Three Birthday Nuggets For Igor

Gregory Moore Rutgers

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Two Constructions of Affine Lie Algebra Representations and Boson–Fermion Correspondence in Quantum Field Theory

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Communicated by the Editor

We establish an isomorphism between the vertex and spinor representations of affine Lie algebras for types $D_l^{(1)}$ and $D_{l+1}^{(2)}$. We also study decomposition of spinor representations using the infinite family of Casimir operators and prove that they are either irreducible or have two irreducible components. We show that the vertex and spinor constructions of the representations can be reformulated in the language of two-dimensional quantum field theory. In this physical context, the two constructions yield the generalized sine-Gordon and Thirring models, respectively, already in renormalized form. The isomorphism of representations implies an equivalence of these two models which is known in quantum field theory as the boson–fermion correspondence

Contents.

Good times @ Yale:

Zamolodchikov's tetrahedral equations

Geometry-Symmetry-Physics Seminar

Dinner discussions: Igor would present his broad and beautiful vision of what is and is not important in the development of math. *Very* original viewpoints.

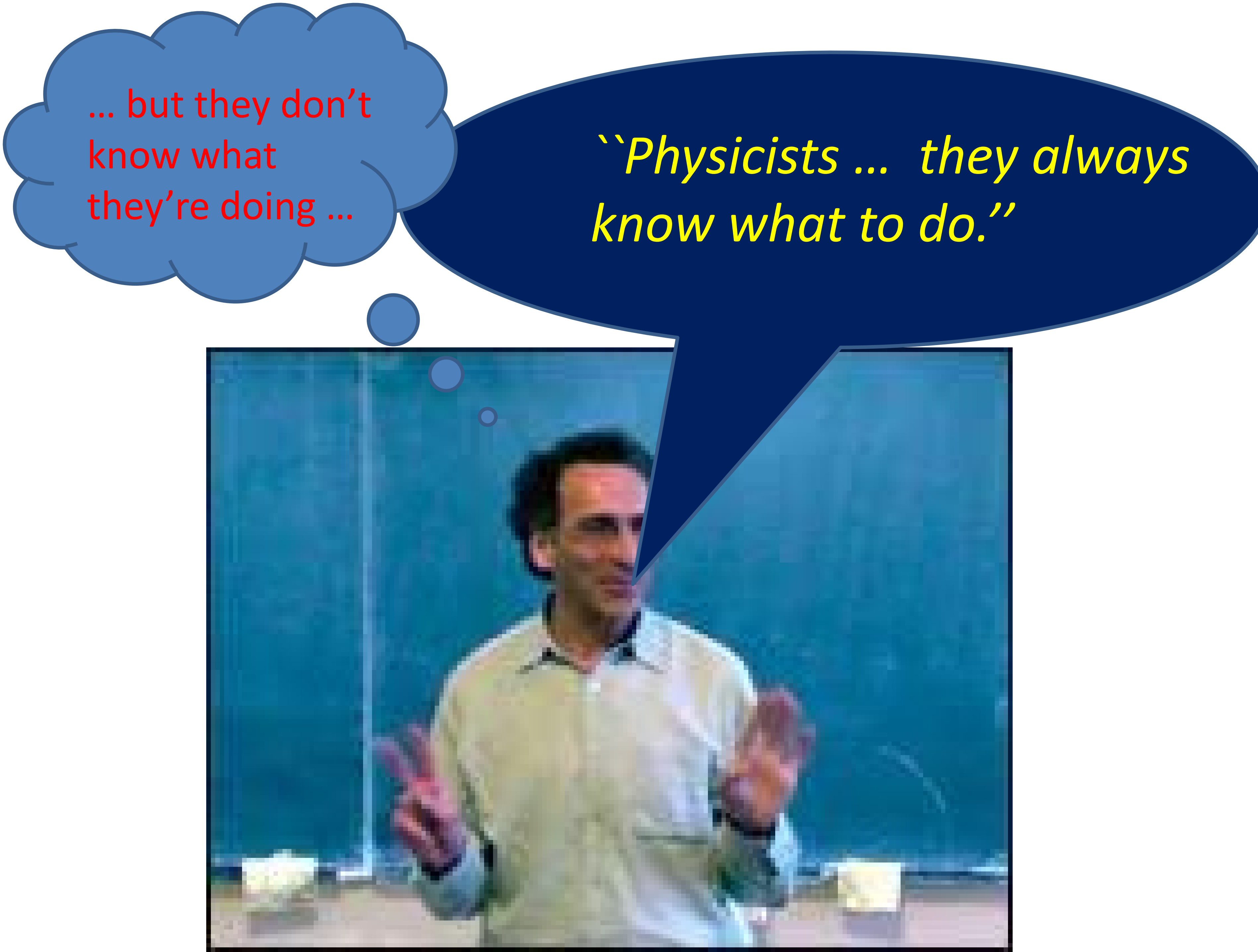
On the work of Igor Frenkel

Introduction

by Pavel Etingof

Igor Frenkel is one of the leading representation theorists and mathematical physicists of our time. Inspired by the mathematical philosophy of Herman Weyl, who recognized the central role of representation theory in mathematics and its relevance to quantum physics, Frenkel made a number of foundational contributions at the juncture of these fields. A quintessential mathematical visionary and romantic, he has rarely followed the present day fashion. Instead, he has striven to get ahead of time and get a glimpse into the mathematics of the future – at least a decade, no less. In this, he has followed the example of I. M. Gelfand, whose approach to mathematics has always inspired him. He would often write several foundational papers in a subject, and then leave it for the future generations to be developed further. His ideas have sometimes been so bold and ambitious and so much ahead of their time that they would not be fully appreciated even by his students at the time of their formulation, and would produce a storm of activity only a few years later. And, of course, as a result, many of his ideas are still waiting for their time to go off.

This text is a modest attempt by Igor's students and colleagues of various generations to review his work, and to highlight how it has influenced in each case the development of the corresponding field in subsequent years.


… but they don't know what they're doing …
``Physicists … they always know what to do.''

NUGGET 1

Moonshine Phenomena, Supersymmetry, and Quantum Codes

A. SOME BACKGROUND

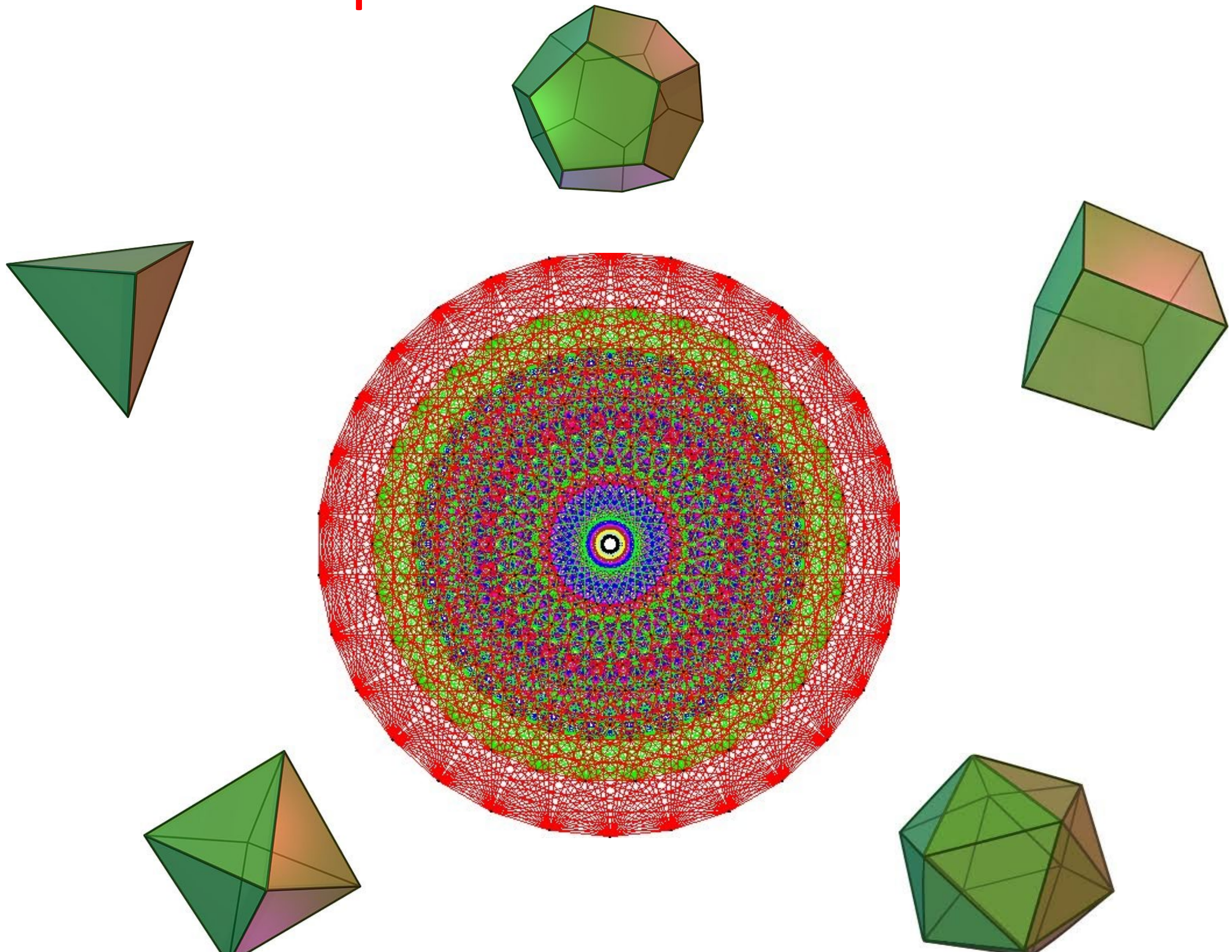
Philosophy – 1/2

We can divide physicists into two types:

Our world is a random choice drawn from a huge ensemble:

Philosophy – 2/2

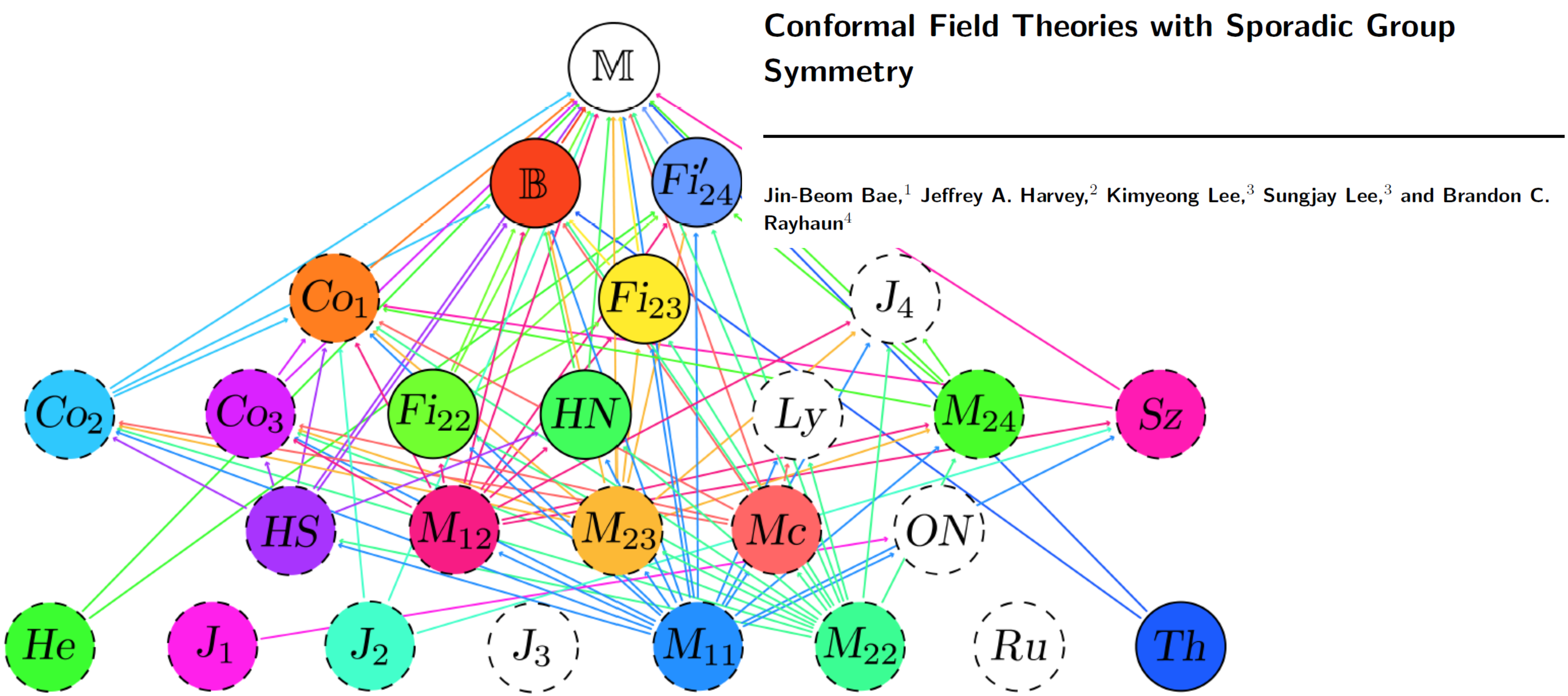
The fundamental laws of nature are based on some beautiful exceptional mathematical structure:



Finite-Simple Groups

Jordan-Holder Theorem: Finite simple groups are the atoms of finite group theory.

$\mathbb{Z}_p$ p = prime A_n $n \geq 5$ $SL_n(\mathbb{F}_p)$ etc.



Conformal Field Theories with Sporadic Group Symmetry

Jin-Beom Bae,[1] Jeffrey A. Harvey,[2] Kimyeong Lee,[3] Sungjay Lee,[3] and Brandon C. Rayhaun[4]

McKay & Conway-Norton 1978-1979

$$J = \sum_n J_n q^n = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

Compare with the dimensions of the 194 irreps of $\mathbb{M}$

R_n = 1, 196883, 21296876, 842609326, 18538750076, 19360062527, 293553734298,…. , $\sim 2.6 \times 10^{26}$

$J_{-1} = R_1$ $\quad$ $J_1 = R_1 + R_2$

$J_2 = R_1 + R_2 + R_3$ $\quad$ $J_3 = 2R_1 + 2R_2 + R_3 + R_4$

A way of writing J_n as a positive linear combination of the R_j for all n is a ``***solution of the Sum-Dimension Game***.''

There are infinitely many such solutions!!

McKay & Conway-Norton 1978-1979

Which, if any, of these solutions is interesting?

Every solution defines an infinite-dimensional $\mathbb{Z}$-graded representation of $\mathbb{M}$

$$V = q^{-1} R_1 \oplus q(R_1 \oplus R_2) \oplus q^2(R_1 \oplus R_2 \oplus R_3) \oplus \cdots$$

Now for every $g \in \mathbb{M}$ we can compute the character:

$$\chi(q; g) \coloneqq Tr_V \, g \, q^N$$

A solution of the Sum-Dimension game is <u>*modular*</u> if the $\chi(q; g)$ is a modular function in $\Gamma_0(m)$ where $g^m = 1$.

Amazing Fact Of Monstrous Moonshine

There is a unique modular solution of the Sum-Dimension game!

Moreover the $\chi(q;g)$ have very remarkable properties (``genus zero'' etc.)

Much of this is explained by 2d conformal field theory - thanks to the foundational work of Frenkel, Lepowsky, and Meurman.

New Moonshine

Eguchi, Ooguri, Tachikawa 2010

There are analogous moonshine phenomena relating the elliptic genus of K3 to M24.

Cheng, Duncan, Harvey (2012,2013)
"Umbral Moonshine"

The New Moonshine Phenomena Remain Unexplained

There is no known analog of the FLM construction explaining umbral moonshine.

Despite 12 years of intense effort by a small, but devoted, community of physicists and mathematicians….

We don't understand something about symmetries of 2d conformal field theories.

It might be something important. Or maybe not.

Why Should Physicists Care? 1/2

CFT explanation of Monstrous Moonshine by Frenkel, Lepowsky, Meurman, & Borcherds drove many developments in 2d CFT, especially RCFT

Techniques introduced to explain moonshine – orbifolds, VOA, holomorphic CFT have played a key role in other aspects of physics and have led to many important advances…

e.g. modular tensor categories are a direct descendent of this research --

Why Should Physicists Care? 2/2

History repeats itself

Lightning does not strike twice

RCFT Approach To FLM

For the original Monstrous Moonshine:

24 free *chiral* bosons with target space the Leech torus := $\mathbb{R}^{24}/\Lambda$

$\Lambda \subset \mathbb{R}^{24}$ is the Leech lattice,

D25-brane

Moreover, target space torus has a very special ``B-field''

$$S = \int d^2\sigma \, (\, G_{\mu\nu} \, \partial_i x^\mu \, \partial^i x^\nu + B_{\mu\nu}\epsilon^{ij}\partial_i x^\mu \partial_j x^\nu)$$

$\mathbb{Z}_2$ −Orbifold

Now gauge the global symmetry:
$\vec{x} \to -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

$$\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^+ \oplus \mathcal{H}_\Lambda^-$$

Nontrivial Gauge Bundle on S^1
Twist Fields

Identify order two points in the torus $\mathbb{R}^{24}/\Lambda$

$$T_2(\Lambda) \coloneqq \Lambda/2\Lambda$$

Orbifold breaks translation symmetry on Leech torus down to $T_2(\Lambda)$

B −field defines a symplectic form on $T_2(\Lambda)$

$$B(\lambda_1, \lambda_2) = (-1)^{\lambda_1 \cdot \lambda_2}$$

Noncommutative Translations - 2/2

Unbroken translation symmetry realized on Hilbert space as a Heisenberg group:

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{H}\left(T_2(\Lambda)\right) \rightarrow T_2(\Lambda) \rightarrow 0$$

$$T(\lambda_1)T(\lambda_2) = \epsilon(\lambda_1, \lambda_2)T(\lambda_1 + \lambda_2)$$

$$\frac{\epsilon(\lambda_1, \lambda_2)}{\epsilon(\lambda_2, \lambda_1)} = (-1)^{\lambda_1 \cdot \lambda_2}$$

Early example of noncommutative geometry on D-branes induced by a B-field

$\mathcal{S}$ is the unique irreducible representation of the Heisenberg group $\mathcal{H}(T_2(\Lambda))$:

Construct it using γ −matrices.

$\mathcal{S}$: ``Spinor representation “

$$\mathcal{H}_T = \mathcal{F} \otimes \mathcal{S} = \mathcal{H}_T^+ \oplus \mathcal{H}_T^-$$

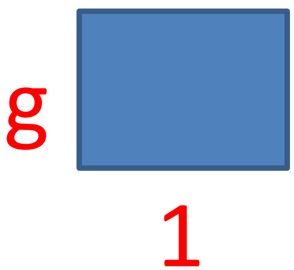
FLM Module

$$\mathcal{H}_{FLM} = \mathcal{H}_{\Lambda}^{+} \oplus \mathcal{H}_{T}^{+}$$

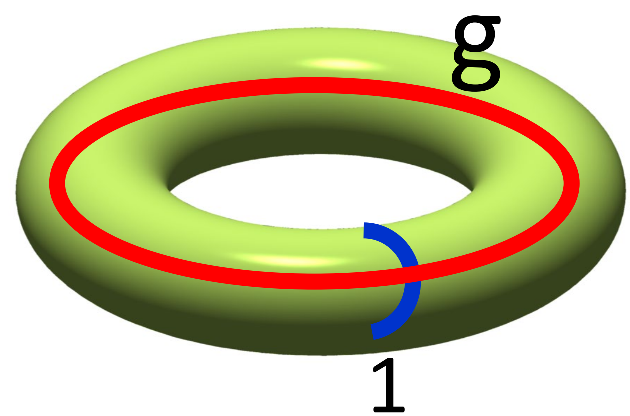
FLM & Borcherds:

The automorphism group of the VOA $\mathcal{H}_{FLM}$ is the Monster Group

Payoff: Conceptual Explanation of Modularity



$$:= Tr_{\mathcal{H}}\, g\, q^{L_0 - \frac{c}{24}} =$$



⟹ Modularity

This is the gold standard for the conceptual explanation of Moonshine-modularity

A truly satisfying conceptual explanation of genus zero properties remains elusive.

Important progress: Duncan & Frenkel 2009; Paquette, Persson, Volpato 2017

B. STATEMENT OF THE PROBLEM

1988:

Beauty and the Beast: Superconformal Symmetry in a Monster Module

L. Dixon[1,]*, P. Ginsparg[2,]** and J. Harvey[3,]***

[1,3] Physics Department, Princeton University, Princeton, NJ 08544, USA

[2] Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c = 24$. In string terminology, their construction corresponds to compactification on a $\mathbf{Z}_2$ asymmetric orbifold constructed from the torus $\mathbf{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c} = 16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

(Super-) Conformal Symmetry:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

Superconformal symmetry ⇒ supercurrent:

$$T(z)\, T_F(w) \sim \frac{\frac{3}{2} T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \cdots$$

$$T_F(z) = \sum_r G_r z^{-r-\frac{3}{2}}$$

$$T_F(z) T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2} T(w)}{z-w} + \cdots$$

$$\mathcal{H}_{B\&B} = \mathcal{H}_{\Lambda} \bigoplus \mathcal{H}_{T}$$

has fields with conformal dimension in $\mathbb{Z} + \frac{1}{2}$

``spin lift''  -  it is a ``2d spin conformal field theory''

What is the actual supercurrent?

Not known.

Not easy.

Today I will fill in this gap.

It is very recent work with R. Singh

C. A LITTLE MORE BACKGROUND

In one of our (several) attempts to explain Umbral Moonshine, Jeff Harvey and I discovered a curious relation between supercurrents in certain superconformal 2d field theories and quantum error correcting codes.

Moonshine, Superconformal Symmetry, and Quantum Error Correction

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Jeffrey A. Harvey,[1] Gregory W. Moore[2]

A WZW Model Equivalent To K3

Amazing result of GTVW: The supersymmetric sigma model on a special K3 surface is isomorphic to the product of 6 copies of (a spin lift of a) ***bosonic*** k=1 SU(2) WZW model !

So it must be possible to write $T_F(z)$ of dimension (3/2,0)

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

Frenkel-Kac-Segal

Witten's Nonabelian Bosonization

Gaussian model: $S = \frac{R^2}{4\pi}\int \partial x \tilde{\partial} x \qquad x \sim x + 2\pi$

$$e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}+w\,R\right)x}(z) \otimes e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}-w\,R\right)\tilde{x}}(\tilde{z})$$

At R=1 we have a theory equivalent to the $SU(2)_1$ WZW model

$$J^3(z) = \frac{1}{\sqrt{2}}\partial x(z), J^{\pm}(z) = e^{\pm i\sqrt{2}x}(z)$$

Gives an $\mathfrak{su}(2) -$ current algebra.

Chiral Fields Of Dimension 3/2

$SU(2)_{k=1}$ =Periodic boson with $R = 1$

$e^{\pm \frac{i}{\sqrt{2}} X(z)}$ SU(2) doublet (``Qbit'')

Conformal dimension $= 1/4$

So in WZW for $SU(2)^6$

$$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6} := \exp\left(\frac{i\sqrt{2}}{2} (\epsilon_1 X_1 + \epsilon_2 X_2 + \cdots + \epsilon_6 X_6) \right) \quad \epsilon_i \in \{ \pm 1 \}$$

$\Rightarrow$ 2^6 vertex operators of conformal dimension = $\left(\frac{1}{4}\right) \times 6 = \frac{3}{2}$

Chiral Fields Of Dimension 3/2

$$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6} := \exp\left(\frac{i\sqrt{2}}{2}(\epsilon_1 X_1 + \epsilon_2 X_2 + \cdots + \epsilon_6 X_6)\right) \quad \epsilon_i \in \{\pm 1\}$$

$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6}$ span a 2^6 dimensional vector space of holomorphic (3/2,0) operators.

Identify this space with the space of states in a system of 6 Qbits.

For any $s \in (\mathbb{C}^2)^{\otimes 6}$ write V_s

Which Ones Are Supercurrents?

The V_s have OPE's:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB} s}{z_{12}}J^A J^B(z_2) + \cdots$$

J^A : generators of $SU(2)^6$ affine Lie algebra, $A = 1, \dots, \; 3 \cdot 6 = 18$

Σ^A, Σ^{AB} generate 1- and 2- Qbit errors

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

N=1 Generator

Using results of GTVW it is V_Ψ for

$$\Psi = [\emptyset] + i\,[123456] + ([1234] + [3456] + 1256]) + i([12] + [34] + [56]) + ([135] + [245] + [236] + [146]) - i([246] + [235] + [136] + [145])$$

$$[135] := |-,+,-,+,-,+\rangle$$

Obtained by tedious translation from the susy for the K3 sigma model….

Is there a *code* governing this quantum state?

Yes!! It is a code over $\mathbb{F}_4$:``hexacode''

Codes Over $\mathbb{F}_4$

Hexacode: $\mathcal{H}_6 \subset \mathbb{F}_4^6$

$\mathcal{H}_6$: A special 3-dimensional subspace of the 6-dimensional vector space $\mathbb{F}_4^6$

$\mathbb{F}_4$ & The Quaternion Group

$$Q \cong \{\pm 1, \pm i\,\sigma^1, \pm i\,\sigma^2, \pm i\,\sigma^3\} \subset SU(2)$$

Group of special unitary bit-flip and phase-flip errors in theory of QEC.

$$Q \xleftarrow{h} \mathbb{F}_4^+$$

$$1 \to \{\pm 1\} \to Q \to \mathbb{F}_4^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(\omega) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad h(\bar{\omega}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$h(x)h(y) = c_{x,y}\, h(x+y)$$

$c_{x,y}$ is a nontrivial cocycle with some nice properties.

N=1 Generator And The Hexacode

For $w = (x_1, x_2, \dots, x_6) \in \mathbb{F}_4^6$ define

$$h(w) := h(x_1) \otimes h(x_2) \otimes \cdots \otimes h(x_6)$$

$$h(w_1)h(w_2) = \chi(w_1, w_2)h(w_1 + w_2)$$

For general $w_1, w_2 \in \mathbb{F}_4^6$ cannot remove signs χ .

But! On the hexacode:

$$h(w_1)h(w_2) = h(w_1 + w_2) \qquad w_1, w_2 \in \mathcal{H}_6 \subset \mathbb{F}_4^6$$

$$P := 2^{-6} \sum_{w \in \mathcal{H}_6} h(w) \qquad \Psi \in Im\, P$$

Consequences: 1/2

V_Ψ generates an N=1 superconformal symmetry:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB}s}{z_{12}}J^A J^B(z_2) + \cdots$$

Σ^A, Σ^{AB} generate 1- and 2- qubit errors

$$\bar{\Psi}\Sigma^A\Psi = 0 \quad \& \quad \bar{\Psi}\Sigma^{AB}\Psi = 0$$

$Im\ P$ is a QEC! $\Rightarrow\ T_F = V_\Psi$

Conway Group Moonshine

[Frenkel, Lepowsky, Meurman; Duncan; Duncan-Mack-Crane]

Susy sigma model with target
$\mathfrak{X}$ = Cartan torus of E8, with special B-field.

$$V_{\Psi}(z_1)V_{\Psi}(z_2)$$
$$\sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{\bar{\Psi}\gamma^{ij}\Psi}{z_{12}^2}\lambda_i\lambda_j + \frac{\bar{\Psi}\gamma^{ijkl}\Psi}{z_{12}}\lambda_i\lambda_j\lambda_k\lambda_l + \cdots$$

$\Psi_{Duncan} \in Im\ P$: error-correcting code associated with the Golay code

$$T_F = V_{\Psi_{\text{Duncan}}}$$

D. SOLUTION OF THE PROBLEM

Now we will use these ideas to fill in the old gap in the Beauty & Beast paper

$$\mathcal{H}_{B\&B} = \mathcal{H}_{\Lambda} \bigoplus \mathcal{H}_{T}$$

$$\mathcal{H}_{T} = \mathcal{F} \bigotimes \mathcal{S}$$

For every spinor $\Psi \in \mathcal{S}$ we have a dimension 3/2 primary field V_Ψ

Jeff and I speculated that once again a supercurrent would be determined from a special spinor determined by a code.

But now we need to know about the OPE of *bosonic* twist fields .....

Much more challenging .....

With a student,
Ranveer Singh,
we have indeed realized the supercurrent in this way

$$V_\Psi(z_1)V_\Psi(z_2) \sim$$

$$\sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{1}{8}\frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{1}{z_{12}} \sum_{\lambda:\lambda^2=4} \kappa(\lambda) e^{i\,\lambda\cdot X(z)} \ldots$$

$$\sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{1}{8}\frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{1}{z_{12}} \sum_{\lambda:\lambda^2=4} \kappa(\lambda) e^{i\,\lambda\cdot X(z)} \dots$$

$$\kappa(\lambda) \sim \langle \Psi, T(\lambda)\Psi \rangle$$

$$T(\lambda) \in \mathcal{H}\left(T_2(\Lambda)\right)$$

Construct an Abelian subgroup $\hat{\mathcal{L}} \subset \mathcal{H}\left(T_2(\Lambda)\right)$

$$P = 2^{-12} \sum_{[\lambda]\in\hat{\mathcal{L}}} T(\lambda)$$

is a rank one projection operator.

Constructing a suitable $\hat{\mathcal{L}} \subset \mathcal{H}\big(T_2(\Lambda)\big)$ requires a lattice $\Lambda_{sc} \subset \Lambda$ such that

$$\lambda_1, \lambda_2 \in \Lambda_{sc} \;\Rightarrow\; \lambda_1 \cdot \lambda_2 = 0 \; mod \; 2$$

$$2\Lambda \underset{2^{12}}{\subset} \Lambda_{sc} \underset{2^{12}}{\subset} \Lambda$$

$$\lambda \in \Lambda_{sc} \Rightarrow \; \lambda^2 = 0 \; mod \; 4$$

Nonzero $\lambda \in \Lambda_{sc} \Rightarrow \lambda^2 > 4$

Choose an isomorphism $T_2(\Lambda) \cong \mathbb{F}_2^{24}$

$$\mathcal{L} \to \mathcal{C} \subset \mathbb{F}_2^{24}$$

Supercurrent = V_Ψ for $\Psi \in Im(P)$

$$\lambda^2 = 4 \Rightarrow \langle \Psi, T(\lambda)\Psi \rangle = 0$$

because of the error correcting properties of $\mathcal{C}$

V_Ψ is a superconformal current in $\mathcal{H}_{B\&B}$

Example of a sublattice Λ_{sc}

Dong, Li, Mason, Norton:
There is an isometric embedding of $L(\sqrt{2})$ into the Leech lattice for every Niemeier lattice L

$$\Lambda_{sc} \cong \quad \Lambda(\sqrt{2})$$

Are there others?
Does $\mathcal{H}_{B\&B}$ have N>1 supersymmetry ?

NUGGET 2

Time Reversal In Chern-Simons-Witten Theory

When does 3d Chern-Simons-Witten theory have a time reversal symmetry?

General theory based on compact group G and a ``level'' $k \in H^4(BG;\mathbb{Z})$

Which (G,k) give T-reversal invariant theories?

Related: When does Reshetikhin-Turaev-Witten topological field theory factor through the unoriented bordism category?

Some nontrivial examples of T-invariant CSW theories appeared in several recent papers

[Seiberg & Witten 2016; Hsin & Seiberg 2016; Cordova, Hsin & Seiberg]

$$G = PSU(N) \qquad\qquad k = N$$

But there is no systematic understanding.

With my student Roman Geiko we have recently carried out a systematic study for

Spin Chern-Simons Theory with torus gauge group $G \cong U(1)^r$

$$S = \frac{1}{4\pi} \int K_{IJ} A_I \, d \, A_J$$

K_{IJ} : $r \times r$ nondegenerate, integral symmetric matrix: determines integral lattice L

Classical T-reversal:

$\exists\ U \in GL(r,\mathbb{Z})$ such that

$$UKU^{tr} = -K$$

(Note: $\sigma(L) = 0$)

But there can be quantum T-reversal symmetries not visible classically.

Rank 2 examples studied by Seiberg & Witten; Delmastro & Gomis

The quantum theory does not depend on all the details of L

What *does* it depend on?

Finite Abelian group $\mathcal{D}(L) := L^{\vee}/L$

a.k.a ``group of anyons''  a.k.a. ``group of 1-form symmetries''

Quadratic Function (spin of anyons) :

$$q_W(x) = \frac{1}{2}(\tilde{x}, \tilde{x} - W) + \frac{1}{8}(W, W) \ \ mod\, \mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}(L)|}} \sum_{x \in \mathcal{D}(L)} e^{2\pi i\, q_W(x)} = e^{2\pi i \frac{\sigma(L)}{8}}$$

Theorem

[Belov & Moore; Freed,Lurie,Hopkins, Teleman]

The quantum theory only depends on the equivalence class of the triple $(\mathcal{D}, q, \bar{\sigma})$

$$q : \mathcal{D} \rightarrow \mathbb{R}/\mathbb{Z} \qquad \bar{\sigma} \in \mathbb{Z}/24\mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} e^{2\pi i \, q(x)} = e^{2\pi i \frac{\bar{\sigma}}{8}}$$

Conversely, every such triple arises from some torus CSW theory

Equivalence of triples

$$(\mathcal{D}, q, \bar{\sigma}) \cong (\mathcal{D}', q', \bar{\sigma})$$

$\exists$ isomorphism $f : \mathcal{D} \to \mathcal{D}'$

$$\exists \ \Delta' \in \mathcal{D}'$$

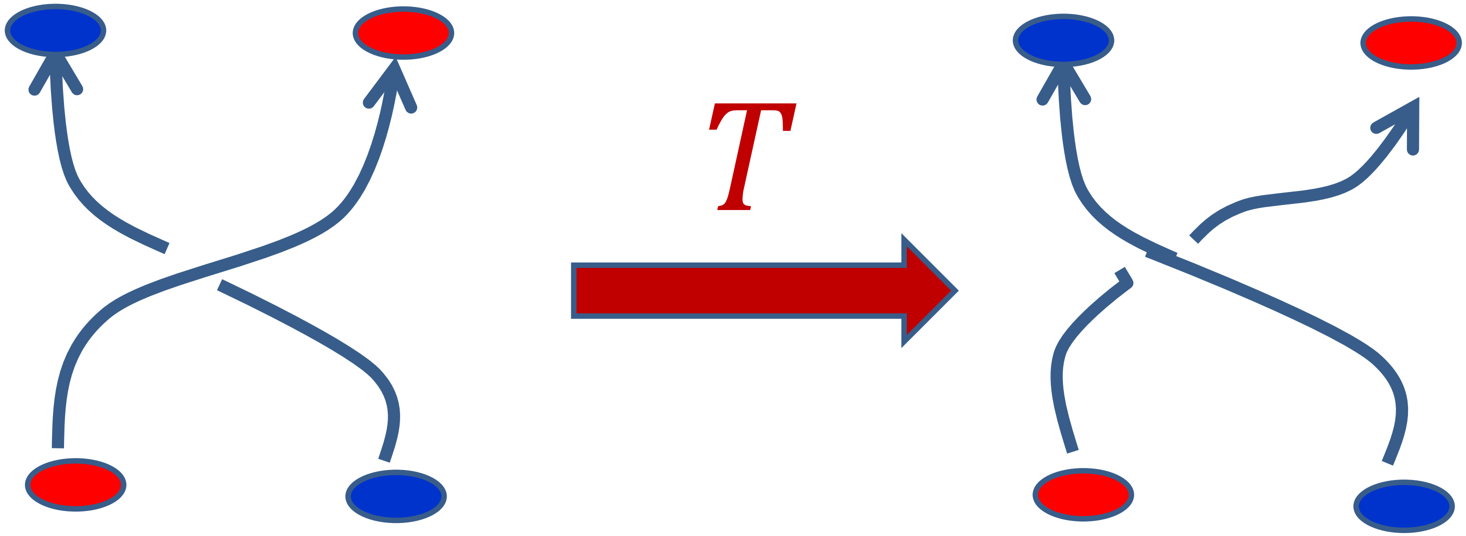
$$q(x) = q'(f(x) + \Delta')$$

T-Reversal Criterion

$$[(\mathcal{D}, q, \bar{\sigma})] = [(\mathcal{D}, -q, -\bar{\sigma})]$$

q: Determines the spin of anyons

b: Determines the braiding of anyons



Simpler Problem: The Witt Group (1936)

$$b(x,y) = q(x+y) - q(x) - q(y) + q(0)$$

Throw away $q, \bar{\sigma}$ and just keep b.

Classify $[(\mathcal{D}, b)]$

$$[(\mathcal{D}_1, b_1)] + [(\mathcal{D}_2, b_2)] \coloneqq [(\mathcal{D}_1 \oplus \mathcal{D}_2, b_1 \oplus b_2)]$$

Abelian monoid $\mathcal{DB}$

Submonoid $\mathcal{Spl}$ Split forms:

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$$

$$\mathcal{D}_1 = \mathcal{D}_1^{\perp}$$

$$\mathcal{Witt} \;\; := \;\; \mathcal{DB}/\mathcal{Spl}$$

Abelian group whose structure is known.

Roughly speaking:

$$\mathcal{Witt} \;\cong (\mathbb{Z}_2)^{\infty} \oplus (\mathbb{Z}_4)^{\infty}$$

$$\mathcal{Spl} \subset \ \mathcal{DB}^T := \{ [\mathcal{D}, b] = [\mathcal{D}, -b] \} \subset \mathcal{DB}$$

Roman computed generators for the (infinite) Abelian subgroup

$$\mathcal{DB}^T / \mathcal{Spl}$$

and then refined it to T –invariant triples

Theorem: A T-invariant triple $[(\mathcal{D}, q, \bar{\sigma})]$ must be a direct sum of

$\mathscr{D}$	b	$\hat{q}$	$\sigma \bmod 8$
$\mathbb{Z}/p^r,\ p \equiv 1 \bmod 4$	X_{p^r} Y_{p^r}	ux^2/p^r vx^2/p^r	$r(p^2-1)/2$ $r(p^2-1)/2+4r$
$\mathbb{Z}/p^r,\ p \equiv 3 \bmod 4$	X_{p^r}	ux^2/p^r	$r(p^2-1)/2$
$\mathbb{Z}/2$	A_2	$x^2/4 - 1/8$	0
$(\mathbb{Z}/2)^2$	E_2	$xy/2$	0
$(\mathbb{Z}/4)^4$	$4A_{2^2}$	$(x_1^2+x_2^2+5x_3^2+5x_4^2)/8$	4
$\mathbb{Z}/2^r \times \mathbb{Z}/2^r,\ r \geqslant 1$	E_{2^r}	$xy/2^r + \alpha(x/2+y/2)$	0
$\mathbb{Z}/2^m \times \mathbb{Z}/2^m,\ m \geqslant 2$	F_{2^m}	$(x^2+xy+y^2)/2^m$	$4(m+1)$
$(\mathbb{Z}/2^m)^4,\ m \geqslant 2$	$4A_{2^m}$	$\sum_{i=1}^4 x_i^2/2^{m+1}$	4
$(\mathbb{Z}/2^m)^2,\ m \geqslant 2$	$A_{2^m}+B_{2^m}$	$x^2/2^{m+1}+3y^2/2^{m+1}$	$4(m+1)$
$(\mathbb{Z}/2^n)^2,\ r \geqslant 3$	$A_{2^n}+D_{2^n}$	$x^2/2^{n+1}+7y^2/2^{n+1}$	0
$(\mathbb{Z}/2^r)^4,\ r \geqslant 3$	$3A_{2^n}+C_{2^n}$	$\sum_{i=1}^3 x_i/2^{n+1}+5y^2/2^{n+1}$	$4n$

Table 3. T-invariant quartets. Here, $\left(\frac{-1}{p}\right)=1$, $\left(\frac{2u}{p}\right)=1$, $\left(\frac{2v}{p}\right)=-1$, $r \geqslant 1$, $m \geqslant 2$, $n \geqslant 3$, $\alpha \in \{0,1\}$. Note, we can add $1/2$ to $\hat{q}$ and 4 to σ in any line to obtain another quartet.

Example: $L \cong A_4$ and $L \cong D_4$ can be primitively embedded into E_8 (Nikulin)

These are positive definite, and cannot be T-invariant classically

Nevertheless, they are quantum T-invariant

Conjecture for the general case:

$$(G,k) \rightarrow CSW(G,k) \rightarrow \quad MTC(G,k)$$

re &

Modular Tensor Categories

There is a mathematical notion of a Witt group of (pointed,nondegenerate) braided fusion categories.

[Davydov, Müger, Nikshych, Ostrik 2010]

CONJECTURE

$CSW(G,k)$ is T-invariant iff

$[MTC(G,k)]$ is order 2 in $\mathcal{W}itt$

Compatible with the physical interpretation of Witt equivalence corresponding to the existence of a topological defect.

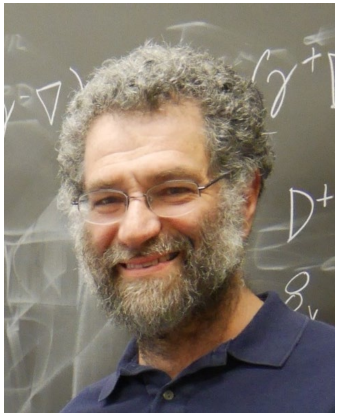
We can also confirm the conjecture for examples of Seiberg et. al. using ``higher central charges''

[Ng, Schopieray, Wang 2018;
Kaidi, Komargodski,Ohmori,Seifnashri, Shao 2021]

NUGGET 3

Two Developments In The Relation Of SYM And Four-Manifold Invariants

Families Of Four-Manifolds - 1/3



Consider a twisted VM in d=4 N=2 SYM on four-manifold $\mathbb{X}$

$$Z[g_{\mu\nu}] = \int [dVM] \exp[-S[\mathrm{VM}; g_{\mu\nu}]]$$

Witten (1988): $Z[g_{\mu\nu}]$ is constant on $MET(\mathbb{X})$

Families Of Four-Manifolds - 2/3

Couple to twisted (truncated) conformal supergravity:

$$Z[g_{\mu\nu}, \Psi_{\mu\nu}] = \int [dVM] \exp[-S[VM; g_{\mu\nu}, \Psi_{\mu\nu}]\,]$$

$Q\, g_{\mu\nu} = \Psi_{\mu\nu}$ Cotangent vector on $MET(\mathbb{X})$

Defines a closed differential form on $MET(\mathbb{X})/Diff^+(\mathbb{X})$

$$[Z[g_{\mu\nu}, \Psi_{\mu\nu}]\,] \in H^*(BDiff^+(\mathbb{X}))$$

New invariants?

Donaldson-Witten a la Baulieu-Singer

$P \to \mathbb{X}$ $\qquad\qquad$ $\mathcal{G} := Aut(P)$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P)$

$$\Big(\Omega^*(\mathcal{A}(P)) \otimes S^*(Lie\mathcal{G}) \Big)^{\mathcal{G}}$$

$Q\, A_\mu = \psi_\mu \qquad Q\, \psi_\mu = -D_\mu \phi \qquad Q\, \phi = 0$

Atiyah & Jeffrey + Baulieu & Singer

Z_{DW} : Pushforward in $\mathcal{G}$ −equivariant cohomology.

$$\mathcal{G}_d := Diff^+(\mathbb{X})$$

$\mathcal{G}_d$ −equivariant cohomology of $MET(\mathbb{X})$

$$Q\, g_{\mu\nu} = \Psi_{\mu\nu} \qquad Q\, \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \qquad Q\Phi^\mu = 0$$

Action e^{-S} is a closed equivariant class in the $\mathcal{G} \rtimes \mathcal{G}_d$ − equivariant cohomology of $MET(\mathbb{X}) \times \mathcal{A}(P)$

Push-forward in $\mathcal{G}$ −equivariant cohomology is a $\mathcal{G}_d$ −equivariant class on $MET(\mathbb{X})$

$$Q\ A_\mu = \psi_\mu \qquad\qquad Q\ g_{\mu\nu} = \Psi_{\mu\nu}$$

$$Q\ \psi_\mu = -D_\mu \phi \quad + \Phi^\sigma F_{\sigma\mu} \qquad Q\ \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu$$

$$Q\ \phi = 0 \quad - \Phi^\sigma \psi_\sigma \qquad\qquad Q \Phi^\mu = 0$$

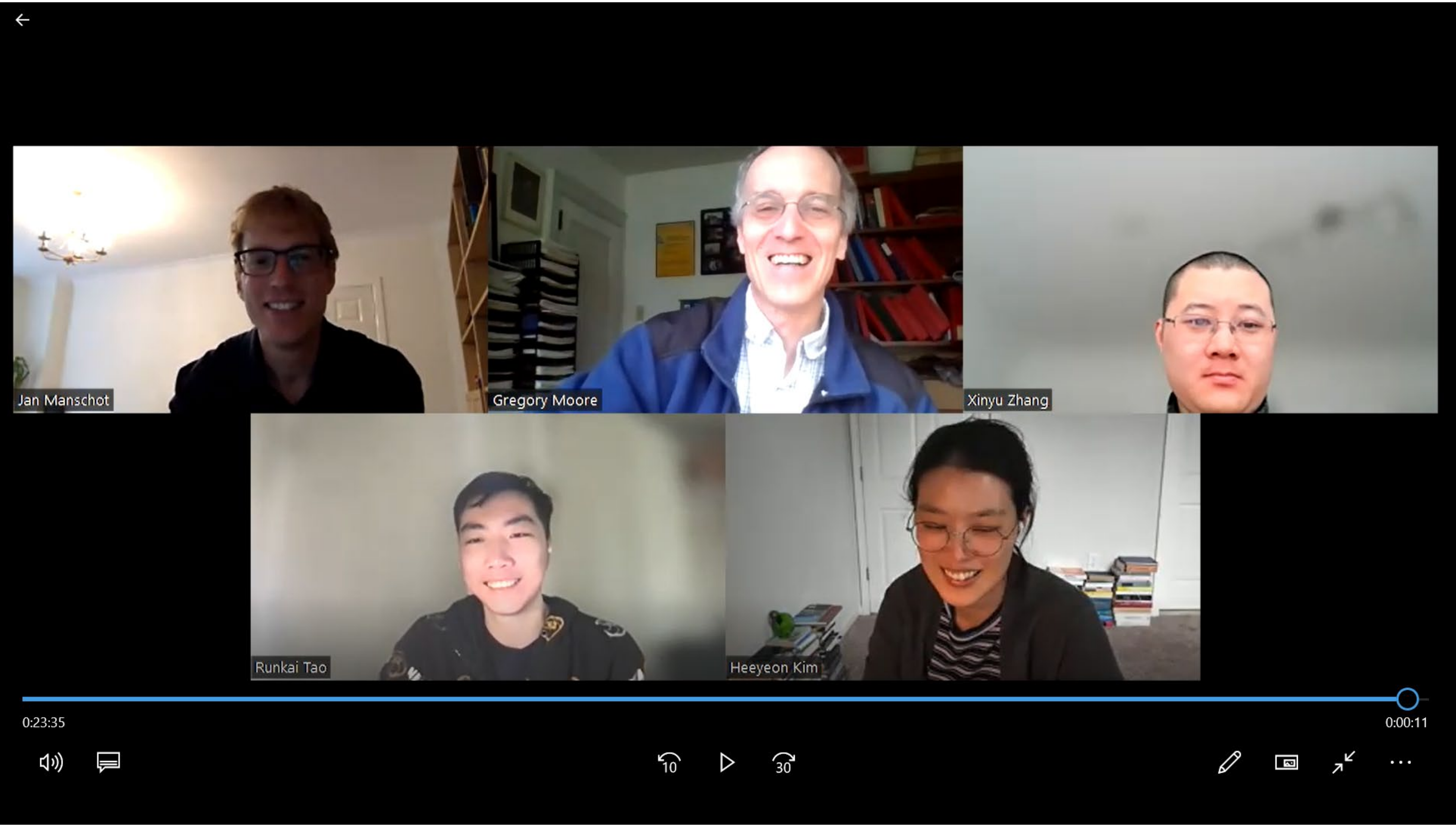
$$S = S_{Witten} \ + \ \int_{\mathbb{X}} vol(g)\ \Psi^{\mu\nu} \Lambda_{\mu\nu} + \ \cdots$$

$$Q_0(\Lambda_{\mu\nu}) = T_{\mu\nu}$$

$+ \cdots =$ heroic computations by Jay & Vivek

$$Q\ \mathcal{O}^{(n)} = d\mathcal{O}^{(n-1)} + \iota_\Phi \mathcal{O}^{(n+1)}$$

``K-Theoretic Donaldson Invariants''



Five Dimensions

Partial Topological Twist of 5d SYM on $\mathbb{X} \times S^1$

Reduces to SQM on the moduli space of instantons:

(Requires that $\mathcal{M}$ be Spin-c)

$$\mathcal{R} := R\,\Lambda$$

$$Z[\mathcal{R}] = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} \hat{A}(T\,\mathcal{M}_k)$$

[Nekrasov (1996); Losev, Nekrasov, Shatashvili; Gottsche et. al. ....]

+ interesting story including observables…

Chern-Simons Observables

$U(1)_{inst}$ symmetry with current $J = Tr(f \wedge f\ )$

Couple to background gauge field A: $n := \left[\frac{F(A)}{2\pi}\right] \in H^2(\mathbb{X}, \mathbb{Z})$

$$\mathcal{O}(n) = \int_{\Sigma(n)\times S^1} Tr\left(a\,da + \frac{2}{3}a^3\right)$$

$$= \int_{\mathbb{X}\times S^1} F(A) \wedge Tr\left(a\,da + \frac{2}{3}a^3\right)$$

$$Z(\mathcal{R}, n) := \langle\, e^{\mathcal{O}(n)} \rangle$$

Five Dimensions

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} e^{c_1(L(n))} \hat{A}(\mathcal{M}_k)$$

Using both the Coulomb branch integral (a.k.a. the U-plane integral) and, independently, localization techniques, we reproduce & generalize

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

Abstract. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

Abstract. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

$$b_2^+(\mathbb{X}) = 1$$

Derived a wall-crossing formula

Differs from GNY.

Agrees with GNY,
suitably interpreted

This raises some puzzles…

$$Z(\mathcal{R},n) = \left[\nu_{\mathcal{R}}(\tau)\ \ C(\tau)^{n^2}\ \ F_n\big(\tau, v(\tau)\big)\right]_{q^0}$$

$$\nu_R = \frac{\vartheta_4^{13-b_2}}{\eta^9}\frac{1}{\sqrt{1-\mathcal{R}^2 u + \mathcal{R}^4}} \qquad u = \left(\frac{\vartheta_2}{\vartheta_3}\right)^2 + \left(\frac{\vartheta_3}{\vartheta_2}\right)^2$$

$$\frac{\vartheta_1\left(\tau, \frac{1}{2}v(\tau)\right)}{\vartheta_4\left(\tau, \frac{1}{2}v(\tau)\right)} = -\mathcal{R} \qquad C(\tau) = \frac{\vartheta_4\left(\tau, \frac{1}{2}v(\tau)\right)}{\vartheta_4(\tau)}$$

$F_n(\tau, z)$: Mock Jacobi form

$$b_2^+(\mathbb{X}) > 1$$

$$Z(\mathcal{R}, n) = \sum_{\xi \in \mu_4} \xi^{-\chi_h} G(\xi\mathcal{R}, n)$$

$$G(\mathcal{R}, n) = \frac{2^{2\chi+3\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

This should generalize to 6d SYM on $\mathbb{X} \times \mathbb{E}$

$$\hat{A}(\mathcal{M}_k) \to Ell(\mathcal{M}_k, q)$$

Conjecture:

Integrals in elliptic cohomology of distinguished classes defined by the susy sigma model with target space $\mathcal{M}_k$ define smooth invariants of four-manifolds

Happy Birthday Igor!!