GGI LECTURE 3:

M5-BRANES, HITCHIN SYSTEMS, & BPS STATES

KEY REFS:

1. KLEMM, LERCHE, MAYR, VAFA, WARNER "SELF-DUAL STRINGS ..." hep-th/9604034

2. E. WITTEN, "SOLUTIONS OF FOUR-DIMENSIONAL FIELD THEORIES VIA M-THEORY," hep-th/9703166

1. INTRODUCTION

FOR A LARGE CLASS OF $d=4, \mathcal{N}=2$ THEORIES THE MODULI SPACE $(\mathcal{M}, g)$ IS ALSO THE MODULI SPACE OF A HITCHIN SYSTEM (WITH SINGULARITIES) ON $\mathbb{P}^1$.

THIS GENERALIZES A RESULT OF CHERKIS + KAPUSTIN AND FOLLOWS FROM THE CONSTRUCTION OF $d=4, \mathcal{N}=2$ THEORIES USING M5-BRANES.

THE AIM OF THIS LECTURE IS TO EXPLAIN THE CONNECTION TO HITCHIN MODULI SPACE, AND HOW BPS STATES LOOK LIKE IN THIS DESCRIPTION.

2. M5 ON A RIEMANN SURFACE

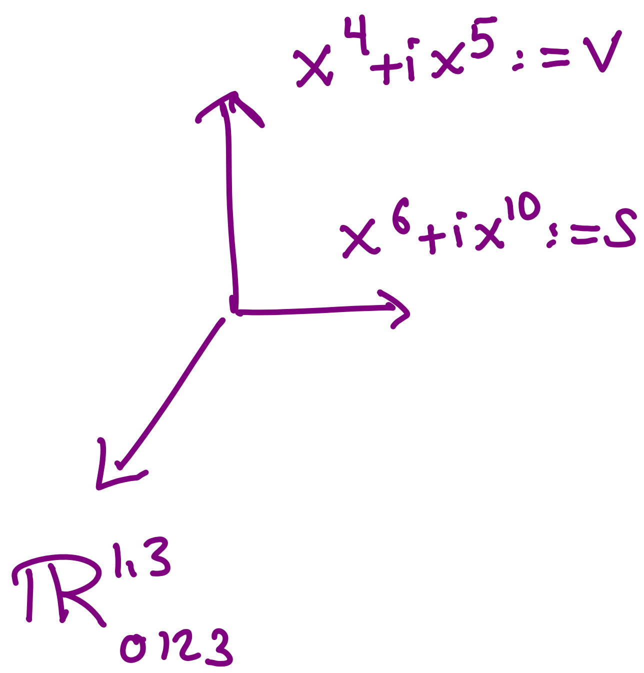
A. M-THEORY / $\underset{0123}{\mathbb{R}^{1,3}} \times \underset{6,10}{C} \times \underset{4,5}{\mathbb{C}} \times \underset{7,8,9}{\mathbb{R}^3}$

METRIC:

$$ds^2 = \sum_{\mu=0,1,2,3} dx^\mu dx_\mu + \sum_{i=4,5,7,8,9} (dx^i)^2 + R_{11}^2 \left[(dx^6)^2 + (dx^{11})^2 \right]$$

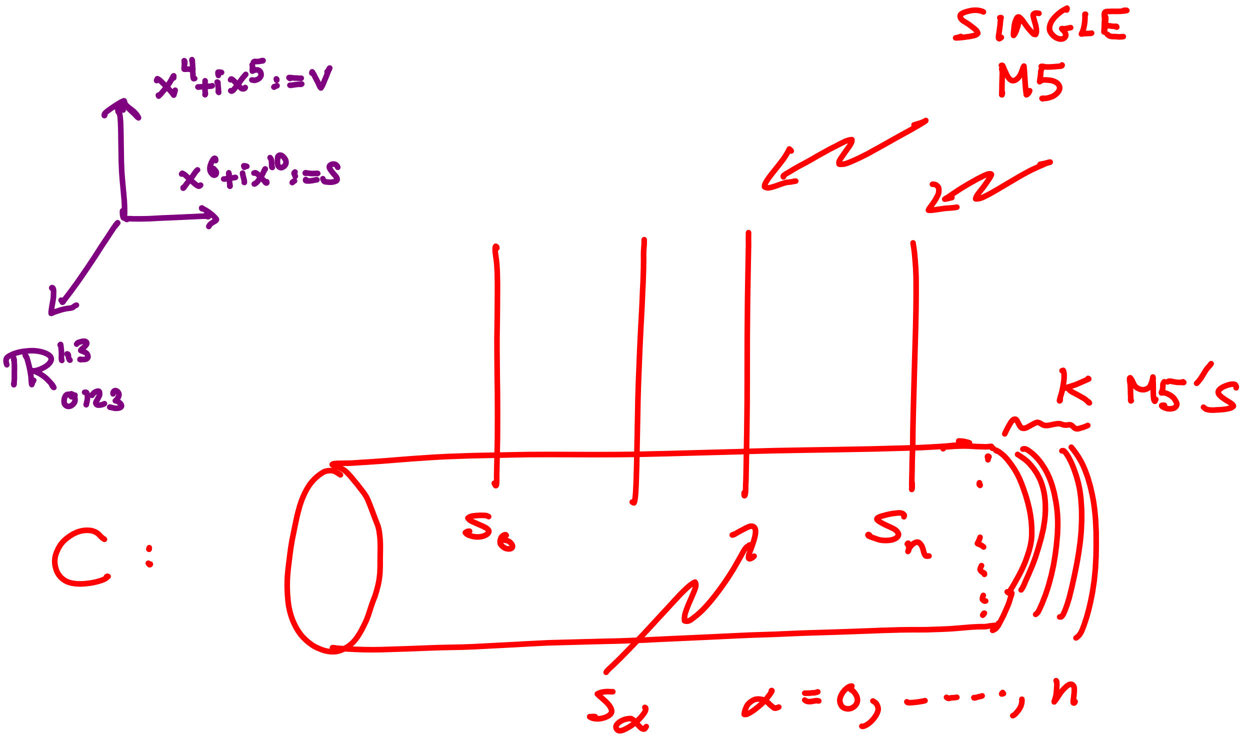
x^6, x^{10} DIMENSIONLESS.

NOW DRAW PICTURES



EVERYTHING AT A SINGLE POINT IN $\mathbb{R}^3_{7,8,9}$ SO SUPPRESS IT.

CONSIDER M5 CONFIGURATION



WRAPS A HOLO. CURVE SO PRESERVES 8 SUPERSYMMETRIES

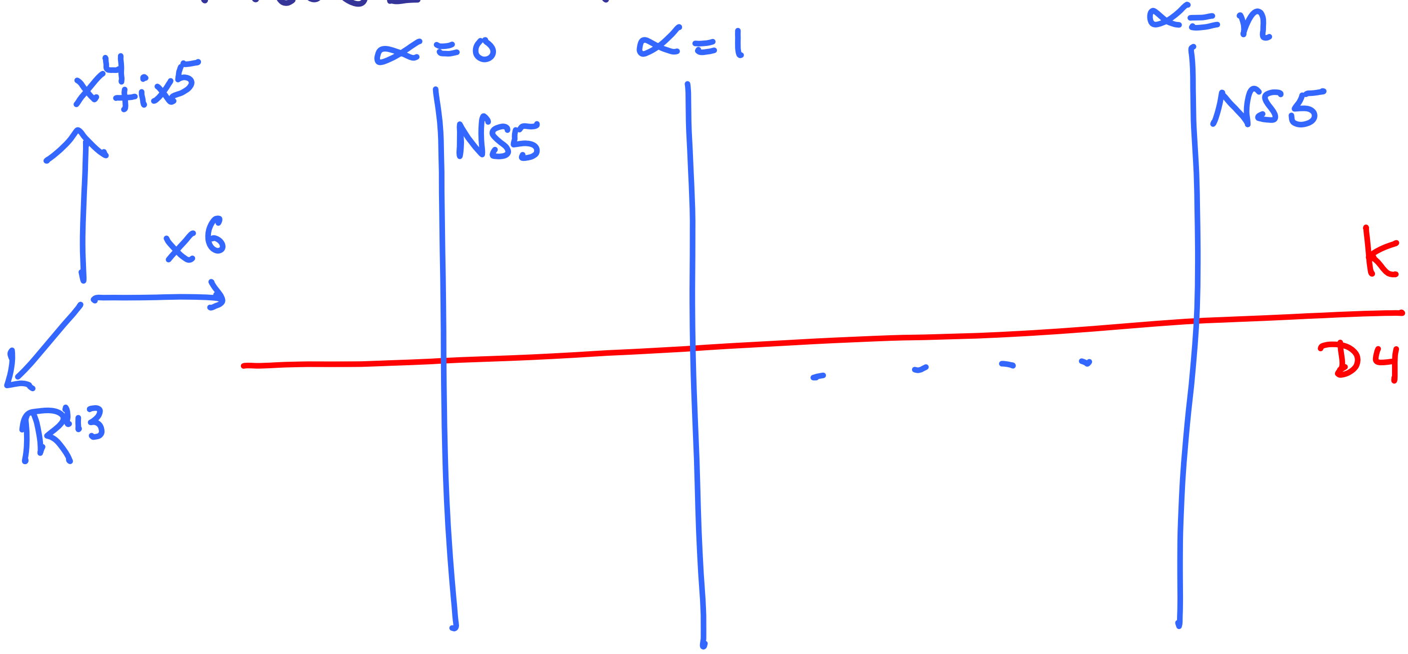
$$C \cong \mathbb{C}^* \qquad t := e^{-(x^6+ix^{10})} \in \mathbb{C}^*$$

$$v := (x^4+ix^5)/\ell$$

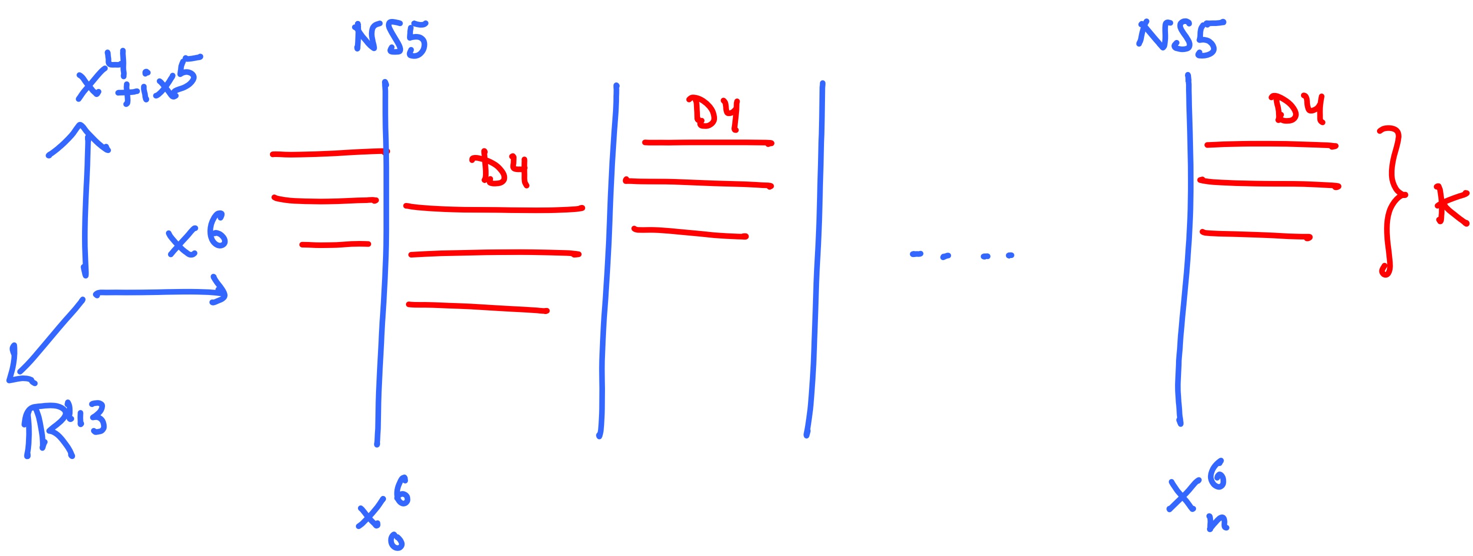
M5 ON CURVE:

$$v^k \prod_{\alpha=0}^{n} (t - t_\alpha) = 0$$

PROJECTION TO IIA SUGRA



THE D4's IN EACH INTERVAL CAN SPLIT APART SEPERATELY MAINTAINING SUPERSYMMETRY:



- TAKE A LIMIT OF LOW ENERGY & GRAVITATIONAL DECOUPLING

$\Longrightarrow$ REPLACE M5 BY SU(K) (2,0) SUPERCONFORMAL THEORY

- AT LONG DISTANCES COMPARED TO $R_{11}(X^6_\alpha - X^6_{\alpha-1})$ WE HAVE AN EFFECTIVE 4D GAUGE THEORY:

<u>4D IR</u> : DESCRIBES LINEAR QUIVER

$$\boxed{K} - \underbrace{(K) - (K) - \cdots - (K)}_{n} - \boxed{K}$$

- THE PICTURE ABOVE IS A PICTURE OF THE BRANE CONFIG. DESCRIBING THE LOW ENERGY DYNAMICS ON THE COULOMB BRANCH:

B. ALGEBRAIC CURVES

LIFTING BACK UP TO M-THEORY WE ARE DEFORMING THE CURVE $v^k \prod_{\alpha=0}^{n}(t-t_\alpha)=0$ AND WRAPPING A SINGLE M5-BRANE ON THAT CURVE:

$$\Sigma = \{(t,v) \mid F(t,v)=0\}$$

$$\subset \mathbb{C}_{6+i10} \times \mathbb{C}_{4+i5} \cong \mathbb{C}^* \times \mathbb{C}$$

THE DIFFERENT ROOTS IN v AT FIXED t DESCRIBE THE POSITIONS OF THE k DIFFERENT D4's $\Rightarrow$ ADD POLYNOMIAL IN v OF DEGREE $< k$:

$$F(t,v) = v^k \prod_{\alpha=0}^{n}(t-t_\alpha) + \sum_{i=1}^{k} p_i(t) v^{k-i} = 0$$

AT GENERIC t THERE ARE K ROOTS. AT $t \to t_\alpha$ LEADING COEFF. DEGENERATES.

FOR GENERIC p_i <u>ONE</u> ROOT $\underset{*}{V}(t)$ GOES TO ∞:

$$V_*(t) \sim \frac{1}{t-t_\alpha}\left(\frac{-p_i(t_\alpha)}{\prod_{\beta\neq\alpha}(t_\alpha-t_\beta)}\right)+\cdots$$

THE OTHER $(K-1)$ ROOTS REMAIN FINITE.

$$\Sigma \xrightarrow{K:1} C$$ IS A BRANCHED COVER. THINK OF THE BRANCHES AS SHEETS OF SINGLY-WRAPPED M5. THE DIVERGENT ROOT AT t_α IS THE TRANSVERSE M5.

IN ADDITION, FOR $x^6 \longrightarrow \pm\infty$,

i.e. $t \to 0$ & $t \to \infty$

WE WANT PRECISELY K ROOTS

$$\Longrightarrow \quad \deg(P_i) \leq n+1$$

$\Longrightarrow$ WE CAN ALSO WRITE:

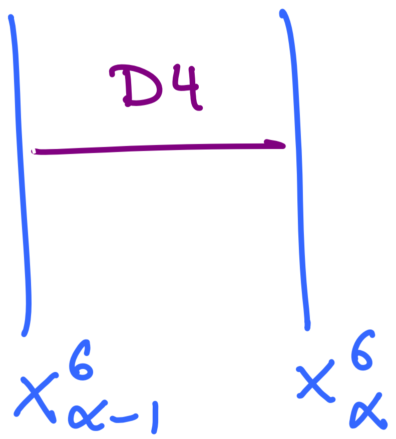
$$F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha}\, q_\alpha(v) = 0.$$

$$\deg(q_\alpha) = K.$$

ROOTS OF $q_0(v)$: POSITIONS FOR $t \to \infty$, etc.

I.C. WEAK COUPLING LIMIT

TO GET SOME PHYSICAL INTUITION CONSIDER THE WEAK COUPLING LIMIT



$$\frac{4\pi^2}{(g^2_{YM})_\alpha} = \left(x^6_\alpha - x^6_{\alpha-1}\right)/g_s$$

$$\sim \mathrm{Re}\left(S_\alpha - S_{\alpha-1}\right)$$

VALID FOR WEAK $g^2_{YM}(SU(k)_\alpha)$

HOLOMORPHY $\Longrightarrow$

$$-i\pi\tau_\alpha = -i\pi\left(\frac{\theta_\alpha}{2\pi} + \frac{4\pi i}{g^2_\alpha}\right) = S_\alpha - S_{\alpha-1}$$

WEAK COUPLING LIMIT

1. $\left| t_\alpha / t_{\alpha-1} \right| = \epsilon_\alpha \longrightarrow 0 \qquad 1 \leq \alpha \leq n$

2. $q_\alpha(v) = t_0 \cdots t_{\alpha-1} \tilde{q}_\alpha(v)$

   HOLDING $\tilde{q}_\alpha(v)$ ORDER 1.

EXERCISE: SHOW THAT IN THIS LIMIT, FOR $|t_\beta| \ll |t| \ll |t_{\beta-1}|$ THE ROOTS ARE APPXT. $\tilde{q}_\beta(v) = 0$.

$$q_\alpha(v) = c_\alpha \left(v^k - \mu^{(\alpha)} v^{k-1} - u_2^{(\alpha)} v^{k-2} - \cdots - u_k^{(\alpha)} \right)$$

gauge couplings

bifund. masses — as we'll see

adjoint vevs of $SU(k)_\alpha$

$\boxed{\text{D: SW CURVE \& DIFFERENTIAL}}$

CLAIM: $\Sigma = \{(t,v) \mid F(t,v) = 0\}$

IS THE SW CURVE OF THE EFFECTIVE 4D GAUGE THEORY.

A COMPUTATION THAT MAKES THIS PLAUSIBLE IS THE K.E. OF THE SCALARS DESCRIBING FLUCTUATIONS OF THE SCALARS IN THE V-DIRECTION.

$$\mathbb{R}^{1,3} \times \mathbb{C}^*_{6,10} \times \mathbb{C}_{4,5}$$

$$\left(x^\mu, t, v(t, \xi_i(x^\mu))\right)$$

$\xi_i \sim$ PARAMETERS OF Σ

SLOWLY VARYING

$\xi_i = 0$: SOME SUSY BASEPOINT

USE THE DBI ACTION FOR THE SINGLE M5:

$$ds^2 = dx^\mu dx_\mu + \ell^2 |dv|^2 + R_{11}^2 \left|\frac{dt}{t}\right|^2 + \cdots$$

$$\frac{2\pi}{\ell^6} \int_{M5} \mathrm{Vol} - \frac{2\pi}{\ell^6} \int \mathrm{Vol} \Big|_{\xi=0}$$

$$= \frac{R_{11}^2}{\ell^2} \int_{\mathbb{R}^{1,3}} dx^{0123} \int_\Sigma \frac{dt\, d\bar{t}}{|t|^2} \frac{\partial v}{\partial \xi_i} \partial_\mu \xi_i \overline{\frac{\partial v}{\partial \xi_j} \partial_\mu \xi_j} + O(\xi^3)$$

$$= \frac{R_{11}^2}{\ell^2} \int_{\mathbb{R}^{1,3}} dx^{0123}\, \partial_\mu \xi_i \partial_\mu \bar{\xi}_j \int_\Sigma \frac{\partial v}{\partial \xi_i} \frac{dt}{t} \wedge \overline{\frac{\partial v}{\partial \xi_j} \frac{dt}{t}}$$

• DEFINITION: NORMALIZABLE DEF'S ξ_i:

$$\int_{\Sigma} \frac{\partial v}{\partial \xi_i}\frac{dt}{t} \wedge \overline{\frac{\partial v}{\partial \xi_j}\frac{dt}{t}} < \infty$$

$\Rightarrow$ ξ_i VARIES $u_i^{(\alpha)}$, BUT NOT THE POSITIONS t_α OR RESIDUES OF POLE OF $V_*(t)$ AT $t = t_\alpha$.

LET: γ_a = BASIS FOR $H_1(\overline{\Sigma}, \mathbb{Z})$

I^{ab} = INTERSECTION FORM

$$\text{ACTION} = \frac{R_{11}^2}{\ell^2} \int_{\mathbb{R}^{1,3}} dx^{0123}\, I^{ab}\, \partial_\mu \Pi_a\, \partial_\mu \overline{\Pi_b}$$

WITH $$\Pi_a = \oint_{\gamma_a} v \frac{dt}{t}$$

CHOOSING A DUALITY FRAME

$$\frac{R_{11}^2}{\ell^2} \int dx^{0123} \, \mathrm{Im}\, \tau_{IJ} \, \partial_\mu a^I \, \partial_\mu \bar{a}^J$$

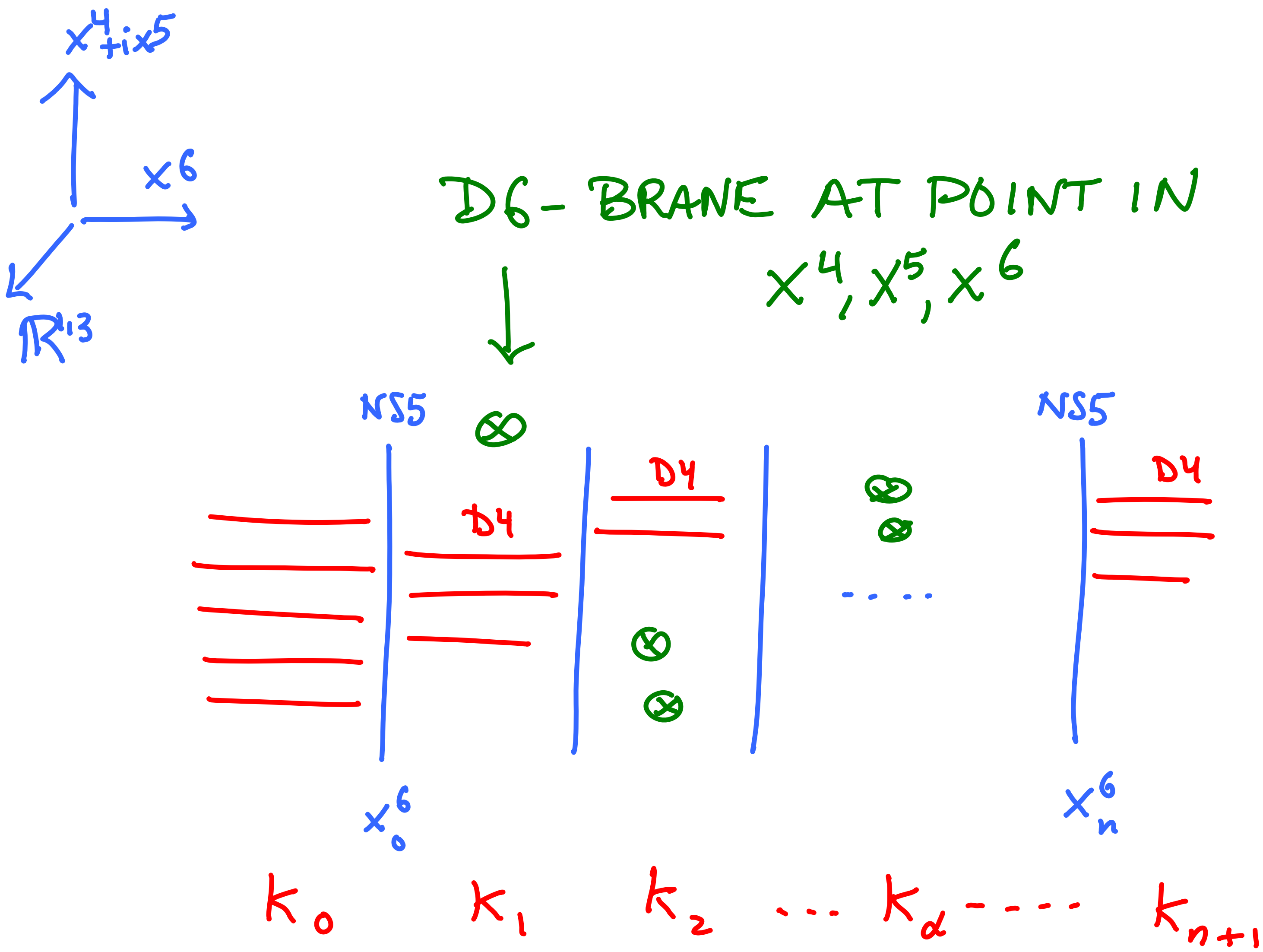
WITH S-W DIFF'L

$$\boxed{\lambda = v \frac{dt}{t}}$$

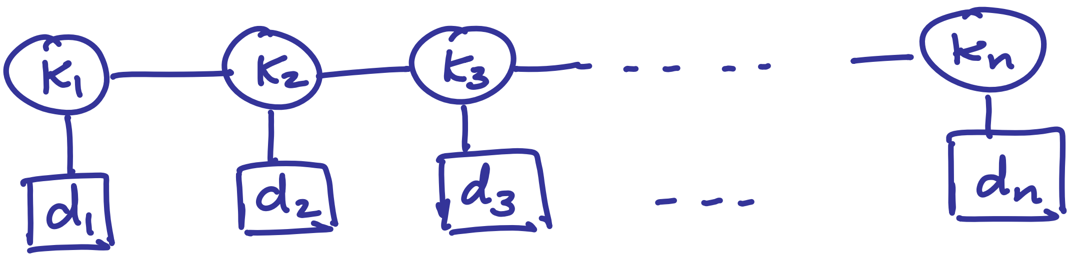
GENERALIZATIONS . . .

<u>Exercise:</u> Compute the residue of λ at $t = t_\alpha$, in the weak coupling limit, And show it is $\mu_\alpha - \mu_{\alpha-1}$. Thereby make a connection of μ_α to bifund. masses.

E: ADDING & DECOUPLING FLAVORS



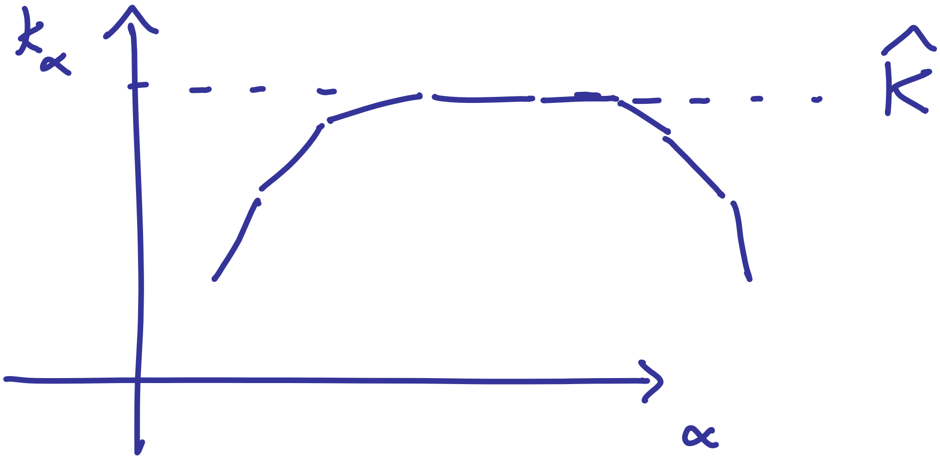
LEADS TO LINEAR QUIVER



ASYMPTOTIC FREEDOM $\Rightarrow$

$$-2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha \leq 0$$

$\Longrightarrow$



D6 LIFT TO MULTI-TAUB-NUT.
$\Sigma \subset TN$ IS HOLOMORPHIC CURVE.

MAIN NEW FEATURE: SOME ROOTS $v_i(t)$ OF $F(t,v)=0$ GO TO ∞ FOR $t \to 0, \infty$.

F. THE GENERAL STORY

- M THEORY ON $\mathbb{R}^{1,3} \times Q \times \mathbb{R}^3_{7,8,9}$

$Q =$ H.K. 4-FOLD

PREVIOUS $Q = T^*C$, $C = \mathbb{C}^*$

- K-WRAPPED M5 ON HOLOMORPHIC CURVE $C \subset Q$, INTERSECTS SINGLY WRAPPED C_α TRANSVERSALLY.

- DECOUPLE GRAVITY $\Rightarrow$ SU(K) (2,0) THEORY ON $\mathbb{R}^{1,3} \times C$ WITH DEFECTS AT $C \cap C_\alpha$. (TWISTED TO PRESERVE $d=4$, $N=2$ SUSY.)

- SMALL FLUCTUATIONS ONLY PROBE INF. NBD. OF $C \subset Q \Rightarrow T^*C$

- SW CURVE OF THE $D=4, N=2$ THEORY IS A HOLOMORPHIC CURVE $\Sigma \subset T^*C$.
- SW DIFFERENTIAL IS RESTRICTION OF CANONICAL $\omega = dx\,dz = d(x\,dz)$

$$\lambda = x\,dz$$

- LOW ENERGY EFFECTIVE THEORY IS COMPACTIFICATION OF SINGLE M5 ON Σ.

$\boxed{\text{3. MAPPING TO HITCHIN SYSTEMS}}$

- WE HAVE NOW SEEN THAT WE GET A $D=4$, $N=2$ THEORY BY COMPACTIFYING A $D=6$ $(2,0)$ THEORY ON C.

- TO GET OUR HYPERKÄHLER σ-MODEL WE THEN COMPACTIFY ON S^1_R

- BUT - BY A KIND OF "FUBINI THEOREM" WE COULD GET THE LOW ENERGY THEORY COMPACTIFYING IN THE OTHER ORDER....

$$
\begin{tikzcd}[column sep=small]
 & \text{6D } (2,0)\ A_{K-1} \,/\, \mathbb{R}^{1,2}\times S^1_R \times C \ \textcolor{magenta}{+\ \text{DEFECTS}} \arrow[dl, "\ell_C \ll \ell_{S^1},\, \ell_{\mathbb{R}^{1,2}}"'] \arrow[dr, "\ell_{S^1} \ll \ell_C,\, \ell_{\mathbb{R}^{1,2}}"] & \\
\text{4D } \mathcal{N}=2 \text{ GAUGE THEORY} \,/\, \mathbb{R}^{1,2}\times S^1_R \arrow[dr, "\ell_{S^1} \ll \ell_{\mathbb{R}^{1,2}}"'] & & \dfrac{\text{5D } U(K) \text{ SYM}}{\mathbb{R}^{1,2}\times C} \arrow[dl, "\ell_C \ll \ell_{\mathbb{R}^{1,2}}"] \\
 & \sigma\text{-MODEL: } \mathbb{R}^{1,2} \longrightarrow \mathcal{M} &
\end{tikzcd}
$$

ON THE OTHER HAND -- REDUCTION

OF THE 5D $U(K)$ $(2,0)$ THEORY ON C $\Rightarrow$

HITCHIN EQS.
$$\begin{cases} F + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_{\bar{A}}\, \varphi = 0 \qquad \partial_A \bar{\varphi} = 0 \end{cases}$$

$\varphi = (\Phi_4 + i\Phi_5)$: $(1,0)$ FORM ON C

$+$ SINGULAR BC's AT S_α

$\Rightarrow$ $\mathcal{M}$ = MODULI SPACE OF HITCHIN SYSTEM.

(Remark: The topological twisting on C makes φ into a $(1,0)$ form on C.)

THE SPECTRAL CURVE

RETURN TO $\quad C \cong \mathbb{C}^*_{6+i10} \ni t = e^{-s}$

$$\varphi = \varphi_s \, ds$$

$$\boxed{\text{* EIGENVALUES OF } \varphi_s = \text{POSITIONS OF D4 BRANES}}$$

THESE MUST BE THE ROOTS OF v IN $F(t, v) = 0$!

INDEED, GIVEN A SOL'N OF THE HITCHIN SYSTEM, AN EASY COMPUTATION SHOWS THAT

$$\frac{\partial}{\partial t}\Big(\det(v + \varphi_s)\Big) = 0$$

(NOTE $\frac{\partial}{\partial t}\varphi_s \neq 0$!)

MORE INVARIANTLY:

$$\boxed{\det\left(v\,ds+\varphi\right)=0}$$

DEFINES THE <u>SPECTRAL CURVE</u> IN T^*C.

WORKING IN $t=e^{-s}$

$$\det\left(v\frac{dt}{t}-\varphi_t\,dt\right)=0.$$

<u>OR</u>

$$\boxed{\det\left(\lambda-\varphi\right)=0}$$

THIS SHOULD DEFINE Σ SO,

COMPARING TO:

$$F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha}\, q_\alpha(v)$$

$$= p_0(t)\, v^k + p_1(t)\, v^{k-1} + \cdots + p_k(t)$$

MAKING IT MONIC WE DIVIDE BY $p_0(t)$ AND IDENTIFY:

$$v^k + R_1(t)\, v^{k-1} + \cdots + R_k(t) = \det(v - t\varphi_t)$$

PREVIOUSLY WE SAW THAT THE ROOTS BEHAVE AS:

1.) $V_i(t) \sim$ FINITE $t \to t_\alpha$

EXCEPT FOR ONE ROOT $V_*(t) \sim \dfrac{\rho_\alpha}{t - t_\alpha}$

2.)

$$V_i(t) \longrightarrow \begin{cases} V_i^{(0)} & t \to 0 \\ V_i^{(\infty)} & t \to \infty \end{cases}$$

THEREFORE, THE S-W DIFF'L BEHAVES LIKE

1.) $\lambda_i \sim$ FINITE ON ALL BRANCHES FOR $t \to t_\alpha$ AND

$$\lambda \sim \frac{m_\alpha}{t - t_\alpha}\, dt \qquad t \to t_\alpha$$

ON ONE BRANCH

2.) FOR $t \to 0, \infty$ THE SW DIFFL HAS POLE $\lambda \longrightarrow \begin{cases} v_i^{(0)} \dfrac{dt}{t} \\ v_i^{(\infty)} \dfrac{dt}{t} \end{cases}$ ON EACH BRANCH AS $t \to 0, \infty$ RESPECTIVELY

BUT $\det(\lambda - \varphi) = 0$

SO THE HIGGS FIELD φ HAS SINGULARITIES

$$\varphi(t) \underset{t\to t_\alpha}{\sim} \frac{dt}{t-t_\alpha}\begin{pmatrix} m_\alpha & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$\varphi(t) \underset{t\to 0}{\sim} \frac{dt}{t}\begin{pmatrix} V_1^{(0)} & & \\ & \ddots & \\ & & V_k^{(0)} \end{pmatrix}$$

$$\varphi(t) \underset{t\to\infty}{\sim} \frac{dt}{t}\begin{pmatrix} V_1^{(\infty)} & & \\ & \ddots & \\ & & V_k^{(\infty)} \end{pmatrix}$$

(WHEN WE ADD FLAVORS OR CONSIDER OTHER GENERALIZATIONS WE HAVE <u>IRREGULAR SINGULAR POINTS</u>.)

AN IMPORTANT SPECIAL CASE: $K=2$

$$\boxed{2} - (2) - (2) - \cdots\cdots - (2) - \boxed{2}$$

$$\Sigma \xrightarrow{2:1} C$$

$$\varphi \sim \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix} \in su(2)$$

(AFTER SUBTRACTING OVERALL CENTER OF MASS OF BRANE SYSTEM)

$$\varphi \sim \frac{dt}{t-t_\alpha} \begin{pmatrix} m_\alpha & \\ & -m_\alpha \end{pmatrix}$$

$$A \sim \cdots\cdots$$

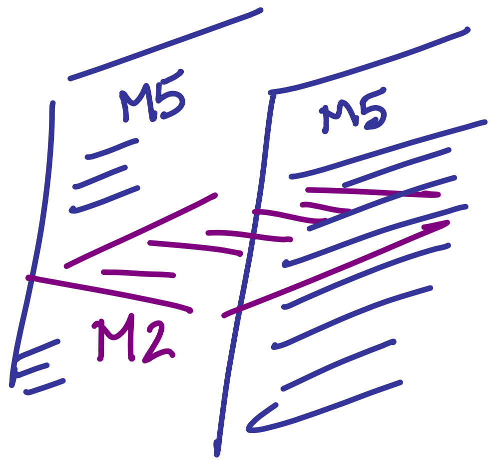
$$\lambda^2 = \sum_\alpha \left(\frac{m_\alpha^2}{(t-t_\alpha)^2} + \frac{c_\alpha}{t-t_\alpha} \right) (dt)^2$$

$c_\alpha \sim$ PARAMETRIZE $\mathcal{B}$.

4. BPS STATES FROM OPEN M2-BRANES

M5 ORIGIN OF THE BPS STATES IS FROM OPEN M2 BRANES BETWEEN || M5

(STROMINGER)



$\Rightarrow$ BPS STRINGS IN THE (2,0) THRY

BPS STATES IN THE HITCHIN FRAMEWORK

THE PHYSICAL DERIVATION LEADS TO THE FOLLOWING RULES FOR DESCRIBING BPS STATES IN THE HITCHIN FRAMEWORK

(KLEMM, LERCHE, VAFA, WARNER)

$\Omega(\gamma, u) = 0$ UNLESS THERE EXISTS A CURVE $\tilde{C} \in \Sigma$ IN HOMOLOGY CLASS γ SO THAT $\pi_*(\tilde{C}) = C$ IS A CURVE IN $\mathbb{C}$ S.T.

$$\boxed{\langle \lambda, \partial_t \rangle \in e^{i\vartheta_*} = \text{CONSTANT}}$$

AND

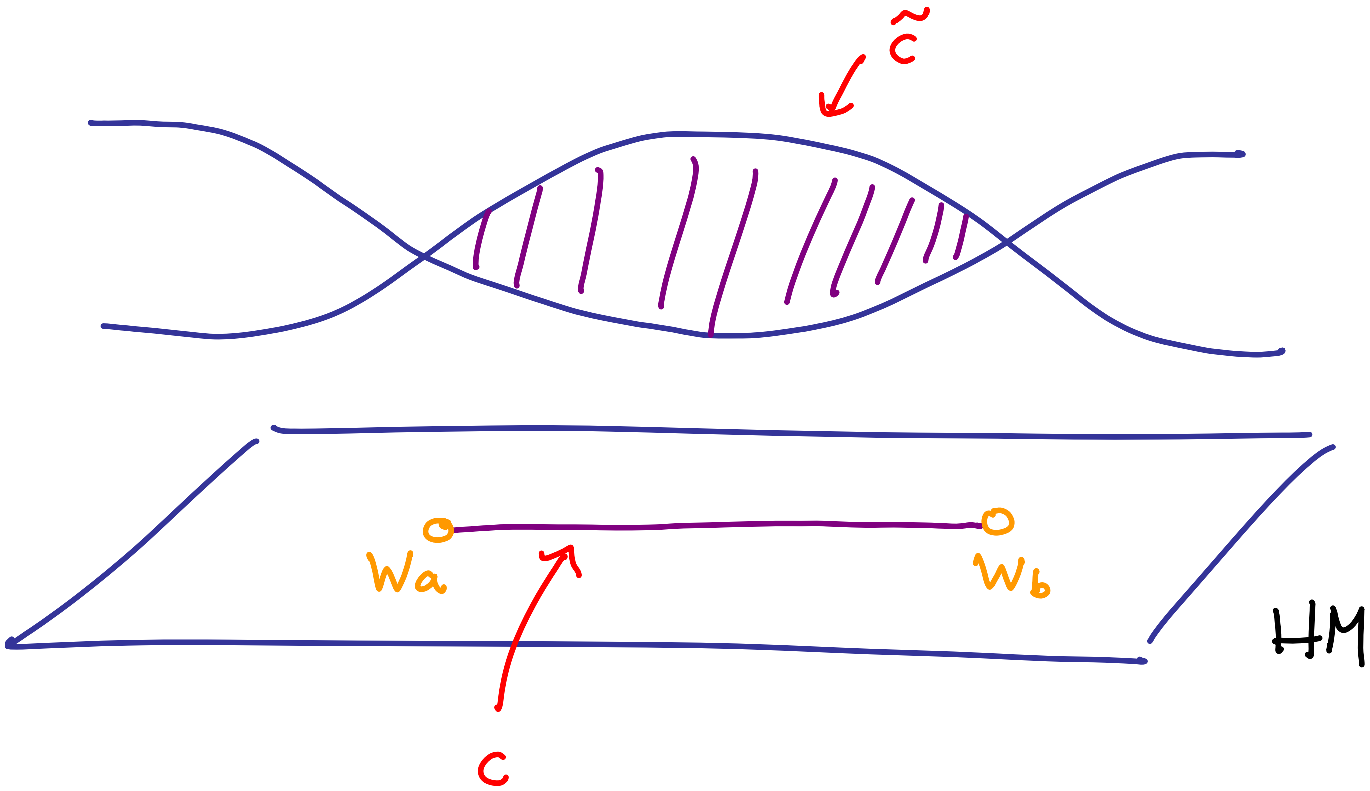
ONE OF TWO THINGS HAPPENS:

1. EITHER,

C BEGINS AND ENDS ON A BRANCH POINT

$\Rightarrow$ HM WITH $\Omega = +1$

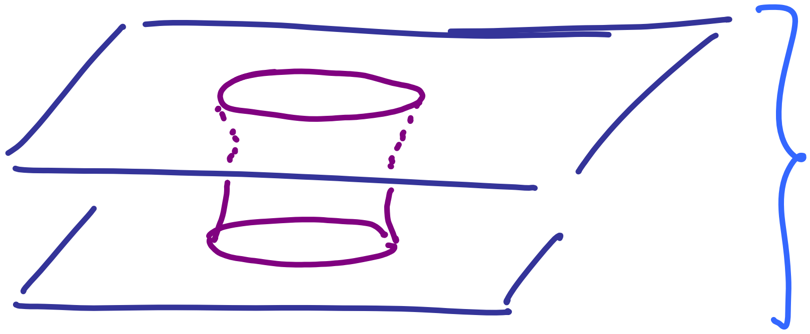
M-THEORY PICTURE: OPEN M2:



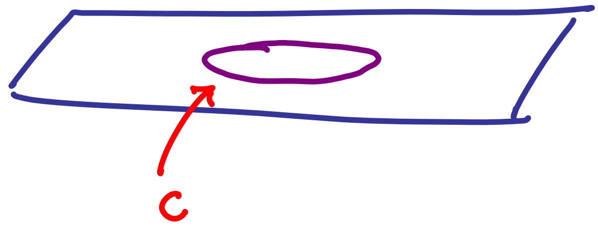
2. OR, C = CLOSED CURVE $\Rightarrow$

VM WITH $\Omega = -2$.

Σ



M-THEORY PICTURE



> N.B. FOR SUCH CURVES $\langle \lambda, \partial_t \rangle = e^{i\vartheta_*}$
>
> $$\vartheta_* = \arg(Z)$$

(REMARK: FOR $SU(K)$, $K > 2$ THE BPS STATES ARE MORE COMPLICATED AND INVOLVE "STRING WEBS.")

5. TWISTOR COORD'S FOR $K=2$

WE HAVE CONSTRUCTED THE $\mathcal{X}_\gamma$ AND VERIFIED THE PROPERTIES 1–6 ABOVE

THE KS TRANSFORMATIONS AND $\zeta \to 0, \infty$ ASYMPTOTICS EMERGE VERY NATURALLY...

CONJECTURALLY A GENERALIZED CONSTRUCTION APPLIES TO $K>2$.

THIS IS THE CONSTRUCTION WE TURN TO NEXT