Review (On Right BB before)

On X:

$$Z^{\xi}_{DW}(p,s) = \left\langle e^{p\mathcal{O} + s^{\alpha}\mathcal{O}(S_{\alpha})} \right\rangle_{\mathcal{T}(\Lambda)} \qquad \begin{matrix} SU(2)\ VM \\ \omega_R = \omega^+ \end{matrix}$$

$$= \frac{1}{2}\Lambda^{-\frac{3}{4}(\chi+\sigma)} \sum_{l,r\geq 0} \frac{p^l s^r}{l!\,r!} \mathbb{P}_D(\wp^l S^r)\,\Lambda^{2l+r}$$

$$= Z^{\xi}_{IR}(p,s) = \left\langle e^{p\mathcal{O}_{IR} + s^{\alpha}\mathcal{O}_{IR}(S_{\alpha}) + s^2 T(u)} \right\rangle_{LEET}$$

Quantum Theory on $\mathbb{R}^4$:

$$\mathcal{M}^{Quant}_{Coul} = \mathbb{C} \ni u \qquad \left(\mathfrak{t}\otimes\mathbb{C}/W \text{ for } G \text{ simple} \right)$$

$|\Omega(u)\rangle$ Poin invt vacuum.

LEET: $|u| \gg \Lambda^2$ Massless fields $U(1)$ $N=2$ VM

$a, A, D, \eta, \chi, \psi$

$a: \mathbb{R}^4 \to \mathbb{C}$

Classical Limit

$$\langle\phi\rangle \sim \begin{pmatrix} \langle a\rangle & \\ & -\langle a\rangle \end{pmatrix} \qquad u \sim \langle a\rangle^2$$

General Theorem of $N=2$ SUSY:

For abelian VM's most general action is determined by holomorphic family of abelian varieties + central charge function.

On Right BB before

$G = SU(2) \longrightarrow U(1)$:

1. Family of E_u $u \in \mathbb{C}$

2. $\Gamma_u := H_1(E_u, \mathbb{Z})$: Lattice of (el, mg) charges

3. $Z : \Gamma \longrightarrow \mathbb{C}$ holo, lin on fibers $\langle dZ, dZ \rangle = 0$

Def^n: "Duality Frame": Max. Lag. splitting

$$\Gamma_u \cong \Gamma_u^{el} \oplus \Gamma_u^{mg}$$

$$= A \cdot \mathbb{Z} \oplus B \cdot \mathbb{Z}$$

Then: Action for that duality frame

LEAVE

$$S_{LEET} = \int i \left[\overline{\tau(a)} (F^+)^2 + \tau(a) (F^-)^2 \right]$$

$$+ \mathrm{Im}\tau \, da * d\bar{a} + \mathrm{Im}\tau D * D + \tau \psi * d\eta + \bar{\tau} \eta d * \psi$$

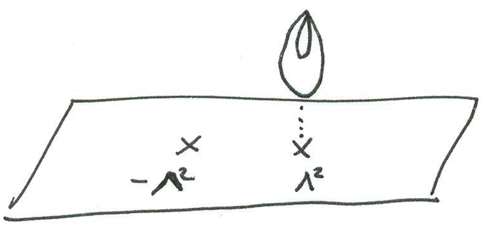
$$+ \tau \psi d \chi - \bar{\tau} \chi d\psi + i \frac{d\bar{\tau}}{d\bar{a}} \eta \chi (F^+ + D) + \ldots$$

Equivalence between duality frames: Abelian S-duality

Seiberg-Witten Proposal

$$E_u : \quad y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)$$

$\mathcal{M}^{\text{Quant}}_{\text{Coul}}$



$$Z_\gamma = \oint_\gamma \lambda_{SW}$$

$$\lambda_{SW} = \frac{dx}{y}(x-u)$$

Better: $\quad y^2 = x(x-u) + \frac{\Lambda^4}{4}x$

Monodromy of Γ: $\quad M_\infty = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \Rightarrow \Gamma_u^{el} = A \cdot \mathbb{Z}$

$$\Rightarrow a = \oint_A \lambda_{SW}$$

Choose B $\quad a_D = \oint_B \lambda_{SW} \longrightarrow 0 \quad @ \; u \longrightarrow \Lambda^2$

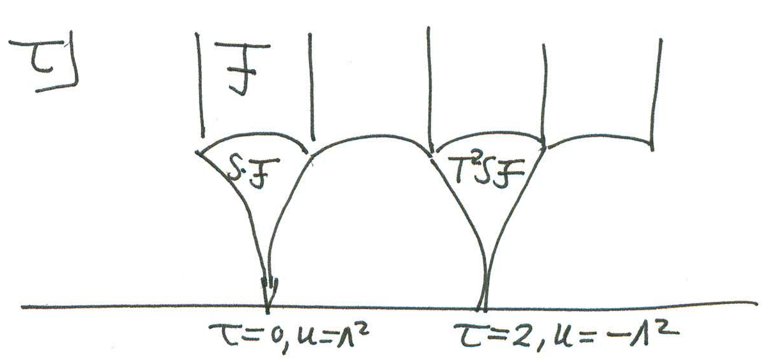
$$\Rightarrow \quad u = \frac{\Lambda^2}{2}\frac{\vartheta_2^4+\vartheta_3^4}{(\vartheta_2\vartheta_3)^2} = \frac{\Lambda^2}{8q^{1/4}}\left(1+20q^{1/2}+\cdots\right)$$

$$a = \frac{\Lambda}{6}\frac{2E_2(\tau)+\vartheta_2^4+\vartheta_3^4}{\vartheta_2\vartheta_3}$$

$\Rightarrow \tau(a)$

Monodromy group of Γ : $\quad \Gamma^0(4)$

u plane $\approx$ modular curve



Start Lecture Proper

At $u = \pm \Lambda^2$ $\quad \mathrm{Im}\,\tau \to 0 \quad$ S_{LEET} singular

Why? S_{LEET} only valid if we keep all light fields below cutoff scale.

At $u = \pm\Lambda^2$ new massless fields emerge!

BPS states $\quad \mathcal{H} \supset \mathcal{H}_u^{BPS} = \bigoplus_{\gamma\in\Gamma_u} \mathcal{H}_{\gamma,u}^{BPS}$

$$\{\psi \mid H\psi = |Z_\gamma(u)|\psi\}$$

$|u| \gg \Lambda^2$ semiclassical.

Magnetic monopole solutions: L^2-kernel of $\not{D}$ on $\mathcal{M}_{monopoles/\mathbb{R}^3}$

- Monopoles are HM $\quad \gamma = B \quad M = \left|\oint_B \lambda_{SW}\right|$

  Dyons HM $\quad \gamma = B + A \quad M = |a + a_D|$

in $\mathcal{U}_{\Lambda^2}$ LEET must be corrected

$$S_{LEET,\Lambda^2} = S_{U(1)VM}(a_D, A_D, D_D, \eta_D, \chi_D, \psi_D) + S_{HM}(M = q \oplus \tilde{q}^*)$$

$\tau_D = -1/\tau$

$$\boxed{F(A_D) = *F(A)}$$

Similar @ $u = -\Lambda^2$

Symmetry $u \longrightarrow -u$

G VM $\quad U(1)_R \xrightarrow[\text{inst.}]{} \mathbb{Z}_{2h^\vee} \longrightarrow \mathbb{Z}_2$

$\mathbb{Z}_{h^\vee}$: acts on $\mathcal{M}^{\text{Quant}}_{\text{Coul.}}$

Put $\overset{SU(2)}{\text{VM}}$ theory on X

~~Z^{ξ}_{IR} = Sum over vacua $\Rightarrow$ integrate / u-plane~~

~~$= Z_{\text{Coul.}} + Z_{\Lambda^2} + Z_{-\Lambda^2}$~~

path integral includes sum over vacua

$$Z^{\xi}_{IR} = \int e^{S_{LEET}}$$

NLσM: $u : \mathbb{R}^4 \longrightarrow \mathcal{M}^{\text{Quant}}_{\text{Coul.}}$

$$S_{LEET,\Lambda^2} = Q(\Psi_s) + \underline{\text{top. terms}}$$

s : SW eqs.

On X new solutions "new vacua"

$$= Z_{\text{Coul.}} + Z_{\text{Higgs}}$$

$\mathcal{M}^{\text{Quantum}}_{\text{Higgs}}$ = Moduli of solutions of SW eqs.

$\boxed{\text{Donaldson} \times \text{Witten} = \text{Coulomb} + \text{Higgs}}$

$$Z_{Higgs} = Z_{H,\Lambda^2} + Z_{H,-\Lambda^2}$$

S_{LEET} on Coulomb branch.

Extra terms on X

$$\Delta S_{LEET} = \int e(u)\, \mathrm{Tr}\, RR^* + p(u)\, \mathrm{Tr}\, R^2 + \frac{i}{4} F \wedge \overline{w_2(X)}$$

$$\int du \ldots E(u)^{\chi} P(u)^{\sigma}$$

$$E(u) = \alpha \left(\frac{du}{da}\right)^{1/2} \qquad P(u) = \beta \cdot \Delta^{1/8} \qquad \Delta = u^2 - \Lambda^4$$

α, β
- explicit comparison $Z_{\cancel{R}M}$
- WC
- blowup

$$Z_{Coul} = \sum_{\substack{\text{Flux sectors} \\ [F] = 4\pi\lambda}} \langle \quad \rangle_{Coul, \lambda}$$

$$\lambda \in \Gamma_\xi = \lambda_0 + \overline{H}^2(X)$$

$$2\lambda_0 = \overline{w_2(P)} = \overline{\xi}$$

Phase in comparing sectors.

Massive fermions have canon. orient. given a.c. str. X

Relative orientation: $(-1)^{w_2(X)\cdot(\lambda_1 - \lambda_2)}$

Z_{DW} real for

$$e^{\pm 2\pi i \lambda_0^2} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)}$$

Extra terms for $S_{LEET, \pm\Lambda^2}$

$$S_{VM}^{(a_D, A_D \cdots)} + S_{HM} = Q(\Psi_s) +$$

$$\int c(u) F_D \wedge F_D + e_h(u) \operatorname{Tr} RR^* + p_h(u) \operatorname{Tr} R^2 + \underline{\tfrac{i}{4} F_D\, \overline{w_2(P)}}$$

$$Z_{H,\Lambda^2} = \sum_{\substack{\text{Flux sectors} \\ [F_D] = 4\pi\lambda \\ \lambda \in \Gamma_W = \frac{1}{2}\overline{w_2(X)} + \bar{H}^2(X)}}$$

$\tau_D = -1/\tau$:

Recall fact about $\vartheta\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\theta)^2 + 2\pi i(n+\theta)\phi}$

$\uparrow$ shift $\qquad \uparrow$ phase

$$\vartheta\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](-1/\tau) = (-i\tau)^{1/2}\, \vartheta\left[\begin{smallmatrix}-\phi\\ \theta\end{smallmatrix}\right](\tau)$$

$$\underbrace{C(u)^{\lambda^2} E_h(u)^{\chi} P_h(u)^{\sigma}}_{\text{in principle can be derived}}$$

Map Operators : @ UV $\longrightarrow$ IR

Coul. $\quad \mathcal{O} = 2u \longrightarrow \mathcal{O}^{(0)}_{IR} = 2u$

$$\mathcal{O}_{IR,C}(S) = \int_S K^2 \mathcal{O}^{(0)}_{IR}$$

$K\, a \sim \psi$

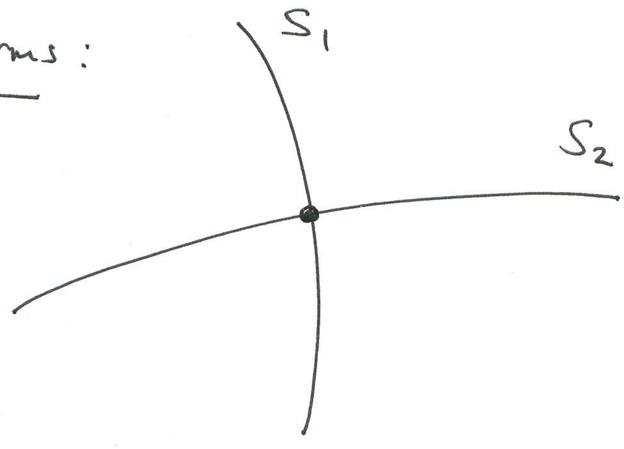
$K\, \psi \sim (F^- + D)$

$$= \int \frac{du}{da}(F^- + D) + \frac{d^2u}{da^2}\psi^2$$

Higgs $\quad \mathcal{O}_{IR,H} = 2u$

$$\mathcal{O}_{IR,H}(S) = \int \frac{du}{da_D}(F_D^- + D) + \frac{d^2u}{da_D^2}\psi_D^2$$

Contact Terms :



$$\left\langle \cdots \int_{S_1} \mathrm{Tr}\left(\phi(x_1) F(x_1) + \cdots\right) \int_{S_2} \mathrm{Tr}\left(\phi(x_2) F(x_2) + \cdots\right) \cdots \right\rangle_{J(\Lambda)}$$

Contractions singular @ $x_1 = x_2$

$t g_{\mu\nu} \quad t \to \infty$

$$\mathcal{O}_{UV}(\Sigma_1)\,\mathcal{O}_{UV}(\Sigma_2) \longrightarrow \mathcal{O}_{IR}(\Sigma_1)\,\mathcal{O}_{IR}(\Sigma_2) + \sum_{P \in \Sigma_1 \cap \Sigma_2} \epsilon_P\, T(u)$$

$$= \mathcal{O}_{IR}(\Sigma_1)\mathcal{O}_{IR}(\Sigma_2) + \Sigma_1 \cdot \Sigma_2 \, T(u)$$

Unknown C, E, P, T

$$Z_{H,\lambda^2} = \sum_{\lambda \in \Gamma_W} \underbrace{\left\langle e^{2pu + \frac{i}{4\pi}\int_S \frac{du}{da_D} F(A_D) + S^2 T(u)} \right\rangle_{\lambda,\lambda^2}}$$

$$e^{2\pi i(\lambda_0^2 + \lambda\cdot\lambda_0)} \int_{\mathcal{M}(\lambda)} e^{2pu + i\frac{du}{da_D} S\cdot\lambda + S^2 T(u)}$$

$$E_h(u)^{\chi} P_h(u)^{\sigma} C(u)^{\lambda^2}$$

$\mathcal{M}(\lambda)$ moduli space of sol'ns to SW w/ spin-c str. λ

$$v\dim = \frac{(2\lambda)^2 - (2\chi + 3\sigma)}{4} = 2\, n(\lambda)$$

$$\mathcal{M}(\lambda) \subset \left(\mathcal{A} \times \Gamma(S^+ \otimes L_\lambda)\right) / \mathcal{G}_{abel.}$$

$$= T^{b_1} \times V \times \underbrace{\Gamma(S^+ \otimes L_\lambda) / U(1)}_{\text{Cone}(\mathbb{CP}^\infty)}$$

global gauge trans

If no $F^+ = 0$ $\quad x \in H^*(\mathbb{CP}^\infty) \Rightarrow$ deg = 2 class on $\mathcal{M}(\lambda)$

<u>TFT</u> generator a_D of H^*_Q gh # 2.

$$a_D \approx x$$
$$\theta \longleftrightarrow \omega_\theta$$

$$\int_{\mathcal{M}(\lambda)} [\text{expansion in } a_D]$$

$$u = \Lambda^2 - 2i\Lambda a_D + \mathcal{O}(a_D^2)$$

$$SW(\lambda) := \int_{\mathcal{M}(\lambda)} a_D^{n(\lambda)}$$

$$Z_{H,\Lambda^2,\lambda} = SW(\lambda) \operatorname*{Res}_{a_D=0} \frac{da_D}{a_D^{1+n(\lambda)}} \left(e^{2pu+\cdots} C^{\lambda^2} P_h^{\sigma} E_h^{\chi} \right)$$

T, C, P, E indept of X.

$Z_{Coul.}$ Path integral of U(1) VM

$$\int da\, d\bar{a}\, d\eta\, d\chi\, d\psi\, dA\, dD\; e^{S_{LEET} + 2pu + \mathcal{O}_{IR,C}(S) + S^2 T(u)}$$

Thm (M–W) If $b_2^+ > 0$ only tree level contributes

($b_2^+ = 0$ only one-loop contributes)

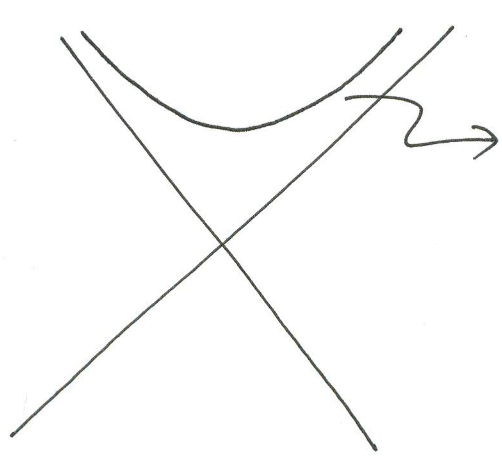
- Integrate out Fermi zm's.

η	$H^0(X,\mathbb{R})$	1
~~ψ~~	~~$H^1(X,\mathbb{R})$~~	~~b_1~~ $\rightarrow \pi_1(X) = 0$
χ	$H^{2,+}(X,\mathbb{R})$	b_2^+

Must orient Fermi path integral.

- η, χ do not appear in obs.
- $Z_{Coul} = 0$ if $b_2^+ \neq 1$!

$H^2(X,\mathbb{R})$



$\omega \cdot \omega = 1$

$\omega \in$ FLC.

$*\omega = \omega$

$$F = \omega F_+ + F^-$$

F_+: Scalar F^-: vector, <0

$\int dA$:

$$\# = \frac{e^{\frac{S_+^2}{8\pi y}\left(\frac{du}{da}\right)^2} e^{2\pi i \lambda_0^2}}{\sqrt{\operatorname{Im}\tau}} \sum_{\lambda \in \Gamma_\xi} e^{-i\pi\bar\tau\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(S,\lambda_-)} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \left(4\pi\lambda_+ + \frac{i}{4\pi y}\frac{du}{da}S_+ + \cancel{D_+}\right)$$

$$Z^{\xi}_{u}(p,s) = \alpha^\chi \beta^\sigma \int da\, d\bar a\, \frac{d\bar\tau}{d\bar a}\left(\frac{du}{da}\right)^{\chi/2} \Delta^{\sigma/8}\, e^{2pu + S^2 T_c(u)}\ \#$$

Rmks

$$* \quad \int da\, d\bar a\, (\cdots) = \int_{\mathbb{C}} du\, d\bar u \left|\frac{da}{du}\right|^2 (\cdots)$$

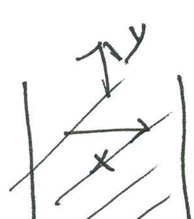
$$= \int_{\mathcal{F}(\Gamma^0(4))} d\tau\, d\bar\tau \left|\frac{da}{d\tau}\right|^2 (\cdots)$$

$$\boxed{\text{Measure single-valued pins down } T_c(u)}$$

* Sing's at 3 cusps $u \to \infty$ ~~$\operatorname{Im}\tau \to \infty$~~ $u \to \pm\Lambda^2$

τ]

Expand



$$\lim_{y_{max}\to\infty} \int^{y_{max}} \frac{dy}{y^{1/2}} \int dx \sum_{\mu,\nu} c_{\mu,\nu}\, q^\mu \bar q^\nu \left(1 + O\left(\frac{1}{y}\right)\right)$$

* only depends on homotopy type of X

<u>Metric Dependence</u> : <u>Integrand</u> depends on metric through period poin ω.

<u>Integral?</u> $\omega(t)$

$$\frac{d}{dt} Z_u^{\xi}(p,s) = \int du\, d\bar{u} \; \underbrace{d\left[\cdots\right]}_{\text{explicit total derivative}}$$

$$= \oint_{u=\infty} + \oint_{u=\Lambda^2} + \oint_{u=-\Lambda^2}$$

Boundary terms do not contribute <u>except</u> when there are abelian instantons:

$$\omega \cdot \lambda = \lambda_+ = 0$$

Dangerous term

$$\underset{\substack{\uparrow \\ \text{Coeff. of} \\ \text{modular} \\ \text{form}}}{c(n)} \int \frac{dy}{y^{1/2}}\, e^{-2\pi \lambda_+^2 y}\, \lambda_+ \simeq c(n)\, \mathrm{sgn}(\lambda_+)$$

$\Rightarrow$ discontinuities across Walls

$$W(\lambda) = \{\omega \mid \omega \cdot \lambda = 0\}$$

$\Delta_\infty Z_u^\xi \neq 0 \qquad \lambda \in \Gamma_\xi = \frac{1}{2}\overline{w_2(P)} + \overline{H}^2(X)$

$\Delta_{\pm\Lambda^2} Z_u^\xi \neq 0 \qquad \lambda \in \Gamma_W = \frac{1}{2}\overline{w_2(X)} + \overline{H}^2(X)$

ΔZ_u^ξ = Coeff'ts of modular forms.

$\Delta_\infty Z_u^\xi$ = Formula of L. Göttsche $\Leftrightarrow$

$$\alpha^\chi \beta^\sigma = \frac{2^{(2+3\sigma)/4}}{\pi}, \quad \chi+\sigma=4$$

$\boxed{\Rightarrow Z_u^\xi \text{ completely explicit}}$ $\qquad \Rightarrow \alpha, \beta$

$$Z_{DW}^\xi(p,s) = Z_u^\xi(p,s) + \underbrace{Z^\xi_{H,\Lambda^2} + Z^\xi_{H,-\Lambda^2}}_{Z_{Higgs}}$$

$b_2^+=1$ $\qquad$ disc. at both sets of walls

Only disc at $W(\lambda)$, $\lambda \in \Gamma_\xi$

$$\Rightarrow \quad 0 = \underbrace{\Delta_{u=\Lambda^2,\lambda} Z_u^\xi}_{\text{known}} + \underbrace{\Delta_\lambda Z^\xi_{H,\Lambda^2}}_{\text{known in terms of } G, P_{hr}, E_n, T_h}$$

Indeed $\quad SW(\lambda)\Big|_{\omega\cdot\lambda=0+} - SW(\lambda)\Big|_{\omega\cdot\lambda=0-} = (-1)^{1+n(\lambda)}$

$\Rightarrow$ Determine C P E T explicitly

$$C(u) = \left(\frac{a_D}{q_D}\right)^{1/2} = 4e^{i\pi/4} + \mathcal{O}(a_D)$$

$$\left\{\begin{array}{l} P_n(u) = e^{i\pi/32}\,2^{5/4} + \mathcal{O}(a_D) \\ E_n(u) = e^{i\pi/8}\,2^{3/4} + \mathcal{O}(a_D) \end{array}\right.$$

determined by periods of degenerating elliptic curve.

$\therefore$ Completely explicit expression for $Z_{DW}^{\xi}(p, S)$ in terms of intersection form and SW invts ; valid for $b_2^+ > 0$

Def: • X is SW sple type if $\mathcal{M}(\lambda) \neq 0$ only for λ s.t. $n(\lambda) = 0$.

• basic classes: λ s.t. $SW(\lambda) \neq 0$.

Suppose $b_2^+(X) > 1$ & SW sple type

Then:

$$Z^{\xi}_{DW}(p,S) \overset{\Lambda=1}{=} 2^{(2\chi+3\sigma)-\chi_h}$$

$$\Big(\underbrace{e^{\frac{1}{2}S^2+2p} \sum_{\lambda\in\Gamma_w} SW(\lambda)\, e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)}\, e^{2S\cdot\lambda}}_{Z_M}$$

$$+ \underbrace{i^{\chi_h} e^{-\frac{1}{2}S^2-2p} \sum_{\lambda\in\Gamma_w} SW(\lambda)\, e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)}\, e^{-i2S\cdot\lambda}}_{Z_D} \Big)$$

Def: X is gen. KM sple type if $\exists\, r\in\mathbb{Z}_+$

$$\left(\frac{\partial^2}{\partial p^2}-4\right)^r Z_{KM} = 0.$$

Thm 1: SW-ST $\Longrightarrow$ KM-ST

~~Def: X is gen. KM sple~~

Thm 2: All X $\pi_1(X)=0$ $b_2^+(X)>1$ are generalized KM ST with

$$r = 1 + \max_{\{\lambda\}} n(\lambda)$$