Three Transverse Intersections Between Physical Mathematics and Condensed Matter Theory

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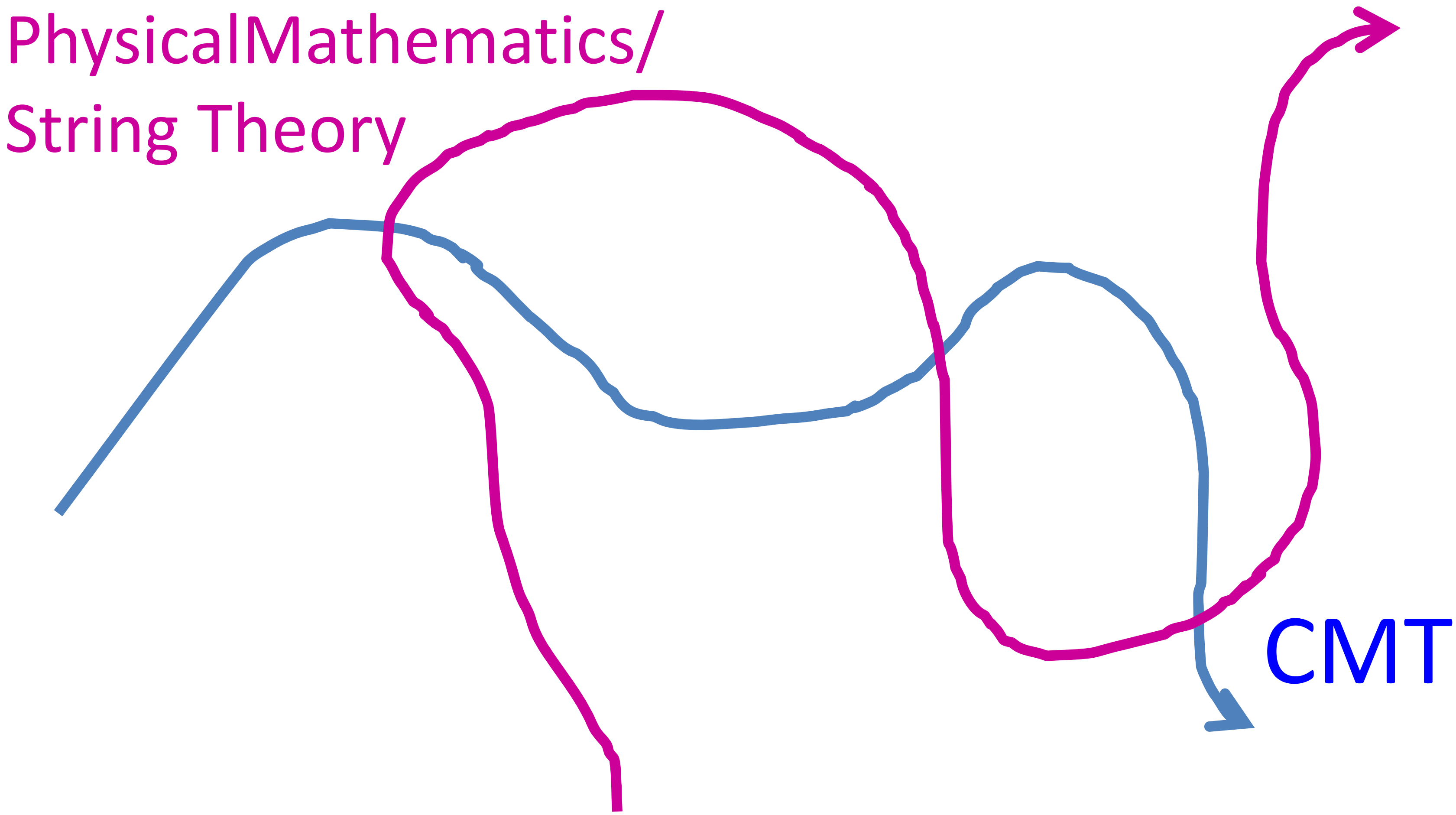
Introduction



Main interaction is AdS/CMT: c.f. S. Sachdev talk

Three Transverse Intersections

1. Twisted K-theory and topological phases of electronic matter.

2. Generalizations of the Chern-Simons edge state phenomenon…

& 3 Corollaries concerning 3D and 4D abelian gauge theories.

3. The relation of higher category theory to classification of defects and locality in topological field theory.

Part I: Topological Band Theory & Twisted K-Theory

(Inspired by discussions with D. Freed, A. Kitaev, and N. Read;
Possible paper by DF and GM.)

There has been recent progress in classifying topological phases of (free) fermions using ideas from K-theory such as Bott periodicity.

This development goes back to the TKNN invariant and Haldane's work on the quantum spin Hall effect in graphene.

The recent developments began with the $\mathbb{Z}_2$ invariant associated to the 2D TR invariant quantum spin Hall system.

The CMT community is way ahead of most string theorists, who refuse to have any interest in torsion invariants.

What really peaked my interest was the work of Kitaev and of Schnyder, Ryu, Furusaki, and Ludwig using K-theory to classify states of electronic matter.

The reason is that there is also a role for K-theory in string theory/M-theory.

I will now sketch that role, because it leads to a generalization of K-theory which might be of some interest in CMT.

(Prescient work of P. Horava in 2000 used the D-brane/K-theory connection to study ``classification and stability of Fermi surfaces.'')

RR Fields

Type II supergravity in 10 dimensions has a collection of differential form fields:

IIA: F_0, F_2, F_4, F_6, F_8, F_{10}

IIB: F_1, F_3, F_5, F_7, F_9

These are generalizations of Maxwell's F_2 in four dimensions: $dF_j=0$.

$$F_j = dC_{j-1} \qquad C_{j-1} \sim C_{j-1} + d\Lambda_{j-2}$$

$$d * F_j = 0$$

Dirac Charge Quantization

Theory of the F_j 's is an abelian gauge theory, and, just like Dirac quantization in Maxwell theory, there should be a quantization condition on the electric/magnetic charges for these fields.

Perhaps surprisingly, the charge quantization condition turns out to involve the K-theory of the 10-dimensional spacetime: $K^1(X)$ (for IIA) and $K^0(X)$ (for IIB). [Minasian & Moore, 1997]

Orientifolds

Witten (1998) pointed out several important generalizations. Among them, in the theory of ``orientifolds'' one should use a version of K-theory invented by M. Atiyah, known as *KR theory*.

KR-Theory

X: A space (e.g. Brillouin torus)

K(X) is an abelian group made from equivalence classes of complex vector bundles over X

Now suppose X is a space with involution.

For example, the Brillouin torus, with $k \to -k$

KR(X) is made from equivalence classes of a pair (T,V) where T is a $\mathbb{C}$-antilinear map: $T: V_k \to V_{-k}$

But as people studied different kinds of spacetimes and orientifolds there was an unfortunate proliferation of variations of K-theories….

Older Classification

Op^-	K-group
$O0^-$	$KR_\pm(S^{9,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O1^-$	$KR^{-1}(S^{8,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O2^-$	$KR(S^{7,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O3^-$	$KH_\pm^{-1}(S^{6,0}) = \mathbb{Z}$
$O4^-$	$KH_\pm(S^{5,0}) = \mathbb{Z}$
$O5^-$	$KH^{-1}(S^{4,0}) = \mathbb{Z}$
$O6^-$	$KH(S^{3,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O7^-$	$KR_\pm^{-1}(S^{2,0}) = \mathbb{Z}$
$O8^-$	$KR_\pm(S^{1,0}) = \mathbb{Z}$

Op^+	K-group
$O0^+$	$KH_\pm(S^{9,0}) = \mathbb{Z}$
$O1^+$	$KH^{-1}(S^{8,0}) = \mathbb{Z}$
$O2^+$	$KH(S^{7,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O3^+$	$KR_\pm^{-1}(S^{6,0}) = \mathbb{Z}$
$O4^+$	$KR_\pm(S^{5,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O5^+$	$KR^{-1}(S^{4,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O6^+$	$KR(S^{3,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O7^+$	$KH_\pm^{-1}(S^{2,0}) = \mathbb{Z}$
$O8^+$	$KH_\pm(S^{1,0}) = \mathbb{Z}$

Table 2: Orientifold K-theory groups for RR fields.

(Bergman, Gimon, Sugimoto, 2001)

An Organizing Principle

Now, in ongoing work with Jacques Distler and Dan Freed, we have realized that a very nice organizing principle in the theory of orientifolds is that of ``***twisted K-theory***.''

I am going to suggest here that it is also a useful concept in organizing phases of electronic matter.

So,
what is ``twisted K-theory'' ?

First, let's recall why K-theory is relevant at all…

K-theory as homotopy groups

Thanks to the work of Ludwig et. al. and of Kitaev CMT people know that

$$\tilde{K}^j(X) = [X, \mathcal{F}_j]$$

$$\mathcal{F}_0 = \lim \frac{U(n+m)}{U(n)\times U(m)} = BU$$

$$\mathcal{F}_1 = \lim U(n) = U$$

These are 2 of the 10 Cartan symmetric spaces which appear in the Dyson-Altland-Zirnbauer classification of free fermion Hamiltonians in d=0 dimensions.

K-theory and band structure

The Grassmannian can be identified with a space of projection operators, so if X = Brillouin torus, the groundstate of filled bands defines a map

$$k \to P(k)$$

$P(k)$ is the projector onto the filled electronic levels.

People claim that the homotopy class of the map P can distinguish between different ``topological phases'' of electronic systems.

Generalization to KR

$\mathcal{F}_j$ has an action of complex conjugation.

So, if X has an action of $\mathbb{Z}_2$, we can define

$$KR(X) = [X, \mathbb{Z} \times \mathcal{F}_0]^{\mathbb{Z}_2}$$

Example: $$TP(k)T^{-1} = P(-k)$$

T = antilinear and unitary, e.g. from time reversal symmetry

AZ class	space of projectors in momentum space	BL class	$N_f^{\min}$	fermionic replica NLσM target space	topological or WZW term
A	$\{ Q(k) \in G_{m,m+n}(\mathbb{C}) \}$	0	1	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	Pruisken
AI	$\{ Q(k) \in G_{m,m+n}(\mathbb{C}) \,\vert\, Q(k)^* = Q(-k) \}$	4_+	2	$\mathrm{Sp}(2N)/\mathrm{Sp}(N)\times\mathrm{Sp}(N)$	N/A
AII	$\{ Q(k) \in G_{2m,2(m+n)}(\mathbb{C}) \,\vert\, (i\sigma_y)Q(k)^*(-i\sigma_y) = Q(-k) \}$	3_+	1	$\mathrm{O}(2N)/\mathrm{O}(N)\times\mathrm{O}(N)$	$\mathbb{Z}_2$
AIII	$\{ q(k) \in \mathrm{U}(m) \}$	1 or 2	1 or 2	$\mathrm{U}(N)\times\mathrm{U}(N)/\mathrm{U}(N)$	WZW
BDI	$\{ q(k) \in \mathrm{U}(m) \vert q(k)^* = q(-k) \}$	9_+	2	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	N/A
CII	$\{ q(k) \in \mathrm{U}(2m) \,\vert\, (i\sigma_y)q(k)^*(-i\sigma_y) = q(-k) \}$	9_-	2	$\mathrm{U}(2N)/\mathrm{O}(2N)$	$\mathbb{Z}_2$
D	$\{ Q(k) \in G_{m,2m}(\mathbb{C}) \,\vert\, \tau_x Q(k)^* \tau_x = -Q(-k) \}$	3_-	1	$\mathrm{O}(2N)/\mathrm{U}(N)$	Pruisken
C	$\{ Q(k) \in G_{m,2m}(\mathbb{C}) \,\vert\, \tau_y Q(k)^* \tau_y = -Q(-k) \}$	4_-	2	$\mathrm{Sp}(N)/\mathrm{U}(N)$	Pruisken
DIII	$\{ q(k) \in \mathrm{U}(2m) \,\vert\, q(k)^T = -q(-k) \}$	5 or 7	1 or 2	$\mathrm{O}(N)\times\mathrm{O}(N)/\mathrm{O}(N)$	WZW
CI	$\{ q(k) \in \mathrm{U}(m) \,\vert\, q(k)^T = q(-k) \}$	6 or 8	2 or 4	$\mathrm{Sp}(N)\times\mathrm{Sp}(N)/\mathrm{Sp}(N)$	WZW

$$Q(k) = 2P(k)-1$$

Schnyder, Ryu, Furusaki, Ludwig 2008

Generalization to Twisted K-Theory

Now suppose we have a ``twisted bundle'' of classifying spaces:

$$\mathcal{F}_j \to \mathcal{B} \to X$$

Sections of $\pi : \mathcal{B} \to X$ generalize maps $X \to \mathcal{F}_j$

Homotopy classes of sections defines twisted K-theory groups of X:

$$K^{\text{twisted}}(X) := \Gamma(\mathcal{B})/\text{homotopy}$$

Roughly speaking: We have ``bundles of the Cartan symmetric spaces'' over the BZ and then the projector to the filled band would define a twisted K-theory element.

Twist Happens

It turns out that CM theorists indeed use the twisted form of KR theory for 3D $\mathbb{Z}_2$ topological insulators:

$$T^2 = -1$$

(In the untwisted original Atiyah KR theory we would have $T^2 = +1$.)

Balents & Joel. E. Moore ; R. Roy ; Kane, Fu, Mele (2006)

Twistings of K-Theory

The possible ``twisted bundles of classifying spaces over X'' is a set, denoted $\mathfrak{Twist}_K(X)$

For $\tau \in \mathfrak{Twist}_K(X)$ we denote $K^{\tau}(X)$

Similarly, if X has a $\mathbb{Z}_2$ action (like k $\rightarrow$ -k) there is a set of twistings of KR theory: $\mathfrak{Twist}_{KR}(X)$ and we denote the twisted KR groups as $KR^{\tau}(X)$.

Isomorphism Classes of Twistings

There is a notion of isomorphism of twistings.

As an abelian group, $KR^{\tau}(X)$ only depends on isomorphism class:

$$[\tau] \in [\mathfrak{Twist}_{KR}(X)]$$

Moreover, $[\mathfrak{Twist}_{KR}(X)]$ is itself an abelian group.

Relation to the Brauer Group

Already for 0-dimensional systems, i.e. K-theory of a point, there is a nontrivial set of twistings:

$$[\mathfrak{Twist}_K(pt)] \cong \mathbb{Z}_2 \quad [\mathfrak{Twist}_{KR}(pt)] \cong \mathbb{Z}_8$$

Model for twistings: Bundles of central simple superalgebras.

Isomorphism classes: $\mathbb{Z}_2$-graded Brauer groups.

Theorem[C.T.C. Wall]: They are cyclic, and generated by the one-dimensional Clifford algebras.

Brauer = Dyson-Altland-Zirnbauer

A recent paper of Fidkowski & Kitaev [1008.4138] explains the connection between the 10 DAZ symmetry classes of free fermion Hamiltonians and Wall's classification of central simple superalgebras.

Therefore, we can identify the DAZ symmetry classes of Hamiltonians with the twistings of K and KR theory associated to a point….

A Speculation

This suggests (to me) that there should be a larger set of ``symmetry classes'' of free fermion systems, when we take into account further discrete symmetries and/or go to higher dimensions.

Proposal:

A. The ``symmetries'' of (free) fermion systems should be identified with isomorphism classes of twistings of KR theory on some appropriate space X.

B. The phases of electronic matter in class $[\tau]$ are classified by $KR^{\tau}(x)$

What do we gain from this?

1. Generalization to Γ-equivariant K-theory is straightforward. In topological band theory it would be quite natural to let Γ be one of the two or three-dimensional magnetic space groups, and to take X to be a quotient of $\mathbb{R}^d$ by Γ

2. So the mathematical machinery suggests new phases

3. There is an Abelian group structure on symmetry classes.

Isomorphism Classes of Twistings

The ***set*** of isomorphism classes of twistings can be written in terms of cohomology:

$$[\mathfrak{Twist}_{KR}(X)] =$$

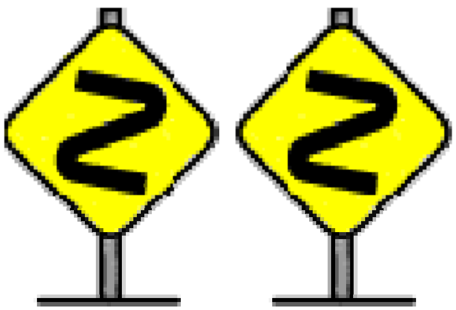
$$H^0(\bar{X};\mathbb{Z}_2) \times H^1(\bar{X};\mathbb{Z}_2) \times H^3(\bar{X};\tilde{\mathbb{Z}})$$

$$\bar{X} = X//\,\mathbb{Z}_2$$

!!Warnings!!

The above formula is deceptively simple:

The abelian group structure on the set is NOT the obvious direct product. Factors get mixed up.



$X//\mathbb{Z}_2$ is a mathematical quotient known as a groupoid ...

and X might also be a groupoid ...

So the cohomology groups are really generalizations of equivariant cohomology.

Example

Let Γ be a discrete group with a homomorphism to $\mathbb{Z}_2$:

$$0 \to \Gamma_0 \to \Gamma \stackrel{\omega}{\to} \mathbb{Z}_2 \to 0$$

ω will tell us if the symmetries are $\mathbb{C}$ –linear or $\mathbb{C}$ -antilinear

$$\Gamma = \Gamma_0 \amalg \Gamma_1$$

$\mathbb{C}$-linear

$\mathbb{C}$ - antilinear

For example Γ might be a magnetic point group.

Example-cont'd

Now one forms a ``double cover''

$$pt//\,\Gamma_0 \to pt//\,\Gamma$$

The cohomology factors have physical interpretations:

$H^0 = \mathbb{Z}_2$ Is there a commuting fermion number symmetry?

$H^1 = Hom(\Gamma, \mathbb{Z}_2)$ A grading on the symmetry group.

$H^3(pt//\,\Gamma;\tilde{\mathbb{Z}})$ Classifies twisted U(1) central extensions of Γ, which become ordinary central extensions of Γ_0 , as is quite natural in quantum mechanics.

Recovering the standard 10 classes

Finally, taking Γ_0 to be trivial so $\Gamma = \mathbb{Z}_2$

our isomorphism classes of twistings becomes:

$$H^0 \times H^1 \times H^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(d,a,h) + (d',a',h') = (d+d', a+a' + dd' , h+h' + a a' + d d' (a+a'))

$$[\mathfrak{Twist}_{KR}(pt)] \cong [\mathfrak{Twist}_{KO}(pt)] \cong \mathbb{Z}_8$$

Algebra	(ϵ, α, t)	(d, a, h)	KO_{-j}	Cartan Label of KO_{-j}	DAZ	(T, C, S)
$C\ell_0 \cong \mathbb{R}$	$(+, +, \mathbb{R})$	$(0, 0, 0)$	$BO \times \mathbb{Z}$	BDI	AI	$(1, 0, 0)$
$C\ell_{-1} \cong \mathbb{C}$	$(-, -, \mathbb{R})$	$(1, 1, 0)$	O	D	CI	$(1, -1, 1)$
$C\ell_{-2} \cong \mathbb{H}$	$(+, -, \mathbb{H})$	$(0, 1, 1)$	O/U	DIII	C	$(0, -1, 0)$
$C\ell_{-3} \cong \mathbb{H} \oplus \mathbb{H}$	$(-, +, \mathbb{H})$	$(1, 0, 0)$	U/Sp	AII	CII	$(-1, -1, 1)$
$C\ell_{-4} \cong \mathbb{H}(2)$	$(+, +, \mathbb{H})$	$(0, 0, 1)$	$BSp \times \mathbb{Z}$	CII	AII	$(-1, 0, 0)$
$C\ell_{-5} \cong \mathbb{C}(4)$	$(-, -, \mathbb{H})$	$(1, 1, 1)$	Sp	C	DIII	$(-1, 1, 1)$
$C\ell_{-6} \cong \mathbb{R}(8)$	$(+, -, \mathbb{R})$	$(0, 1, 0)$	Sp/U	CI	D	$(0, 1, 0)$
$C\ell_{-7} \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$	$(-, +, \mathbb{R})$	$(1, 0, 1)$	U/O	AI	BDI	$(1, 1, 1)$
$C\ell_{-8} \cong \mathbb{R}(16)$	$(+, +, \mathbb{R})$	$(0, 0, 0)$	$BO \times \mathbb{Z}$	BDI	AI	$(1, 0, 0)$

A Question/Challenge to CMT

Thus, in topological band theory, a natural generalization of the 10 DAZ symmetry classes would be

$$[\tau] \in \left[\mathfrak{Twist}_{KR}(\mathbb{R}^3 /\!/ \Gamma_0)\right]$$

And a natural generalization of the classification of topological phases for a given ``symmetry type'' $[\tau]$ would be

$$KR^{\tau}(\mathbb{R}^3 /\!/ \Gamma_0)$$

Can such ``symmetry types'' and topological phases actually be realized by physical fermionic systems?

Part II: Generalizations of Chern-Simons edge states

The ``edge state phenomenon'' is an old and important aspect of the quantum Hall effect, and its relation to Chern-Simons theory will be familiar to everyone here.

We will describe certain generalizations of this mathematical structure, for the case of abelian gauge theories involving differential forms of higher degrees, defined in higher dimensions, and indeed valued in (differential) generalized cohomology theories.

These kinds of theories arise naturally in supergravity and superstring theories.

The general theory of self-dual fields (edge states) leads to three corollaries, which are of potential interest in CMT

A Simple Example

U(1) 3D Chern-Simons Theory

$$\exp\left[2\pi i N \int_Y A dA\right] \qquad N \in \mathbb{Z}$$

Normalization!

$$F \in \Omega^2_{\mathbb{Z}}(Y)$$

$$A \to A + \omega \qquad \omega \in \Omega^1_{\mathbb{Z}}(Y)$$

``Holographic'' Dual

Chern-Simons Theory on Y $\cong$ 2D RCFT on $M = \partial Y$

Holographic dual = ``chiral half'' of the Gaussian model

$$\pi R^2 \int_M d\phi * d\phi \qquad \phi \sim \phi + 1$$

$$\text{Conformal blocks for } R^2 = p/q$$
$$= \text{CS wavefunctions for } N = pq$$

The Chern-Simons wave-functions $\Psi(A|_M)$ are the conformal blocks of the chiral scalar current coupled to A:

$$\Psi(A) = Z(A) = \langle \exp \int_M A d\phi \rangle$$

Holography & Edge States

$\mathrm{Quantization\ on}\ Y = D \times \mathbb{R}$ is equivalent to

quantization of the chiral scalar on $\partial Y = S^1 \times \mathbb{R}$

Gaussian model for $R^2 = p/q$ has level 2N = 2pq current algebra.

$$\mathrm{Quantization\ on}\ S^1 \times \mathbb{R}\ \mathrm{gives}$$
$$\mathcal{H}(S^1) = \mathrm{representations\ of}\ \widehat{LU(1)}_{2N}$$

What about the odd levels? In particular what about k=1 ?

We will return to this question.

Two Points We Want to Make

1. There are significant generalizations in string theory and Physical Mathematics.

2. Even for three-dimensional and four-dimensional abelian gauge theories there are some interesting subtleties and recent results.

Generalizations

The EOM for a chiral boson in 1+1 dimensions can be written as F=*F where F = d ϕ is a one-form ``fieldstrength.''

This is consistent with the wave equation d*F =0.

It is also consistent with having a real fieldstrength because *(*F)=F.

In general, for an oriented Riemannian manifold of dimension n, acting on j-forms $\Omega^j(M)$:

$$*^2 = \mathrm{sign}(\mathrm{det} g_{\mu\nu})(-1)^{j(n-j)}$$

Generalizations - II

So we can impose a self-duality constraint F =* F on a real fieldstrength F, with dF = 0, when **=1.

Example 1: A 3-form fieldstrength in six dimensions

$$F \in \Omega^3(M_6)$$

as occurs in the 5-brane and six-dimensional (2,0) theory:

Example 2: Total RR fieldstrength in 10-dimensional IIB sugra:

$$F \in \Omega^{\mathrm{odd}}(M_6)$$

Generalizations-III

We can also have several independent fields valued in a real vector space V:

$$F \in \Omega^*(M;V)$$

$$F^i = \mathcal{I}^{ij} * F^j \qquad \mathcal{I}^2 = \pm 1$$

For example the low energy Seiberg-Witten solution of N=2 , d=4 susy theories is best thought of as a self-dual theory.

Holographic Duals

These abelian gauge theories all have holographic duals involving some Chern-Simons theory in one higher dimension. They appear in various ways:

1. AdS/CFT: There is a term in the IIB Lagrangian:

$$\int_{M_5} B_2 dC_2$$

which is dual to free U(1) Maxwell theory on the boundary. There are several other examples of such ``singleton modes.''

2. The 7D theories are useful for studying the M5-brane and (2,0) theories. The 11D theory is useful for studying the RR fields.

General Self-Dual Abelian Gauge Theory

To formulate the general theory of self-dual fields, valid in arbitrary topology turns out to require some sophisticated mathematics,

``differential generalized cohomology theory.''

Just to get a sense of the subtleties involved let us return to the quantization of U(1) Chern-Simons theory at level N. Recall this leads to level 2N current algebra:

$$\widehat{LU(1)}_{2N}$$

What about the odd levels? In particular k=1?

Why not just put N= ½ ?

$$\exp\left[2\pi i \frac{1}{2}\int_Y AdA\right]$$

Not well-defined.

Spin-Chern-Simons

But if Y has a spin structure α, then we can give an unambiguous definition :

$$e^{2\pi i q_\alpha(A)} = \exp\left[i\pi \int_Y AdA\right] = \exp\left[2\pi i \int_Z \frac{1}{2} F^2\right]$$

Z = Spin bordism of Y.

Price to pay: The theory depends on spin structure:

$$q_{\alpha+\epsilon}(A) = q_\alpha(A) + \frac{1}{2}\int_Y \epsilon \wedge F \qquad \epsilon \in H^1(Y;\mathbb{Z}/2\mathbb{Z})$$

The Quadratic Property

The spin Chern-Simons action satisfies the property:

$$q_\alpha(A+a_1+a_2) - q_\alpha(A+a_1) - q_\alpha(A+a_2) + q_\alpha(A)$$

$$= \int_Y a_1 da_2 \mod \mathbb{Z}$$

(Which would follow trivially from the heuristic formula q_α = ½ $\int$ A d A, but is rigorously true.)

Quadratic Refinements

Let A, B be abelian groups, together with a bilinear map

$$b : A \times A \to B$$

A **quadratic refinement** is a map $q : A \to B$

$$q(x_1 + x_2) - q(x_1) - q(x_2) + q(0) = b(x_1, x_2)$$

$q(x) = \frac{1}{2} b(x, x)$ does not make sense when B has 2-torsion

As is the case for $B = \mathbb{R}/\mathbb{Z}$

So it is nontrivial to define $q_\alpha(A)$

General Principle

An essential feature in the formulation of self-dual theory always involves a choice of certain **quadratic refinements**.

The Free Fermion

Recall the Gaussian model for $R^2 = p/q$ is dual to the U(1) CST for N=pq, with current algebra of level 2N=2pq

Indeed, for $R^2 = 2$ there are four reps of the chiral algebra:

$$1, e^{\pm \frac{i}{2}\phi}, e^{i\phi}$$

It is possible to take a ``squareroot'' of this theory to produce the theory of a single self-dual scalar field. It is equivalent to the theory of a free fermion:

$$\psi = e^{i\phi}$$

The chiral free fermion is the holographic dual of level ½, and from this point of view the dependence on spin structure is obvious.

General 3D Abelian Spin Chern Simons

General theory with gauge group $U(1)^r$

Gauge fields: $A_i, i = 1, \ldots, r$

$$\exp[i\pi \int k_{ij} A_i dA_j]$$

k_{ij} define an integral lattice Λ

If Λ is even then the theory does not depend on spin structure.

If Λ is not even then the theory in general will depend on spin structure.

This is the effective theory used to describe the Haldane-Halperin hierarchy of abelian FQHE states. (Block & Wen; Read; Frohlich & Zee)

The classification of *classical* CSW theories is the classification of integral symmetric matrices.

But, there can be nontrivial quantum equivalences…

A Canonically Trivial Theory

Witten (2003): The U(1) x U(1) theory with action

$$\exp[2\pi i \int A_1 dA_2]$$

i.e. $$k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is canonically trivial.

Classification of quantum spin abelian Chern-Simons theories

Theorem: (Belov and Moore) For G= $U(1)^r$ let Λ be the integral lattice corresponding to the classical theory. Then the *quantum theory* only depends on

a.) $\mathcal{D} = \Lambda^*/\Lambda$, the ``discriminant group''

b.) The quadratic function q: $\mathcal{D} \to \mathbb{R}/\mathbb{Z}$

c.) $\sigma(\Lambda)$ mod 24

These data satisfy the Gauss-Milgram identity:

$$|\mathcal{D}|^{-1/2} \sum_{\gamma\in\mathcal{D}} e^{2\pi i q(\gamma)} = e^{2\pi i \sigma/8}$$

Moreover: quantum theories exist for all such (σ, $\mathcal{D}$,q) satisfying Gauss-Milgram.

Example:

Thus, there are other interesting quantum equivalences:

For example, if Λ is one of the 24 even unimodular lattices of rank 24 then the 3D CSW topological field theory is ***trivial***:

One dimensional space of conformal blocks on every Riemann surface.

Trivial representation of the modular group on this one-dimensional space.

Relation to Finite Group Gauge Theory

Recently, further quantum equivalences were discovered:

if $$\mathcal{D} = \Lambda^*/\Lambda = L \times L^*$$

where L is a maximal isotropic subgroup,

then 3D CSGT is equivalent to a 3D CSGT with ***finite gauge group***, *L*

Freed, Hopkins, Lurie, Teleman; Kapustin & Saulina; Banks & Seiberg

(Conjecture (Freed & Moore): This theorem generalizes nicely to all dimensions 3 mod 4)

Maxwell Theory in 3+1 Dimensions

Finally, another interesting corollary of the general theory of a self-dual field applies to ordinary Maxwell theory in 3+1 dimensions:

Theorem [Freed, Moore, Segal]: The groundstates of Maxwell theory on a 3-manifold Y form an irreducible representation of a Heisenberg group extension:

$$0 \to U(1) \to \mathrm{Heis} \to T \times T \to 0$$

$$T = \mathrm{Tors}(H^2(Y;\mathbb{Z}))$$

Example: Maxwell theory on a Lens space

$$Y = S^3/\mathbb{Z}_k \qquad H^2(Y;\mathbb{Z}) \cong \mathbb{Z}_k$$

$$1 \to \mathbb{Z}_k \to \mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \to \mathbb{Z}_k \times \mathbb{Z}_k \to 1$$

This has unique irrep ***P*** = clock operator, ***Q*** = shift operator

$$\boxed{PQ = e^{2\pi i/k} QP}$$

Groundstates have definite electric ***or*** magnetic flux

$$|e\rangle = \sum_{m=1}^{k} e^{2\pi i e m/k} |m\rangle$$

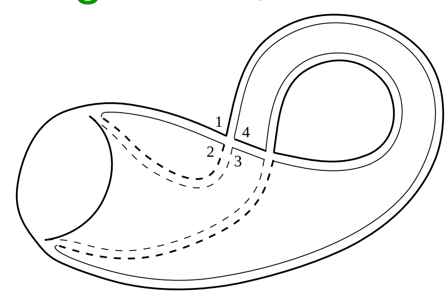
This example already appeared in string theory in Gukov, Rangamani, and Witten, hep-th/9811048. They studied AdS5xS5/Z3 and in order to match nonperturbative states concluded that in the presence of a D3 brane one cannot simultaneously measure D1 and F1 number.

An Experimental Test

Since our remark applies to Maxwell theory: Can we test it experimentally?

Discouraging fact: No region in $\mathbb{R}^3$ has torsion in its cohomology

With A. Kitaev and K. Walker we noted that using arrays of Josephson Junctions, in particular a device called a ``superconducting mirror,'' we can ``trick'' the Maxwell field into behaving as if it were in a 3-fold with torsion in its cohomology.



To exponentially good accuracy the groundstates of the electromagnetic field are an irreducible representation of Heis($\mathbb{Z}_n$ x $\mathbb{Z}_n$)

See arXiv:0706.3410 for more details.

Part III: Defects and Locality in TFT

Defects play a crucial role in both CMT and in Physical Mathematics

Recently experts in TFT have been making progress in ``extended TFT'' (ETFT) which turns out to involve defects and is related to a deeper notion of locality.

Topological Field Theory

A key idea of the Atiyah-Segal definition of TFT is to encode the most basic aspects of locality in QFT.

Axiomatics encodes <u>some</u> aspects of QFT locality:
It is a caricature of QFT locality of n-dimensional QFT:

X: A closed (n-1)-manifold → $\mathcal{H}(X)$: Space of quantum states

Z: $X_0 \to X_1$: A cobordism → Quantum transition amplitudes $\mathcal{A}(Z): \mathcal{H}(X_0) \to \mathcal{H}(X_1)$

Can we enrich this story?

Yes!

1. Defects.

2. Extended locality.

Defects in Local QFT

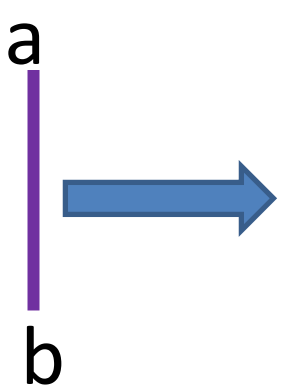
Pseudo-definition: Defects are local disturbances supported on positive codimension submanifolds

- dim =0: Local operators
- dim=1: ``line operators''
- etc
- codim =1: Domain walls

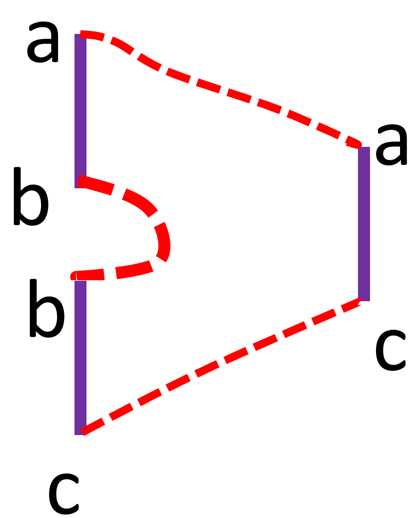
N.B. A boundary condition (in space) in a theory T can be viewed as a domain wall between T and the empty theory. So the theory of defects subsumes the theory of boundary conditions.

Boundary conditions and categories

Let us begin with 2-dimensional TFT. Here the set of boundary conditions can be shown to be objects in a category (Moore & Segal)

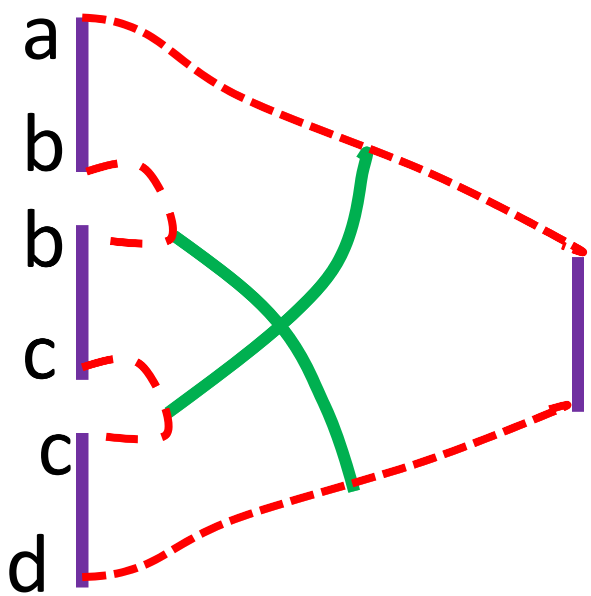


$$\mathcal{O}_{ab}$$



$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \rightarrow \mathcal{O}_{ac}$$

Why are boundary conditions objects in a category?

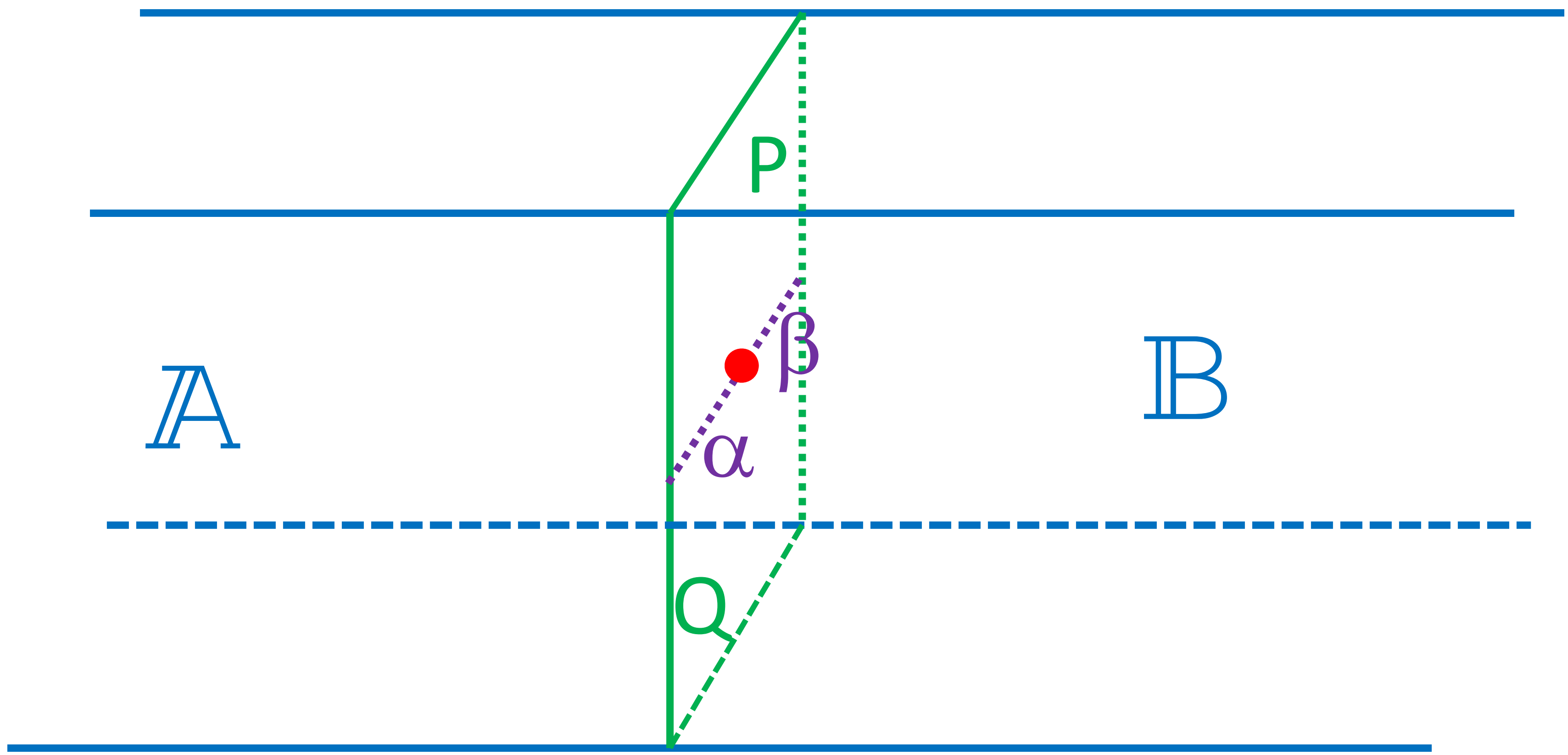


So the product on open string states is associative

Therefore: $a \in \mathrm{Obj}(\mathcal{C})$ and $\mathcal{O}_{ab} = \mathrm{Hom}(a,b)$

Defects Within Defects

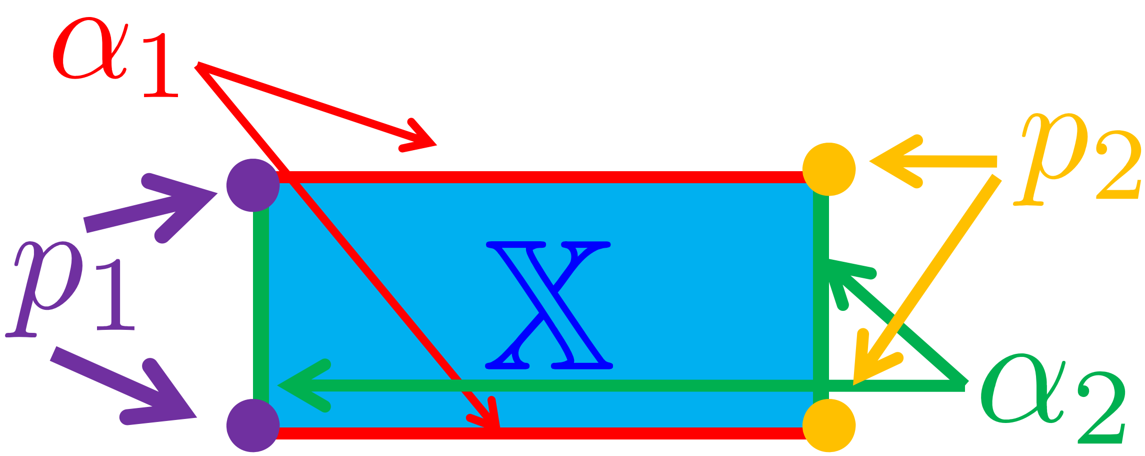
Now – In higher dimensions we can have defects within defects….

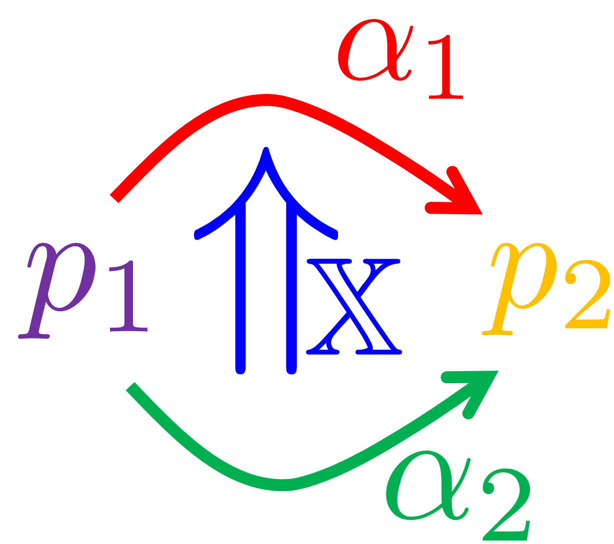


n-Categories

Definition: An n-category is a category C whose morphism spaces are n-1 categories.

Bord_n: Objects = 0-manifolds; 1-Morphisms = 1-manifolds; 2-Morphisms = 2-manifolds (with corners); …



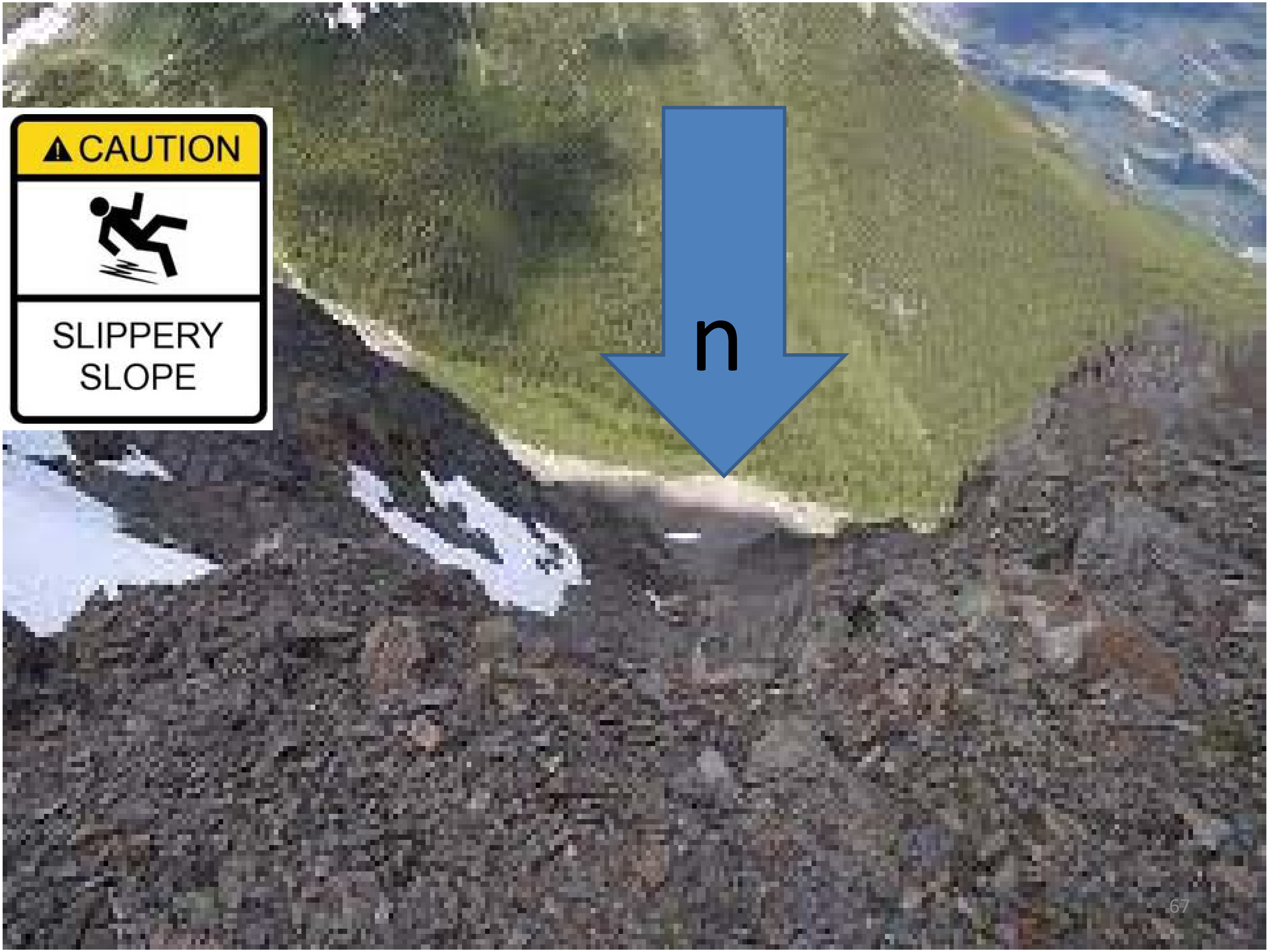


Defects and n-Categories

Conclusion: Spatial boundary conditions in an n-dimensional TFT are objects in an (n-1)-category:

k-morphism = (n-k-1)-dimensional defect in the (n-1)-dimensional spatial boundary.

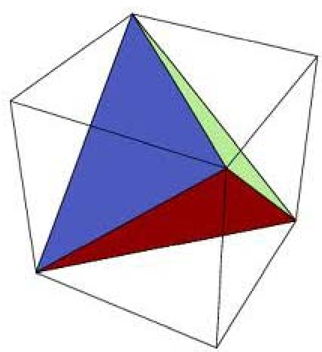
(Kapustin, ICM 2010 talk)


CAUTION
SLIPPERY
SLOPE
n

Locality

The Atiyah-Segal definition of a topological field theory is slightly unsatisfactory:

In a truly local description we should be able to build up the theory from a simplicial decomposition.



What is the axiomatic structure that would describe such a completely local decomposition?

D. Freed; D. Kazhdan; N. Reshetikhin; V. Turaev; L. Crane; Yetter; M. Kapranov; Voevodsky; R. Lawrence; J. Baez + J. Dolan ; G. Segal; M. Hopkins, J. Lurie, C. Teleman,L. Rozansky, K. Walker, A. Kapustin, N. Saulina,…

Answer: Extended TFT

Definition: An n-extended field theory is a ``homomorphism'' from $Bord_n$ to a (symmetric monoidal) n-category.

Example 1: 2-1-0 TFT:

$$F(M_2) \in \mathbb{C}$$ Partition Function

$$F(M_1) \in VECT$$ Hilbert Space

$$F(M_0) \in \mathrm{CAT}$$ Boundary conditions

Example 2: 3-2-1-0 TFT (e.g. Chern-Simons):

$$F(M_3) \in \mathbb{C}$$ Partition Function (Reshetikhin-Turaev-Witten invariant)

$$F(M_2) \in VECT$$ Hilbert Space (of conformal blocks)

$$F(M_1) \in \mathrm{CAT}$$ Category of integrable reps of LG

$$F(M_0) \in 2 - \mathrm{CAT}$$ Current topic of research

The Cobordism Hypothesis

$F(M_n) \in \mathbb{C}$ Partition Function

$F(M_{n-1}) \in VECT$ Hilbert Space

$F(M_{n-2}) \in CAT$ Boundary conditions

$$F(M_{n-k}) \in k - CAT$$

Cobordism Hypothesis of Baez & Dolan: An n-extended TFT is entirely determined by the n-category attached to a point.

For TFTs satisfying a certain finiteness condition this was proved by Jacob Lurie. Expository article. Extensive books.

Generalization: Theories valued in field theories

DEFINITION: An m-dimensional theory $\mathfrak{H}$ valued in an n-dimensional field theory F , where n= m+1, is one such that

$$\mathfrak{H}(N_j) \in F(N_j) \qquad j = 0,1,\ldots, m$$

The ``**partition function**'' of $\mathfrak{H}$ on N_m is a ***vector in a vector space***, and not a complex number . The Hilbert space…

1. The chiral half of a RCFT.
2. The abelian tensormultiplet theories

Conclusions

We discussed three transverse intersections of PM & CMT

A suggested generalization of the K-theory approach to the classification of topological states of matter

Some potentially relevant theorems about 3 and 4 dimensional abelian gauge theories

Most speculative of all: Applications of higher category theory to classification of defects.

It would be delightful if any of these mathematical results had real physical applications!!