Comments on the decay of localized Tachyons

IHES, June 15, 2004

based on papers with

J. Harvey, D. Kutasov, E. Martinec, A. Parnachev

- Localized tachyons and RG Flows, hep-th/0111154
- On Decay of K-theory, hep-th/0212059
- Localized Tachyons and the Quantum McKay Correspondence, hep-th/0403016

Introduction

Open string tachyon condensation is well-studied.

Two important points we learned

A. The K-theory class of D-branes is an invariant of open string RG flow: Two D-branes related by open string tachyon condensation have the same K-theory class.

B. There is an analog of Zamolodchikov's c-function, g_{open} which decreases monotonically under RG flow and defines an action principle (BSFT):

$$G_{ij}\beta^j = \frac{\partial}{\partial\lambda^i}S \qquad\qquad S \sim Z_{\mathrm{disk}}$$

$\Rightarrow\natural$ question: Do A,B have analogs for closed string tachyons?

From the RG point of view we expect to flow to one of the $c \leq 1$ models - but it is hard to make that precise.

Intermediate case: condensation of *localized* closed string tachyons. Adams, Polchinski, Silverstein.

Orbifold fixed locus $\sim$ D-brane.

This talk: Summarizes what I know about A & B in the closed case.

Some Models with localized closed string tachyons

1. Type II strings on Minkowski crossed with the orbifold

$$\mathbb{C}^d/\Gamma$$

Where $\Gamma \subset U(d)$ is a discrete group not in $SU(d)$.

In some cases the lift of Γ to the spin group still allows GSO to project out the bulk tachyon, but there are tachyons in the twisted sectors $\Rightarrow$ localized tachyons.

2: "Melvin models" or "twisted tori"

$$(\mathbb{C}^d \times \mathbb{R}^{d'})/\Gamma$$

where Γ is an infinite discrete group acting by translation on $\mathbb{R}^{d'}$ such that only a finite quotient acts effectively on $\mathbb{C}^d$, e.g. $(\mathbb{C} \times \mathbb{R})/\mathbb{Z}$ with

$$\begin{aligned} g_0 : z &\to e^{2\pi i\gamma} z \qquad z \in \mathbb{C} \\ y &\to y + R \qquad y \in \mathbb{R} \end{aligned}$$

$\exists$ localized tachyons for $R^2 < |\gamma|$.

Methods Used

The methods used to study these models are

• D-brane probes (APS)

• $\mathcal{N} = 2$ worldsheet supersymmetry and the chiral ring (Harvey et. al.)

• Gauged linear sigma model (GLSM) and toric geometry (Vafa, Minwalla et. al., Martinec & Moore)

• String field theory (Okawa & Zwiebach)

There is some consensus on the picture for $d = 1, 2$.

Recent work of Morrison, Narayan, & Plesser discusses analogous results for $d = 3$.

Qualitative pictures of the RG flow

In general, flow to the IR leads to a (partial) resolution of the singularity.

$\mathbb{C}/\mathbb{Z}_n \to \mathbb{C}/\mathbb{Z}_{n'}$ for $n' < n$: Eventually decays to $\mathbb{C}$.

For $\mathbb{C}^2/\mathbb{Z}_n$, the picture depends on what you perturb by.

Main claim: A generic perturbation by tachyons in the chiral ring flows to the Hirzebruch-Jung minimal resolution $\mathcal{X}$, of the singularity.

$\mathcal{X}$ generalizes the ALE space of the susy case.

$H_2(\mathcal{X}) = \mathbb{Z}^r$, $r \leq n-1$

$$-C = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ 1 & -a_2 & 1 & \cdots & 0 \\ 0 & 1 & -a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots\cdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_r \end{pmatrix}, \qquad a_\alpha \geq 2$$

ALE: $r = n-1$, $a_\alpha = 2$.

Cycles with $a_\alpha > 2$ expand to infinity, $a_\alpha = 2$ don't grow $\Rightarrow$ disconnected ALE universes.

What happened to the D-branes?

Already we can sharpen question A:

There are n fractional branes at the orbifold point. What happens to these boundary states under RG flow?

Do they just disappear?

Given the $\mathcal{N} = 2$ supersymmetry we have a category of topological branes.

It seems unlikely that this should be destroyed by RG flow preserving $\mathcal{N} = 2$ supersymmetry.

We expect the category of D-branes to "evolve smoothly" under RG flow.

The objects of the D-brane category at the orbifold point are $K_\Gamma(\mathbb{C}^d) \cong R(\Gamma)$.

For *susy* orbifolds: $\mathbb{C}^2/\Gamma, \Gamma = A, D, E$ there is indeed an isomorphism of $K_\Gamma(\mathbb{C}^2)$ with the K-theory of the resolution: This is the *McKay correspondence*.

Question: Is there a generalization to non-supersymmetric (non-crepant) orbifolds and their resolutions?

But at first sight, it looks hopeless: $d = 1$ flows to $\mathbb{C}$, which has only the $D0$ brane. For $d = 2$, r $D2$'s + 1 $D0$, give $r + 1 < n$ branes.

Chiral Ring for $d = 1$

We now describe more precisely how one can arrive at the above qualitative pictures of the RG flow.

One route is via the chiral ring of the worldsheet $\mathcal{N} = 2$ susy.

The j^{th} twisted sector $z \sim \omega^j z$ has a ! chiral primary

$X_j = \sigma_{j/n}\ \exp\left[i(j/n)(H - \bar{H})\right] \quad ; \qquad j = 1, 2, \cdots, n-1$

$\psi = e^{iH}; \ \ \bar{\psi} = e^{i\bar{H}}$

$$\Delta_j = \frac{1}{2} R_j = \frac{j}{2n}$$

OPE: $X_1^n \sim \frac{1}{V} \bar{\psi}\psi \sim 0$

$\Rightarrow$ at the orbifold point the chiral ring

$$\mathbb{C}[X]/\partial W, \qquad\qquad W(X) = X^n.$$

The perturbed theory has $W = X^n + \sum \lambda_j X^{n-j}$.

X^m are relevant, so in the IR the dominant term is

$$W \to X^{n'}$$

$n' < n$ most relevant operator.

This is the APS picture.

Chiral ring for $d = 2$

The story is much more complicated for $d = 2$

$$(Z_1, Z_2) \sim (\omega Z_1, \omega^p Z_2)$$

We take $(n, p) = 1$. Lift to the spin group $SU(2) \times SU(2)$ depends on $p \in (-n, n)$.

$(-1)^F = (-1)^{p\pm1} \Rightarrow$ GSO projects out bulk tachyon for p odd.

The chiral ring is now made of products for the two factors.

In the s-twisted sector

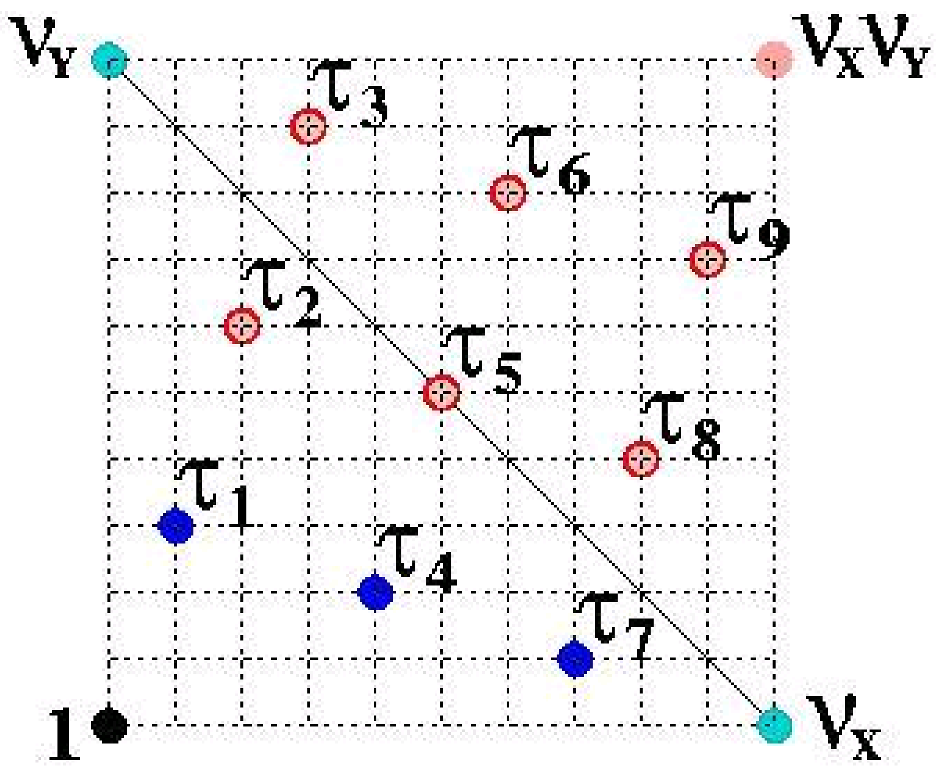
$$\mathcal{T}_s = X_s^{(1)} X^{(2)}_{n\{\frac{sp}{n}\}}$$

$\{x\} = x - [x] =$ fractional part of x.

Chiral ring - example

It is useful to label these operators by $U(1) \times U(1)$ R-charge $(\frac{s}{n}, \{\frac{sp}{n}\})$ and plot them in the R-charge plane.

Example: $n = 10, p = 3$:



- 10 elements in the chiral ring
- Chiral ring multiplication described by vector addition.
- 3 generators of the chiral ring.
- Marginal operators on the diagonal
- Relevant operators below the diagonal.

General algorithm for the chiral ring

The general structure of the chiral ring is as follows:

$$\frac{n}{p_1} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots 1/a_r}}} := [a_1, \ldots, a_r].$$

$p_1 = p, p > 0$ and $p_1 = p + n, p < 0$.

Let $v_0 = (0, n), v_1 = (1, p_1)$ and define v_j by

$$a_j v_j = v_{j-1} + v_{j+1}$$

Recursion terminates at $v_{r+1} = (n, 0)$.

Result: If $a_\alpha \geq 2$ then the generators of the chiral ring are the chiral primaries associated with the vectors v_α, $\alpha = 1, \ldots, r$.

If we let $a_\alpha \geq 1$ then the continued fraction is not unique:

$$x - \frac{1}{y} = [x, y] = [x + 1, 1, y + 1]$$

This corresponds to adding a vector

$$v_* = v_\alpha + v_{\alpha+1}$$

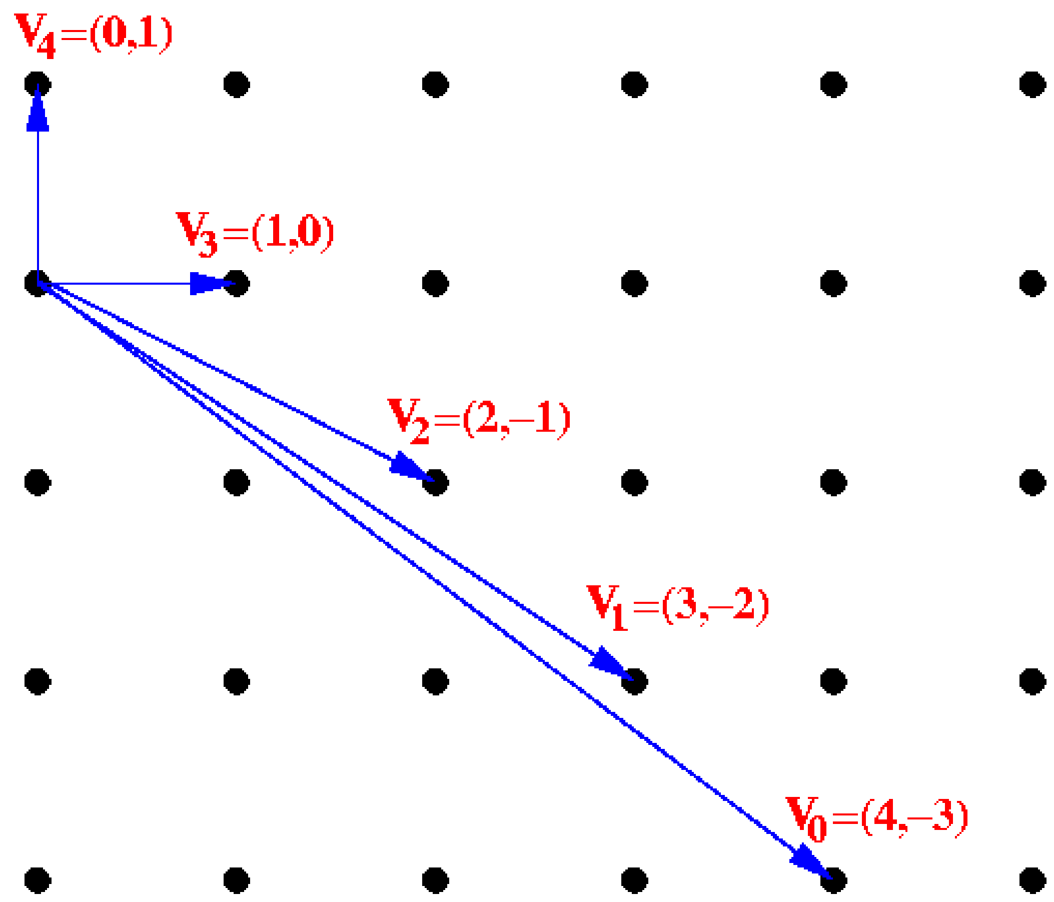
In this way, elements in the chiral ring which are not generators can also be associated with entries in a continued fraction expansion of n/p_1.

Toric data for the resolution

The R-charge vectors in the chiral ring are equivalent to the data of a toric variety which is a resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_{n(p)}$.

$$v_\alpha \to \frac{1}{n}\begin{pmatrix} n & 0 \\ -p & 1 \end{pmatrix} v_\alpha$$

Produces



⇒Fan for a toric variety resolving the singularity.

- All $a_\alpha \geq 2$: *Minimal resolution*
- Some $a_\alpha = 1$: Nonminimal resolution.

Toric resolution ⇒ write a GLSM

GLSM -review

$U(1)^r$ gauge theory: Σ_α = twisted chiral superfields,

$X_i, i = 1, \ldots, r+d$ = charged chiral superfields,

charge matrix: $Q_{\alpha,i}$.

$$\mathcal{L} = \int d^4\theta \left(\bar{X}_i e^{2Q_{\alpha i}V_\alpha} X_i - \frac{1}{2e_\alpha^2}\bar{\Sigma}_\alpha \Sigma_\alpha \right) - \frac{1}{2}\left(\int d^2\tilde{\theta}\, t_\alpha \Sigma_\alpha + \text{c.c.} \right) ,$$

$t_\alpha = \zeta_\alpha - i\theta_\alpha$, have RG flow:

$$t_{\alpha,\text{eff}}(\mu) = t_{\alpha,\text{bare}} + b_\alpha \log \frac{\mu}{\Lambda}$$

$$b_\alpha = \sum_{i=1}^{r+d} Q_{\alpha i}$$

$Q_{\alpha i}$ follow from relations in the fan.

For our case

$$Q_{\alpha i} = -a_\alpha \delta_{\alpha i} + \delta_{\alpha+1,i} + \delta_{\alpha-1,i}$$

so we have the basic trichotomy:

$$\boxed{\begin{array}{rl} b_\alpha = 2 - a_\alpha > 0 & \quad a_\alpha = 1 \\ = 0 & \quad a_\alpha = 2 \\ < 0 & \quad a_\alpha > 2 \end{array}}$$

GLSM Vacua

$$U_{\text{classical}} = \sum_{\alpha=1}^{r} \frac{e_\alpha^2}{2}\left(M_\alpha(X) - \zeta_\alpha\right)^2 + \sum_{\alpha,\beta=1}^{r} \bar{\sigma}_\alpha \sigma_\beta \sum_{i=1}^{r+d} Q_{\alpha i} Q_{\beta i} |X_i|^2$$

Higgs branch: $\langle \Sigma_\alpha \rangle = 0$, but $\langle X_i \rangle \neq 0$.

$$\mathcal{S}_{\vec{\zeta}} = \{\vec{X} | \sum_i Q_{\alpha,i}|X_i|^2 = \zeta_\alpha\} \subset \mathbb{C}^{d+r}$$

The Higgs branch of vacua is

$$\mathcal{X}_{\vec{\zeta}} = \mathcal{S}_{\vec{\zeta}}/U(1)^r$$

We will describe it soon.

$b_\alpha \neq 0 \Rightarrow$ Coulomb branch of vacua: $\langle \Sigma_\alpha \rangle \neq 0$:

$$\widetilde{W}_{\text{eff}} = -\sum_{\alpha=1}^{r} \Sigma_\alpha \left(t_{\alpha,\text{eff}}(\mu) + \sum_{i=1}^{r+d} Q_{\alpha i} \log\left(\frac{1}{e\mu} \sum_{\beta=1}^{r} Q_{\beta i} \Sigma_\beta \right) \right)$$

Example: $r = 1$, $n/p = a/1 = [a]$,

$$\langle \Sigma_1^{(\ell)} \rangle = \Lambda\, c \,\exp\left[\frac{t_{1,\text{bare}} + 2\pi i \ell}{a-2}\right] = \mu\, c\, \exp\left[\frac{t_{1,\text{eff}}(\mu) + 2\pi i \ell}{a-2}\right] \quad , \qquad \ell = 1, ..., a-2\,.$$

$\ell = 1, \ldots, a-2$.

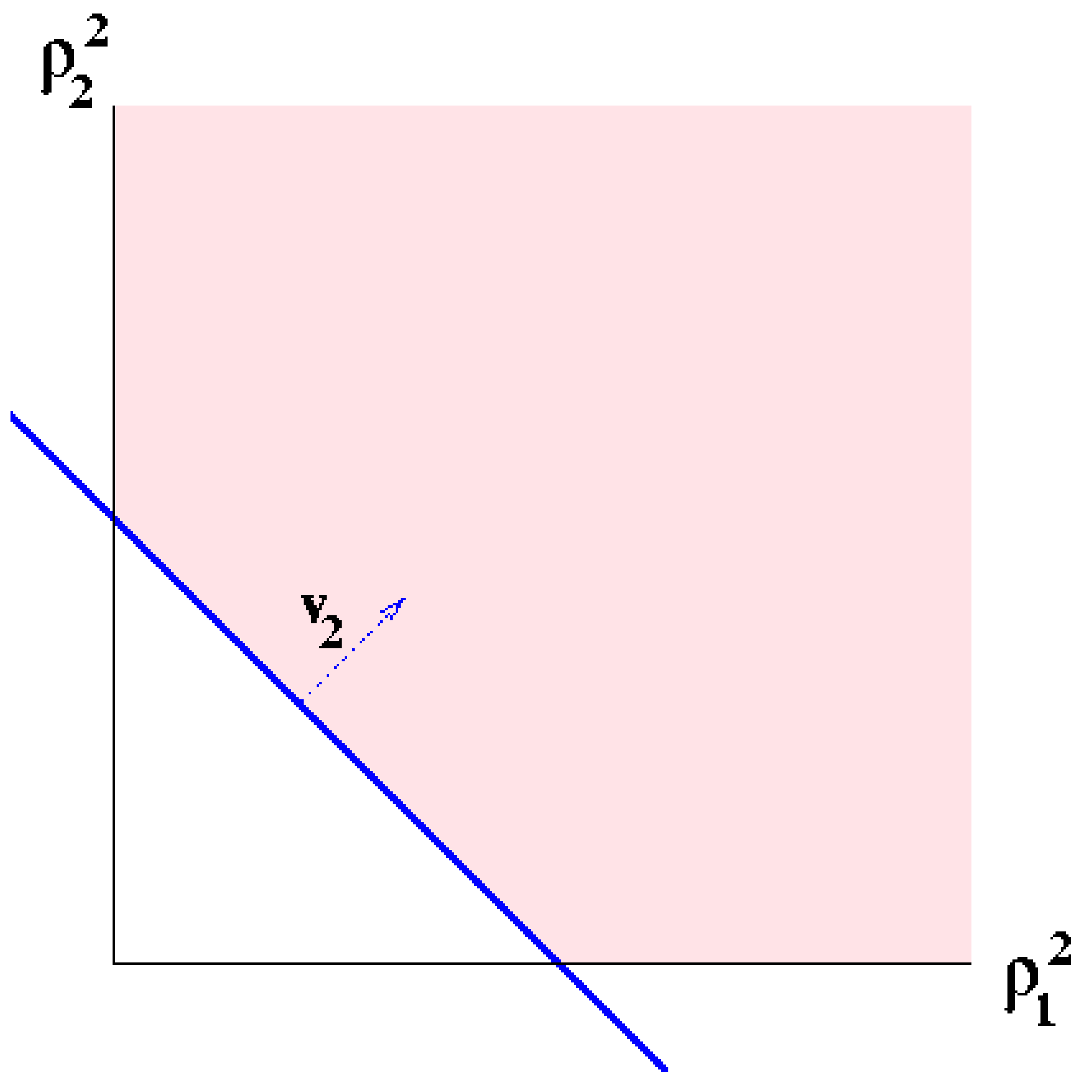
Description of the Higgs branch: $r = 1$

Consider again $r = 1$, $n/p = [a]$.

$$|X_0|^2 + |X_2|^2 - \zeta = a|X_1|^2$$

$$(X_0, X_1, X_2) \sim (e^{i\theta} X_0, e^{-ia\theta} X_1, e^{i\theta} X_2)$$

$\boxed{\zeta > 0:}$ $\mathcal{O}(-a) \to \mathbb{P}^1 = T^2$ fibration over a the polyhedron



$\boxed{\zeta < 0} \Rightarrow X_1 \neq 0 \Rightarrow U(1)$ gauge symmetry is broken

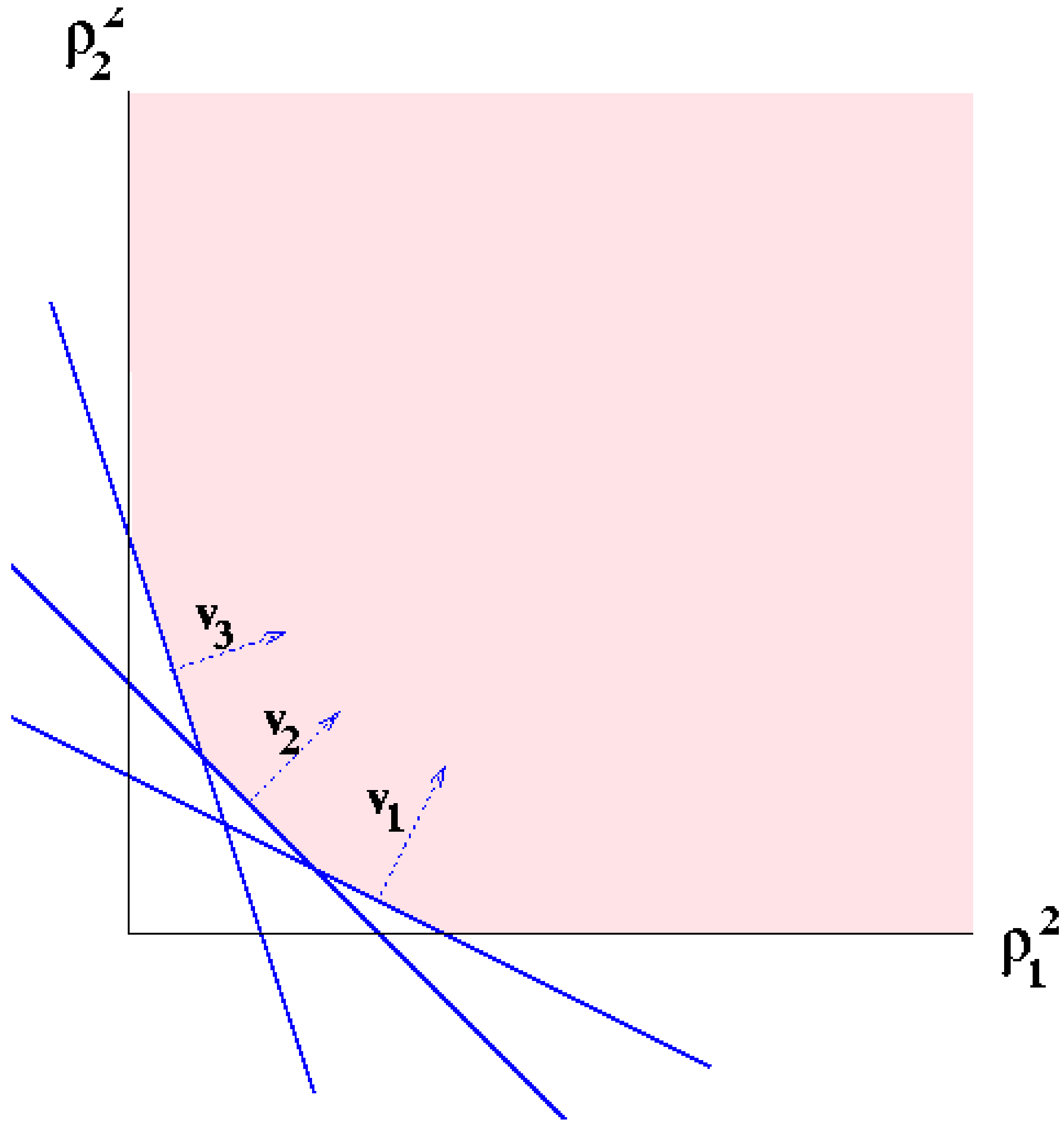
$$U(1) \to \mathbb{Z}/a\mathbb{Z}$$

Unbroken group acts $(X_0, X_2) \sim (\omega X_0, \omega X_2) \Rightarrow \mathbb{C}^2/\mathbb{Z}_{a(1)}$ orbifold.

RG flow: $\zeta(\mu) \sim (2 - a) \log(\mu/\Lambda)$

Velocity of wall $\propto (1, 1)$ with speed $1 - \Delta = \frac{a-2}{a}$.

Description of the Higgs branch in general



$\boxed{\vec{\zeta} > 0:}$

- Successive pairs $v_i, v_{i+1} \Rightarrow$wall.
- Each wall is $\mathbb{P}^1 = S^3/U(1)$. These cycles generate $H_2(\mathcal{X}_{\vec{\zeta}})$

$$-C = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ 1 & -a_2 & 1 & \cdots & 0 \\ 0 & 1 & -a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots\cdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_r \end{pmatrix}$$

- for $\vec{\zeta} < 0$, all X_α obtain vevs breaking the gauge symmetry

$$U(1)^r \to \mathbb{Z}_n$$

So Higgs branch $= \mathbb{C}^2/\mathbb{Z}_{n(p)}$.

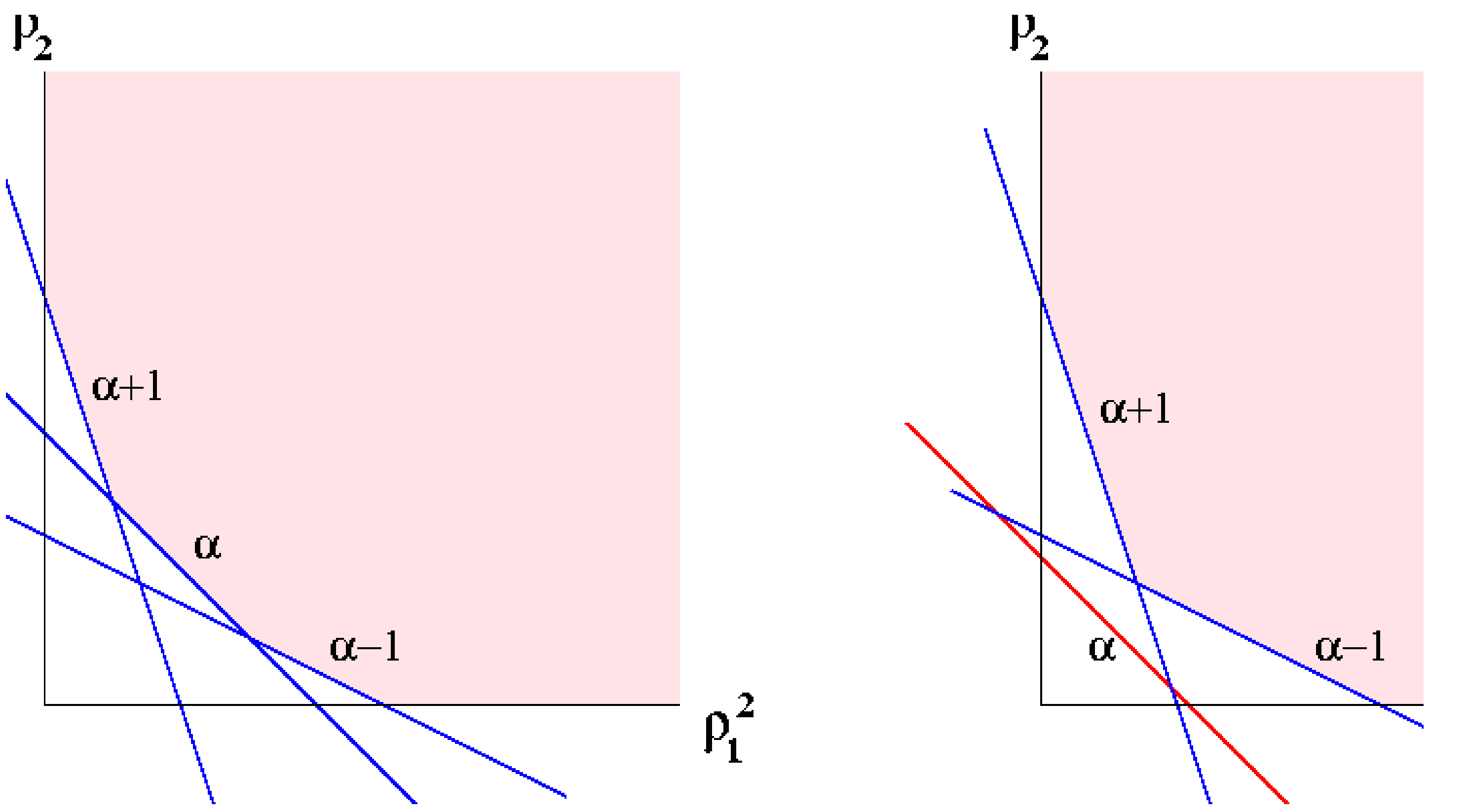
RG Flow: The walls move with with speed $1 - \Delta_\alpha$ in orthogonal direction.

Nonminimal resolutions

The above picture applies to nonminimal resolutions.

Boundaries corresponding to irrelevant operators shrink.

Boundaries corresponding to relevant operators with $a_\alpha = 1$ are overtaken:



$\Rightarrow$ Far IR always produces the *minimal* resolution.

Generic flow: Isolated universes of ALE singularities:

$$[\cdots,\underbrace{2,\ldots,2}_{\ell_1},\cdots,\underbrace{2,\ldots,2}_{\ell_2},\cdots] \longrightarrow [\underbrace{2,\ldots,2}_{\ell_1}]\oplus[\underbrace{2,\ldots,2}_{\ell_2}]\oplus\cdots$$

Accounting for the branes

General proposal: The "missing branes" are accounted for by considering the Coulomb vacua.

We take the naive point of view that the objects in the category of LG topological branes has one $\mathbb{Z}$ summand for each nondegenerate critical point.

$\mathbb{C}/\mathbb{Z}_n \to \mathbb{C}$: There is only $D0$ brane. Generically, there are $(n-1)$ Coulomb vacua associated to roots of W':

$$n = (n-1) + 1$$

Accounting for $\mathbb{C}^2/\mathbb{Z}_{n(p)}$: For the minimal resolution:

Higgs branch: 1 $D0$ brane + r $D2$ branes wrapping a basis of cycles $\mathcal{C}_\alpha$ of the HJ space.

In the nonsusy case: $r + 1 < n$.

Coulomb branch is best described by the twisted chiral superpotential (Morrison-Plesser, Hori-Vafa):

$$\widetilde{W} = u_0^n + u_{r+1}^n + \sum_{\alpha=1}^{r} \lambda'_\alpha u_0^{p_\alpha} u_{r+1}^{q_\alpha} \ ,$$

where $v_\alpha = (q_\alpha, p_\alpha)$.

Result (Martinec & Moore): For generic λ'_α there are $(n-r-1)$ distinct massive vacua of this superpotential.

The special representations

We can say more: We can identify D-branes on the Higgs branch with certain fractional branes at the orbifold point.

The D-branes on the Higgs branch generate the (compact) K-theory of the HJ resolution:

Writing $\mathcal{X}_{\vec{\zeta}} = \mathcal{S}_{\vec{\zeta}}/U(1)^r$ we associate r tautological line bundles corresponding to the fundamental representations:

$$\rho_j : \Big(\exp[i\theta_1], \dots, \exp[i\theta_r]\Big) \to e^{i\theta_j}$$

Now "continue" from $\vec{\zeta} > 0$ to $\vec{\zeta} < 0$.

Then the vev's of the charged fields X_α break $U(1)^r \to \mathbb{Z}_n$.

The generator $\hat{g}$ of $\mathbb{Z}_n$ is embedded in $U(1)^r$ as

$$\hat{g} \to \Big(\exp[2\pi i \frac{p_1}{n}], \exp[2\pi i \frac{p_2}{n}], \dots, \exp[2\pi i \frac{p_r}{n}]\Big)$$

Therefore, we conclude that the D2 branes singly wrapping $\mathcal{C}_\alpha$ "correspond" to the fractional branes in the representations $\rho_f^{p_\alpha}$ of $\mathbb{Z}_n$, where $\rho_f(\hat{g}) = \omega$.

In the math literature, these are known as the "special representations." (Ito; Ishii; Riemenschneider)

Intersection matrix

There is a natural quadratic form on the orbifold K-theory

$$\mathrm{Ind}_{\Gamma}(\not{D}_{E^*\otimes F})$$

which corresponds to the pairing of branes (Douglas & Fiol)

$$(a,b) = I_{ab} = \mathrm{tr}_{R,ab}(-1)^F q^{L_0-\frac{c}{24}}$$

For $\mathbb{C}^d/\Gamma$, let $\pi_* : R(\Gamma) \to \mathbb{Z}$ project to the trivial rep.

$$(\rho_1,\rho_2) = \pi_*\Big(\bar{\rho}_1 \otimes \rho_2 \otimes (S^+ - S^-)\Big)$$

For the special case $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ this works out to be

$$\boxed{I = \mathcal{S}^{(p-1)/2} + \mathcal{S}^{-(p-1)/2} - \mathcal{S}^{(p+1)/2} - \mathcal{S}^{-(p+1)/2}}$$

$\mathcal{S} = n \times n$ shift matrix: $\mathcal{S}_{ij} = \delta_{i,j+1}$.

?? How can this be related to the intersection matrix:

$$-C = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ 1 & -a_2 & 1 & \cdots & 0 \\ 0 & 1 & -a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots\cdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_r \end{pmatrix}$$

Quantum McKay Correspondence

Result (Parnachev & Moore): Let e_i denote the fractional branes corresponding to representations $\rho_f^{\otimes i}$ of $\mathbb{Z}_n$.

Denote

$e_\alpha = e_{p_\alpha}$: Special representations

e_ν: Remaining representations

Then, there is a basis

$$h_0 = e_0 + \cdots + e_{n-1}$$

$$h_\alpha = e_\alpha + \sum_{\nu=1}^{r'} u_\alpha{}^\nu e_\nu \qquad \alpha 1, \ldots, r$$

$$c_\nu = e_\nu$$

where the intersection form is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & -C' \end{pmatrix}$$

The $u_\alpha{}^\nu$ are *integers.*

Technical point: $p < 0$, a_α even $\leftrightarrow$ GSO keeps all $\mathcal{T}_\alpha$.

Conjecture: The K-theory of a toric resolution of a singularity $\mathbb{C}^d/\Gamma$ is a direct summand of $K_\Gamma(\mathbb{C}^d)$. The complementary space is generated by the K-theory classes of the topological LG models associated to the nondegenerate critical points on the Coulomb branch of the associated GLSM.

Analog of the g-function?

Recall the open string case: There is an analog:

$$g_{\text{open}} = \langle 0|B(a)\rangle \ \sim Z_{\text{disk}}$$

This has been argued to be a decreasing quantity under RG flow. (Affeck & Ludwig; Kutasov, Marino, Moore; Friedan & Konechny).

At the conformal fixed points $g_{\text{open}} = \langle 0|B(a)\rangle$ has a nice interpretation in terms of the regularized dimension:

$$Z_{aa}(q) := \mathrm{Tr}_{\mathcal{H}_{aa}} q^{L_0 - c/24}$$

$$\lim_{q\to 1} Z_{aa}(q) = \dim \mathcal{H}_{aa} \to (g_{\text{open}})^2 e^{\frac{\pi c}{6\tau_2}}$$

For the closed string case, the analog should be Zamolodchikov's c-function:

$$c(t) = 2z^4\langle T(z)T(0)\rangle - 4z^3\bar{z}\langle T(z)\Theta(0)\rangle - 6|z|^4\langle \Theta(z)\Theta(0)\rangle$$

$$\frac{dc}{dt} = -12\beta^i\beta^j G_{ij} \ , \qquad G_{ij} = \langle \phi_i(z)\phi_j(0)\rangle |z|^4$$

ϕ_i: Perturbation from CFT.

$$\text{But } \dot{c} = 0$$

However, for the noncompact orbifolds we have

$$\langle \phi_i(z)\phi_j(0)\rangle = 0$$

because these are *normalized correlators*:

$$\langle \prod \phi \rangle = \frac{\langle\langle \prod \phi \rangle\rangle}{\langle\langle 1 \rangle\rangle}$$

so $\dot{c} = 0$!

We would like to introduce a volume regulator so that

$$c_V(t) = 2z^4 \langle\langle T(z)T(0)\rangle\rangle_V - 4z^3\bar{z}\langle\langle T(z)\Theta(0)\rangle\rangle_V - 6|z|^4\langle\langle \Theta(z)\Theta(0)\rangle\rangle_V$$

$$\frac{dc_V}{dt} = -12\beta^i\beta^j\langle\langle \phi_i(z)\phi_j(0)\rangle\rangle_V |z|^4 < 0$$

Then for $V \to \infty$:

$$c_V(t) = Vc_Z + c_\partial + o(V)$$

Then $\dot{c}_Z = 0$ and

$$\lim_{V\to\infty} \frac{dc_V}{dt} = \frac{dc_\partial}{dt}$$

$\Rightarrow$ c_∂ is a decreasing quantitiy.

However, it is hard to make this precise and compute c_∂. (Note for $d = 2$ there is topology change at infinity - so the volume regulator must be subtle.)

Other proposals

So, there are various proposals in the literature ...

One guess is to look at the high energy density of localized states. For $\mathbb{C}^d/\Gamma$ orbifolds:

$$\mathcal{H} = \mathcal{H}_{\text{untwisted}} \oplus \mathcal{H}_{\text{twisted}}$$

$$\boxed{\dim \mathcal{H}_{\text{twisted}} = \lim_{q \to 1} Z_{\text{twisted}}(q) \sim g_{\text{cl}} e^{\frac{\pi c}{6\tau_2}}}$$

If g_{cl} is a good analog of g_{open}^2 then we expect the "g_{cl}-conjecture"

$$g_{\text{cl}}(UV) > g_{\text{cl}}(IR)$$

to hold under localized closed string condensation.

For $\mathbb{C}^d/\Gamma$ orbifolds

$$g_{cl} = \frac{1}{|\Gamma|} \sum_{\gamma \neq 1} \frac{1}{|\det(1 - R(\gamma))|^2}$$

For $\mathbb{C}/\mathbb{Z}_n$ this works nicely

$$g_{\text{cl}} = \frac{1}{12}(n - \frac{1}{n})$$

Indeed, decreases under the flows we described

$g_{\rm cl}$ for $d=2$

For $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ the situation is much less clear ...

The formula is

$$g_{cl}(n,p) = \frac{1}{n}\sum_{s=1}^{n-1}\frac{1}{[4\sin(\pi s/n)\sin(\pi p s/n)]^2}$$

This can be shown to be a rational number. There is an algorithm to compute it, and it depends on the partial quotients a_α. But it is very complicated.

$\sim \frac{1}{720}\frac{n^3}{p^2}$ for $n\to\infty$, p fixed.

While $g_{\rm cl}$ decreases in many examples of flows, there are an infinite number of apparent counterexamples to $g_{\rm cl}(UV) > g_{\rm cl}(IR)$.

$$\mathbb{C}^2/\mathbb{Z}_{2\ell(3)} \to \mathbb{C}^2/\mathbb{Z}_{\ell(1)} \oplus \mathbb{C}^2/\mathbb{Z}_{\ell(-3)}$$

via a marginal deformation.

$$\frac{(2\ell)^3}{9} < \frac{\ell^3}{1}$$

So - maybe the conjecture is just wrong.

Possible way out

But... where do the new localized degrees of freedom come from?

Recently Kutasov, Parnachev, and I reconsidered it and found some evidence that $g_{\rm cl}$ indeed decreases as long as we consider perturbations by generators of the chiral ring, or products of two generators of the chiral ring.

Why is this interesting?

Even in the susy case the *geometrical* target $\mathbb{C}^2/\mathbb{Z}_n$ can lead to a singular CFT if the "B-field through shrunken cycles" is zero.

In the proposed counterexamples to the $g_{\rm cl}$-conjecture a minimal cycle in the daughter singularity appears which does not correspond to such a cycle in the parent.

Example: $\ell = 3s+1$:

$$\frac{2\ell}{3} = [2s+1, 3] \qquad\qquad \frac{\ell}{\ell-3} = [\underbrace{2,\ldots,2}_{s-1}, 4]$$

Is it possible that the flows to daughter singularities actually produce singular CFT's?

Melvin Models & Delocalization of States

We'll finish with some tangentially related remarks.

They illustrate the point that in some orbifolds the distinction between localized and delocalized is blurred.

Consider the Melvin geometry $\mathbb{R}^{1,6} \times (\mathbb{C} \times \mathbb{R})/\mathbb{Z}$

$$\begin{aligned} g_0 : z &\to e^{2\pi i\gamma} z \qquad z \in \mathbb{C} \\ y &\to y + R \qquad y \in \mathbb{R} \end{aligned}$$

What are the localized states?

Classical mass formula for the groundstate in w-twist sector $\mathcal{H}_w$:

$$\alpha' M^2(r) = (2\pi R w)^2 + (2r \sin \pi \parallel w\gamma \parallel)^2$$

$$\begin{aligned} \gamma = p/q \qquad & \mathcal{H}_{loc} = \oplus_{w \neq 0 \mathrm{modq}} \mathcal{H}_w \\ \gamma \neq p/q \qquad & \mathcal{H}_{loc} = \oplus_{w \neq 0} \mathcal{H}_w \end{aligned}$$

$\parallel x \parallel = \mathrm{Min}(\{x\}, 1 - \{x\})$

Nearly delocalized states

γ irrational $\Rightarrow \parallel w\gamma \parallel$ can be arbitrarily small for large w.

This leads to divergences in defining sums over twisted sectors.

Example:

$$Z_{\rm loc} \quad \sim \quad \sqrt{R^2 \tau_2} e^{\frac{2\pi}{\tau_2}} \sum_{s\neq 0}^{\infty} e^{-\pi \tau_2 R^2 s^2} \frac{1}{(\sin \pi s \gamma)^2}$$

$(\sin \pi s\gamma)^{-2}$: Volume of delocalization.

1. There are certain Liouville numbers γ for which $Z(\tau) = \infty$.

2. If $\gamma = [a_0, a_1, a_2, \ldots]$ with $a_i < M$ then

$$\parallel q\gamma \parallel \geq \frac{\nu(\gamma)}{q}$$

$$\frac{\dim \mathcal{H}_{\rm loc}(\gamma_1)}{\dim \mathcal{H}_{\rm loc}(\gamma_2)} = \frac{\nu(\gamma_2)}{\nu(\gamma_1)}$$

Entropy of delocalization and Lyapunov exponent

3. Regulate the divergence by

$$\gamma_n = p_n/q_n = [a_0, \ldots, a_n]$$

$$\dim \mathcal{H}_{\text{loc}}(\gamma_n) = \frac{1}{48}(q_n - \frac{1}{q_n})$$

But $q_n \sim c e^{\frac{1}{2}\lambda(\gamma) n}$, where $\lambda(\gamma)$:= *Lyapunov exponent.*

So

$$\boxed{\frac{\log \dim \mathcal{H}_{\text{loc}}(\gamma_1)}{\log \dim \mathcal{H}_{\text{loc}}(\gamma_2)} = \frac{\lambda(\gamma_1)}{\lambda(\gamma_2)}}$$

Facts:

1. (Khinchin): For almost every γ

$$\lambda(\gamma) = \lambda_0 = \frac{\pi^2}{6 \log 2}.$$

2. (Pollicott & Weiss):

$$\lambda(\gamma) \in [2 \log \frac{1+\sqrt{5}}{2}, \infty)$$

Thus, the degree of delocalization depends sensitively on γ.

Conclusions - Open Problems

We have a coherent picture of the decay of localized closed string tachyons.

It suggests an interesting mathematical conjecture: The quantum McKay correspondence.

However, there are several open problems and unresolved puzzles:

1. Extension to $d \geq 3$: RG flow and Quantum McKay correspondence

2. Understanding the Coulomb branch branes better.

3. Resolving the puzzles about g_{cl}.

4. Can we make sense of closed string tachyon decay for γ irrational? Is it related to the Gauss map?