# SPLIT POLAR ATTRACTORS

WORK DONE WITH F. DENEF

hep-th/??.mmnnn, JUST IN TIME FOR THE SSC

MIT, Sept. 27 ; HARVARD, Sept. 29, 2006

## OUTLINE

1. INTRODUCTION
   - A. BPS STATE COUNTING IS INTERESTING
   - B. FRAMEWORK : IIA / $CY_3$
   - C. SUMMARY OF MAIN RESULTS
   - D. SKETCH OF FORMAL DERIVATION OF THE OSV CONJECTURE

2. DESCRIBING BPS STATES
   - A. MULTICENTERED SOLUTIONS
   - B. WALL-CROSSING FORMULA
   - C. SPLIT ATTRACTOR FLOWS
   - D. A SURPRISING EXAMPLE

3. APOTHEOSIS OF DONALDSON, THOMAS, AND McMAHON

..... CONTINUATION FOR THE CONNAISSEUR ....

4. BPS COUNTING FUNCTION
   A. MODULARITY
   B. SINGLETON DECOMP.
   C. FORMULA FOR $\Omega(P,Q)$
4. FAREYTAIL EXPANSION
5. POLAR STATES AS BOUNDSTATES
6. CHARGE LIMITS
7. EXTREME POLAR STATES
8. FORMAL DERIVATION OF OSV
9. CORRECTIONS & CUTOFFS

# 1. Introduction

A. BPS states are important for

* Black hole entropy from microstates
* Quantum corrections to $S_{eff}$
* Duality
* Math. structures:

  - BPS algebras
  - automorphic forms
  - enumerative geometry of derived cat's
  - knot invariants

B. Framework: Type IIA / $M_4 \times X_{CY}$

$S_{eff}$ = 4D $\mathcal{N}=2$ SUGRA $\Rightarrow$

$(h^{1,1}+1)$ $U(1)$'s ; $t \in \mathcal{M}_{VM}$ SCALARS

WRAP D-BRANES $\Rightarrow$ CHARGE: $\Gamma \in K^0(X)$

IGNORE QUANTIZATION: $K^0(X)\otimes\mathbb{R} = H^{ev}(X,\mathbb{R})$ :

| | | | |
|---|---|---|---|
| D6 | D4 | D2 | D0 |
| $p^0$ | $P$ | $q_2$ | $q_0$ |
| $H_6$ | $H_4$ | $H_2$ | $H_0$ |
| $H^0$ | $H^2$ | $H^4$ | $H^6$ |

Often identify $H^6(X,\mathbb{Z}) \cong \mathbb{Z}$

E/M Splitting: $\Gamma = (P, Q)$

$P \in H^0 \oplus H^2$ = Magnetic

$Q \in H^4 \oplus H^6$ = Electric

FOR $t \in \mathcal{M}_{VM}$ & CHARGE $= \Gamma$

THERE IS A FINITE DIM'L SPACE OF BPS STATES

$$\mathcal{H}_{BPS}(\Gamma; t) = \mathcal{H}_{HM} \otimes \mathcal{H}'_{BPS}$$

$$\Omega(\Gamma; t) := -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}_{BPS}(\Gamma, t)} (2J_3)^2 (-1)^{2J_3}$$

$$= \mathrm{Tr}_{\mathcal{H}'_{BPS}} (-1)^{F'}$$

THE GOAL IS TO UNDERSTAND THE BEHAVIOR OF $\Omega(\Gamma; t)$:

a.) ASYMP'S FOR "LARGE" $\Gamma$

b.) RELATION TO BLACK HOLE ENTROPY

c.) DEPENDENCE ON $t \in \mathcal{M}_{VM}$

# C. SUMMARY OF MAIN RESULTS

1. $\Omega(\Gamma, t=i\infty)$ FOR CHARGES SUPPORTING BLACK HOLES ARE DETERMINED ENTIRELY BY A FINITE SET OF DEGENERACIES $\Omega(\Gamma_{POLAR})$ OF "SPLIT POLAR STATES"

2. WALL-CROSSING FORMULA FOR $\Omega(\Gamma; t) \Rightarrow$ FORMULA FOR $\Omega(\Gamma_{POLAR})$

3. A FORMAL DERIVATION OF THE OSV CONJECTURE

4. APPARENT COUNTEREXAMPLE TO THE OSV CONJECTURE

5. A FRAMEWORK FOR DERIVING A REFINED CONJECTURE.

# D. FORMAL DERIVATION OF OSV CONJECTURE

ASSUME $p^0 = 0$ ; $P$ = AMPLE.

## STEP 1: MODULAR D4 PARTITION FUNCTION

ON $\mathbb{R}^3 \times S^1 \times X_{CY}$ WITH FLAT RR FIELDS

$$C_0 := \oint_{S^1} C_1 \qquad C_2 := \oint_{S^1} C_3$$

$$Z_{D4} := \sum_{Q} \Omega(P,Q)\, e^{-\beta H_{BPS}(\Gamma;t) - i C_2 \cdot q_2 - i C_0 q_0}$$

IS SUITABLY MODULAR IN

$$\tau := C_0 + i \frac{\beta}{g_s^{IIA}}$$

DUE TO U-DUALITY

## STEP 2: SINGLETON DECOMPOSITION

$$Z_{D4}(\tau, \bar{\tau}, C_2) = \sum_{\gamma} \underbrace{\Theta_{\gamma}(\tau, \bar{\tau}; C_2)}_{\text{Theta Function}} H_{\gamma}(\tau)$$

$\Rightarrow$ $\exists$ A VECTOR OF MODULAR FORMS $H_\gamma(\tau)$ AND A MAP $Q \longrightarrow \gamma_Q$

$$\boxed{\Omega(P, Q; t = i\infty) = \oint H_{\gamma_Q}(\tau) e^{2\pi i \tau \hat{q}_0}}$$

$$\hat{q}_0 := q_0 - \frac{1}{2}(q_2)^2$$

$$= q_0 - \frac{1}{2}(D_{ABC}P^C)^{-1}(q_2)_A(q_2)_B$$

STEP 3: *FAREYTAIL*

$H_\gamma(\tau)$ IS DETERMINED ENTIRELY BY ITS POLAR PART:

$$H_\gamma(\tau) \sim \sum \Omega(\Gamma_{\gamma,\hat{q}_0})\, e^{-2\pi i \hat{q}_0 \tau}$$

$$\underbrace{H_\gamma = \underbrace{H_\gamma^-}_{\hat{q}_0 > 0:\ \text{POLAR STATES}}}_{\text{FINITE!}} + \underbrace{H_\gamma^+}_{\hat{q}_0 < 0:\ \text{BLACK HOLES}}$$

$$H_\gamma(\tau) \sim \sum_{SL(2,\mathbb{Z})} \mathcal{M}_{\gamma\gamma'}\, H^-_{\gamma'}\left(\frac{a\tau+b}{c\tau+d}\right)$$

$\Longrightarrow$

$$\mathcal{Z}_{D4}(\tau, C_2) \sim \sum_{SL(2,\mathbb{Z})} \mathcal{Z}^-\left(\frac{a\tau+b}{c\tau+d}, \frac{C_2}{c\tau+d}\right)$$

## STEP 4: WALL-CROSSING

POLAR STATES ARE ALWAYS BOUNDSTATES : $\Gamma = \Gamma_1 + \Gamma_2$

$$\Omega(\Gamma; t) \sim \sum_{\Gamma = \Gamma_1 + \Gamma_2} \langle \Gamma_1, \Gamma_2 \rangle \Omega(\Gamma_1) \Omega(\Gamma_2)$$

## STEP 5: EXTREME POLAR STATES

LARGE VALUES OF $\hat{q}_0 \Rightarrow$

BOUNDSTATE = $\begin{array}{c} (D6\ D4\ D2\ D0) \\ + \\ \overline{(D6\ D4\ D2\ D0)} \end{array}$

$$\mathcal{Z}_{\overline{D4}} \sim \mathcal{Z}_{\overline{D6}} \, \overline{\mathcal{Z}_{\overline{D6}}}$$

$\mathcal{Z}$ VERY TRICKY!

STEP 6: IDENTIFY

$$\mathcal{Z}_{D6} \sim Z_{DT} \sim Z_{GW} := Z_{TOP}$$

TO GET OSV-LIKE STATEMENT

$$\sum \Omega(P,Q) e^{q_2 \phi + q_0 \phi^0} \sim \left| Z_{TOP}(g_s, t) \right|^2$$

$$g_s \sim \frac{i}{\phi^0} \qquad t^A \sim \frac{\phi^A}{\phi^0} + i\,\frac{P^A}{\phi^0}$$

LOTS OF SUBLTETIES!!!

# 2. DESCRIBING THE BPS STATES

## A: MULTICENTERED SOLUTIONS

D-BRANES ARE OBJECTS IN A CATEGORY

FOR SUSY IIA/CY — PROBABLY BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES

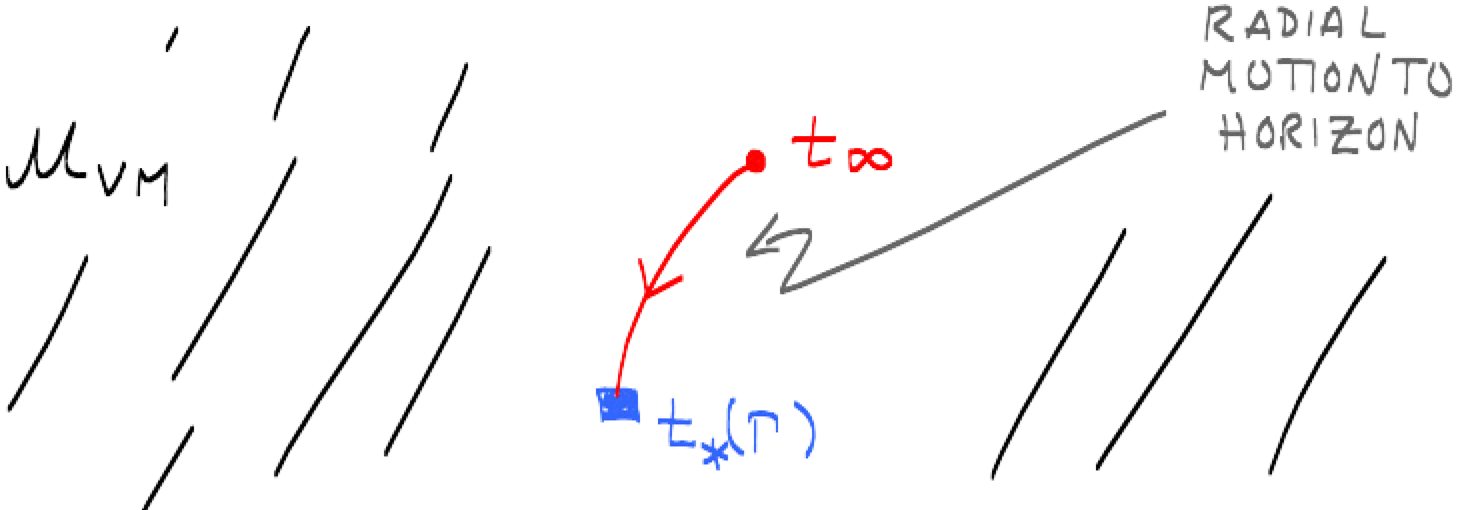
WEAK COUPLING, $J \to \infty$ : $\exists$ BEAUTIFUL SUGRA DESCRIPTION: OUR MAIN TOOL.

ATTRACTOR MECHANISM:

$\Gamma$, $t_\infty$ & SPHERICAL SYMMETRY

$\Longrightarrow$ UNIQUE BPS SOLUTION

SCALAR FIELDS $t = t(r)$



ATTRACTOR FLOW = GRADIENT FLOW FOR

$$\log |Z(\Gamma;t)|^2$$

$$Z(\Gamma;t) := \langle \Gamma, \omega \rangle$$

$\langle -, \cdot \rangle :=$ SYMPLECTIC PRODUCT ON $K^0(X)$:

$$\int \mathrm{ch} E \, \mathrm{ch} \overline{E'} \, \hat{A} = \int -p^0 q_0' + p q' - q p' + q_0 p^{0\prime}$$

$$\omega \approx e^t / \sqrt{(\mathrm{Im}\, t)^3} \qquad t = B + iJ$$

$$Z \approx \left(\tfrac{1}{6} p^0 t^3 - \tfrac{1}{2} p t^2 + q t - q_0\right) \cdot \left(\mathrm{Im}\, t^3\right)^{-1/2}$$

HORIZON AREA $= 4\, S(\Gamma) = 4\pi \, |Z(\Gamma, t_*(\Gamma))|^2$

$$S(\Gamma) = \sqrt{\mathcal{D}(\Gamma)}, \quad \underbrace{\mathcal{D}: H^{ev}(X, \mathbb{R}) \to \mathbb{R}}_{\text{DISCRIMINANT}}$$

CONJECTURE: $\log |\Omega(\Gamma; t_*(\Gamma))| \sim S(\Gamma)$

FOR "LARGE" $\Gamma$

FOR $p^0 = 0$ & $\underline{P}$ AMPLE

$$S(\Gamma) = 2\pi \sqrt{\frac{-\hat{q}_0 \chi(P)}{6}}$$

$$\chi(P) := P^3 + c_2 \cdot P = \text{EULER CHAR. OF SURFACE } \Sigma \in |P|$$

$$\hat{q}_0 := q_0 - \frac{1}{2}\left(D_{ABC}P^C\right)^{-1} q_A q_B$$

$\chi(P) > 0$ FOR $P$ AMPLE $\Rightarrow$

$\mathcal{D}(\Gamma) < 0$ FOR $\hat{q}_0 > 0$ !!

ATTRACTOR FLOW IS SINGULAR !

$t_*(\Gamma)$ DOES NOT EXIST

( $\mathcal{D}(\Gamma) < 0 \iff Z(\Gamma; t)$ HAS A ZERO.

FLOW IS SINGULAR NEAR A ZERO. )

SO, NO BPS STATE ..... ?

DENEF: NOT NECESSARILY! RELAX SPHERICAL SYMMETRY $\Rightarrow$

## BPS EQUATIONS

(1.) $ds^2 = -e^{2U}(dt+\Theta)^2 + e^{-2U} d\vec{x}^2$

$U = U(\vec{x})$

(2.) $2e^{U}\,\mathrm{Im}\left(e^{-i\alpha}\omega\right) = -H$

$=$ HARMONIC FUNCTION : $\mathbb{R}^3 \longrightarrow H^{ev}(X,\mathbb{R})$

$\alpha(\vec{x}) = \arg\left(Z(\Gamma; t(\vec{x}))\right)$

(3.) $*_3 d\Theta = \langle dH, H\rangle$

(4.) $\vec{E} + i\vec{B} = \ldots\,.$

$$H(\vec{x}) = \sum_j \frac{\Gamma_j}{|\vec{x}-\vec{x}_j|} - 2\,\mathrm{Im}\left(e^{-i\alpha}\omega\right)_\infty$$

$$\Rightarrow \quad \sum_{\substack{j \\ j\neq i}} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i-\vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_\infty$$

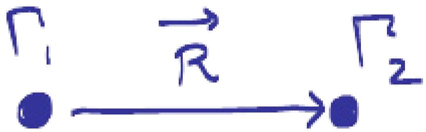
SUGRA VALID $\Longleftrightarrow$

$$t(\vec{x}) \in \mathcal{M}_{VM} \qquad \forall\, \vec{x} \in \mathbb{R}^3$$

$$\mathcal{D}(H(\vec{x})) \geq 0$$

NOW WE LOOK AT AN EXAMPLE OF SOME IMPORTANCE TO THE OSV CONJECTURE

## EXAMPLE 1: TWO CENTERS




$$R = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\operatorname{Im}(Z_1 \bar{Z}_2)_\infty}$$

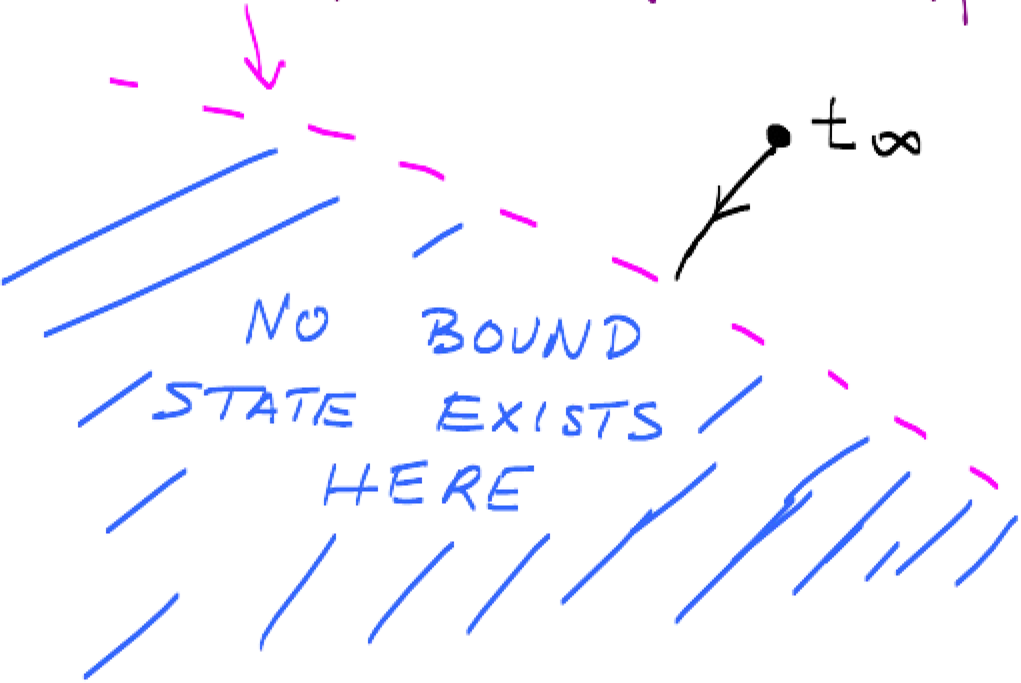
$$\vec{J} = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \hat{R} \quad \text{(USEFUL LATER)}$$

- $\Gamma_1, \Gamma_2$ CAN THEMSELVES BE COMPLICATED BOUND STATES

- NOTE THAT BY CHANGING $t_\infty$ WE CAN MAKE $\operatorname{Im}(Z_1 \bar{Z}_2)\big|_{t_\infty} \to 0$ WHILE $|Z_1 + Z_2|_{t_\infty} \neq 0$

ILLUSTRATES THE KEY POINT OF MARGINAL STABILITY:

$$MS(\Gamma_1,\Gamma_2) := \left\{ t \in \mathcal{M}_{VM} \;\middle|\; \frac{Z_1}{Z_2} \in \mathbb{R}_+ \right\}$$

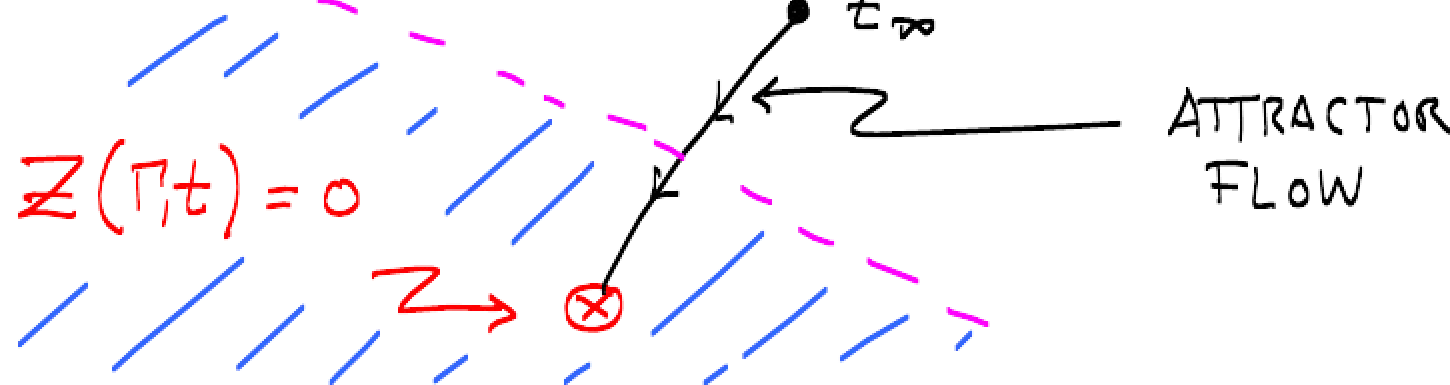



CHANGE BC's @ $r = \infty \Rightarrow$

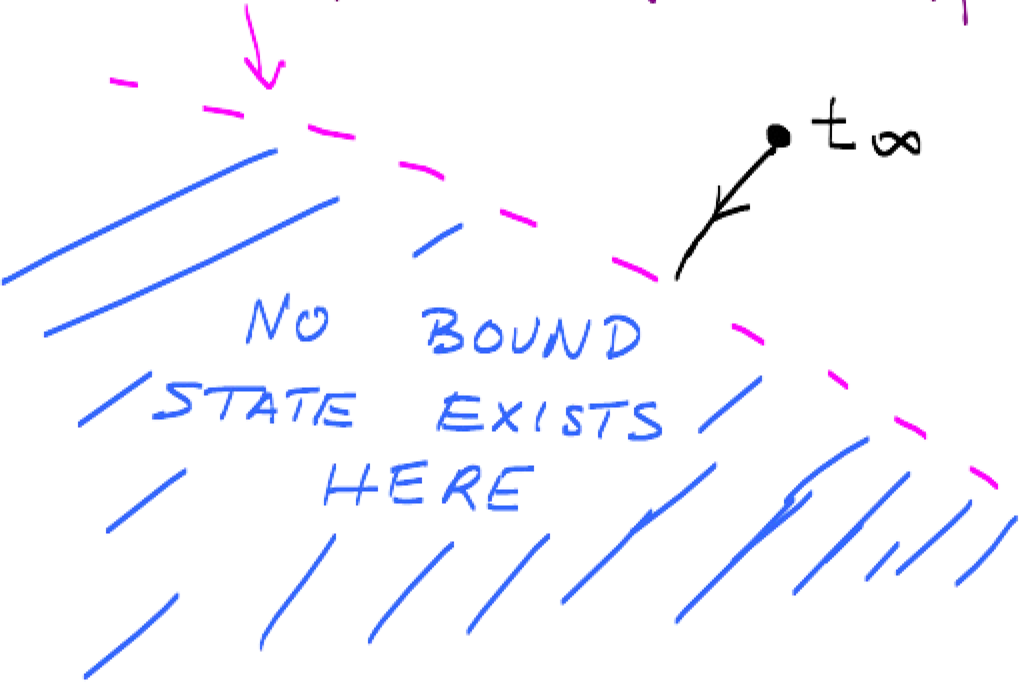
$R_{12} \longrightarrow \infty$

RECALL:

$$\mathcal{D}(\Gamma) < 0 \iff Z(\Gamma;t) \text{ HAS A ZERO IN } \mathcal{M}_{VM}$$



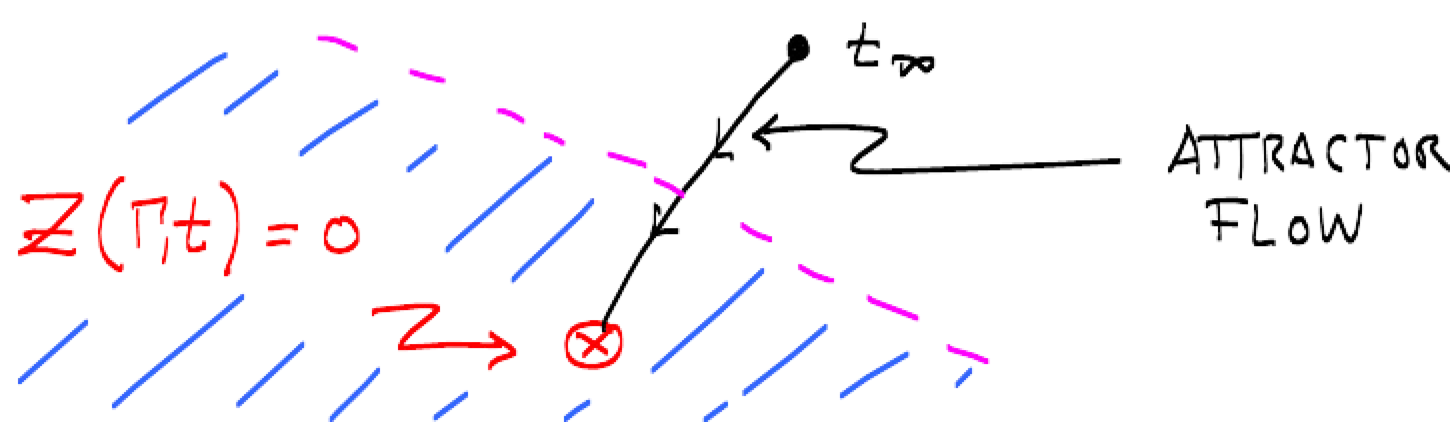
$$MS(\Gamma_1,\Gamma_2) := \left\{ t\in\mathcal{M}_{VM} \,\middle|\, \frac{Z_1}{Z_2}\in\mathbb{R}_+ \right\}$$




CHANGE BC's @ $r=\infty \Rightarrow$ $R_{12}\longrightarrow\infty$

RECALL:

$$\mathcal{D}(\Gamma)<0 \iff Z(\Gamma;t) \text{ HAS A ZERO IN } \mathcal{M}_{VM}$$



SO, IF $\exists\, t_0$ WITH $Z(\Gamma, t_0) = 0$,

AS FOR POLAR STATES, THEN

$$\Omega(\Gamma; t) = \sum_{\Gamma = \Gamma_1 + \Gamma_2} (-1)^{I_{12} - 1} |I_{12}| \, \Omega(\Gamma_1, t_{MS}) \, \Omega(\Gamma_2, t_{MS})$$

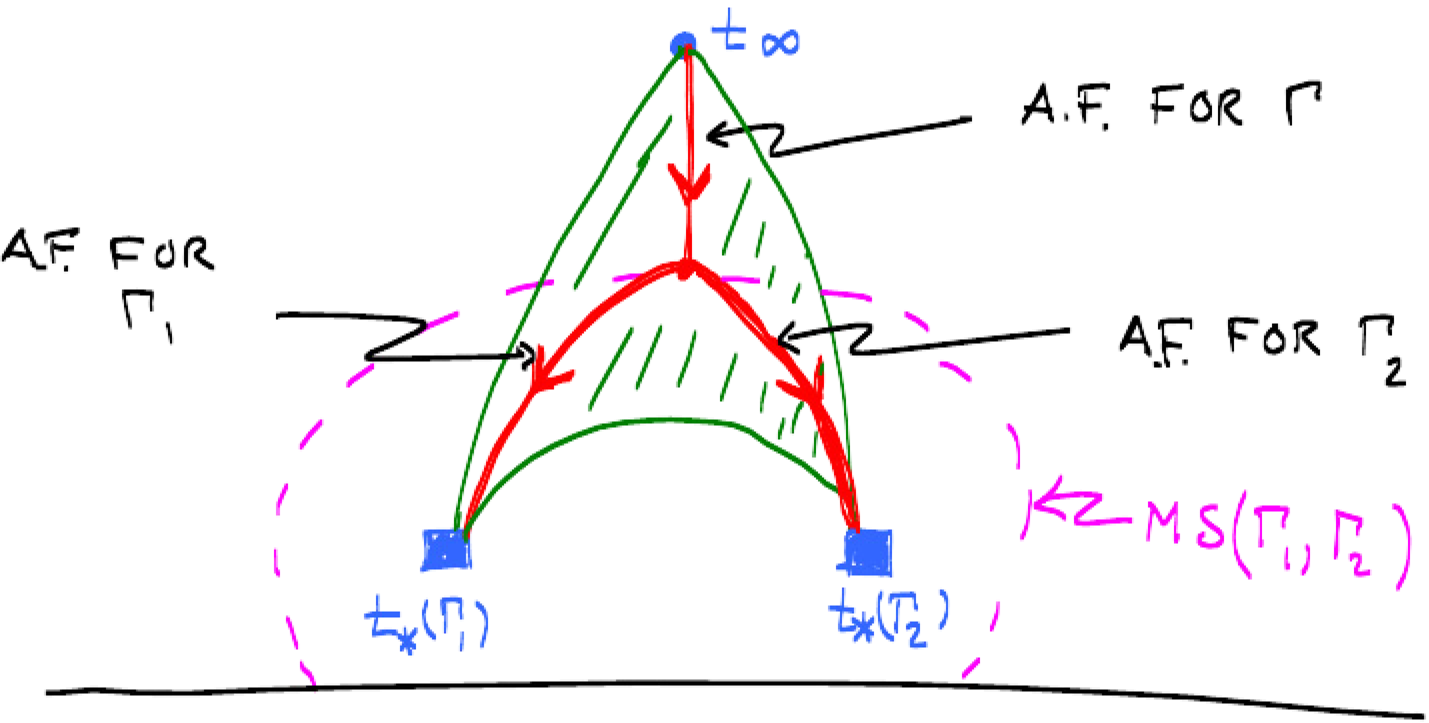
- $\exists$ UNDERSTANDING FROM QUIVER MODULI SPACES

$$\begin{array}{ccc} \mathbb{CP}^{I-1} & \to & \mathcal{M}(\Gamma) \\ & & \downarrow \\ & & \mathcal{M}(\Gamma_1) \times \mathcal{M}(\Gamma_2) \end{array}$$

- SIMILAR TO RECENT FORMULA OF JOYCE

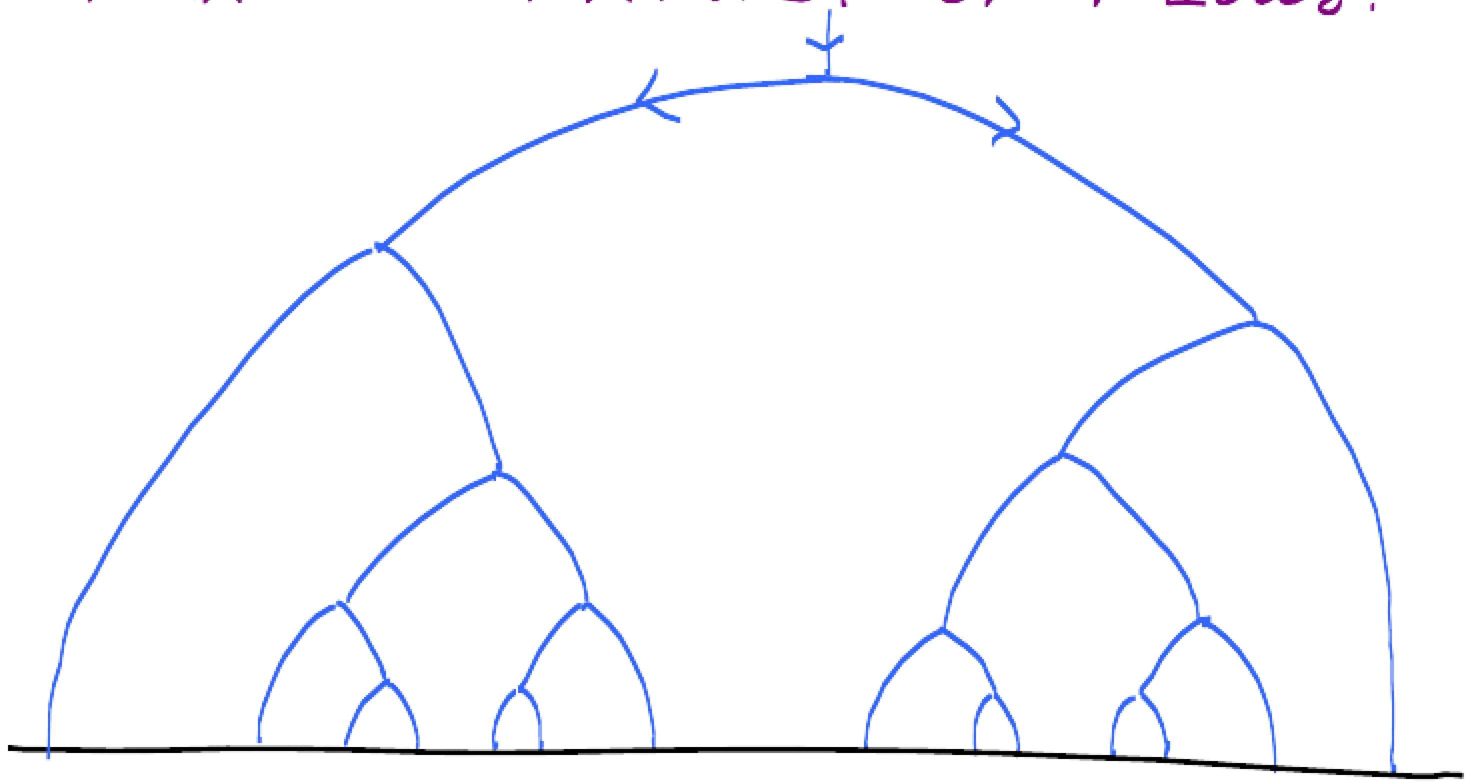
# C : SPLIT ATTRACTOR FLOWS

PLOT THE IMAGE OF $t(\vec{x}) \in \mathcal{M}_{VM}$ FOR A TWO-CENTERED SOL'N :




**DEF'N**: A "SPLIT ATTRACTOR FLOW" IS A PIECEWISE ATTRACTOR FLOW TREE CONNECTED AT WALLS OF MARGINAL STABILITY AND TERMINATNG ON REGULAR ATTRACTOR POINTS.

THESE S.A.F.'S CAN BE VERY INTRICATE. FOR EXAMPLE, WE CONSTRUCTED AN INFINITE FRACTAL FAMILY OF FLOWS:



## SPLIT ATTRACTOR CONJECTURE (DENEF)

(a.) (COMPONENTS OF MODULI OF) MULTICENTERED SOLUTIONS ARE IN $1 \leftrightarrow 1$ CORRESPONDENCE WITH S.A.F.'S.

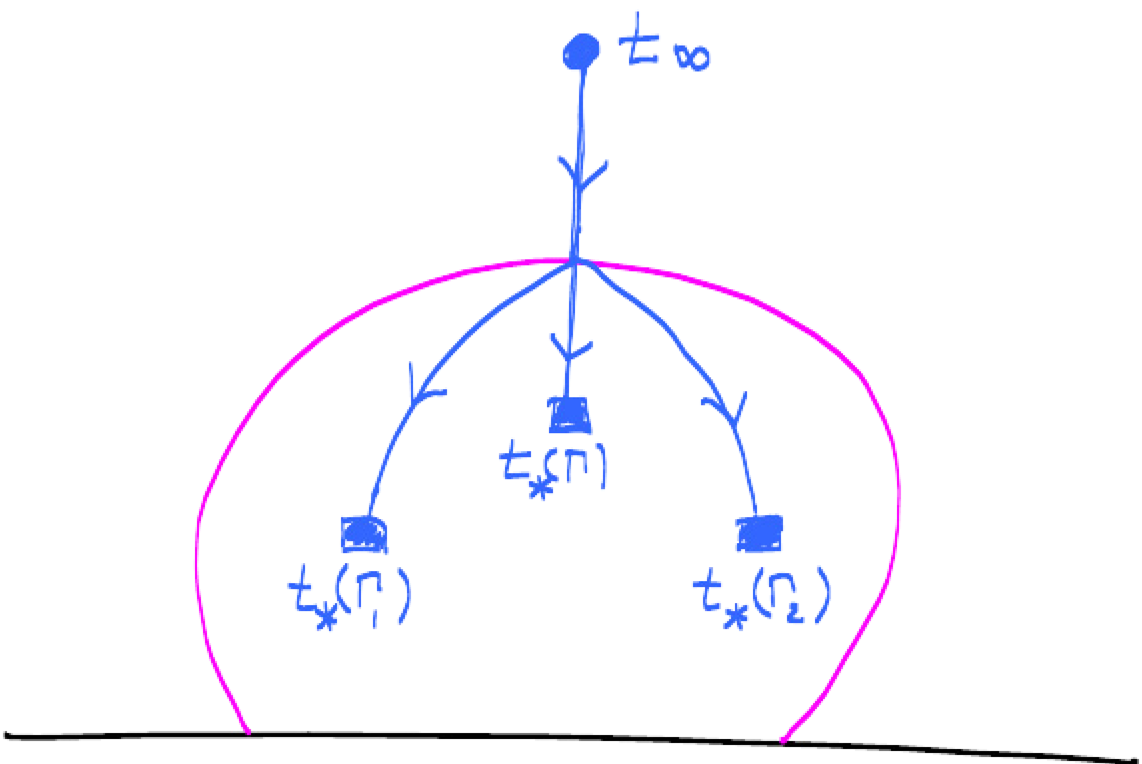
(b.) FOR A FIXED $(t_\infty, \Gamma)$ THERE ARE A <u>FINITE</u> NUMBER OF S.A.F.'S

## D: A SURPRISING EXAMPLE

$$\Gamma = (0, P, 0, q_0) = \Gamma_1 + \Gamma_2$$

$$\Gamma_1 = \left(r, \tfrac{1}{2}P, q_2, \tfrac{1}{2}q_0\right) \quad \Gamma_2 = \left(-r, \tfrac{1}{2}P, -q_2, \tfrac{1}{2}q_0\right)$$

FOR AN APPROPRIATE RANGE OF $t_\infty, q_2, q_0$ $\exists$ BOTH SINGLE-CENTERED AND TWO-CENTERED SOLUTIONS



SO... COMPARE ENTROPIES

$$S(\Gamma) \quad \text{vs.} \quad S(\Gamma_1) + S(\Gamma_2)$$

IN FACT,

$\exists$ FAMILY OF CHARGES

$$\lambda\Gamma = \lambda(0, P, 0, q_0) = \left(r, \tfrac{\lambda}{2}P, q_1(\lambda), \tfrac{\lambda}{2}q_0\right)$$

$$+ \left(-r, \tfrac{\lambda}{2}P, -q_2(\lambda), \tfrac{\lambda}{2}q_0\right)$$

$$S(\lambda\Gamma) = \lambda^2 S(\Gamma)$$

BUT

$$\boxed{S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda P)^3}{r} \sim \lambda^3}$$

$S(\Gamma) :=$ SINGLE-CENTERED ENTROPY

$$\sim \log |\Omega(\Gamma; t_*(\Gamma))|$$

# SOME TECHNICAL DETAILS

1. CONSTRUCT A FAMILY OF 2-CENTERED

$$\Gamma_1^\lambda = \left(r, \frac{p}{2}, q_2(\lambda), \lambda^{-2}\frac{q_0}{2}\right)$$

$$\Gamma_2^\lambda = \left(-r, \frac{p}{2}, -q_2(\lambda), \lambda^{-2}\frac{q_0}{2}\right)$$

$\Gamma_i^\lambda$ CAN BE 1-CENTERED BH'S OR CAN THEMSELVES BE POLAR

2. ATTRACTOR FORMALISM HAS A SCALING SYMMETRY UNDER

$$T_\lambda(p^0, \underline{P}, q_2, q_0) = (p^0, \lambda \underline{P}, \lambda^2 q_2, \lambda^3 q_0)$$

$$S(T_\lambda \Gamma) = \lambda^3 S(\Gamma)$$

3. APPLY TO $\quad T_\lambda \Gamma_1^\lambda + T_\lambda \Gamma_2^\lambda = \lambda \Gamma$

# CONSEQUENCES

1. 2-CENTERED SOLUTION DOMINATES THE ENTROPY!!

2. APPEARS TO CONTRADICT OSV PREDICTION FOR $\Omega(\Gamma; t = i\infty)$

3. IN FACT, WE ARGUE BELOW THAT, FOR LARGE P, THERE IS A PHASE TRANSITION IN $g_{top} \sim 1/\phi^0$ AND

$$\log |\Omega(\Lambda P, \Lambda Q; t = i\infty)| \underset{\Lambda \to \infty}{\sim} c\Lambda^3$$

QUITE GENERALLY!

# 3. APOTHEOSIS OF DONALDSON, THOMAS & McMAHON

## HALO STATES

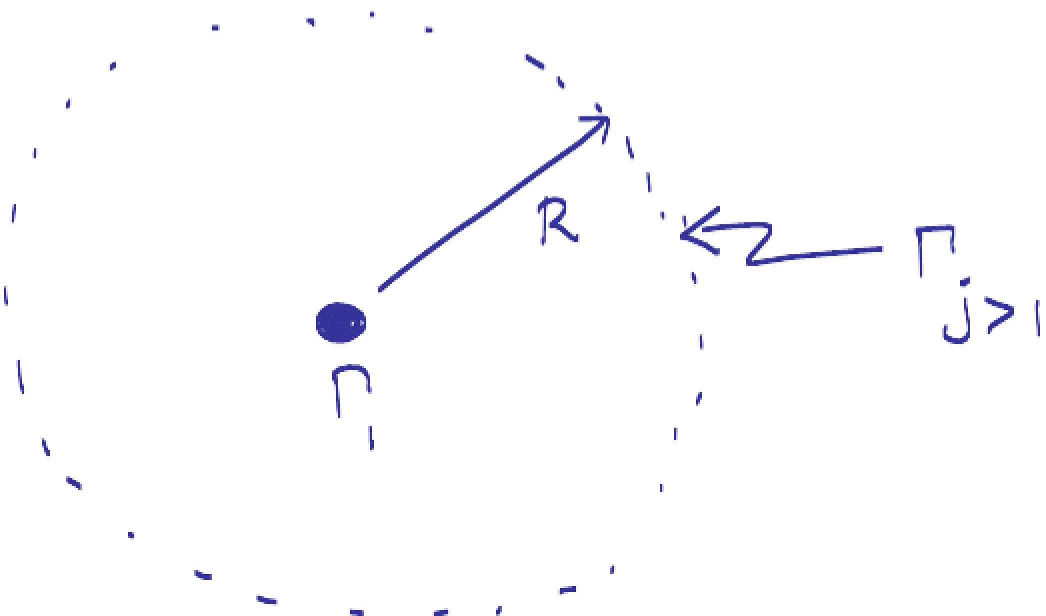
SUPPOSE

$$\Gamma_j = \lambda_j \Gamma_2 \qquad \lambda_j > 0, \; j = 2, \dots, N$$

ARE ALL MUTUALLY LOCAL

INTEGRABILITY CONDITIONS SAY

$$i > 2 \qquad \frac{\langle \Gamma_i, \Gamma_1 \rangle}{|\vec{x}_i - \vec{x}_1|} = 2 \frac{\mathrm{Im}\left( Z(\Gamma_i) \overline{Z(\Gamma)} \right)}{|Z(\Gamma)|}$$



CONSIDER THE CHARGE OF AN IDEAL SHEAF $\Gamma = (1, 0, -\beta, n)$

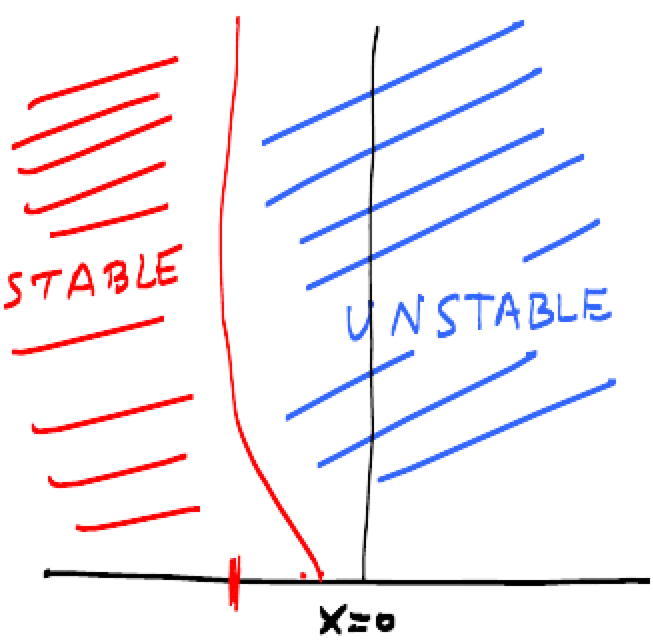
$\beta = P.D.[\sigma] \quad \sigma \subset X$ HOLOMORPHIC CURVE

DEPENDING ON B-FIELD, THESE STATES CAN BIND TO D2D0 BOUNDSTATES:

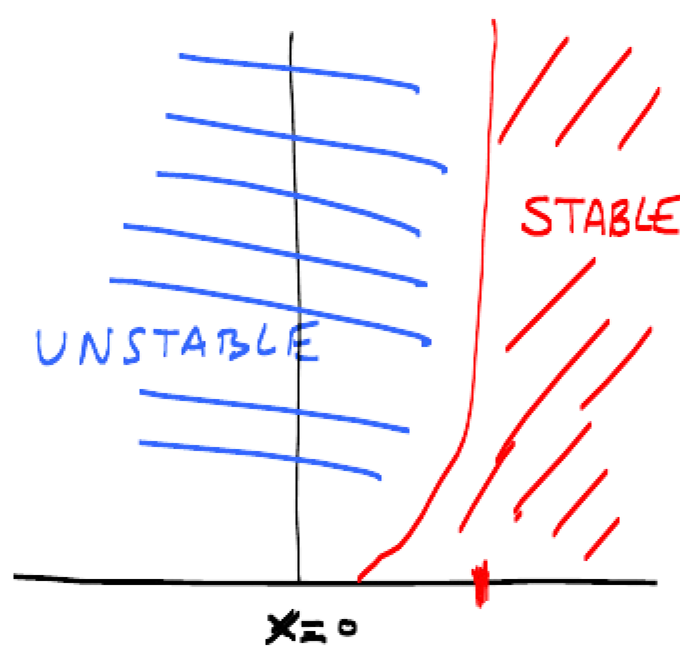
$$\Gamma_2 = (0, 0, -\beta_h, n_h)$$

THESE D2D0'S FORM HALO STATES AROUND A "CORE"

SET $t = (x+iy)P$. THE MS CURVE $\mathrm{Im}\, Z_1 Z_2^* = 0$ IS $(P\cdot\beta_h > 0)$




$x = n_h/2P\cdot\beta_h, \quad n_h < 0$



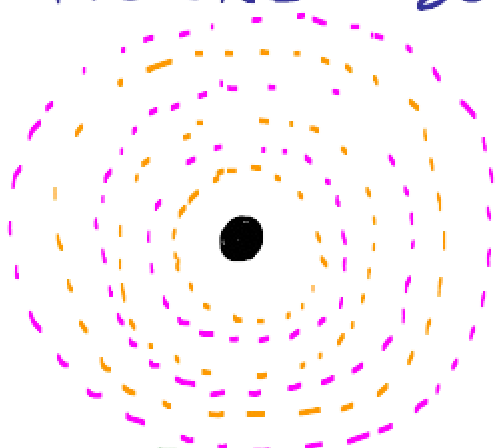

$n_h > 0, \quad x = n_h/2P\cdot\beta_h$

# CONSEQUENCES

1. $\Omega(\Gamma(\beta,n);t) \neq$ DT INVARIANTS IN GENERAL

2. $\Gamma(\beta,n)$ FORM HALO BOUNDSTATES WITH D2D0 STATES PROPORTIONAL TO $(0,0,-\beta_h,n_h)$.

GENERAL PICTURE: BOHR MODEL



INCREASE $B = xP$, $x \longrightarrow +\infty$ : FOR $x > 0$ ALL $n_h < 0$ STATES HAVEDECAYED.

AS $x \longrightarrow +\infty$ WE MOVE INTO THE STABLE REGION FOR ALL $n_h > 0$, AND EVER LARGER "ATOMS" BECOME STABLE

# GENERATING FUNCTIONS

$$Z_{D6}(u,v;t) := \sum_{n,\beta} \Omega(\Gamma(\beta,n);t)\, u^n v^\beta$$

SET $t = (x+iy)P$

CROSSING A WALL $x_{ms} = \dfrac{n_h}{2P\cdot\beta_h}$

$$Z \rightarrow (1-u^{n_h}v^{-\beta_h})^{-|n_h|\Omega(\beta_h)} Z$$

$$\Omega(\beta_h) = \sum_{j_L j_R} (-1)^{2j_L+2j_R}(2j_L+1)(2j_R+1)\, N_{\beta_h j_L, j_R}$$

$$= \chi\,[\text{MODULI OF D2 BRANES}]$$

REMARK:

$$\Omega(q_2=0) = \chi(X) \Longrightarrow \beta_h = 0$$

CONTRIBUTION @ LARGE x IS

$$(M(u))^{-\chi(X)}$$

# CONJECTURE

$$\lim_{x\to\infty}\lim_{y\to\infty} Z_{DGD2D0}(u,v;t) = Z'_{DT}(u,v)$$

$$\lim_{x\to-\infty}\lim_{y\to\infty} Z_{DGD2D0}(u,v;t) = Z'_{DT}(u^{-1},v)$$

- $\exists$ M-THEORY ARGUMENT JUSTIFYING THIS PICTURE

# CONCLUSIONS (SHORT TALK)

1. WE HAVE A FRAMEWORK FOR DERIVING AN OSV-LIKE FORMULA, BUT THERE APPEAR TO BE MANY CORRECTIONS TO THE ORIGINAL PROPOSAL

2. IN PARTICULAR, $\exists$ APPARENT COUNTEREXAMPLE

3. IT IS STILL POSSIBLE THAT

$$\Omega\left(\Gamma; t_*(\Gamma)\right) \sim \oint d\phi \, e^{q\cdot\phi} |Z_{top}|^2$$

# 3. BPS COUNTING FUNCTIONS

COUNT D4D2D0 BOUNDSTATES

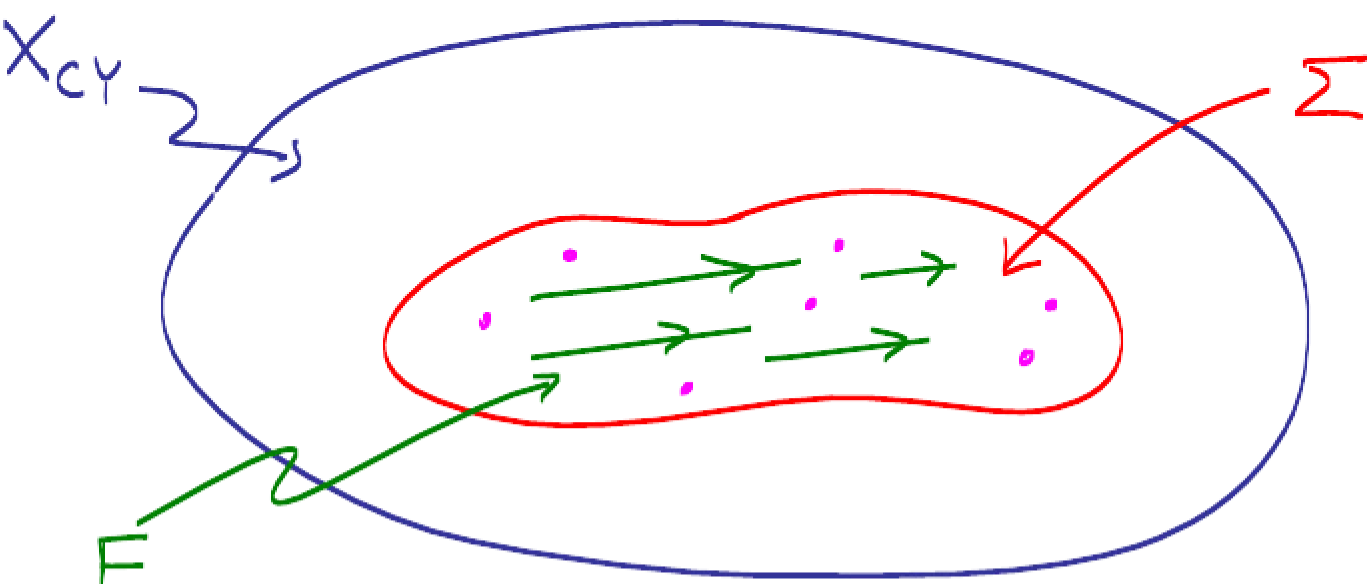
D4 WRAPS HOLO. SURFACE $\Sigma \in |P|$

MODULI SPACE OF $\Sigma$ : DIM $= I_p - 1 = \frac{p^3}{6} + \frac{c_2 \cdot p}{12} -$

$$Z_{D4} := \mathrm{Tr}_{\mathcal{H}_{D4}} (-1)^{F'} e^{-\beta H - i C_2 \cdot q_2 - i C_0 q_0}$$

MICROSCOPIC DESCRIPTION OF BPS STATES:

- D4 WRAPS $\Sigma$ ONCE
- $U(1)$ FLUX $F \in H^2(\Sigma; \mathbb{Z})$ TURNED ON
- $N$ POINTLIKE INSTANTONS



$$\text{SUSY} \Longleftrightarrow F^{2,0} = 0$$

FIXES MODULI OF $\Sigma$: "OPEN STRING VACUA"

$$\Sigma \in NL(P,F) = \{ \Sigma \mid F \in H^{1,1} \}$$

NOTE

- $b_2(P) = \chi(P) + 2 = P^3 + c_2 \cdot P + 2 \gg b_2(X)$
- "GENERIC" $F \Rightarrow NL(P,F) =$ FINITE SET OF POINTS

$$d(F,N) = \chi \left[ \begin{tikzcd} \mathrm{Sym}^N \Sigma \arrow[r, hook] \arrow[d] & \mathcal{M}(P,F,N) \arrow[d] \\ \underset{\text{smooth}}{\Sigma} \arrow[r, hook] & NL(P,F) \end{tikzcd} \right]$$

$d(F,N) =$ DONALDSON-THOMAS INVARIANT

$$q_0 = q_0(F,N) = \frac{\chi(P)}{24} + \frac{1}{2}F^2 - N$$

$$(q_2)_A = \int_P J_A \cdot F \qquad A = 1, \ldots, h^{1,1}(X)$$

$$Z_{D4} = \sum_{\substack{F \in H^2(\Sigma, \mathbb{Z}) \\ N \geq 0}} d(F,N) e^{-\frac{\beta}{g_s}|Z| - i C_2 q_2 - i C_0 q_0}$$

U-DUALITY $\Rightarrow$ MODULAR

LARGE RADIUS:

2

$$\mathfrak{Z}_{D4}(\tau, \bar\tau, C_2) := \sum_{F,N} d(F,N) e^{2\pi i \tau \left(N - \frac{1}{2}F_-^2 - \frac{\chi(p)}{24}\right)} \cdot e^{-2\pi i \bar\tau F_+^2 - 2\pi i F \cdot C}$$

IS A JACOBI FORM

NOTE: $F^{2,0} = 0 \Rightarrow F \in H^{1,1}(\Sigma)$ $SIG = (+1, (-1)^{h-1})$

$J$ $\quad$ $F_-$

2 RELATION TO OSV'S $Z_{BH}$

PUT $\tau = \bar\tau = i\phi^0 \qquad \phi^0 > 0$

$C = i\Phi$

$$\mathfrak{Z}_{D4}\Big|_{\substack{OSV \\ SUBST.}} = \sum_{F,N} d(F,N) e^{2\pi\phi^0 q_0 + 2\pi \Phi \cdot q_2}$$

$$= Z_{BH}^{OSV}$$

# SINGLETON DECOMPOSITION

SEPERATE OUT EXACT $\bar{\tau}$, $C_2$ DEPENDENCE

$$Z_{D4}(\tau, \bar{\tau}, C_2) = \sum_{\gamma} \Theta_{\gamma}(\tau, \bar{\tau}, C_2)\, H_{\gamma}(\tau)$$

$\Theta_{\gamma}$ = THETA FUNCTION ~ P.F. OF FREE U(1) GAUGE FIELD

$\gamma \in L_X^* / L_X$ "'t Hooft Flux"

$$L_X = i^* H^2(X, \mathbb{Z}) \subset H^2(P, \mathbb{Z})$$

PROOF: "SPECTRAL FLOW"

$$d(F+S, N) = d(F, N) \qquad S \in L_X$$

NONTRIVIAL INFO IS IN:

$$H_\gamma(\tau) = \sum_{\substack{F \in L_x^\perp + \gamma^\perp \\ N \geq 0}} d(F,N)\, e^{-2\pi i \tau \hat{q}_0}$$

$$\hat{q}_0 = \hat{q}_0(F,N) = \frac{\chi(P)}{24} + \frac{1}{2}(F^\perp)^2 - N$$

RECALL: $F^{2,0} = 0 \Rightarrow (F^\perp)^2 \leq 0$

FINITE # OF POLAR TERMS WITH NEGATIVE POWERS OF $q = e^{2\pi i \tau}$

# FORMULA FOR $\Omega(P,Q;t=i\infty)$

GAUGE INVARIANT $\mathcal{F} = F - B$

- $\Omega(e^S\Gamma, B+iJ) = \Omega(\Gamma, B-S+iJ)$
  $S \in H^2(X,\mathbb{Z})$
- $\Omega(\Gamma; B+i\infty) = \Omega(\Gamma; i\infty)$
  (FOR $p^0=0$ !! )

$$H_\gamma(\tau) = \sum \Omega(\Gamma_{\gamma,\hat{q}_0})\, e^{-2\pi i\tau \hat{q}_0}$$

$\Gamma_{\gamma,\hat{q}_0}$ : IN FUND. DOMAIN FOR $\Gamma \to e^S\Gamma$,

$$q_2 \in \imath_*(L_X+\gamma)$$

$$\boxed{\Omega(P,Q;i\infty) = \oint d\tau\, H_\gamma(\tau)\, e^{2\pi i\tau \hat{Q}_0}}$$

$$Q_2 \in \imath_*(L_X+\gamma)$$

# 4. FAREYFACTS

SUPPOSE $\quad f(A\cdot\tau) = j(A,\tau)\, f(\tau)$

$\forall\, A\,, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$j(A,\tau) = \omega_A\,(c\tau+d)^{w} \qquad w \leq 0$$

DECOMPOSE INTO POLAR/NONPOLAR

$$f(\tau) = \sum_{n\geq 0} c_n\, e^{2\pi i(n-\Delta)\tau}$$

$$= \underbrace{f^{-}(\tau)}_{n-\Delta<0} + f^{+}(\tau)$$

THEN:

$$f(\tau) = \sum_{A}{}' \, j(A,\tau)^{-1} f^{-}(A\cdot\tau)$$

WHEN SUITABLY REGULARIZED

APPLY TO $H_\gamma(\tau)$ OF $WT = -1 - \frac{h}{2}$

$$H_\gamma(\tau) = H_\gamma^-(\tau) + \mathcal{M}_{\gamma\gamma'}(S)\, H_{\gamma'}^-(-1/\tau) +$$

$$+ \sum{}' \mathcal{M}_{\gamma\gamma'}(A)\, H_{\gamma'}^-(A\cdot\tau)$$

THUS: ALL $\Omega(P,Q;i\infty)$ ARE DETERMINED FROM THE FINITE SET OF $\Omega(\Gamma_{\gamma,\hat{q}_0})$ FOR $\hat{q}_0 > 0$

$$H_\gamma^-(\tau) = \underbrace{I_p\, e^{-2\pi i\tau \frac{\chi(P)}{24}} + \cdots + \Omega(\Gamma_{\gamma,\hat{q}_0})\, e^{-2\pi i\tau \hat{q}_0}}_{\hat{q}_0 > 0}$$

FOR $\hat{Q}_0 < 0$

$$\Omega(P,Q;i\infty) = \oint d\tau\, e^{2\pi i\tau \hat{Q}_0}\left[H_\gamma^-\left(-\frac{1}{\tau}\right) + \cdots\right]$$

# 5. POLAR STATES AS BOUND STATES

RECALL $S = 2\pi \sqrt{\frac{-\hat{q}_0 \chi(P)}{24}}$

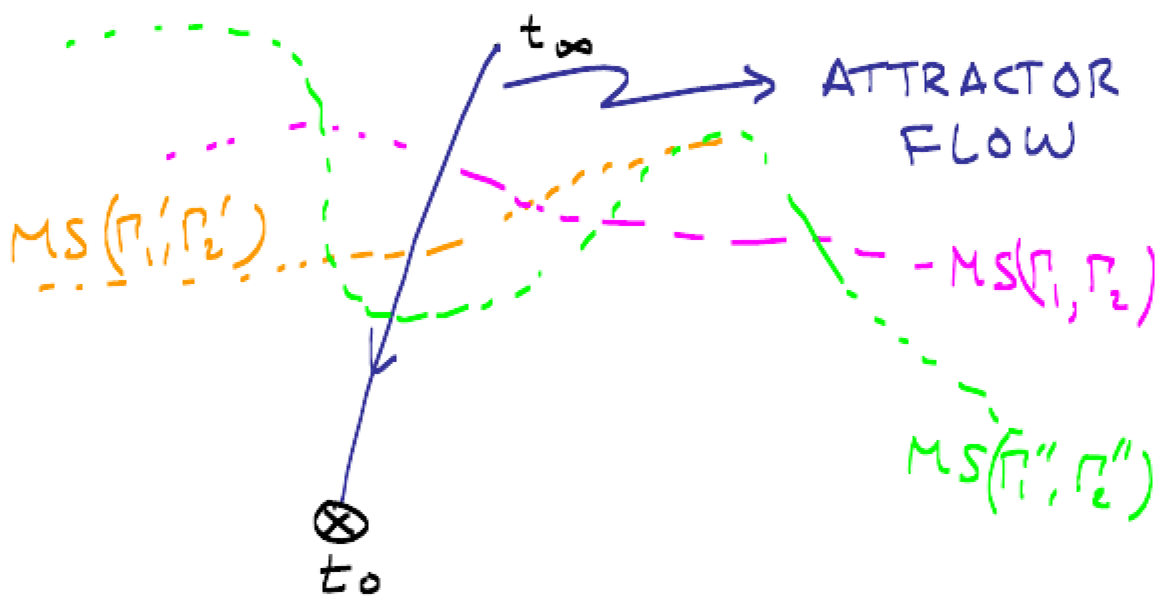
$\hat{q}_0 > 0, \ \chi(P) > 0 \Rightarrow \mathcal{D} < 0$

$\Rightarrow$ POLAR STATES ARE SPLIT STATES

WALL CROSSING & $\mathcal{H}(\Gamma; t_0) = 0$

FOR $Z(\Gamma, t_0) = 0 \Rightarrow$

$$\Omega(\Gamma; i\infty) = \sum_{\Gamma = \Gamma_1 + \Gamma_2} (-1)^{I_{12}-1} |I_{12}| \Omega(\Gamma_1; t_{MS}) \Omega(\Gamma_2; t_{MS})$$



SINCE $|\Omega(\Gamma; t_*(\Gamma))| \sim e^{S(\Gamma)}$

THIS REDUCES THE COMPUTATION OF BLACK HOLE ENTROPY TO THE ENUMERATION OF SPLIT ATTRACTOR FLOWS...

# 6. CHARGE LIMITS

$$\oint d\tau \, e^{2\pi i \tau Q_0} \, e^{-2\pi i \hat{q}_0 A \tau} \sim e^{\frac{4\pi}{c}\sqrt{\hat{q}_0 |\hat{Q}_0|}}$$

$$\tau_{s.p.} = i \frac{1}{c}\sqrt{\frac{\hat{q}_0}{|\hat{Q}_0|}} - \frac{d}{c}$$

LARGE $\hat{q}_0 |\hat{Q}_0| \Rightarrow$ DOMINANT TERMS FROM $c = 1$

$$H^-_\gamma(-1/\tau)\Big|_{OSV} = I_p \, e^{\frac{2\pi}{\phi^0}\frac{\chi(P)}{24}} + \sum_{0 < \hat{q}_0 < \frac{\chi(P)}{24}} \Omega(\Gamma_\gamma, \hat{q}_0) \, e^{\frac{2\pi \hat{q}_0}{\phi^0}}$$

FOR $|\hat{Q}_0| \gg \chi(P)$ : STRONG COUPLING

$\phi^0 \to 0$ ; $g_{top} \sim 1/\phi^0 \to \infty$

THIS IS THE MSW LIMIT

OSV CONFIRMED — BUT NO GW —

FOR $\Gamma^{\Lambda} = (\Lambda P, \Lambda Q)$ $\quad \Lambda \longrightarrow \infty$

$$\phi^0 \sim \Lambda \longrightarrow +\infty \quad \text{WEAK COUPLING}$$

BUT THEN ALL POLAR TERMS CONTRIBUTE !

IN PARTICULAR

$$\Omega(\Gamma^{\Lambda}_{\gamma,\hat{q}_0}) \sim e^{c(\gamma,\hat{q}_0)\Lambda^3}$$

FOR SUFFICIENTLY SMALL $\hat{q}_0$ !!

$$\Rightarrow \quad \log|\Omega(\Gamma^{\Lambda};i\infty)| \underset{\Lambda \to \infty}{\sim} c(\Gamma)\Lambda^3 \quad !!$$

THERE IS A PHASE TRANSITION IN $g_{TOP}$

$\Rightarrow$ TO SEE THE GW INVARIANTS

$$P \to \Lambda P \qquad \Lambda \longrightarrow \infty$$

$$\phi^0 < \phi^0_{cr} = \frac{\pi}{12} \frac{P^3}{\max c_{\gamma,\hat{q}_0}}$$

# 7. EXTREME POLAR STATES

$$H_\gamma^-\left(-\frac{1}{\tau}\right)\Big|_{OSV} = \sum_{0<\hat{q}_0\leq\frac{\chi(P)}{24}} \Omega(\Gamma_{\gamma,\hat{q}_0})\, e^{\frac{2\pi}{\phi^0}\hat{q}_0}$$

FOR $\Gamma^\Lambda = (\Lambda P, Q_\Lambda)$ $\quad \chi(\Lambda P)\sim\Lambda^3 P^3$

FOR $\phi^0 < \phi^0_{CR}$ DOMINANT TERMS

REQUIRE $\hat{q}_0 = \frac{\Lambda^3 P^3}{24} + o(\Lambda^3)$

NOTE

$$\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^\perp)^2 - N$$

FORM A "DISCRETUUM" FOR LARGE P

WE NEED A DESCRIPTION OF

THESE EXTREME POLAR STATES

# EXTREME POLAR STATE CONJECTURE

IF $\Gamma^\Lambda = (0, \Lambda P, q_2^\Lambda, q_0^\Lambda)$ IS A FAMILY OF CHARGES WITH

$$\hat{q}_0(\Gamma^\Lambda) = \frac{\Lambda^3 P^3}{24} + o(\Lambda^3)$$

THEN THE ONLY S.A.F.'S HAVE

$$\Gamma^\Lambda = \Gamma_1 + \Gamma_2$$

OF THE FORM

$$\Gamma_1 = e^{S_1}(1 - \beta_1 + n_1 dV) := e^{S_1}\Gamma(\beta_1, n_1)$$

$$\Gamma_2 = -e^{S_2}(1 - \beta_2 + n_2 dV) := -e^{S_2}\Gamma(\beta_2, n_2)$$

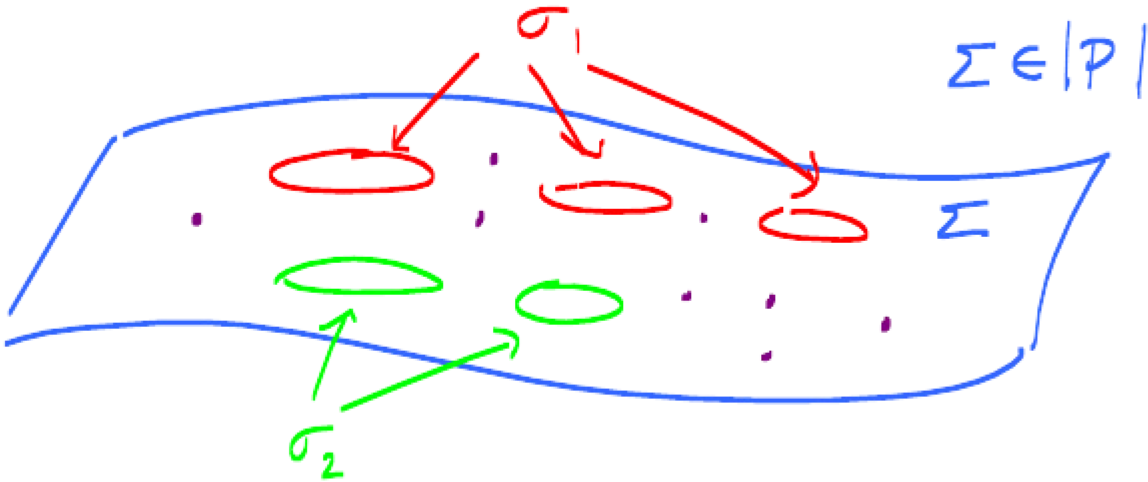
a.) $S_1 - S_2 = \Lambda P$

b.) $\beta_i = [\sigma_i]$ FOR HOLO. CURVES

$\sigma_i \subset X$, $0 \leq (\beta_i)_A = o(\Lambda^2)$

c.) $|n_i| = o(\Lambda^3)$

# HEURISTIC PICTURE




$$F = \iota^* S + [\sigma_1]_\Sigma - [\sigma_2]_\Sigma$$

D4 SPLITS INTO D6 $\overline{\text{D6}}$

COMPARE $g_0, g_2 \Longrightarrow$

$$\Gamma(\beta_1, n_1) = \mathrm{ch}\, I_1^*$$

$$\Gamma(\beta_2, n_2) = -\mathrm{ch}\, I_2$$

$I_1, I_2$ = IDEAL SHEAVES

CONJECTURE SUPPORTED BY SEVERAL NONTRIVIAL EXAMPLES

# 8. FORMAL DERIVATION OF OSV

1. COMBINE FAREYTAIL EXPANSION WITH SINGLETON DECOMPOSITON

$$Z(\tau,\bar\tau,C) = {\sum_A}' j(A,\tau)^{-1} Z^{-}(A\cdot(\tau,\bar\tau,C))$$

$Z^{-}$ = SUM OF TERMS WITH

$$\hat q_0 = q_0 - \frac{1}{2}(q_2)^2 > 0$$

2. KEEP $C=1$ TERMS; MAKE OSV SUBSTITUTION :

$$\sum_\alpha \left\{ \phi^0 e^{-\frac{\pi}{\phi^0}\Phi^2} \sum_{\hat q_0>0} \Omega(P,Q)\, e^{\frac{2\pi}{\phi^0}q_0 + \frac{2\pi i}{\phi^0} q\cdot\Phi} \right\}_{\phi^0 \to \phi^0 + i\alpha}$$

3. FOR $\phi^0 < \phi^0_{cr}$, $P \to \infty$

APPLY EPS CONJECTURE

$$\sum_{\phi^0 \to \phi^0 + i\alpha} \sum_{S} \sum_{\beta_i, n_i} \left( I_P - P \cdot (\beta_1 + \beta_2) + n_1 - n_2 \right) \cdot$$

$$\cdot \Omega\left(\Gamma(\beta_1, n_1); t = e^{\frac{2\pi i}{3}} P\right) \Omega\left(\Gamma(\beta_2, n_2); t = e^{\frac{i\pi}{3}} P\right) \cdot$$

$$\cdot \; e^{\frac{2\pi}{\phi^0} q_0 + \frac{2\pi i}{\phi^0} q_2 \Phi}$$

$$P = S_1 - S_2 \qquad S = \frac{1}{2}(S_1 + S_2)$$

$$q_2 = SP - (\beta_1 - \beta_2)$$

$$q_0 = \frac{\chi(P)}{24} + \frac{1}{2} S^2 P + S(\beta_2 - \beta_1) - \frac{1}{2} P(\beta_1 + \beta_2) + n_1 - n_2$$

$\Longrightarrow$ FORMAL FACTORIZATION

4. INTRODUCE

$$Z_{D6\overline{D6}}(\phi^0; \Phi; w) = \sum_{\Gamma_1, \Gamma_2} e^{\frac{2\pi}{\phi^0} q_0 + \frac{2\pi i}{\phi^0} q_2 \cdot \Phi} \, w^{\langle \Gamma_1, \Gamma_2 \rangle},$$

$$\times \Omega\left(\Gamma(\beta_1, n_1); t = e^{\frac{2\pi i}{3}} p\right) \Omega\left(\Gamma(\beta_2 n_2); t = e^{\frac{i\pi}{3}} p\right)$$

SUM OVER

$$\Gamma = e^{S_1} \Gamma(\beta_1, n_1) - e^{S_2} \Gamma(\beta_2, n_2)$$

5. COMPUTE :

$$Z_{D6\overline{D6}} = \sum_{S \in H^2(X, \mathbb{Z})} e^{\frac{2\pi}{\phi^0} \chi(P) - \frac{\pi}{\phi^0} (\Phi - iS)^2 P} \times$$

$$\times \, \mathcal{Z}_{D6}\left(w e^{\frac{2\pi}{\phi^0}}, w^{-P} e^{-\frac{2\pi i}{\phi^0}(\Phi - iS) - \frac{\pi}{\phi^0} P}; t = e^{\frac{2\pi i}{3}} P\right)$$

$$\times \, \mathcal{Z}_{D6}\left(w^{-1} e^{-\frac{2\pi}{\phi^0}}, w^{-P} e^{\frac{2\pi i}{\phi^0}(\Phi - iS) + \frac{\pi}{\phi^0} P}; t = e^{\frac{\pi i}{3}} P\right)$$

# FORMAL OSV

- $$\mathcal{Z}(\tau,\bar\tau;C)\Big|_{OSV} = \frac{\partial}{\partial w}\Big|_{w=1} \sum_{\phi^0\to\phi^0+i\alpha} Z_{D6\overline{D6}}(\phi^0,\Phi;w)$$

- RECALL

$$\lim_{x\to\infty}\lim_{y\to\infty} \mathcal{Z}_{D6}(u,v;(x+iy)P) = \mathcal{Z}'_{DT}(u,v)$$

$$= \mathcal{Z}'_{GW}(u,v)$$

$$\lim_{x\to-\infty}\lim_{y\to\infty} \mathcal{Z}_{D6}(u,v;(x+iy)P) = \mathcal{Z}'_{DT}(\bar u,v)$$

$$= \overline{\mathcal{Z}'_{GW}(u,v)}$$

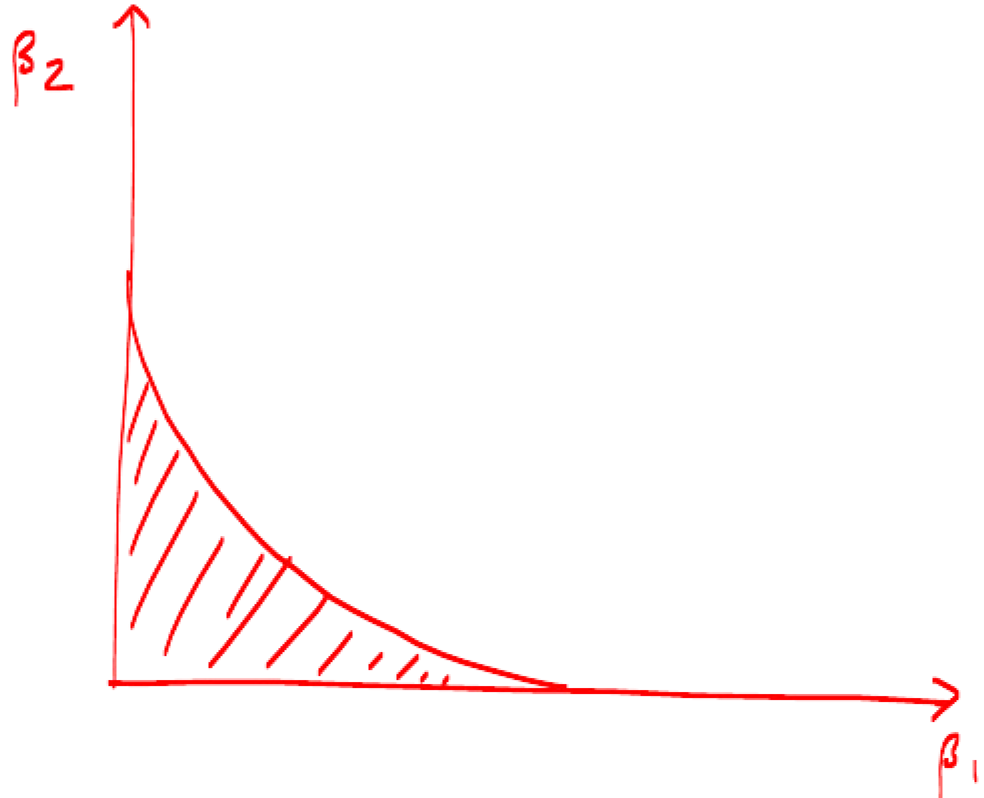
# 9. CORRECTIONS & CUTOFFS

1. OTHER TERMS IN FAREY EXPANSION

2. $Z_{D6}$ EVALUATED AT $t = e^{\frac{2\pi i}{3}} p$

3. NONFACTORIZATION OF DOMAIN: $\Gamma_1$ AND $\Gamma_2$ ARE CORRELATED

NUMERICAL STUDY OF $D6\overline{D6}$ SPLIT ATTRACTORS ON THE QUINTIC



PUT A CUTOFF: CHOOSE $\varepsilon > 0$

SMALL AND

$$0 \leq (\beta_i)_A \leq K_1 \Lambda^{2-\varepsilon}$$

$$|n_i| \leq K_2 \Lambda^{3-\varepsilon}$$

THIS RENDERS ALL SUMS OVER

$D6\overline{D6}$ STATES FINITE