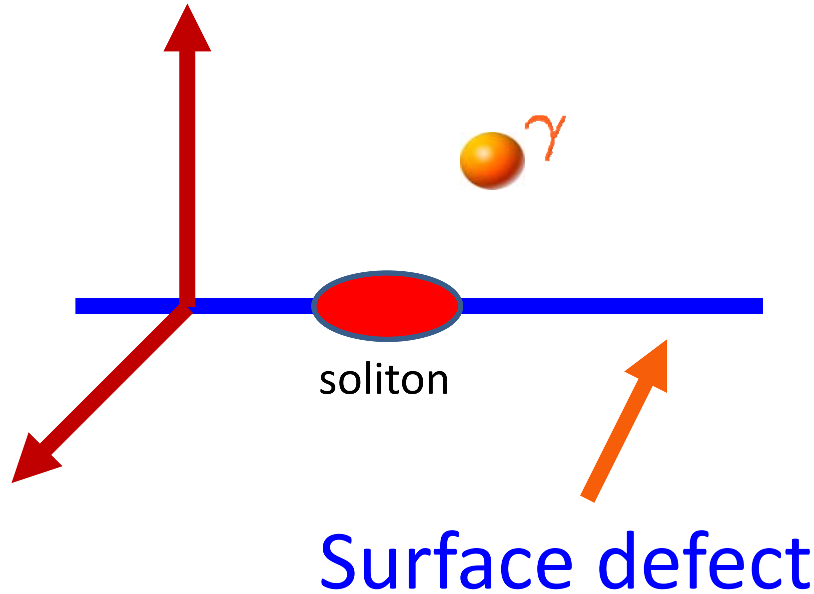


$z \in C$

For each $z \in C$ we have a ***surface defect*** $\mathbb{S}_z$

$\mathbb{S}_z$ is a 1+1 dimensional QFT in $\mathbb{M}^{1,3}$.
It couples to the ambient four-dimensional theory.

$\mathbb{S}_z$ has BPS solitons and they have an $\mathcal{N}$=2 central charge as well.



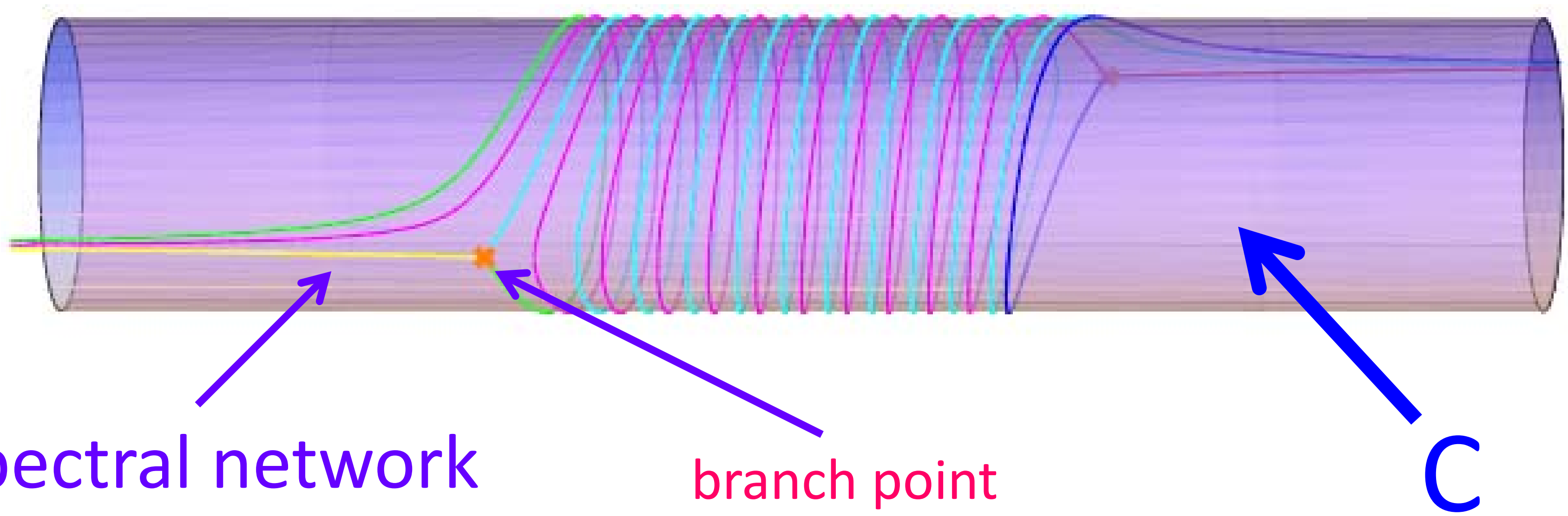
The behavior of d=2 BPS solitons on the surface defects $\mathbb{S}_z$ turns out to *encode* the spectrum of d=4 BPS states.

The key construction involves ``*spectral networks*''

What are Spectral Networks ?

(For mathematicians)

Spectral networks are combinatorial objects associated to a covering of Riemann surfaces $\Sigma \longrightarrow C$, with differential λ on Σ



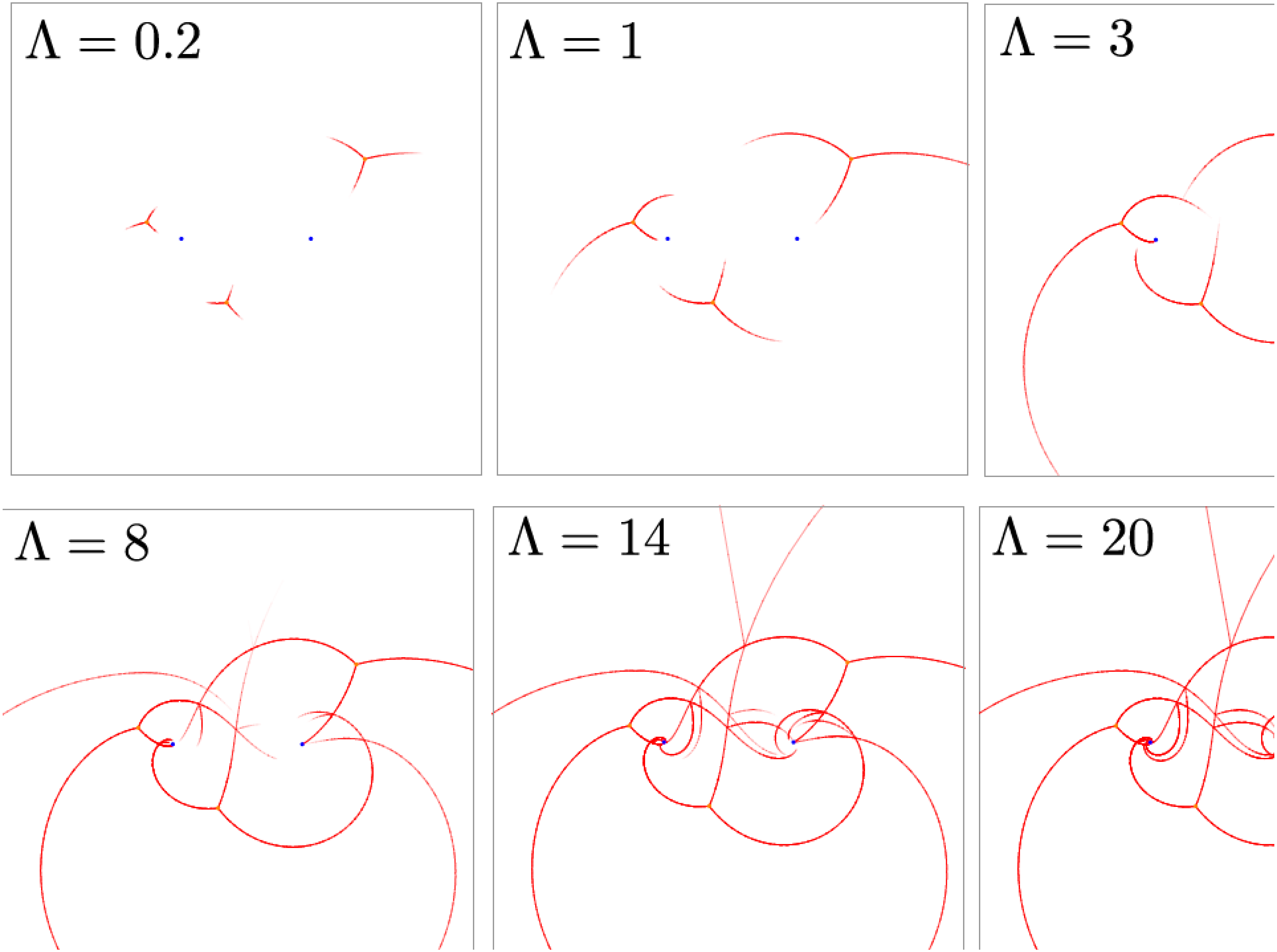
(For physicists)

Spectral networks are defined, physically, by considering BPS solitons on the two-dimensional surface defect $\mathbb{S}_z$

Choose a phase $\zeta = e^{i\vartheta}$

SN: The set of points $z \in C$ so that there are solitons in $\mathbb{S}_z$ with $\mathcal{N}$=2 central charge of phase ϑ

Can be constructed using local rules



$\Lambda = 0.2$
$\Lambda = 1$
$\Lambda = 3$
$\Lambda = 8$
$\Lambda = 14$
$\Lambda = 20$

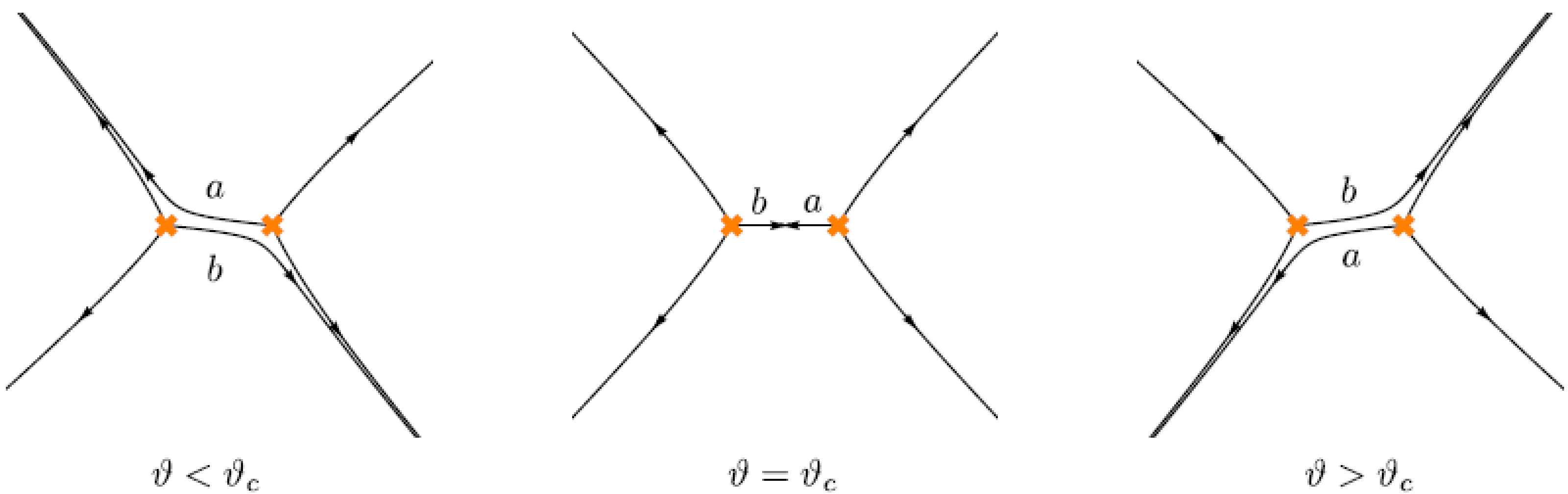
When we vary the phase ϑ the network changes continuously except at certain critical phases ϑ_c

The critical networks encode facts about the *four-dimensional* BPS spectrum.

For example, ϑ_c turns out to be the phase of $Z_\gamma(u)$ of the d=4 BPS particle.

Movies:
http://www.ma.utexas.edu/users/neitzke/spectral-network-movies/

Make your own: [Chan Park & Pietro Longhi]
http://het-math2.physics.rutgers.edu/loom/



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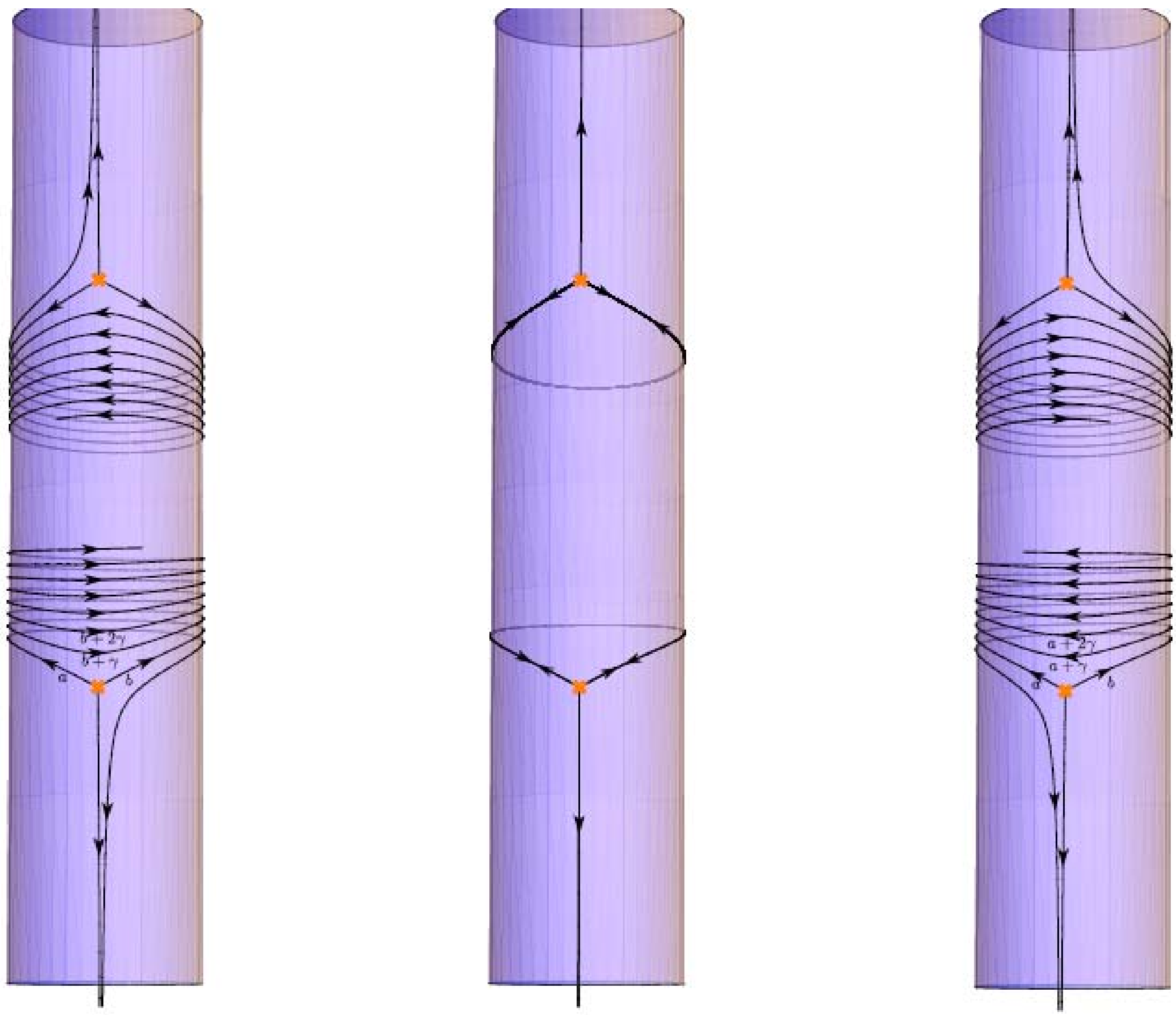
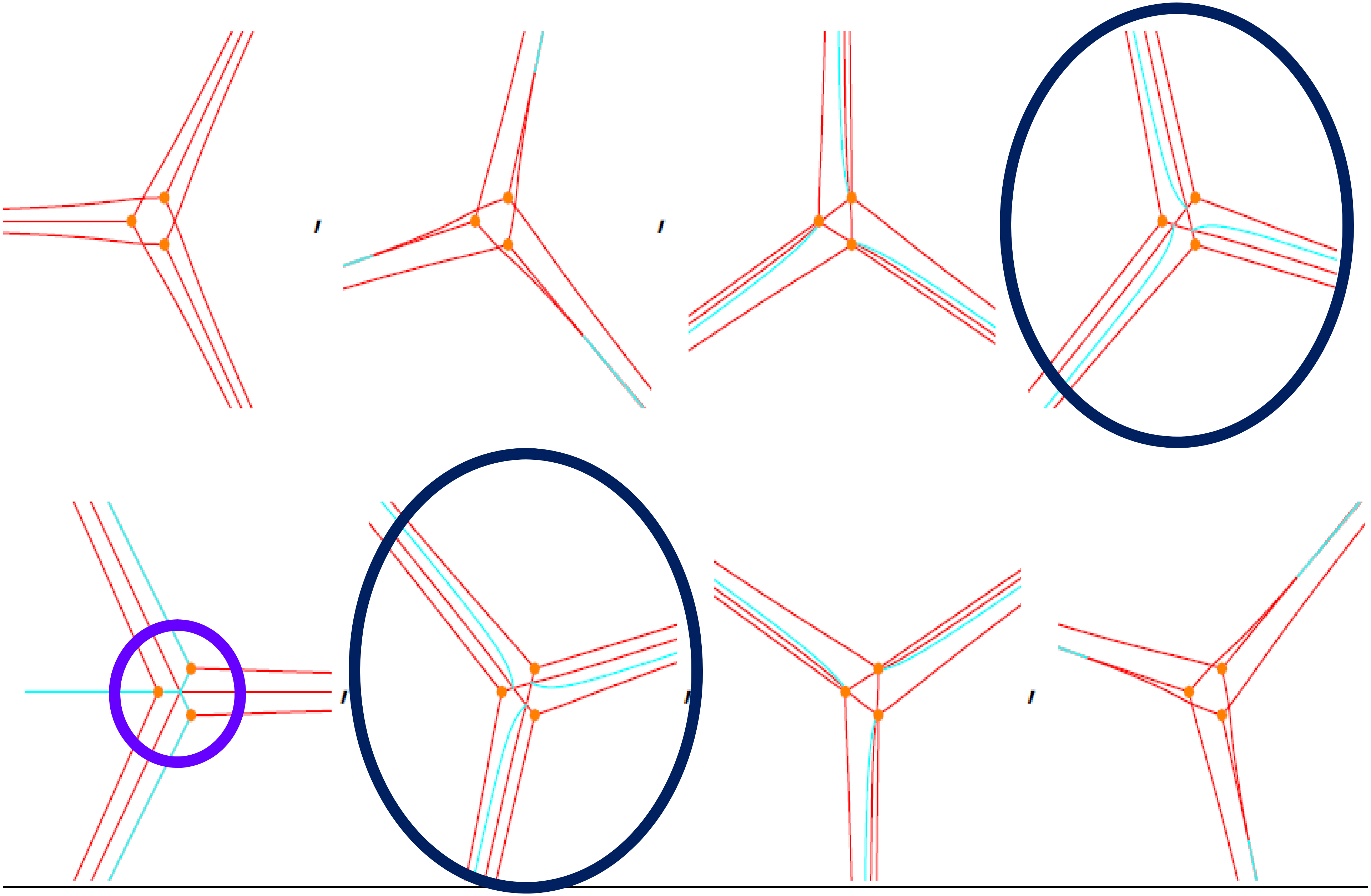


$\vartheta < \vartheta_c$ $\vartheta = \vartheta_c$ $\vartheta > \vartheta_c$



Finding the BPS Spectrum

One can write very explicit formulae for the BPS indices $\Omega(\gamma)$ in terms of the combinatorics of the change of the spectral network.

GMN, Spectral Networks, 1204.4824

GMN, Spectral Networks and Snakes, 1209.0866

Galakhov, Longhi, Moore: Include spin information

Mathematical Applications of Spectral Networks

They construct a system of coordinates on moduli spaces of flat connections on C which generalize the cluster coordinates of Thurston, Penner, Fock and Goncharov.

WKB asymptotics for first order matrix ODE's:

$$\left(\hbar \frac{d}{dz} + A\right) \Psi = 0$$

(generalizing the Schrodinger equation)

Spectral network = generalization of Stokes lines

Conclusion: 3 Main Messages

1. Seiberg and Witten’s breakthrough in 1994, opened up many interesting problems. Some were quickly solved, but some remained stubbornly open.

But the past eight years has witnessed a renaissance of the subject, with a much deeper understanding of the BPS spectrum and the line and surface defects in these theories.

Conclusions: Main Messages

2. This progress has involved nontrivial and surprising connections to other aspects of Physical Mathematics:

Hyperkähler geometry, cluster algebras, moduli spaces of flat connections, Hitchin systems, instantons, integrable systems, Teichmüller theory, …

1 2 3 S-Duality and the modular groupoid

4 5 6 7 8

9 10 11 12 13 14 AGT: Liouville & Toda theory

15 16 Higgs_branches

17 18 Cluster algebras

19 20 Holographic duals

21 N=4 scattering

22 23 24 Ω-backgrounds, Nekrasov partition functions, Pestun localization.

25 26 $Z(S^3 \times S^1)$ Scfml indx

27 28 29 30 Nekrasov-Shatashvili: Quantum Integrable systems

Rest rooms

31 32 33 34 35 Three dimensions, Chern-Simons, and mirror symmetry

Conclusions: Main Messages

3. There are nontrivial superconformal fixed points in 6 dimensions.

(They were predicted many years ago from string theory.)

We have seen that the mere existence of these theories leads to a host of nontrivial results in quantum field theory.

Still, formulating 6-dimensional superconformal theories in a mathematically precise way remains an outstanding problem in Physical Mathematics.

A Central Unanswered Question

Can we construct $S[\mathfrak{g}]$?

MIND THE GAP


NOT
That's all Folks!

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We will now show how susy line defects give a physical interpretation & derivation of the Kontsevich-Soibelman wall-crossing formula.

Gaiotto, Moore, Neitzke; Andriyash, Denef, Jafferis, Moore

Supersymmetric Line Defects

Our line defects will be at $\mathbb{R}_t \times \{ 0 \} \subset \mathbb{R}^{1,3}$

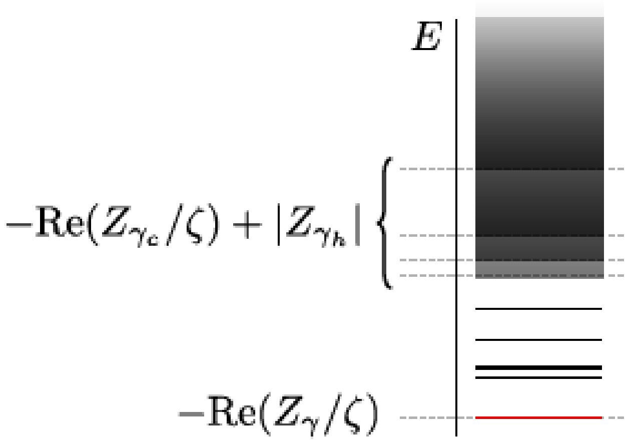
A supersymmetric line defect L requires a choice of phase ζ :

Example: $L_\zeta = \mathrm{Tr}_R \mathrm{P}\exp \int_{\mathbb{R}_t \times \vec{0}} \left(\zeta^{-1} \varphi + A + \zeta \bar{\varphi} \right)$

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma + \gamma_0} \mathcal{H}_{L,\gamma}$$

Physical picture for charge sector γ: As if we inserted an infinitely heavy BPS particle of charge γ

Framed BPS States



$$E \geq -\mathrm{Re}(Z_\gamma/\zeta)$$

Framed BPS States are states in $\mathcal{H}_{L,\gamma}$ which saturate the bound.

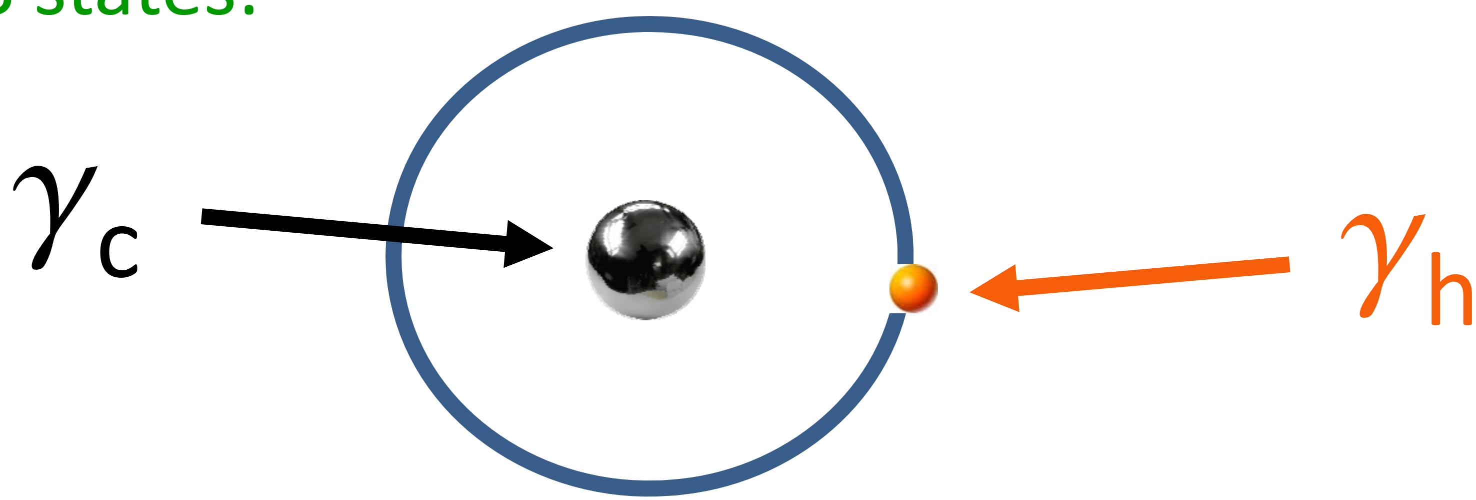
$$\overline{\underline{\Omega}}(L_\zeta;\gamma) := \mathrm{Tr}_{\mathcal{H}_{L_\zeta,\gamma}}(-1)^{2J_3}$$

So, there are *two* kinds of BPS states:

Ordinary/vanilla: $\Omega(\gamma; u)$

Framed: $\overline{\underline{\Omega}}(L_{\zeta}; \gamma)$

Vanilla BPS particles of charge γ_h can bind to framed BPS states in charge sector γ_c to make new framed BPS states:



Framed BPS Wall-Crossing 1/2

Particles of charge γ_h bind to a ``core'' of charge γ_c at radius:

$$r = \frac{\langle \gamma_h, \gamma_c \rangle}{2\mathrm{Im}(Z_{\gamma_h}(u)/\zeta)}$$

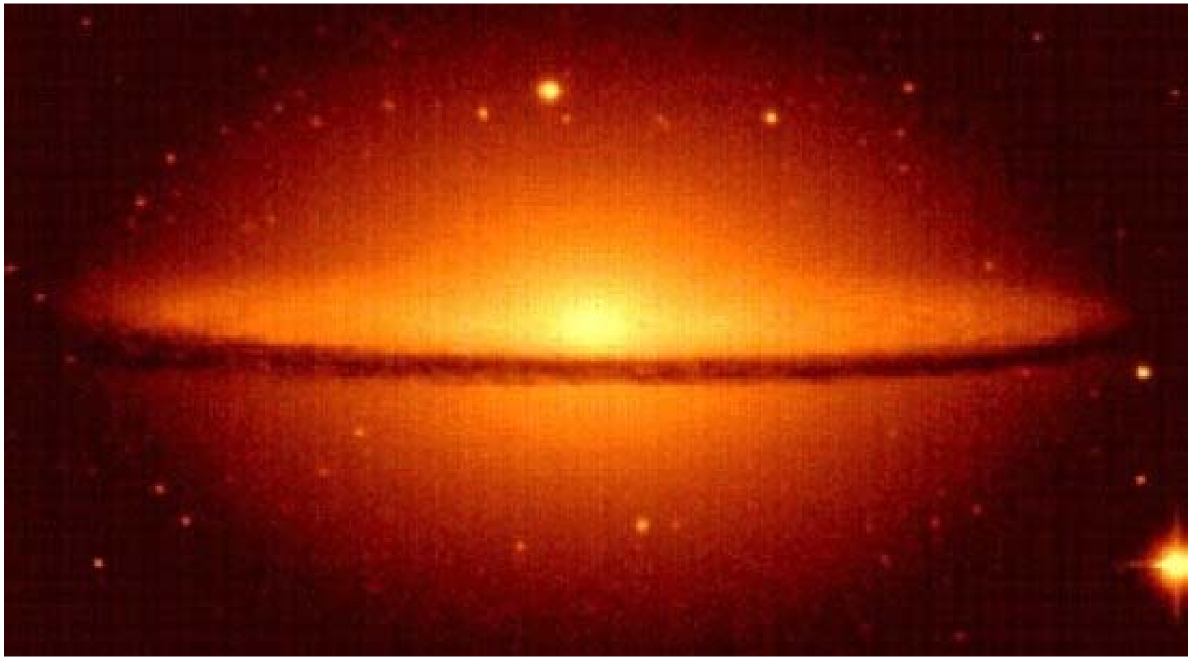
So crossing a ``***BPS wall***'' defined by:

$$W_{\gamma_h} := \{u | Z_{\gamma_h}(u)/\zeta \in \mathbb{R}_-\}$$

the bound state comes (or goes).

Halo Picture

But, particles of charge γ_h, and indeed n γ_h for any n>0 can bind in arbitrary numbers: they feel no relative force, and hence there is an entire *Fock space* of boundstates with halo particles of charges n γ_h.



Framed BPS Wall-Crossing 2/2

So across the BPS walls

$$W_{\gamma_h} := \{u | Z_{\gamma_h}(u)/\zeta \in \mathbb{R}_-\}$$

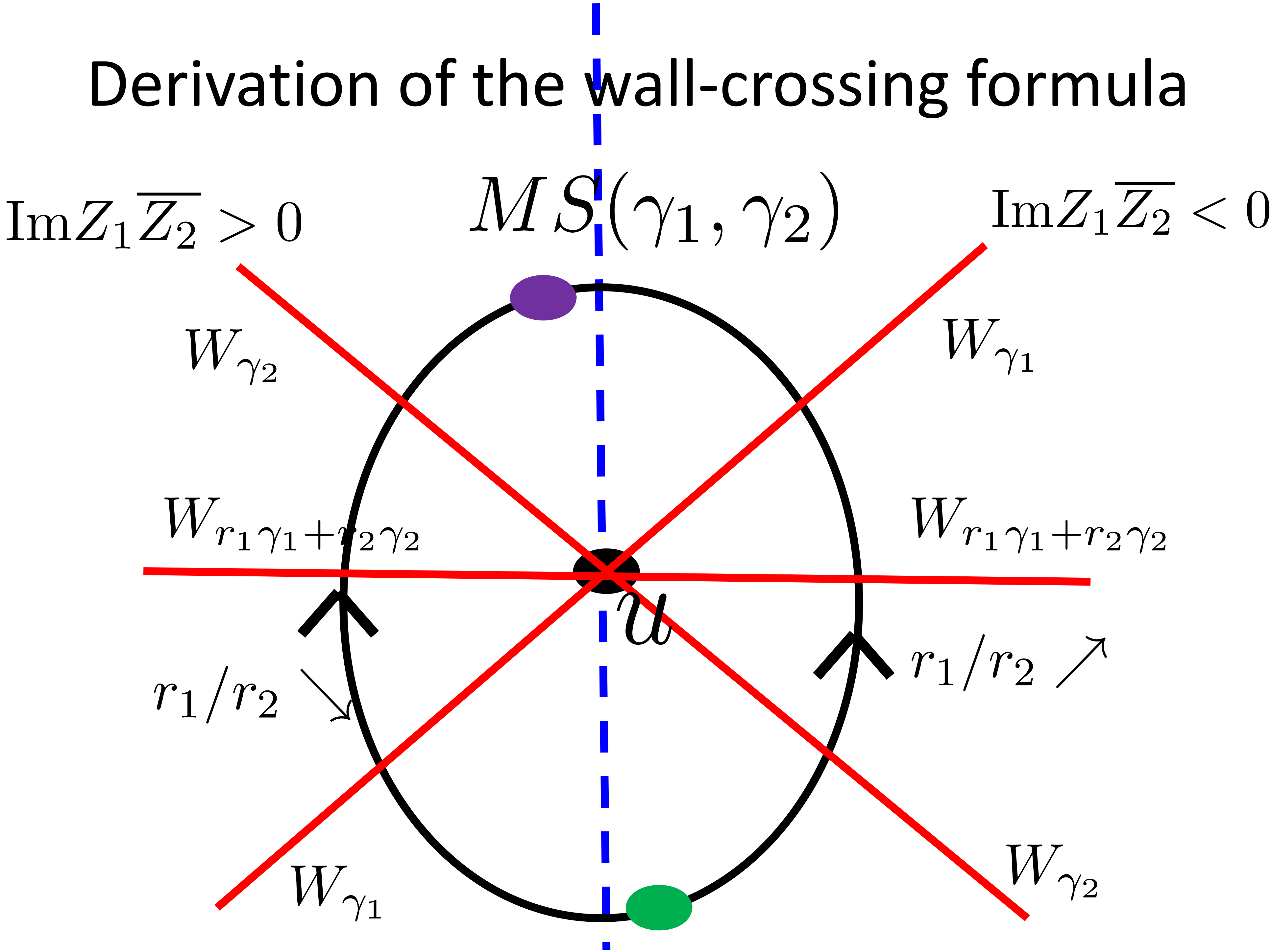
entire Fock spaces of boundstates come/go.

Introduce ``Fock space creation operator'' for Fock space of a particle of charge γ_h: K_{γ_h}

Suppose a path in $\mathcal{B}$ crosses walls $W_{\gamma_1}, W_{\gamma_2}, \ldots$

$$K_{\gamma_1}^{\Omega(\gamma_1)} K_{\gamma_2}^{\Omega(\gamma_2)} \cdots$$

Derivation of the wall-crossing formula



The Kontsevich-Soibelman Formula

$$\prod_{\searrow} K_{r_1\gamma_1+r_2\gamma_2}^{\Omega(r_1\gamma_1+r_2\gamma_2;-)} = \prod_{\nearrow} K_{r_1\gamma_1+r_2\gamma_2}^{\Omega(r_1\gamma_1+r_2\gamma_2;+)}$$

$$K_{\gamma_2}^{\Omega(\gamma_2;u_-)} \cdots K_{\gamma_1}^{\Omega(\gamma_1;u_-)}$$

$$= \quad K_{\gamma_1}^{\Omega(\gamma_1;u_+)} \cdots K_{\gamma_2}^{\Omega(\gamma_2;u_+)}$$

A Good Analogy

$$S_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_2;+)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_1+\gamma_2;+)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_1;+)}$$

$$S_- = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_1;-)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_1+\gamma_2;-)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{\Omega(\gamma_2;-)}$$

$$S_+ = S_-$$

$$\Omega(\gamma_1;+) = \Omega(\gamma_1;-) \qquad \Omega(\gamma_2;+) = \Omega(\gamma_2;-)$$

$$\Omega(\gamma_1+\gamma_2;+) = \Omega(\gamma_1+\gamma_2;-) + \Omega(\gamma_1;-)\Omega(\gamma_2;-)$$