Update On Susy Field Theory And Invariants Of Smooth Four-Manifolds

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1 **Motivation**

2 The Paradigm: SU(2) N=2 SYM & Donaldson invariants

3 Other d=4 N=2 Theories $_5$

4 d=5: ``K-theoretic Donaldson invariants''

5 Family Donaldson invariants

Glorious History Of
4d Field Theory & Four-Manifold Topology

Instantons (BPST) (1975) $\Rightarrow$

Donaldson invariants (1982) $\Rightarrow$

TQFT (1988) $\Rightarrow$

Seiberg-Witten Invariants (1994)

$\Rightarrow$ Revolution of 1995

A paradigmatic example of the modern interplay between the physics and mathematics

Physical Mathematics And The Future
http://www.physics.rutgers.edu/~gmoore/

Today: Continue the line of development from 1988-1998

But not all questions are answered…

X: $\mathrm{d} = 4$, Smooth, compact, oriented, $\partial X = \emptyset$.

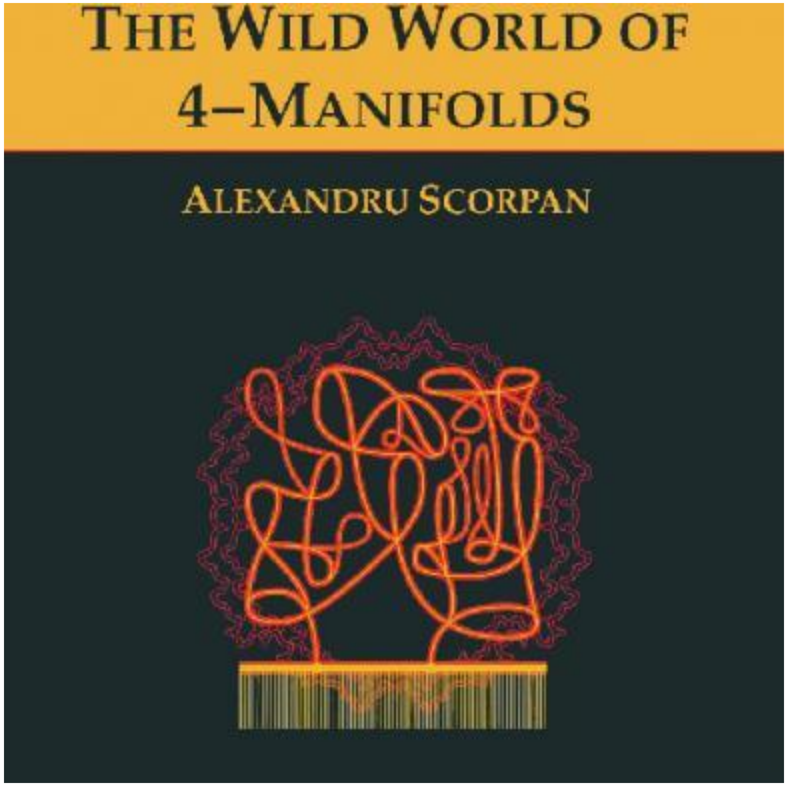
For **simplicity**: Connected & $\pi_1(X) = 0$

We assume
(essential in Donaldson & Seiberg-Witten theory)
that X admits an almost complex structure

$\Longleftrightarrow$ $b_2^+(X)$ *is* odd

Misses ``half'' the world of four-manifolds!

We do not know anything even close to a complete topological invariant.



What About Other N=2 Theories?

There are infinitely many other four-dimensional N=2 supersymmetric quantum field theories.

Topological twisting should make sense for any $\mathcal{N}=2$ theory $\mathcal{T}$.

(but $\mathcal{T}$-dependent details remain to be worked out)

Given the successful application of $\mathcal{N}=2$ SYM for $G=SU(2)$ to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N}=2$ field theories?

Yes

Learn more about existing invariants

Marino-Moore-Peradze (1998):
Argyres-Plesser-Seiberg-Witten
description of AD1 theory $\Rightarrow$
Superconformal simple type.

Marino-Moore: SU(N) Donaldson invariants:
Applications to 3d Floer homology
(MM 1998; Daemi & Xie 2020)

Generalizations

Other 4d theories

5d theories

6d theories

Coupling to background supergravity…

These *might* lead to truly new invariants that are independent of the Donaldson/SW invariants …

…. or not …

Today’s talk: Some of these generalizations are already leading to some interesting issues in QFT…

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4d N=2 SU(2) SYM On X

$P \to X$: Principal $SO(3)$ bundle

$w_2(P) \in H^2(X, \mathbb{Z}_2)$: ``'t Hooft flux''

``background 1-form symmetry gauge field''

Witten (1988): ``Topological twisting''

Spinors $\to$ differential forms

$$Q_{\alpha}^{A}, \bar{Q}_{\dot{\alpha}}^{A} \quad \to \quad Q, K, \ldots$$

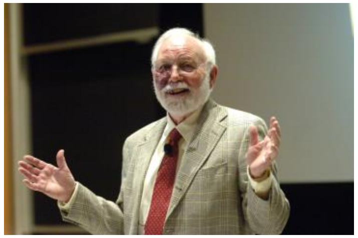
$$Q^2 = 0 \qquad \{K, Q\} = d$$

$$Q\,A_\mu = \psi_\mu \qquad Q\,\psi_\mu = -D_\mu\phi \qquad Q\,\phi = 0$$

$$A \in \mathcal{A}(P) \qquad \phi \in \Omega^0(X, ad\,P \otimes \mathbb{C})$$

An important viewpoint in the section on families below

Baulieu & Singer, 1988



$\mathcal{G} := Aut(P)$ Group of gauge transformations

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P)$

Q –closed observables: $\mathcal{O}(pt) = Tr(\phi^2(pt))$

$$\{ K, Q \} = d \Rightarrow \quad \mathcal{O}_j := K^j \mathcal{O}$$

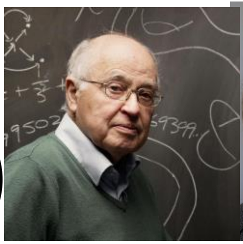
$\mathcal{O}(\Sigma_j) := \int_{\Sigma_j} \mathcal{O}_j$ only depends on $[\Sigma_j] \in H_j(X)$

Function on $H_*(X)$: $Z_W(\Sigma) = \langle e^{\mathcal{O}(\Sigma)} \rangle$

Witten (1988): For a suitable background $SU(2)_R$ connection

$Z_W(\Sigma)$ independent of metric $g_{\mu\nu}$ on X

Witten (1988) & Atiyah& Jeffrey(1990)



$Z_W(\Sigma)$ path integral localizes to an integral over

$$\mathcal{M} \subset \mathcal{A}(P)/\mathcal{G}$$

$$\mathcal{M} := \{A \in \mathcal{A}(P) : F(A)^+ = 0\}/\mathcal{G}$$

$$F^+ := \frac{1}{2}(F + * F)$$

Donaldson Invariants

Donaldson: $\mu: H_*(X) \to H^*(\mathcal{M})$

$S \subset X$: smooth surface.

$\mathcal{M}(S)$: subspace where the Dirac equation on S coupled to ∇_P has a solution

Poincare dual to $\mathcal{M}(S)$ defines $\mu(S) \in H^2(\mathcal{M};\mathbb{Q})$

$$Z_D(\Sigma) = \int_{\mathcal{M}} e^{\mu(\Sigma)} = \sum_{k,r} \int_{\mathcal{M}_k} \frac{\mu(\Sigma)^r}{r!}$$

$\mathcal{M}$ depends on $g_{\mu\nu}$, but $Z_D(\Sigma)$ does not

Main Statement

$$Z_W(\Sigma) = Z_D(\Sigma)$$

$$=: Z_{DW}\ (\Sigma)$$

Evaluation Of $Z_{DW}(\Sigma)$

$Z_{DW}(\Sigma)$ independent of $g_{\mu\nu}$ on X

Consider metric $L^2 g_{\mu\nu}$ in the limit $L \to \infty$

$\Rightarrow$ Use Seiberg-Witten LEET

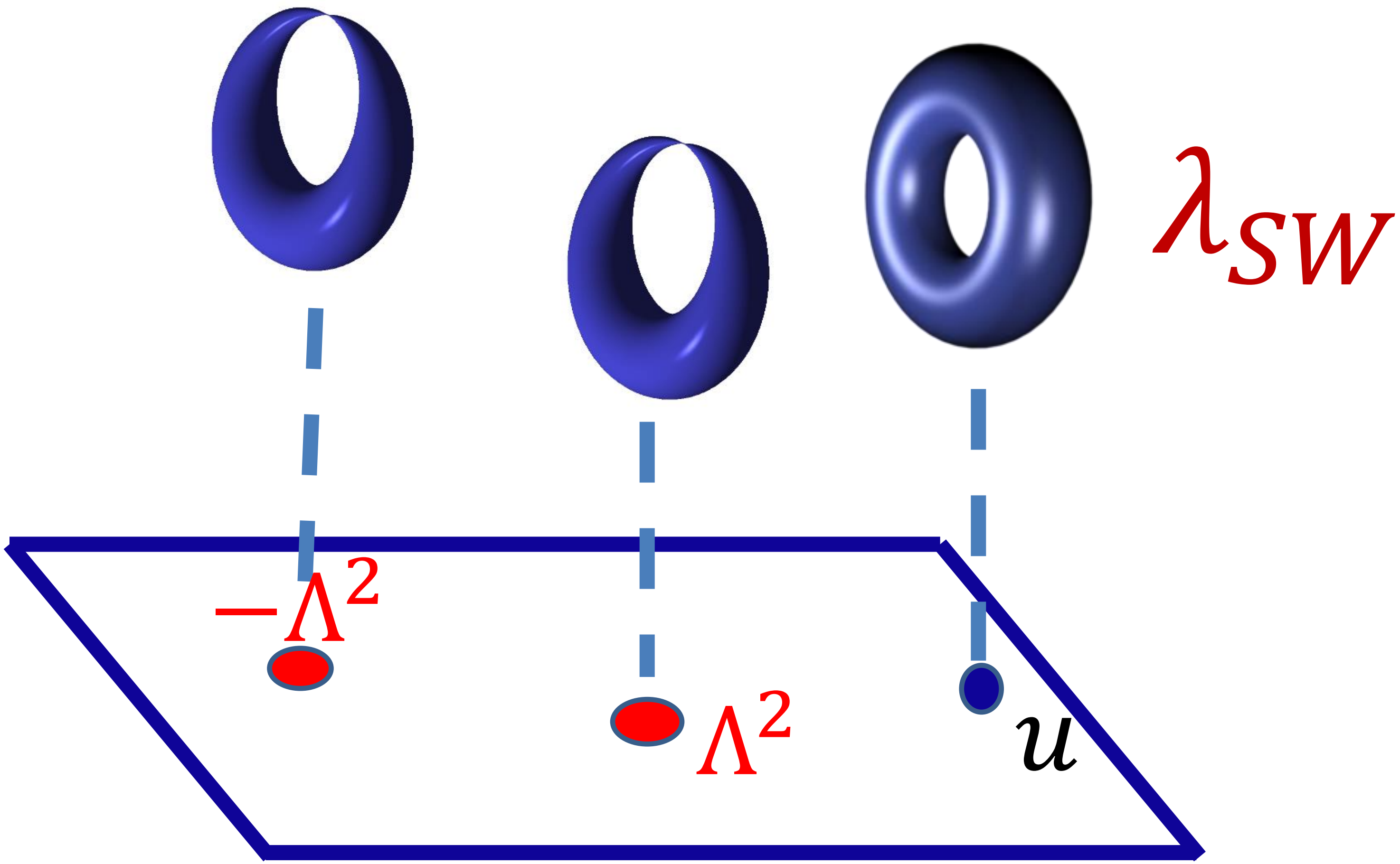
$$Z_{DW}(\Sigma) = Z^J_{Coul}(\Sigma) + Z^J_{SW}(\Sigma)$$

Witten 94
Moore-Witten 97

Period point $J \in H^2(X, \mathbb{R})$: $J = * J$

$Z^J_{Coul}(\Sigma)$

Integral over the Coulomb branch of vacua on $\mathbb{R}^4$, parametrized by $u = \langle Tr\phi^2 \rangle$, and computed using SW LEET for U(1) VM



Singularities at $u = \pm\Lambda^2$ spoil topological invariance.

Restore it with integral over ``Higgs branch vacua''

$$Z_{SW}^{J}(\Sigma) = \sum_{c \in Spin^c(X)} SW^J(c)\ f_c(\Sigma)$$

$f_c(\Sigma)$: Trigonometric function of Σ computed using classical intersection theory and LEET.

$SW^J(c) \in \mathbb{Z}$ Seiberg-Witten invariants

$Spin^c(X) \leftrightarrow\ \mathrm{w}_2(\mathrm{X})^{\mathrm{lift}} + 2\,\mathrm{H}\hat{}2(\mathrm{X},\mathbb{Z})$

$SW^J(c)$ Counts the number of solutions to SW equations:

A_{ab} ``$U(1)$ connection'' $M \in \Gamma(W^+)$

$$F(A_{ab})^+ = \bar{M}M \qquad \gamma \cdot D\ M = 0$$

Theorem (Witten 94): When $b_2^+(X) > 1$, $SW^J(c)$ is independent of J and is only nonzero for finitely many spin-c structures.

$$Z^{J}_{Coul}(\Sigma) = \int [dVM] e^{-S_{LEET} + \mathcal{O}(\Sigma)}$$

$$U(1)\, VM = (A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, a)$$

$$\begin{aligned}
\mathcal{L}_0^{\mathrm{IR}} = & \frac{i}{16\pi}(\overline{\tau}\, F_+ \wedge F_+ + \tau\, F_- \wedge F_-) + \frac{1}{2\pi}(\mathrm{Im}\,\tau) da \wedge \star d\overline{a} - \frac{1}{8\pi}(\mathrm{Im}\,\tau) D \wedge \star D \\
& - \frac{1}{16\pi}\tau\, \psi \wedge \star d\eta + \frac{1}{16\pi}\overline{\tau}\eta \wedge d \star \psi - \frac{1}{8\pi}\tau\psi \wedge d\chi - \frac{1}{8\pi}\overline{\tau}\chi \wedge d\psi \\
& + \frac{i\sqrt{2}}{16\pi}\frac{d\overline{\tau}}{d\overline{a}}\eta\chi \wedge (F_+ + D) + \frac{i\sqrt{2}}{2^7\pi}\frac{d\tau}{da}\psi \wedge \psi \wedge (F_- + D) \\
& + \frac{i}{3 \cdot 2^{11}\pi}\frac{d^2\tau}{da^2}\psi \wedge \psi \wedge \psi \wedge \psi - \frac{i\sqrt{2}}{3 \cdot 2^5\pi}\mathcal{Q}\left(\frac{d\overline{\tau}}{d\overline{a}}\chi_\mu{}^\rho \chi^{\mu\sigma}\chi_{\rho\sigma}\right)\sqrt{g}\, d^4x\ ,\qquad (
\end{aligned}$$

$\tau(a)$: Complicated function determined by SW LEET

Claim (Moore-Witten 97):

$$b_2^+(X) > 1 \;\Rightarrow Z^J_{Coul}(\Sigma) = 0$$

$$b_2^+(X) = 1 \;\Rightarrow Z^J_{Coul}(\Sigma)\text{: Tree level exact}$$

$$b_2^+(X) = 0 \;\Rightarrow Z^J_{Coul}(\Sigma)\text{: One loop exact}$$

N.B. Not a localization evaluation of $Z^J_{Coul}(\Sigma)$!!

1997 derivation overlooked some subtleties, now being revisited with V. Saxena

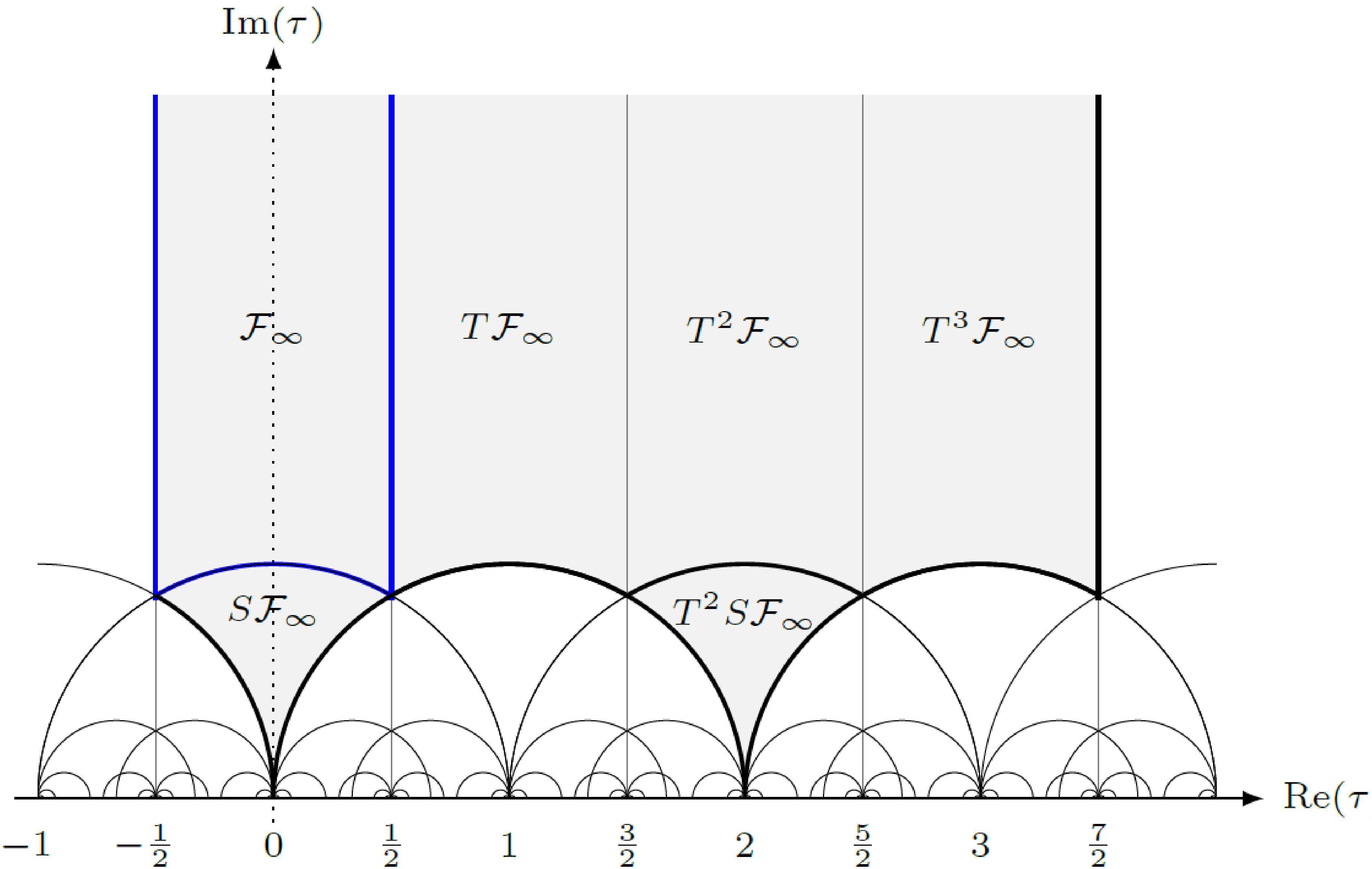
u-Plane Integral

SW94: Coulomb branch has a modular parametrization:

$$u(\tau) = \frac{\vartheta_2^4 + \vartheta_4^4}{2\,\vartheta_2^2\vartheta_4^2} = \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} + \cdots$$

$$q = e^{2\pi i \tau}$$

Coulomb branch $\cong UHP/\Gamma^0(4)$



$$\Gamma^0(4) = \{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} mod 4\} \subset SL(2, \mathbb{Z})$$

$$Z^J_{Coul}(\Sigma) = \int_{\mathcal{F}(\Gamma^0(4))} d\tau d\bar{\tau}\ \mathcal{H}(\tau) \frac{\partial}{\partial \bar{\tau}}\ G^J(\tau, \bar{\tau}, \Sigma)$$

$G^J(\tau, \bar{\tau}, \Sigma)$ Comes from the photon path integral

Not holomorphic in τ $(or\ u)$

Continuously metric dependent.

$G^J(\tau, \bar{\tau}, \Sigma)$: Is a mock Jacobi form

G. Korpas, J. Manschot, G. Moore, I. Nidaiev (2019)

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Topological Twisting

For an astutely chosen background $SU(2)_R$-symmetry connection:

$$A^{R-symmetry} \sim \omega^{+,LC}$$

$$S_{N=2\,SYM} = \{Q,V\} + \int \tau_{uv} \; Tr\, F \wedge F$$

It is useful to rephrase topological twisting in terms of ``reduction of structure group''

Reduction Of Structure Group

$\varphi: G_1 \to G_2$ Homomorphism of groups

Given a principal G_1 bundle $P_1 \to M$ we can form an associated principal G_2 bundle

$$P_1 \times_\varphi G_2 = \{ [(p_1, g_2)] \} \quad (p_1 g_1, g_2) \sim (p_1, \varphi(g_1) g_2)$$

Transition functions $g_{\alpha\beta}: U_{\alpha\beta} \to G_1$ map to new transition functions $\varphi\left(g_{\alpha\beta}\right): U_{\alpha\beta} \to G_2$

Reduction Of Structure Group (RSG)

We say a principal G_2 bundle $P_2 \to M$ ``admits a Reduction of Structure Group to G_1 via φ " if it is in the image of this map (functor) (up to isomorphism)

P_2 has transition functions $\tilde{g}_{\alpha\beta}: U_{\alpha\beta} \to G_2$

We can find functions $g_{\alpha\beta}: U_{\alpha\beta} \to G_1$ such that

$\tilde{g}_{\alpha\beta} = \varphi\left(g_{\alpha\beta}\right)$ AND $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}(x) = 1_{G_1}$

Examples Of RSG

1. A Riemannian metric on an n-manifold M_n ***IS*** an RSG of the frame bundle $Fr(M_n)$ via

$$\varphi: O(n) \to GL(n, \mathbb{R})$$

2. An orientation ***IS*** an RSG of the frame bundle $Fr(M_n)$ via

$$\varphi: GL^{+}(n, \mathbb{R}) \to GL(n, \mathbb{R})$$

Examples Of RSG

3. An almost complex structure
(compatible with a metric + orientation)
IS an RSG of the bundle of oriented ON frames via

$$\varphi: U\left(\frac{n}{2}\right) \to SO(n)$$

4. A $Spin^c$-structure ***IS*** an RSG of the bundle of oriented ON frames via

$$\varphi: Spin^c(n) := \frac{Spin(n) \times U(1)}{\mathbb{Z}_2} \to SO(n)$$

Reduction Of Structure Group (RSG)

RGS extends to category of bundles with connection:

∇_1 on P_1

$$\Rightarrow \quad \nabla_2 = \varphi_*(\nabla_1) \; on \; P_1 \times_\varphi G_2$$

Topological Twisting As RSG

[Manschot-Moore; Cushing,Moore,Rocek,Saxena; D. Freed, unpublished]

Background fields for $\mathcal{N} = 2 \;\; SU(N)$ SYM:

A connection for $G^{phys} = \frac{Spin(4) \times SU(2)_R}{\langle\, (-1,-1,-1\,)\rangle}$

$$\varphi\colon \;\; SO(4) = \frac{(SU(2)_+ \times SU(2)_-)}{\langle(-1,-1)\rangle} \to \;\; G^{phys}$$

$$\varphi(\,[(u_1, u_2)]) := [\,(u_1, u_2, u_1)]$$

$\varphi_*(\nabla^{LC}) =$ background fields (for ``0-form symmetry'') of the physical theory

Example: $SU(2), \mathcal{N} = 2^*$ Symmetry group is

$$G^{phys} = \left(SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1) \right) / \mathcal{Z}$$

$$\mathcal{Z} = \langle (-1,-1,-1,-1) \rangle \cong \mathbb{Z}_2$$

There is ***NO*** homomorphism from $SO(4)$ to G^{phys}

(compatible with constraints on the morphism of Lie algebras)

There ***IS*** a homomorphism $Spin^c(4) \to G^{phys}$

The twisted theory depends nontrivially on the $Spin^c$ *structure* on the 4-fold [Manschot & Moore 2021]

WIP with Vivek Saxena and Ranveer Singh aims to generalize the picture to arbitrary Lagrangian N=2 theories.

Conjecture: Topological theory depends on a $\frac{Spin(4)\times U(1)^{N_f}}{\mathbb{Z}_2}$ structure + restrictions on background gauge fields for ``1-form symmetries''

Goal: Further generalization to arbitrary class S theories.

What Do The Other (Lagrangian) Theories Compute?

The path integral for topologically twisted Lagrangian theories localizes to intersection theory on moduli space of the Nonabelian Seiberg-Witten equations

(instanton moduli space is a special case)

How Twisted Lagrangian Theories Generalize Donaldson Invariants

$$Z(\Sigma) = \langle e^{\mathcal{O}(\Sigma)} \rangle_{\mathcal{T}} \;\; = \int_{\mathcal{M}} e^{\mu(\backslash \mathrm{Sigma})} \mathcal{E}(\mathcal{V})$$

But now $\mathcal{M}$: is the moduli space of:

$$F^{+} = \mathcal{D}(M, \bar{M}) \qquad\qquad \gamma \cdot D\, M = 0$$

$$M \in \Gamma(W^{+} \otimes V)$$

W^{+}: Rank 2 ``spin'' bundle; V depends on matter rep

``Nonabelian Seiberg-Witten equations''

[Labastida-Marino 1997; Losev-Shatashvili-Nekrasov1997]

Defining the integral over $\mathcal{M}$ requires a choice of orientation

Orientability should be determined by the mod-two index of the deformation operator

$$d_A^+ \oplus d_A^* \oplus \gamma \cdot D : \Omega^1(X, adP) \oplus \Gamma(W^+ \otimes V) \\ \rightarrow \quad \Omega^{2,+}(X, ad\ P\ ) \oplus \Omega^0(X, adP) \oplus \Gamma(W^- \otimes V\ )$$

View the determinant bundle of the deformation complex as the (real) state space of a 5d invertible theory

We *hope* the theory is trivializable…

An orientation is a trivialization of this invertible theory.

Question (WIP with D. Freed): Is there a useful description of the 5d invertible theory for the moduli space of the nonabelian Seiberg-Witten equations for general compact group and quaternionic representation?

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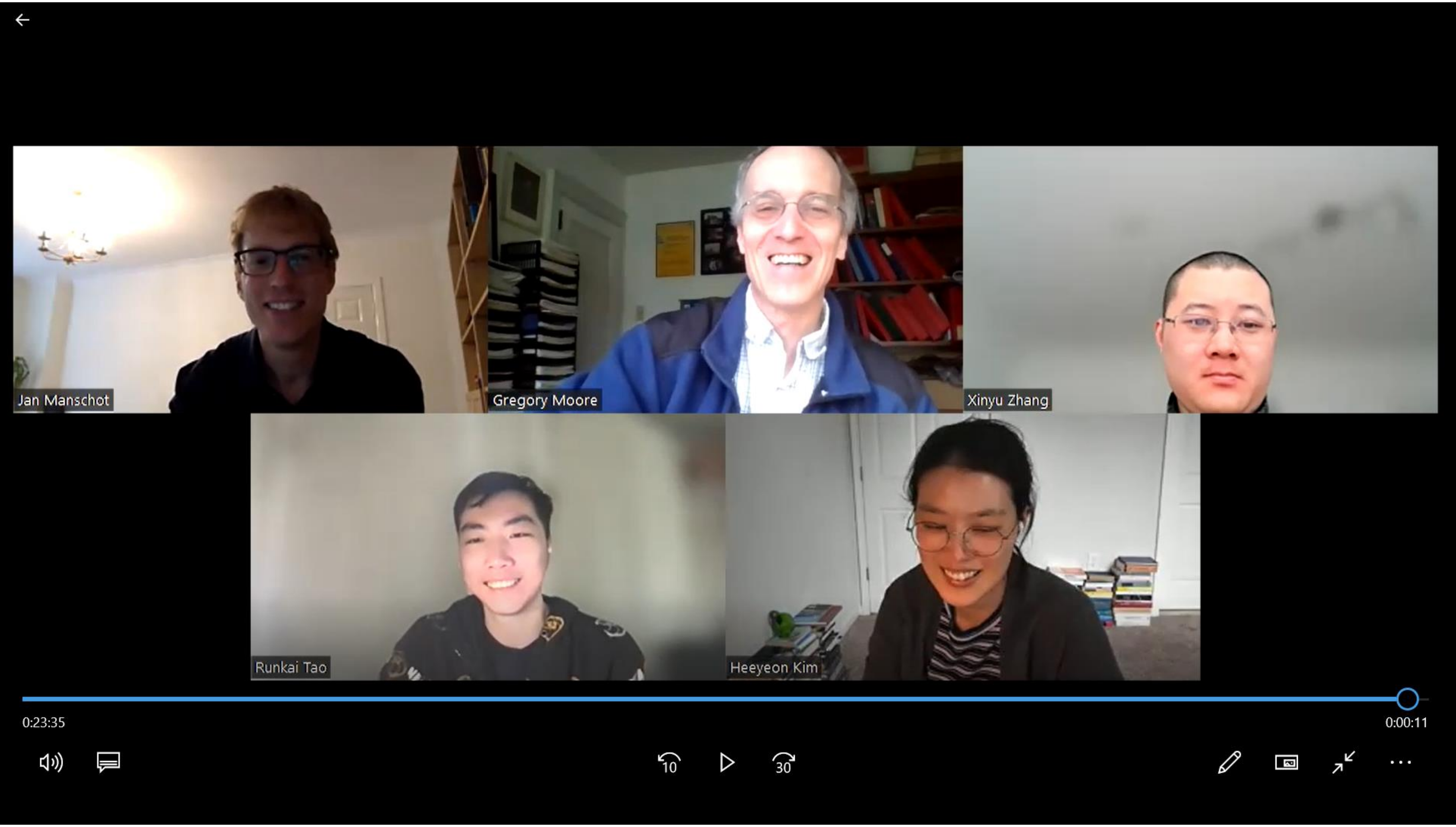
``K-Theoretic Donaldson Invariants’’



$\mathcal{N} = 1$ 5D SYM

Vectormultiplet: $V = (A_m, \sigma, \lambda_\alpha^A, D_{AB})$

$$S[V] = g_{5d,SYM}^{-2} \int_{X_5} tr\, F * F + tr\, D\sigma * D\, \sigma + \cdots$$

$$J_{TOP} = tr\, F^2 \Rightarrow V^{(I)} = \left(A_m^{(I)}, \sigma^{(I)}, \ldots\right)$$ Seiberg 1995

$$S\left[V; V^{(I)}\right] = \int_{X_5} A^{(I)} \wedge tr\, F^2 + \int \sigma^{(I)}\, tr\, F * F + \cdots$$

$$\sigma^{(I)} \sim g_{5d,SYM}^{-2}$$

Now take $X_5 = X \times S^1$

$\theta \sim \oint_{S^1} A^{(I)}$ const. on X

$F^{(I)}$ pulled back from X

We have a partial topological twist based on reduction of structure group

$$\varphi: \mathbb{Z}_2 \times SO(4) \to (Spin(5) \times SU(2)_R)/\mathbb{Z}_2$$

Background fields: $\varphi_*(\nabla^{LC})$

Topological on X but a nontopological, spin, theory on S^1

$$Q^2 = \partial_t$$

SQM With Target $\mathcal{M}$

Topological on X $\Rightarrow$ Can shrink X $\Rightarrow$
Describe the twisted theory in terms of
SQM on S^1 with target space
the moduli space of instantons
[Nekrasov, 1996]

But $\mathcal{M}$ is not spin in general,
so the theory will be anomalous

Potential Global Anomalies

$$w_2(\mathcal{M}) \neq 0$$

1D: $Pfaff(\gamma \cdot D_{S^1})$ not well-defined on $L\mathcal{M}$

5D: $Pfaff(\gamma \cdot D_{X\times S^1})$ not well-defined on $\mathcal{A}/\mathcal{G}$

All controlled by ``the same'' 6d mod-two index.

Discussions are in progress with D.Freed and E. Witten to give a useful formula for it.

Anomaly Cancellation

$$S[V;V^{(I)}] = \int_{X\times S^1} F(A^{(I)}) \wedge CS(A_{dyn}) + \cdots$$

$\Rightarrow$ SQM($\mathcal{M}$) couples to a ``line bundle'' $\mathrm{L}_{\mathrm{CS}}(n) \to \mathcal{M}$

$$n := \left[\frac{F^{(I)}}{2\pi}\right] \in H^2(X,\mathbb{Z})$$

Working hypothesis: For suitable n $S^+_{\mathcal{M}} \otimes L(n)$ exists

Evidence: One can show

X admits an ACS $\Rightarrow$ $\mathcal{M}_k$ is spin-c

$$S_{twisted} = \{\, Q, V \,\} + \log \mathcal{R}^4 \int_X tr \frac{F^2}{8\pi^2}$$

$$\mathcal{R}^4 = \exp\left[-8\,\pi^2 \frac{R}{g^2_{5d,YM}} + i\,\theta \right]$$

$$m_{inst.part.} = \frac{1}{R} \log \mathcal{R}^2$$

$$\mathcal{R} = R\,\Lambda$$

Λ dimensional scale in the physical theory

$g^2_{5d,sym} = \infty$ corresponds to the E_1 5d superconformal theory [Seiberg 1995]

And our formulae below indeed have special properties at $\mathcal{R}^4 = 1$.

At least formally the path integral should compute

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \; Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{L(n)}\;)\}$$

$$d_k = \dim_{\mathbb{R}} \mathcal{M}_k \; = 4h^{\vee} k \; - \dim G \, \frac{\chi + \sigma}{2}$$

$$Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{L(n)}\ )\}$$

In good cases, this is the index of the Dirac operator $D_{L(n)}$

$\Rightarrow$ ``K-theoretic Donaldson invariants''

All this should generalize to (anomaly-free) 6d SYM theories on $X \times \mathbb{E}$

$$Index\left(D_{L(n)}\right) \to Ell(\sigma(\mathcal{M}_k))$$

Five Dimensions

$$Z(\mathcal{R},n)`` = " \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} ch\big(L(n)\big)\,\hat{A}(\mathcal{M}_k)$$

[Nekrasov, 1996; Losev, Nekrasov, Shatashvili, 1997]

We study $Z(\mathcal{R},n)$ using both
the Coulomb branch integral
and, independently, toric localization,
for X a toric Kahler manifold

2006:

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

ABSTRACT. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

2019:

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

ABSTRACT. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

Using the physical techniques we derived results

Differ from GNY!

Agree with GNY!

(Suitably interpreted.)

This raises a puzzle…

Total partition function is a sum of two terms

$$Z^J(\mathcal{R}, n) = \mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n) + Z^J_{SW}(\mathcal{R}, n)$$

$\mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n)$: 4d Coulomb branch integral

One can deduce Z^J_{SW} from $\mathrm{Z}^J_{\mathrm{Coul}}$

SW special Kahler geometry is subtle

For 5d SYM gauge group of rank 1:

Coulomb branch = $\mathbb{C}$

parametrized by : $U = \langle\, Pexp \oint_{S^1} (\sigma + i\, A_{5d,ym})\,\rangle$

$$= e^{Ra} + e^{-Ra} + instanton\, corr's$$

a: cylinder valued

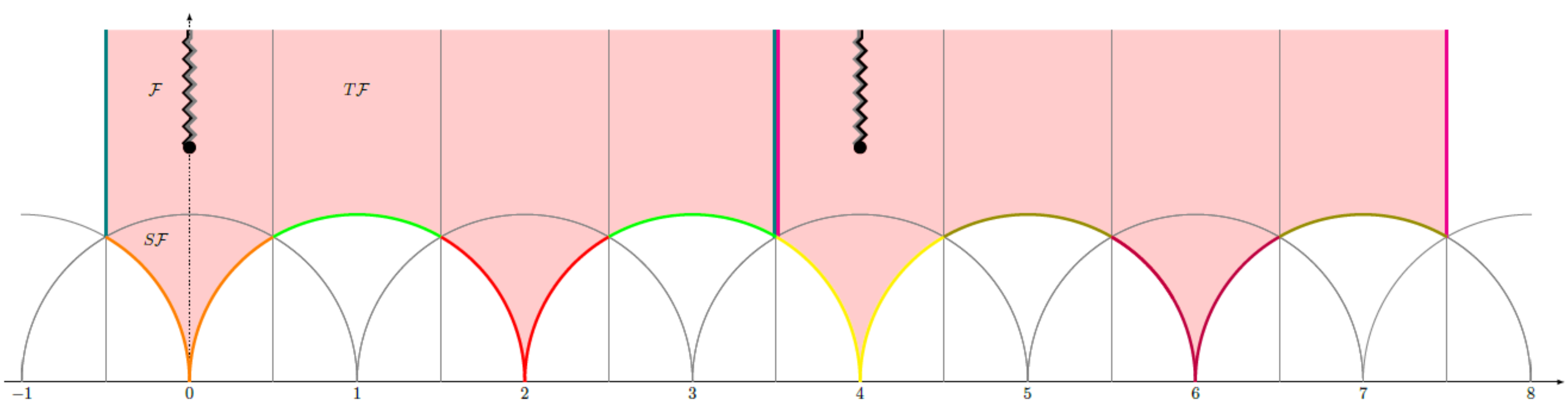
$\mathcal{F}(a)$ is known [Nekrasov 1996, 2000,…]

Modular Parametrization Of U −plane

The Coulomb branch is a branched double cover of the modular curve for $\Gamma^0(4)$

$$\left(\frac{U}{R}\right)^2 + u(\tau) = 8 + 4(\mathcal{R}^2 + \mathcal{R}^{-2})$$

$$u(\tau) = \frac{\vartheta_2(\tau)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_3(\tau)^2}{\vartheta_2(\tau)^2}$$ Hauptmodul for $\Gamma^0(4)$



$$Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \nu\, C^{n^2} \;\; \Psi^{J}\left(\tau, \frac{n}{2}\,\zeta\right)$$

$$\nu(\tau, \mathcal{R}) = \frac{\vartheta_4^{13-b_2}}{\eta^9} \; \frac{1}{\sqrt{1 - 2\,\mathcal{R}^2 u(\tau) + \mathcal{R}^4}}$$

$$C(\tau, \mathcal{R})$$

Suitably modular invariant and holomorphic ``contact term''

$$\zeta(\tau, \mathcal{R}) \sim \frac{\partial^2 \mathcal{F}}{\partial a\, \partial m_{inst}}$$

$$\Psi^J(\tau, z) = \sum_{k \in H^2(X,\mathbb{Z})} \left(\frac{\partial}{\partial \bar{\tau}} E_k^J \right) q^{-\frac{k^2}{2}} e^{-2\pi i\, k\cdot z} (-1)^{k\cdot K}$$

$$E_k^J = Erf\left(\sqrt{Im \tau}\, \left(k + \frac{Im\, z}{Im\, \tau} \right) \cdot J \right)$$

$$z \to \frac{n}{2}\, \zeta(\tau, \mathcal{R})$$

Not holomorphic.

Metric dependent .

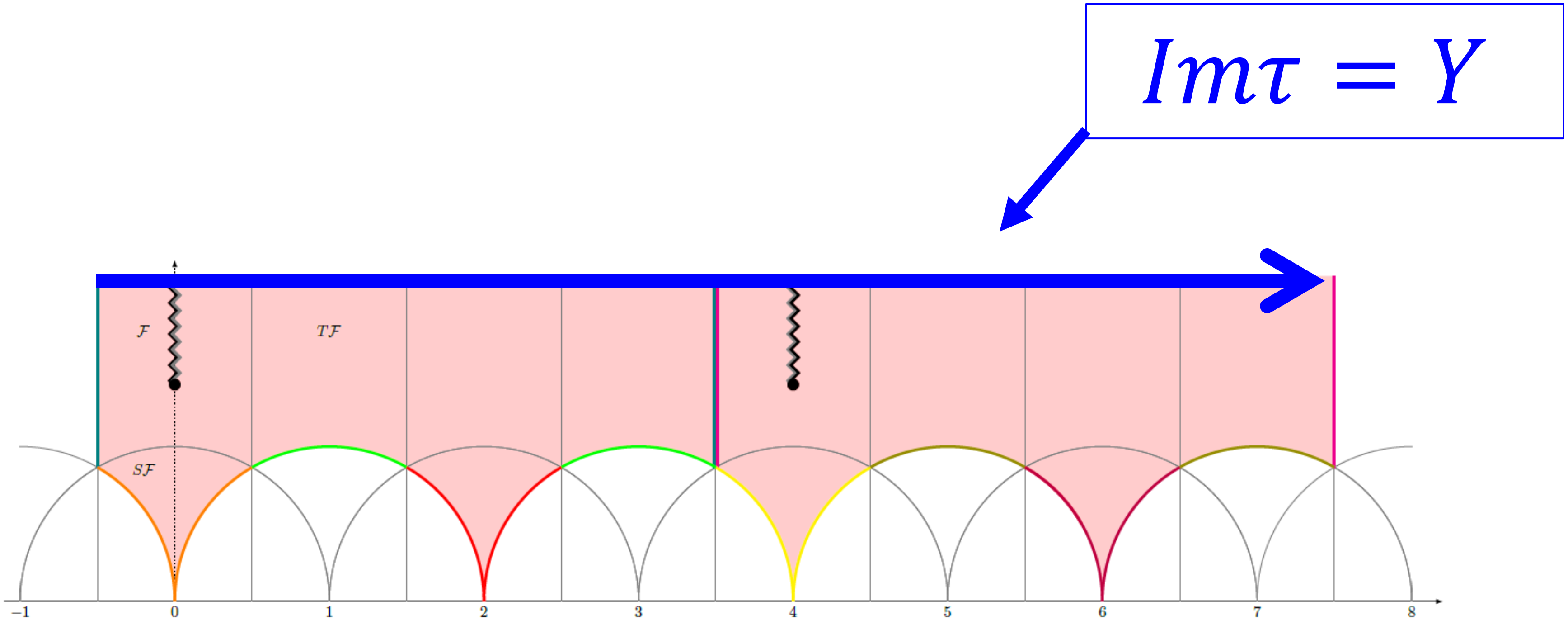
Measure As A Total Derivative

$$Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \mathcal{H}(\tau) \;\; \Psi^{J}\left(\tau, \frac{n}{2}\,\zeta\right)$$

$\exists$ suitably modular invariant and nonsingular $\hat{G}(\tau, \bar{\tau})$

$$\frac{\partial}{\partial \bar{\tau}} \hat{G} = \Psi^{J}$$

(It can be hard to find explicit formulae for $\hat{G}$: one needs the theory of mock modular forms, and their generalizations.)



$$Z^{J}_{\mathrm{Coul}}(n,\mathcal{R}) = \lim_{Y\to\infty} \int d\tau_1\, \mathcal{H}\, \hat{G}\,\Big|_{\tau=\tau_1+i\,Y}$$

Examples Of Explicit Results

$$X = \mathbb{CP}^2$$

$$Z_{Coul}(n,\mathcal{R}) = \Big[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\Big]_{q^0}$$

$$G(\tau,\mathcal{R}) = -\frac{e^{i\,\pi n\frac{\zeta(\tau,\mathcal{R})}{2}}}{\vartheta_4(\tau)} \sum_{\ell\in\mathbb{Z}} (-1)^{\ell}\ \ \frac{q^{\frac{\ell^2}{2}-\frac{1}{8}}}{1-e^{i\,\pi n\,\zeta(\tau,\mathcal{R})}q^{\ell-\frac{1}{2}}}$$

Examples Of Explicit Results

Wall Crossing Formula:

$$Z^{J}_{\mathrm{Coul}} - Z^{J'}_{\mathrm{Coul}} = \left[\nu \, C^{n^2} \Theta^{J,J'}(\tau, \mathcal{R}) \right]_{q^0}$$

$$\Theta^{J,J'} = \sum_{k \in H^2(X,\mathbb{Z})} \left(s^{J}_{k} - s^{J'}_{k} \right) q^{-\frac{k^2}{2}} e^{-2\pi i \, k\cdot n \, \zeta(\tau,\mathcal{R})} \, (-1)^{k\cdot K}$$

$$s^{J}_{k} := sign\left(\sqrt{Im \tau} \, \left(k + \frac{Im \, \zeta(\tau,\mathcal{R})}{Im \, \tau} \right) \cdot J \right)$$

If we take these formulae literally, we get results that are very different from GNY

We get finite Laurent polynomials in $\mathcal{R}$ with terms involving negative powers of $\mathcal{R}$

It looks nothing like:

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \; Index\left(D_{L(n)}, \mathcal{M}_k\right)$$

$\nu, C, G, \Theta^{J,J'}$ are functions of τ and of $\mathcal{R}$

Subtle order of limits: $\mathcal{R} \to 0$ vs. $\mathrm{Im}\,\tau \to \infty$

Example: $u(\tau) \sim \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} - \frac{31}{4} q^{\frac{3}{4}} + \mathcal{O}\left(q^{\frac{5}{4}}\right)$

$$\nu(\tau, \mathcal{R}) = \frac{\vartheta_4^{13-b_2}}{\eta^9} \frac{1}{\sqrt{1 - 2\,\mathcal{R}^2 u(\tau) + \mathcal{R}^4}}$$

Similarly for $\zeta(\tau,\mathcal{R}) \sim \dfrac{\partial^2 \mathcal{F}}{\partial a\, \partial m_{inst}}$

$$\frac{2v}{\kappa} = \frac{2}{\pi}\frac{1}{\vartheta_2(\tau)\,\vartheta_3(\tau)}\int_0^{\mathcal{R}} \frac{dx}{\sqrt{1-2\mathrm{u}\,x^2+x^4}}.$$

$$2v/\kappa = -\frac{\tau}{2} + \frac{1}{\pi i}\Big(\log(-\mathcal{R}^2) + 2(\mathcal{R}^2-\mathcal{R}^{-2})\,q^{1/4} - 3(\mathcal{R}^4-\mathcal{R}^{-4})\,q^{1/2}$$
$$+(4\mathcal{R}^{-2} - 4\mathcal{R}^2 + 20/3\,\mathcal{R}^6 - 20/3\,\mathcal{R}^{-6})\,q^{3/4} + \ldots\Big)$$

$$\frac{2v}{\kappa} = \frac{2}{\pi}\frac{1}{\vartheta_2(\tau)\,\vartheta_3(\tau)}\sum_{\substack{n\geq 0,\\ n>k>0}} \binom{-1/2}{n}\binom{n}{k}\frac{(-2\mathrm{u})^k\,\mathcal{R}^{4n-2k+1}}{4n-2k+1},$$

$$Z_{\mathrm{Coul}}(n,\mathcal{R}) = \Big[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\Big]_{q^0}$$

$$Z^{J}_{\mathrm{Coul}} - Z^{J'}_{\mathrm{Coul}} = \Big[\,\nu\, C^{n^2}\Theta^{J,J'}\Big]_{q^0}$$

If we *first* expand the expressions in [....] in $\mathcal{R}$ around $\mathcal{R} = 0$ *then* take the constant q^0 term at each order in $\mathcal{R}$ we find remarkable and nontrivial agreement with similarly complicated results in GNY.

Using toric localization and the 5d instanton partition function we derived exactly the same formula for wall-crossing @ ∞

This would be another entire seminar….

The Puzzle: The naïve physical interpretation suggests we should take the constant term in the q-expansion

$$Z_{\mathrm{Coul}}(n,\mathcal{R}) = \Big[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\Big]_{q^0}$$

$$Z^{J}_{\mathrm{Coul}} - Z^{J'}_{\mathrm{Coul}} = \Big[\ \nu\ C^{n^2}\Theta^{J,J'}\Big]_{q^0}$$

But to get answers that agree with mathematical results we *first* expand in $\mathcal{R}$ and *then* take the constant term in q.

Using the wall-crossing behavior of $\mathrm{Z}^{J}_{\mathrm{Coul}}(\mathcal{R}, n)$
at the strong coupling cusps allows
one to ***derive*** Z^{J}_{SW} $\Rightarrow$ partition function for $b_2^+ > 1$

$$G(\mathcal{R}, n) = \frac{2^{2\chi+3\,\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

$Z(\mathcal{R}, n)$ = Terms in the power series with $\mathcal{R}^d$ with $d = \frac{\chi+\sigma}{4} \ mod\, 4$

Agrees with, and generalizes, GKW Conjecture 1.1

E_1 Theory

$$\lim_{\mathcal{R}\to 1} Z^J_{Coul} \neq \int_{\mathcal{F}} d\tau d\bar{\tau} \lim_{\mathcal{R}\to 1} \left(\nu_R C^{n^2} \Psi^J\right)$$

Strong coupling walls are SHIFTED from the walls defined by spin-c structures!

But this is nicely explained by the coupling to the background $V^{(I)}$

$\Rightarrow$ perturbed SW equations !

$$F^+(A) + Im(\zeta) F^+\left(A^{(I)}\right) = \bar{M} M$$

So far, we did not use any K-theory in describing the ``K-theoretic Donaldson invariants''

It would be very desirable to do so, because the 6d version, analogously formulated, could be quite interesting:

Conjecture:

Integrals in elliptic cohomology of distinguished classes defined by the susy sigma model with target space $\mathcal{M}_k$ define smooth invariants of four-manifolds

1. Motivation

2. The Paradigm: SU(2) N=2 SYM & Donaldson invariants

3. Other d=4 N=2 Theories

5

4. d=5: ``K-theoretic Donaldson invariants''

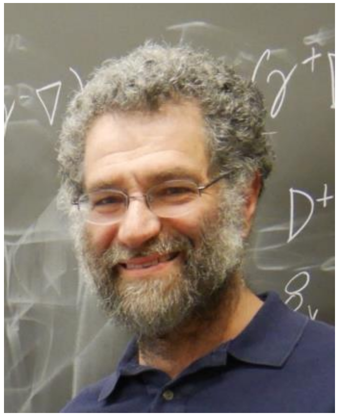
5. **Family Donaldson invariants**

Family Donaldson Invariants

There is an interesting generalization to invariants for families of four-manifolds.

Mentioned by Donaldson long ago.

A modest amount of work has been done in the math literature .



Families Of Metrics

Couple twisted theory to a family of metrics: $g_{\mu\nu}(x;s)$

$s \in \mathcal{P}$: Parameters of the family.

$Z_{DW}(g_{\mu\nu}(x;s))$ is independent of s.

A suitable coupling to background supergravity gives a partition function which is a closed differential form $\mathrm{Z}_{\mathrm{i}_1 \dots i_p} \mathrm{ds}^{i_1} \wedge \cdots ds^{i_p}$.

Periods of these forms are the family Donaldson invariants.

Universal Family

$$\mathcal{P} = Met(X)/Diff^{+}(X)$$

$$\pi_j\left(\frac{Met(X)}{Diff^{+}(X)}\right) \cong \pi_{j-1}\left(Diff^{+}(X)\right)$$

$$\pi_1\left(\frac{Met(X)}{Diff^{+}(X)}\right) \cong \pi_0\left(Diff^{+}(X)\right)$$

$\pi_0\left(Diff^{+}(X)\right)$: 4d mapping class group

Donaldson-Witten a la Baulieu-Singer

$$P \to \mathbb{X} \qquad\qquad \mathcal{G} := Aut(P)$$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P)$

$$\left(\Omega^*(\mathcal{A}(P)) \otimes S^*(Lie\mathcal{G}) \right)^{\mathcal{G}}$$

$$Q\, A_\mu = \psi_\mu \qquad Q\, \psi_\mu = -D_\mu \phi \qquad Q\, \phi = 0$$

Atiyah & Jeffrey + Baulieu & Singer

Z_{DW} : Pushforward in $\mathcal{G}$ −equivariant cohomology.

$$\mathcal{G}_d := Diff^+(\mathbb{X})$$

$\mathcal{G}_d$ −equivariant cohomology of $MET(\mathbb{X})$

$$Q\, g_{\mu\nu} = \Psi_{\mu\nu} \qquad Q\, \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \qquad Q \Phi^\mu = 0$$

Action e^{-S} is a closed equivariant class in the $\mathcal{G} \rtimes \mathcal{G}_d$ − equivariant cohomology of $MET(\mathbb{X}) \times \mathcal{A}(P)$

Push-forward in $\mathcal{G}$ −equivariant cohomology is a $\mathcal{G}_d$ −equivariant class on $MET(\mathbb{X})$

Thanks to heroic computations by JC and VS we have explicit actions e^{-S} obtained by coupling to truncated & twisted $N = 2$ conformal supergravity

Coupling To Twisted Truncated Background Supergravity

$$S[g, \Psi, \Phi] = S_{DW} + \int \sqrt{g} \left(\Psi^{\mu\nu} \Lambda_{\mu\nu} + \Phi^{\mu} Z_{\mu} + \Psi^{\mu\sigma} \Psi^{\nu}{}_{\sigma} \; \Upsilon_{\mu\nu} \right)$$

$$\Lambda_{\mu\nu} = Im\,\tau_{IJ} \left(F^{-,I}_{\rho\mu} \, \chi^{\rho,J}_{\nu} \right) + \cdots$$

$$Z_{\mu} = \mathcal{F}_{IJK} \, \psi^{I}_{\mu} \; F^{+,J}_{\rho\sigma} \; \chi^{\rho\sigma\,K} + \cdots$$

$$\Upsilon_{\mu\nu} = Im\,\tau_{IJ} \; \chi^{I}_{\mu\rho} \, \chi^{\rho,J}_{\nu} + \cdots$$

$\gamma \subset \frac{Met(X)}{Diff^{+}(X)}$ nontrivial cycle from some nontrivial element of $\pi_0\left(Diff^{+}(X)\right)$

$$\oint_{\gamma} ds \int_X \mathrm{vol}(g) \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}s} \langle \Lambda^{\mu\nu} \rangle$$

$$Q\left(\Lambda_{\mu\nu}\right) = T_{\mu\nu}^{SYM} + \cdots$$

This raises several questions:

$\Lambda^{\mu\nu}$ is NOT Q-closed!!!

Does our period integral localize to integrals on bundles of moduli spaces of instantons?

Does tree-level exactness (of LEET) persist?

No restriction on b_2^+. No assumption of ACS.

Does it see the other half of the world of four-manifolds?

Questions & Future Directions

Topological data for twisting the general d=4 N=2 theory?

Invertible theory governing orientation of nonabelian SW moduli

Global anomalies of 5D SYM in topological twisting background

Puzzles regarding physical derivation of K-theoretic Donaldson invariants

Generalization to elliptic invariants from 6d theories on $X \times E$

Puzzles concerning the family generalization of Donaldson invariants

Other puzzles and directions I did not have time to mention ....


NOT
That's all Folks!