BPS STATES, HITCHIN SYSTEMS, AND THE WKB APPROXIMATION

WORK IN PROGRESS WITH

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BONN WORKSHOP ON MIRROR SYMMETRY

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BASED ON

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& 0906.????

OUTLINE

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1. INTRODUCTION

- IN THIS TALK WE STUDY "BPS DEGENERACIES" OR "BPS INDICES", DENOTED Ω

- MATHEMATICALLY THEY ARE RELATED TO DONALDSON-THOMAS INVTS.

- IN THE PAST 3 YEARS THERE HAS BEEN MUCH PROGRESS ON "WALL-CROSSING" FOR THESE INVTS.

- W.C.F. TELL US HOW Ω CHANGES AS FUNCTIONS OF MODULI

- THAT STILL LEAVES OPEN THE PROBLEM OF COMPUTING Ω.

- IN THIS TALK I WILL DESCRIBE A NEW METHOD TO COMPUTE Ω IN A CLASS OF $N=2, D=4$ FIELD THEORIES.

- THESE ARE "LINEAR SU(2) QUIVER GAUGE THEORIES"

$$\boxed{2} - \underbrace{(2) - (2) - \cdots - (2)}_{G = SU(2)^r} - \boxed{2}$$

GAUGE GROUP

- SIMILAR METHODS PROBABLY EXTEND TO A MUCH LARGER CLASS OF FIELD THEORIES.

2. $\mathcal{N}=2$, $D=4$ FIELD THEORY

- $\exists$ "MODULI SPACE OF VACUA" COULOMB BRANCH $\mathcal{B} \cong \mathbb{C}^r$ WITH SPECIAL KÄHLER METRIC
- IR PHYSICS : $U(1)^r$ GAUGE THEORY
- CHARGE LATTICE FORMS A LOCAL SYSTEM OVER $\mathcal{B}$ WITH MONODROMY AROUND $\mathcal{B}_{sing}$.

$$0 \to \Gamma_{flav.} \longrightarrow \Gamma \longrightarrow \overline{\Gamma} \longrightarrow 0$$

FIBER AT $u \in \mathcal{B}$:

Γ_u "POISSON": INTEGRAL ANTISYMMETRIC $\langle\cdot,\cdot\rangle$

$\overline{\Gamma}_u$ SYMPLECTIC, RANK $= 2r$

- $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ <u>CENTRAL</u> <u>CHARGE</u>

SEIBERG-WITTEN THEORY & RIEMANN SURFACES

- THE SPECIAL KÄHLER GEOMETRY IS PRESENTED USING A FAMILY OF NON-COMPACT RIEMANN SURFACES

$$\Sigma \longrightarrow \mathcal{B}$$

EQUIPPED WITH A FIBREWISE MEROMORPHIC DIFFERENTIAL λ

- Γ IS (A SUBQUOTIENT OF) $H_1(\Sigma, \mathbb{Z})$

- $$Z_\gamma(u) = \oint_\gamma \lambda$$

BPS STATES & BPS INDEX

- $\mathcal{H}^{BPS}_{\gamma, u} := \{\psi \mid E = |Z_\gamma(u)|\}$

$\Omega(\gamma; u) = \mathrm{Tr}_{\mathcal{H}}(-1)^F :=$ "BPS INDEX"

$$\boxed{\text{PROBLEM: COMPUTE THE } \Omega(\gamma; u)}$$

KEY INGREDIENT:

- BPS STATES CAN FORM BPS BOUNDSTATES.
- A BOUNDSTATE WITH CONSTITUENT CHARGES γ_1, γ_2 CAN ONLY DECAY WHEN u CROSSES

$$MS(\gamma_1, \gamma_2) := \{u \mid Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+\}$$

$\Omega(\gamma; u)$ ONLY PIECEWISE CONSTANT

$\Rightarrow$ WALL-CROSSING FORMULAE

3. THE KONTSEVICH–SOIBELMAN FORMULA

DATA:

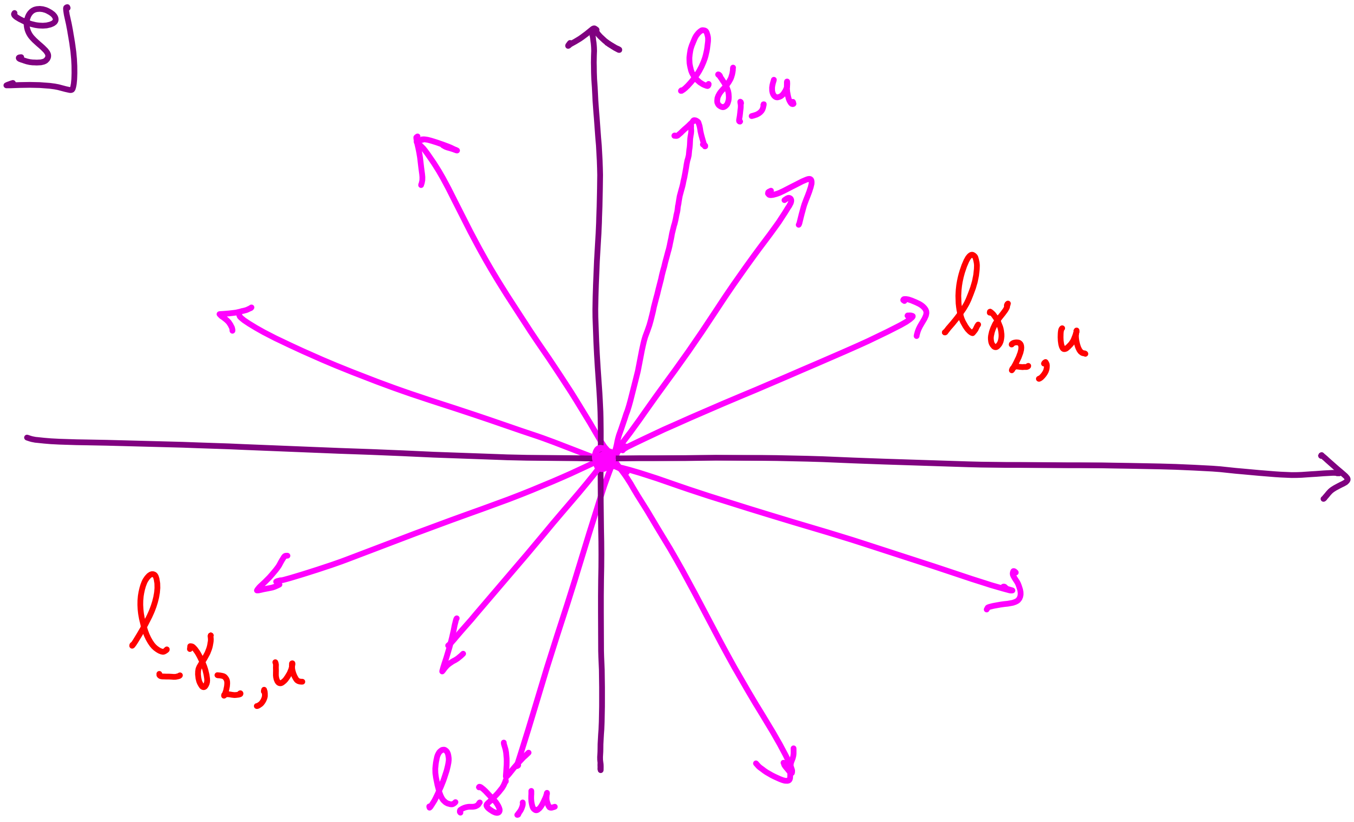
1. $\Gamma \to B$, POISSON, WITH Q.R. σ
2. CENTRAL CHARGE FUNCTION $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$
3. PIECEWISE CONSTANT $\Omega : \Gamma \to \mathbb{Z}$

FIRST INGREDIENT: BPS RAYS:

FOR $u \in B$, $\gamma \in \Gamma_u$

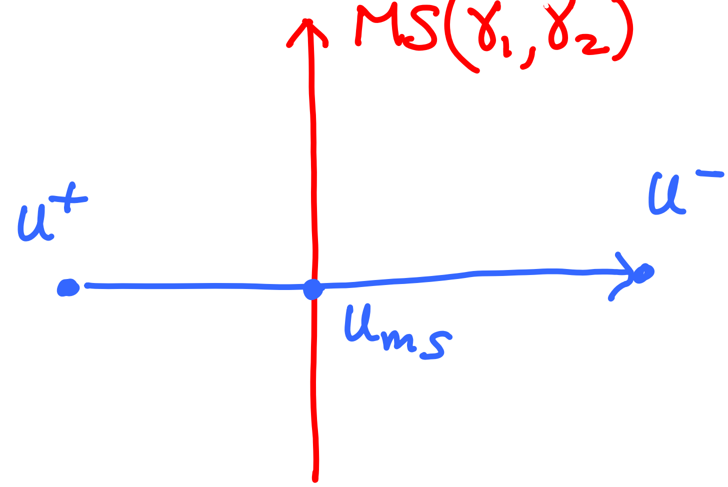
$$\ell_{\gamma,u} := Z_\gamma(u)\mathbb{R}_- = \left\{ \zeta \,\middle|\, \zeta / Z_\gamma(u) \in \mathbb{R}_- \right\}$$

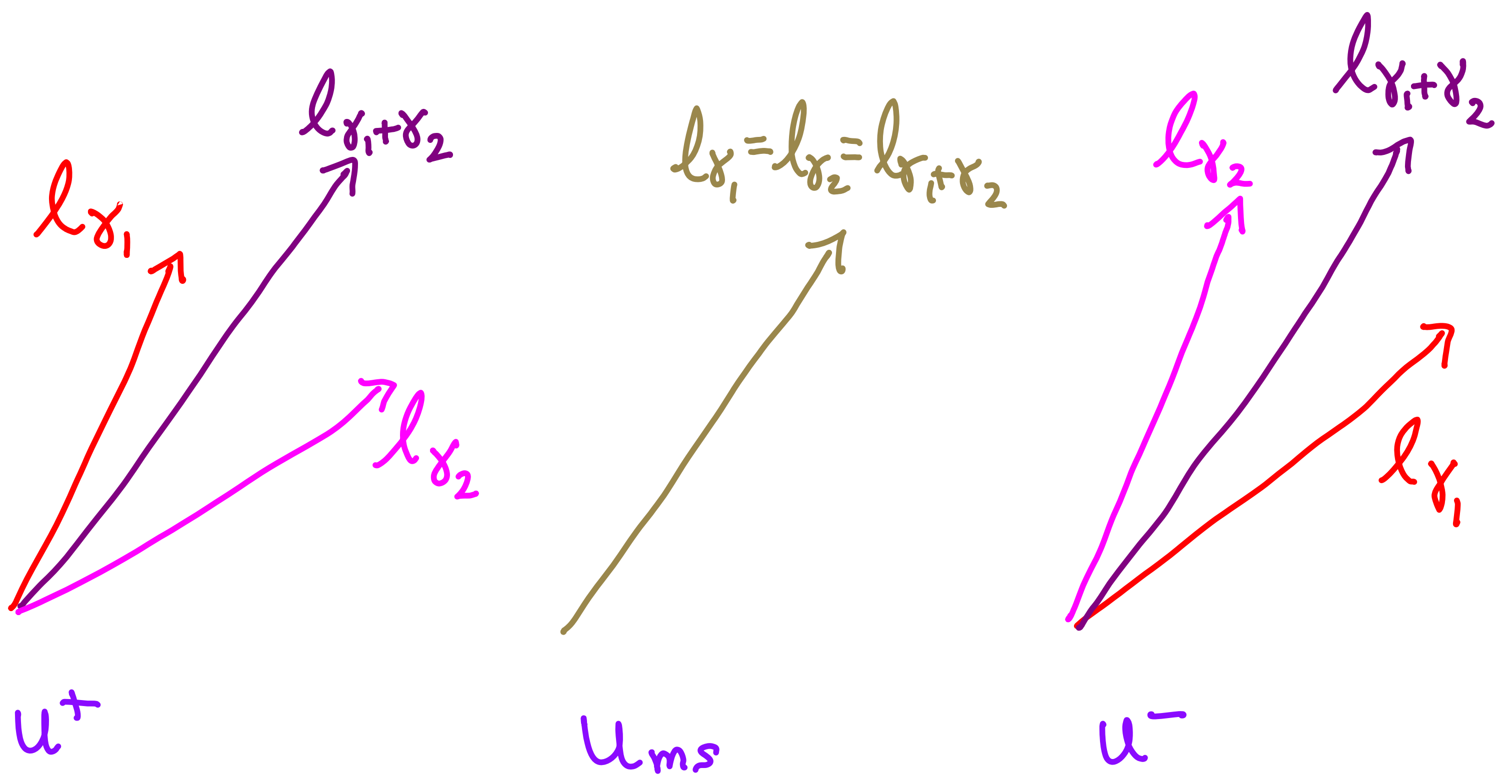
$\boxed{\zeta}$



AS u VARIES THE SLOPES OF THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1,\gamma_2)$ BPS RAYS WILL COALESCE





SECOND INGREDIENT: LOCAL SYSTEM OF POISSON TORI

- INTRODUCE $T := \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$

$$T_u \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$$

$\gamma \in \Gamma_u \implies$ FUNCTION $X_\gamma : T_u \to \mathbb{C}^*$

"HOLOMORPHIC FOURIER MODES"

- FIBREWISE POISSON STRUCTURE

$$\{X_{\gamma_1}, X_{\gamma_2}\} = \langle \gamma_1, \gamma_2 \rangle X_{\gamma_1} X_{\gamma_2}$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A (POISSON) MORPHISM:

$$K_\gamma : X_{\gamma'} \to X_{\gamma'} \left(1 - \sigma(\gamma) X_\gamma\right)^{\langle \gamma', \gamma \rangle}$$

ABOUT THE SIGN :

K & S INTRODUCE A LIE ALGEBRA

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$$

DEFINE A GROUP ELEMENT

$$U_\gamma := \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

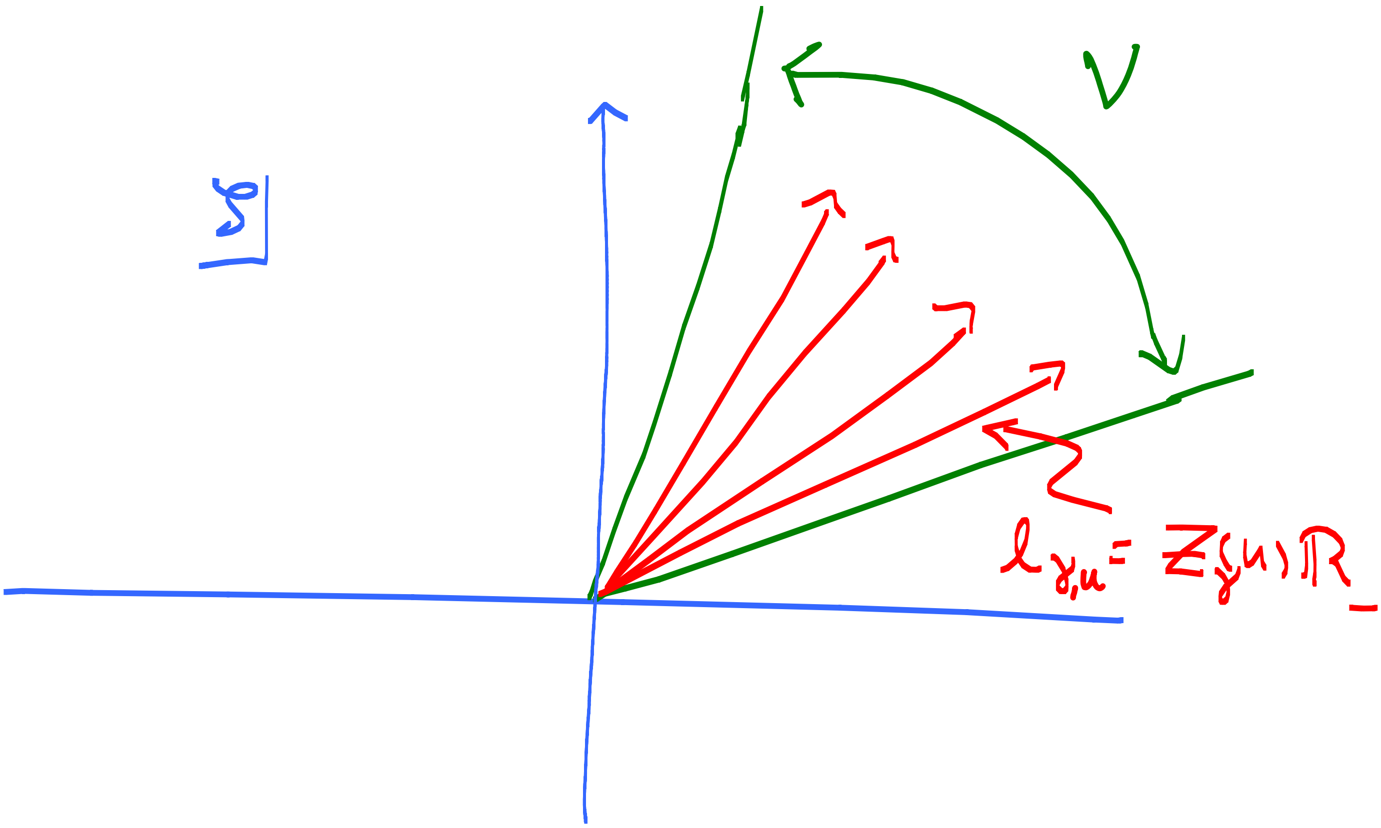
AND WORK WITH U_γ INSTEAD OF K_γ

CHOOSE A <u>QUADRATIC REFINEMENT</u>

$$\frac{\sigma(\gamma_1+\gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle\gamma_1,\gamma_2\rangle}$$

$\sigma(\gamma) e_\gamma$ GENERATE THE LIE ALGEBRA OF SYMPLECTIC VECTOR FIELDS.

THIRD INGREDIENT: CONVEX CONE V



AND THEN DEFINE:

$$\mathcal{A}_V := \overrightarrow{\prod_{-Z_\gamma \in V}} K_\gamma^{\Omega(\gamma;u)}$$

THE PRODUCT IS TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

$$\mathcal{A}_V := \prod^{\rightarrow}_{-Z_\gamma \in V} K_\gamma^{\Omega(\gamma;u)}$$

A_V DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma;u)$ DEPEND ON u ...

DEFINITION: "Ω SATISFIES THE KSWCF"

IF:

A_V IS CONSTANT IN u AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR V.

4. HK MANIFOLDS + TWISTOR COORD'S

I WILL NOW BRIEFLY REVIEW PAPER I.
(FOR SIMPLICITY PUT $\Gamma_{flavor} = 0$)

"THEOREM" GIVEN Ω SATISFYING THE KS W.C.F. ONE CAN CONSTRUCT A "NICE" FAMILY OF H.K. METRICS ON:

$$\overline{\mathcal{M}} = \overline{\Gamma}^{*} \otimes \mathbb{R}/\mathbb{Z}$$

→ SKETCH OF THE CONSTRUCTION ←

- COMPACTIFYING $\mathcal{N}=2, d=4$ THEORY ON $\mathbb{R}^3 \times S^1_R \Rightarrow$ LOW ENERGY σ-MODEL

$$\varphi : \mathbb{R}^3 \to \overline{\mathcal{M}}$$

$\Rightarrow \overline{\mathcal{M}}$ HAS A "NICE" HK METRIC.

- $\overline{\mathcal{M}}$ HAS A CANONICAL FAMILY OF METRICS – THE "SEMI-FLAT METRICS" – BUT THESE ARE SINGULAR.

IN PHYSICS "QUANTUM CORRECTIONS BY BPS INSTANTONS" SMOOTHS OUT THESE SINGULARITIES.

- THE SEMI-FLAT METRIC IS DESCRIBED BY TWISTOR COORDINATES:

$$\mathcal{X}_\gamma : \overline{\mathcal{M}} \times \mathbb{C}^* \longrightarrow \mathbb{C} \qquad \gamma \in \Gamma$$

$$\theta_\gamma : \Gamma^* \otimes \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$\mathcal{X}_\gamma^{sf} := \exp\left[+ \frac{\pi R}{\zeta} Z_\gamma + i\,\theta_\gamma + \pi R \zeta \bar{Z}_\gamma \right]$$

(NEITZKE & PIOLINE)

$\zeta \in \mathbb{C}^* \subseteq$ TWISTOR SPHERE

USING THESE WE CONSTRUCT

A FAMILY OF HOLO. MAPS

$$\zeta \in \mathbb{C}^*, \quad \mathcal{M}^{\zeta} \xrightarrow{\mathcal{X}^{s.f}(\zeta)} \overline{\mathbb{T}} := \overline{\Gamma}^* \otimes \mathbb{C}^*$$

VIA $\quad \mathcal{X}_\gamma^{s.f.}(\cdot, \zeta) := \mathcal{X}^{s.f.}(\zeta)^* (X_\gamma)$

$\overline{\mathbb{T}}$ HAS A FIBERWISE SYMPLECTIC STRUCTURE

$$\varpi^{\overline{\mathbb{T}}} = \frac{1}{2} \epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}}$$

$$\epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

<u>THEN</u> $\quad \varpi_\zeta^{s.f.} := \mathcal{X}^{s.f.}(\zeta)^* \varpi^{\overline{\mathbb{T}}} =$ HOLO. SYMPLECTIC FORM

$$= \frac{1}{2i\zeta} \omega_+^{sf} + \omega_3^{sf} + \frac{\zeta}{2i} \omega_-^{sf}$$

$\Longrightarrow$ SEMI-FLAT HK STRUCTURE.

• THE QUANTUM-CORRECTED H.K. STRUCTURE IS DEFINED USING A SIMILAR PROCEDURE:

FROM A SET OF $\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^* \longrightarrow \mathbb{C}$

CONSTRUCT A MAP: $\mathcal{X}_\gamma(\cdot, \zeta) = \mathcal{X}(\zeta)^*(X_\gamma)$

$$\mathcal{M}^\zeta \xrightarrow{\mathcal{X}(\zeta)} \mathcal{T} = \Gamma^* \otimes \mathbb{C}^*$$

AND EXTRACT HK STRUCTURE FROM:

$$\varpi_\zeta = \mathcal{X}(\zeta)^* \varpi^{\mathcal{T}}$$

$$= \frac{1}{2i\zeta}\omega_+ + \omega_3 + \frac{\zeta}{2i}\omega_-$$

• USING A PHYSICAL INTERPRETATION OF THE $\mathcal{X}_\gamma$ WE SHOW THEY SATISFY A SYSTEM OF ISOMONODROMIC DIFFL. EQS. ("WARD IDENTITIES") WHICH IS EQUIVALENT TO A R.H. PROBLEM

DEFINING PROPERTIES OF $\mathcal{X}_\gamma$

1.) $\mathcal{X}_\gamma(\cdot,\zeta) : \mathcal{M}^\zeta \longrightarrow \mathbb{C}$ HOLO.

2.) $\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$

3.) $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$

4.A.) $\mathcal{X}_\gamma(\cdot,\zeta) \underset{R\to\infty}{\sim} \exp\left(\frac{\pi R}{\zeta} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right)$

4B.) $\lim_{\zeta\to 0} \mathcal{X}_\gamma(\cdot,\zeta)\, e^{-\frac{\pi R}{\zeta} Z_\gamma(u)}$ FINITE

5.) $\mathcal{X}_\gamma(\cdot,\zeta)$ IS <u>PIECEWISE</u> HOLO IN ζ:

HOLO. IN ANGULAR SECTORS BETWEEN BPS RAYS AND TRANSFORMS BY

$\prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma';u)}$ AS ζ CROSSES THE BPS RAY $\ell_{\gamma,u}$.

- RIEMANN-HILBERT PROBLEM IS SOLVED BY:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf}(\zeta) \cdot \exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \int_{\ell_{\gamma', u}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')\right) \right\}$$

- CAN BE SOLVED ITERATIVELY AT $R \gg 1$.

- ONE GOAL OF THIS TALK IS TO GIVE A NEW CONSTRUCTION OF THE $\mathcal{X}_\gamma$'S.

- AN IMPORTANT COROLLARY OF THE NEW CONSTRUCTION IS AN ALGORITHM FOR COMPUTING THE Ω'S.

5. FROM M5-BRANES TO HITCHIN EQS

NOW LETS NARROW DOWN THE CLASS OF $\mathcal{N}=2, d=4$ THEORIES

- IIB STRING VIEWPOINT: WE HAVE A NON-CPT. CY = CURVE C OF ADE SINGULARITIES.

  BPS STATES ~ D3's WRAP SLAGS S^3 or $S^2 \times S^1$.

  WE WANT Ω's FOR THESE.

- A CHAIN OF PHYSICAL REASONING LEADS TO THE CONCLUSION THAT THE Ω's CAN BE COMPUTED BY STUDYING HITCHIN SYSTEMS ON C WITH SINGULARITIES.

... VERY ROUGHLY:

a.) ADE SINGULARITY $\xrightarrow{\text{"T-DUAL"}}$ 5-BRANE

b.) DECOUPLE GRAVITY $\Longrightarrow$

CONSIDER A_{K-1} (2,0) THEORY ON $\mathbb{R}^{1,3} \times C$ WITH "DEFECTS."

c.) LOW ENERGY LIMIT GIVES $N=2, D=4$ THEORY ON $\mathbb{R}^{1,3}$

d.) HK STRUCTURE FROM FURTHER COMPACTIFICATION OF THIS $N=2, D=4$ THEORY ON $\mathbb{R}^{1,2} \times S^1_R$

BUT! WE COULD ARRIVE AT THE SAME LOW ENERGY THEORY ON $\mathbb{R}^{1,2}$ COMPACTIFYING IN THE OTHER ORDER:

$$
\begin{tikzcd}[column sep=small, row sep=large]
& \text{6D } (2,0)\ A_{K-1} \,/\, \mathbb{R}^{1,2}\times S^1_R \times C + \text{DEFECTS} \arrow[dl, "\ell_C \ll \ell_{S^1},\, \ell_{\mathbb{R}^{1,2}}"'] \arrow[dr, "\ell_{S^1} \ll \ell_C,\, \ell_{\mathbb{R}^{1,2}}"] & \\
\text{4D } \mathcal{N}=2 \text{ GAUGE THEORY} \,/\, \mathbb{R}^{1,2}\times S^1_R \arrow[dr, "\ell_{S^1} \ll \ell_{\mathbb{R}^{1,2}}"'] & & \dfrac{\text{5D SU(K) SYM}}{\mathbb{R}^{1,2}\times C} \arrow[dl, "\ell_C \ll \ell_{\mathbb{R}^{1,2}}"] \\
& \sigma\text{- MODEL: } \mathbb{R}^{1,2} \longrightarrow \mathcal{M} &
\end{tikzcd}
$$

6. THE HITCHIN SYSTEM

$$G = SU(K)$$

$E \to C$ HAS CONNECTION A AND

HIGGS FIELD $\quad \varphi \in \Omega^{0,1}(C \,;\, \mathrm{End}\, E)$

BPS EQS = HITCHIN EQS:

$$\begin{cases} F + R^2 [\varphi, \bar\varphi] = 0 \\ \bar\partial_A \varphi = 0 \qquad \partial_A \bar\varphi = 0 \end{cases}$$

ALLOW SINGULARITIES AT $z_i \in C$:

REGULAR + IRREGULAR SING'S

RSP: IN LOCAL COORD'S NEAR z_i

$$\varphi \sim \frac{1}{2} \rho \frac{dz}{z - z_i} + \cdots\cdots$$

$$A \sim \frac{\alpha}{2i} \left(\frac{dz}{z - z_i} - \frac{d\bar z}{\bar z - \bar z_i} \right) + \cdots\cdots$$

SO NOW WE IDENTIFY:

$$\mathcal{M} = \text{HITCHIN MODULI SPACE}$$

HITCHIN FIBRATION: $(A, \varphi) \to \mathrm{CharPoly}(\varphi)$

$$\mathcal{B} = \bigoplus_d H^0(C, K_C^{\otimes d})$$

$\mathcal{B}$: BASE OF A FAMILY OF SPECTRAL CURVES

$$\Sigma = \{(x, z) \mid \det(x\,dz - \varphi(z)) = 0\} \subset T^*C$$

= SEIBERG-WITTEN CURVE OF THE $d=4, N=2$ THEORY RESULTING FROM COMPACTIFYING THE A_{K-1} (2,0) THEORY ON C

$\lambda = x\,dz$ = S-W DIFFERENTIAL

($d\lambda = dx\,dz$ = CANONICAL SYMPLECTIC FORM ON T^*C)

FOR SIMPLICITY WE HENCEFORTH TAKE GAUGE GROUP $G = SU(2)$

1. $\Sigma \xrightarrow{2:1} C$

$\lambda^2 =$ QUADRATIC DIFF'L ON $C = \frac{1}{2} \mathrm{Tr}\, \varphi^2$

LOCALLY: $\varphi \sim \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$

WE ASSUME THAT λ^2 HAS FIRST ORDER ZEROES $w_a \in C$.

THESE ARE THE <u>BRANCH POINTS</u> OF $\Sigma \to \mathbb{CP}^1$

ALSO CALLED "<u>TURNING POINTS</u>"

2. IN ADDITION THERE ARE SINGULAR POINTS z_i:

$$\varphi = \frac{1}{z - z_i} \begin{pmatrix} m_i & \\ & -m_i \end{pmatrix} + \cdots$$

$$A = \begin{pmatrix} m_i^{(3)} & \\ & -m_i^{(3)} \end{pmatrix} \left(\frac{dz}{z - z_i} - \frac{d\bar{z}}{\bar{z} - \bar{z}_i} \right) + \cdots$$

BPS STATES IN THE HITCHIN FRAMEWORK

THE PHYSICAL DERIVATION LEADS TO THE FOLLOWING <u>RULES</u> FOR DESCRIBING BPS STATES.

(KLEMM, LERCHE, VAFA, WARNER)

$\Omega(\gamma, u) = 0$ <u>UNLESS</u> THERE EXISTS A CURVE $\tilde{C} \in \Sigma$ IN HOMOLOGY CLASS γ SO THAT $\pi_*(\tilde{C}) = C$ IS A CURVE IN C S.T.

$$\boxed{\langle \lambda, \partial_t \rangle \in e^{i\vartheta_*} = \text{CONSTANT}}$$

<u>AND</u>

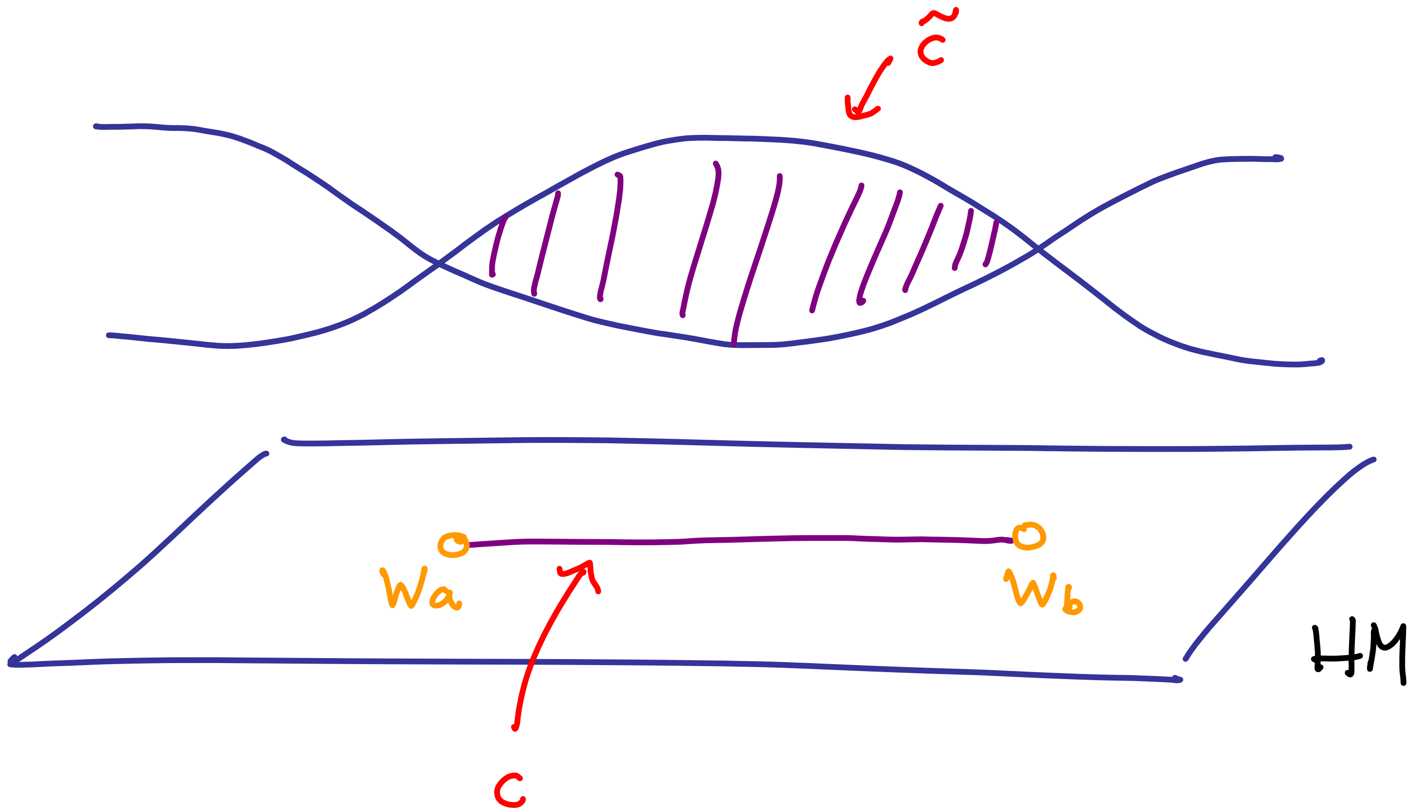
ONE OF TWO THINGS HAPPENS:

1. EITHER,

C BEGINS AND ENDS ON A BRANCH POINT

$\Rightarrow$ HM WITH $\Omega = +1$

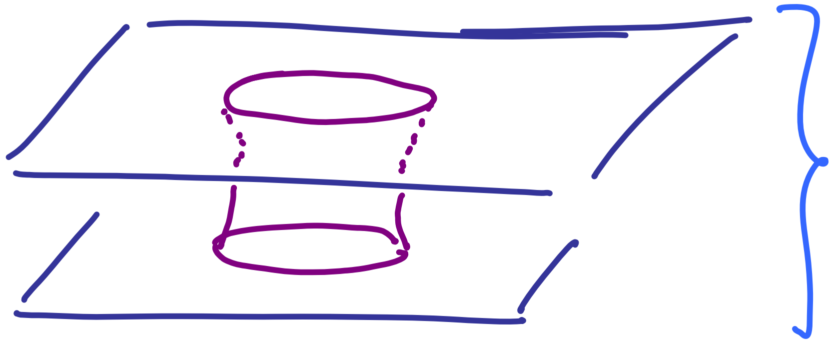
M-THEORY PICTURE: OPEN M2:



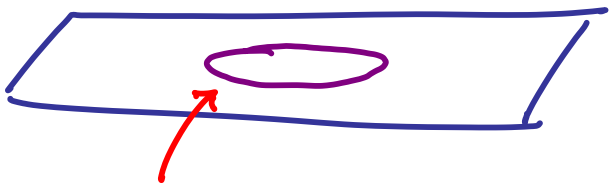
2. OR, C = CLOSED CURVE $\Rightarrow$

VM WITH $\Omega = -2$.

Σ



M-THEORY PICTURE



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N.B. FOR SUCH CURVES $\langle \lambda, \partial_t \rangle = e^{i\vartheta_*}$

$$\vartheta_* = \arg(Z)$$

(REMARK: FOR $SU(K)$, $K>2$ THE BPS STATES ARE MORE COMPLICATED AND INVOLVE "STRING WEBS.")

HITCHIN SYSTEMS & FLAT CONN'S

THE H.E.'S $\Longrightarrow$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

IS FLAT.

NEAR REG. SING. POINT z_i

$$\mathcal{A} \sim \left(\frac{R}{\zeta}\frac{\rho}{2} + \frac{\alpha}{2i}\right)\frac{dz}{z-z_i} + \left(R\zeta\frac{\bar{\rho}}{2} - \frac{\alpha}{2i}\right)\frac{d\bar{z}}{\bar{z}-\bar{z}_i}$$

SO B.C.'S FIX MONODROMY OF $\mathcal{A}$.

AROUND z_i:

$$M_i \sim \begin{pmatrix} \mu_i & \\ & \mu_i^{-1} \end{pmatrix}$$

$$\mu_i = \exp\left[2\pi i\left(\tfrac{1}{2}\zeta^{-1}Rm_i - m_i^3 - \tfrac{1}{2}\zeta R\bar{m}_i\right)\right]$$

THEOREM OF C. SIMPSON $\Longrightarrow$

IDENTIFY $\mathcal{M}^{\zeta}$, $\zeta \in \mathbb{C}^*$, WITH MODULI OF FLAT $SL(2,\mathbb{C})$ CONNECTIONS WITH PRESCRIBED MONODROMY AT z_i

MOREOVER, THE HOLO. SYMPLECTIC FORM ON $\mathcal{M}^{\zeta}$ HAS THE SIMPLE FORM:

$$\boxed{\varpi_{\zeta} = \int_C \mathrm{Tr}(\delta A \, \delta A)}$$

STRATEGY:

- FOCK & GONCHAROV CONSTRUCTED NICE COORDINATE SYSTEMS ON $\mathcal{M}^5$

- WE WILL USE THESE FUNCTIONS TO CONSTRUCT SYSTEMS OF COORD'S $\mathcal{X}_\gamma^\vartheta(\cdot,\zeta)$ ON $\mathcal{M}$, $\vartheta \in \mathbb{R}/\mathbb{Z}$

RELATION TO PREVIOUS TWISTOR COORD'S

$$\mathcal{X}_\gamma(\cdot,\zeta) = \mathcal{X}_\gamma^{\vartheta=\arg\zeta}(\cdot,\zeta)$$

THEN, AT LARGER R,

THE $\mathcal{X}_\gamma(\cdot,\zeta)$ SATISFY

DEFINING PROPERTIES $1 \to 5$.

6. FOCK-GONCHAROV COORD'S AND CLUSTER TMNS

$\mathcal{U}$ = A FLAT $SL(2,\mathbb{C})$ CONNECTION WITH MONODROMY M_i AROUND z_i

$$M_i \sim \begin{pmatrix} \mu_i & \\ & 1/\mu_i \end{pmatrix}$$

A. DECORATED TRIANGULATIONS

<u>DEF</u>: A "DECORATED TRIANG." T IS AN IDEAL TRIANGULATION OF C WITH VERTICES AT z_i TOGETHER WITH A CHOICE OF MONODROMY EIGENVALUE μ_i OR $1/\mu_i$ AT EACH z_i

B. FLIPS AND POPS

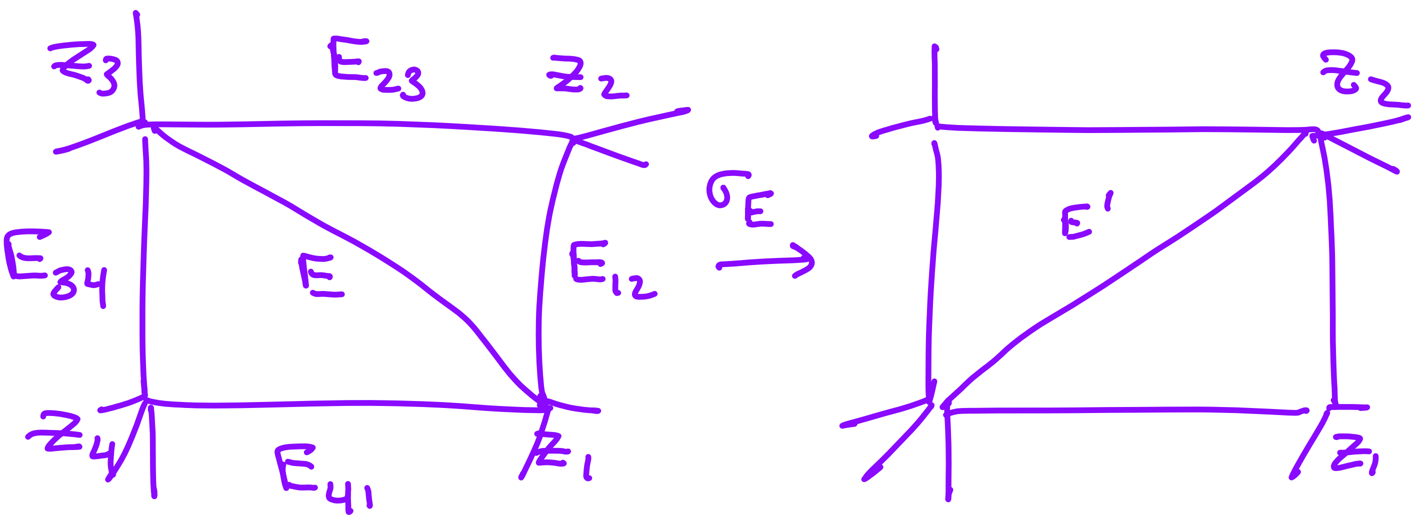
DEFINE A GROUPOID:

OBJECTS = DECORATED TRIANGULATIONS

MORPHISMS ARE GENERATED BY FLIPS + POPS

FLIP: σ_E FOR $E \in \mathcal{E}(T)$

Q_E: $Q_{E'}$:



T T'

POP: π_i FOR $z_i \in V(T)$:

EXCHANGE: $\mu_i \longleftrightarrow \mu_i^{-1}$

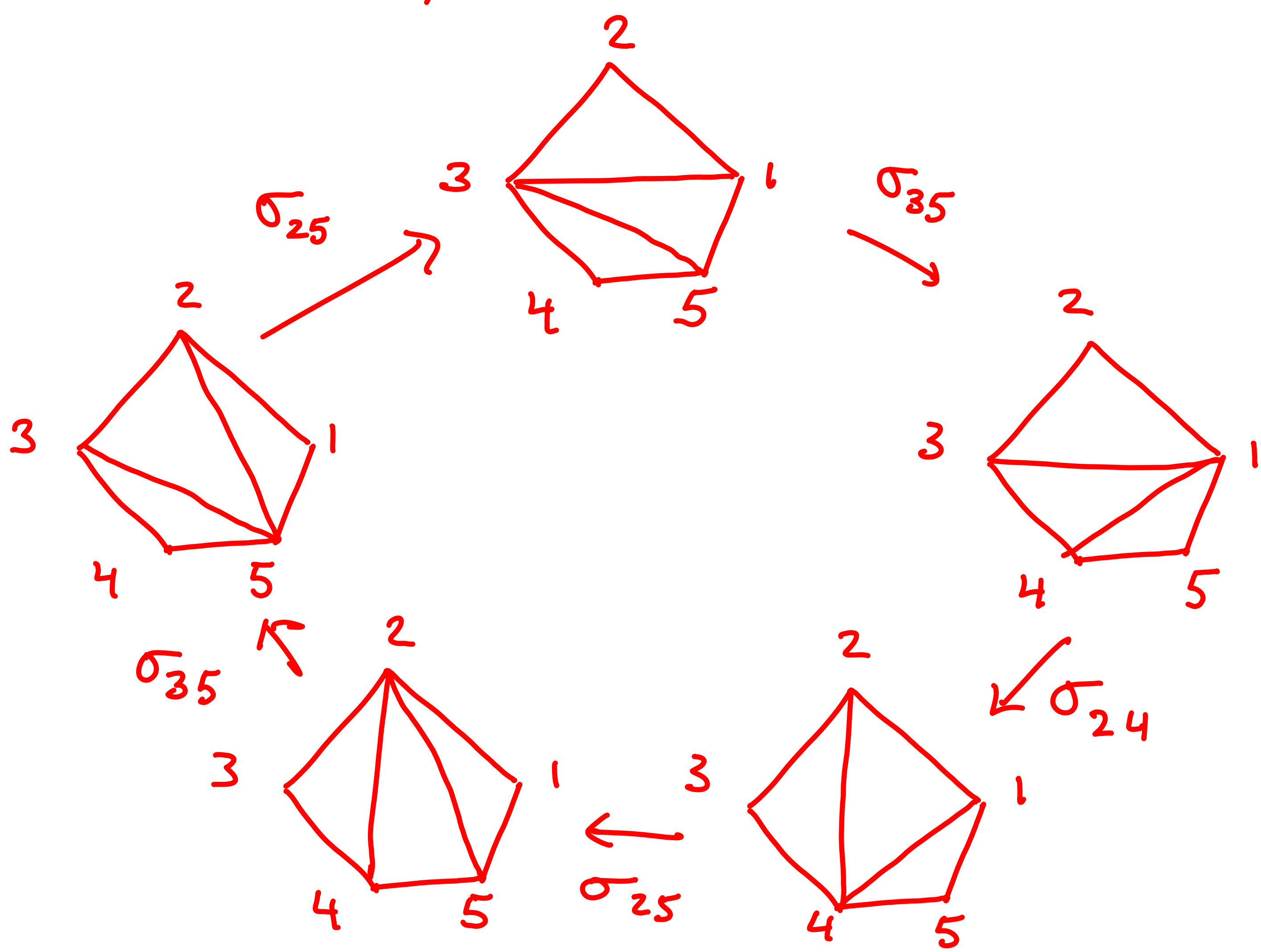
RELATIONS ON FLIPS & POPS

1. $\sigma_E^2 = 1$ AND $\pi_i^2 = 1$

2. POPS COMMUTE

3. $\sigma_E, \sigma_{E'}$ COMMUTE IF $Q_E, Q_{E'}$ DO NOT SHARE A TRIANGLE

4. IF $Q_E, Q_{E'}$ SHARE A TRIANGLE



LATER WE WILL ENHANCE OUR GROUPOID TO INCLUDE "LIMIT TRIANG'S" AND "TWISTS"

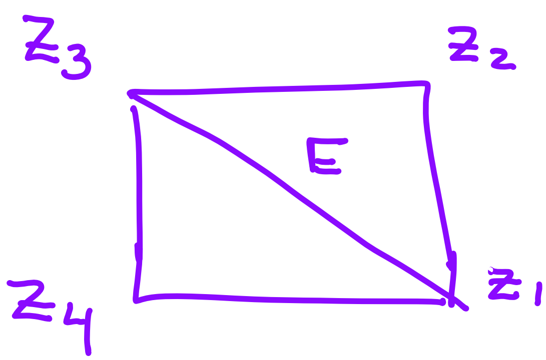
C. FG COORDINATES

GIVEN A DECORATED TRIANGULATION OF C, F & G DEFINE A COLLECTION OF FUNCTIONS ON $\mathcal{M}$:

$$\mathcal{X} : T \longrightarrow \{\mathcal{X}_E^T\}_{E \in \mathcal{E}(T)}$$

DEFINITION:

Q_E:



- CHOOSE FLAT SECTIONS s_i OF SPECIFIED MONODROMY NEAR z_i
- $$\boxed{\mathcal{X}_E^T := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}}$$

$$\boxed{\chi_E^T := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}}$$

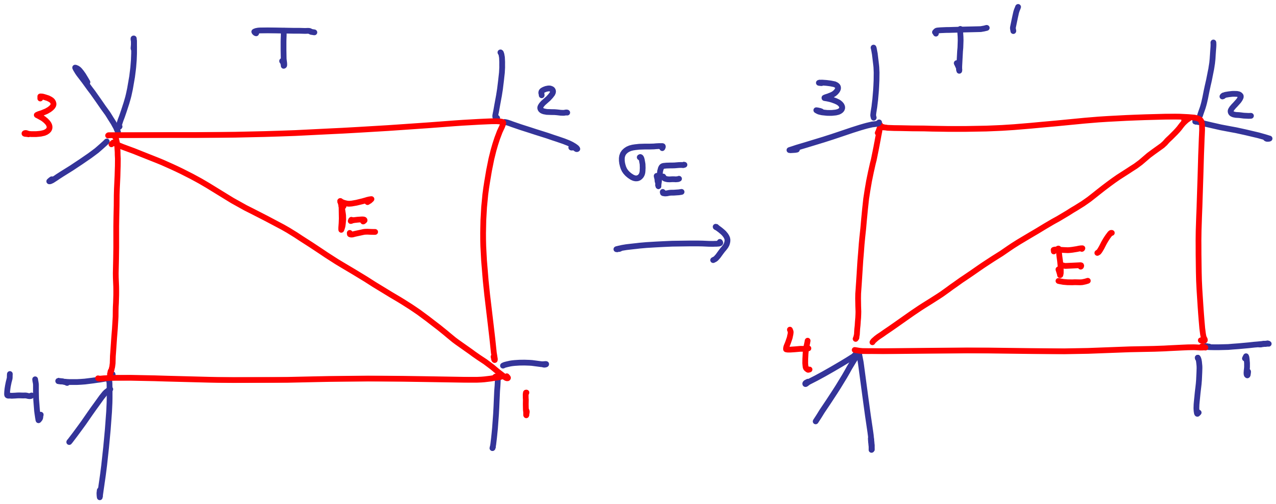
- $s_i \wedge s_j \in \Lambda^2 E$ = LINE BUNDLE
- PARALLEL TRANSPORT TO ANY POINT $Q \in Q_E$
- NORMALIZATION OF s_i CANCELS

<u>THEOREM</u>: (F; G) $\{\chi_E^T\}_E$ PROVIDE HOLO. COORDINATES ON OPEN SET U_T OF $\mathcal{M}$.

D. COORDINATE TMN'S

NOW DESCRIBE THE COORD. TMNS AS WE CHANGE THE DECORATED TRIANGULATION $T \to T'$

TRANSFORMATION UNDER FLIPS

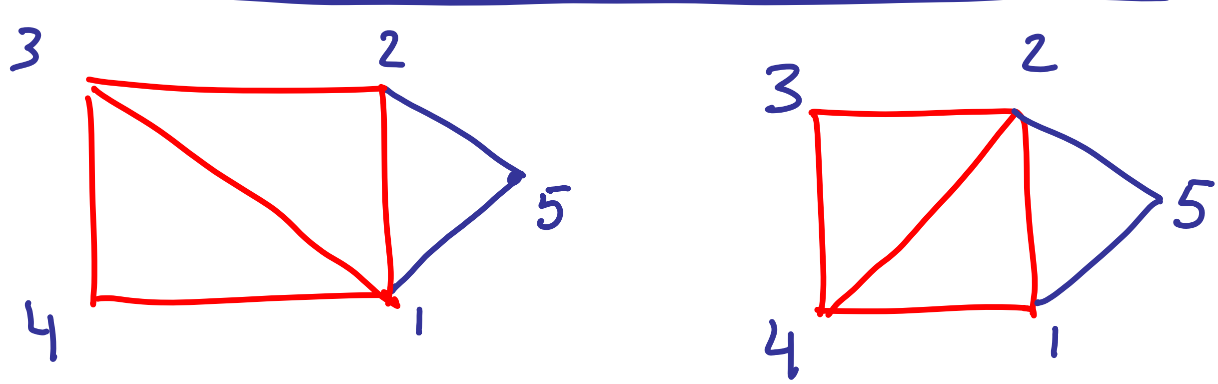


ONLY THE EDGES IN RED CHANGE

$$\chi^{T'}_{E'} = -\frac{s_4\wedge s_1\ \ s_2\wedge s_3}{s_1\wedge s_2\ \ s_3\wedge s_4} = \frac{1}{\chi^{T}_{E}}$$

$$\chi^{T'}_{E_{12}} = \chi^{T}_{E_{12}}\left(1+\chi^{T}_{E}\right)$$

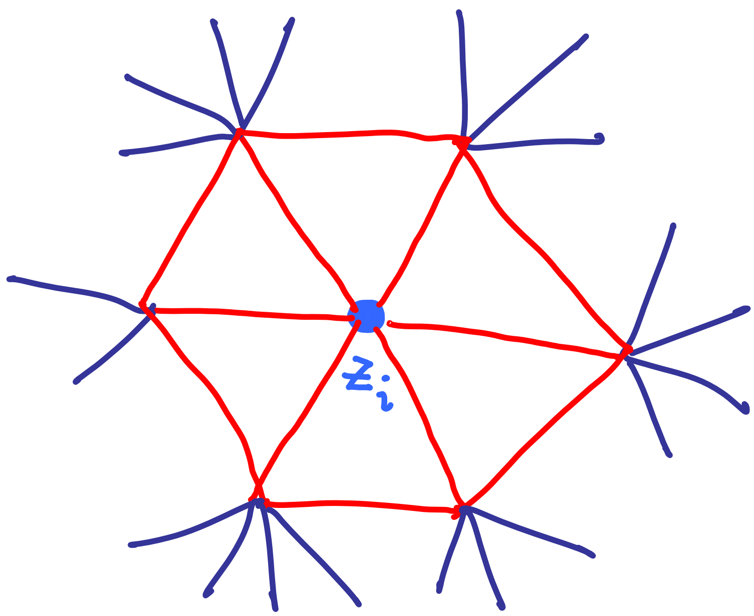
...



"CLUSTER TRANSFORMATIONS"

TRANSFORMATION UNDER POPS

A POP AT VERTEX z_i CHANGES THE EDGE COORDINATES IN RED



IT IS POSSIBLE TO WRITE EXPLICIT FORMULAE FOR THE POP TRANSFORMATION, BUT THEY ARE COMPLICATED....

IMPORTANTLY!

IT TURNS OUT THAT THE PRODUCT OF ALL POPS $\prod_v \pi_v$ IS RELATIVELY SIMPLE...

E. SYMPLECTIC STRUCTURE

- USING THE SYMPLECTIC STRUCT $\tilde{\omega}_{\mathcal{T}}$ ONE CAN SHOW

$$\{ \chi^T_E , \chi^T_{E'} \} = \langle E, E' \rangle \, \chi^T_E \, \chi^T_{E'}$$

- TRANSFORMATIONS UNDER FLIPS & POPS ARE POISSON

7. WKB TRIANGULATIONS

RECALL THAT A KEY PROPERTY OF $\mathcal{X}_\gamma(\zeta)$ ARE THE $\zeta \to 0$ ASYMPT'S:

$$\lim_{\zeta \to 0} \mathcal{X}_\gamma(\zeta)\, e^{-\frac{\pi R}{\zeta} Z_\gamma(u)} \sim \text{FINITE}$$

$\Rightarrow$ WE NEED TO USE VERY SPECIAL TRIANGULATIONS FOR WHICH WE CAN PROVE SUCH ASYMPTOTICS.

<u>IDEA</u>: USE THE WKB APPROXIMATION TO DESCRIBE THE FLAT SECTIONS:

$$(d + A)\, s = 0$$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

FOR $\zeta \to 0$, $\zeta \sim \hbar$

$$\left(\zeta d + R\varphi + O(\zeta)\right) s = 0$$

RECALL: $\varphi \sim \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$

WKB:

$$s \underset{\zeta \to 0}{\sim} \exp\left(-\frac{R}{\zeta}\int^{z} \lambda \;\sigma^3\right) s_0$$

Const. $\downarrow$ (s_0)

FROM THIS GET ASYMPT'S OF FG COORD.

HOWEVER THE WKB APPXT. IS NOTORIOUSLY SUBTLE.

EXPONENTIALLY SMALL CORRECTIONS CAN GROW IN Z AND INVALIDATE COMPUTATIONS

FOR VALIDITY OF WKB APPXT. WE MUST RESTRICT TO VERY SPECIAL TRIANGULATIONS $T(\vartheta, \lambda)$ WHOSE EDGES ARE <u>WKB CURVES</u>

<u>DEF</u>: <u>WKB CURVE WITH ANGLE ϑ</u>:

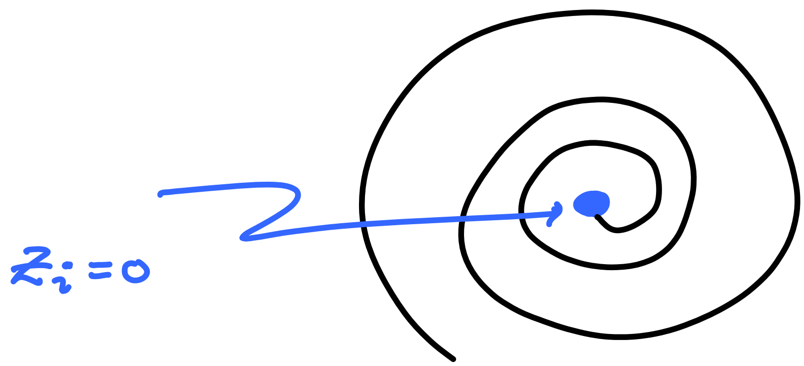
CURVE ON C WITH

$$\boxed{\langle \lambda, \partial_t \rangle = \pm e^{i\vartheta}}$$

$\Longrightarrow$ WKB FOLIATION OF C.

NOTE: WKB CURVES GET TRAPPED BY SINGULARITIES:

$$\lambda = \frac{m}{2}\frac{dz}{z} \Longrightarrow z(t) = z_0 \exp\left(-\frac{e^{i\vartheta}}{m} t\right)$$



THREE KINDS OF WKB CURVES:

GENERIC: BOTH ENDS ON z_i, z_j

SEPARATING: CONNECTS BRANCH POINT w_a TO SINGULAR POINT z_i

FINITE: CLOSED, OR BOTH ENDS ON TURNING POINTS w_a, w_b

NOTE THAT OUR RULE FOR BPS STATES WAS THAT $\exists\ \vartheta_*$ FOR WHICH THERE IS A FINITE WKB CURVE

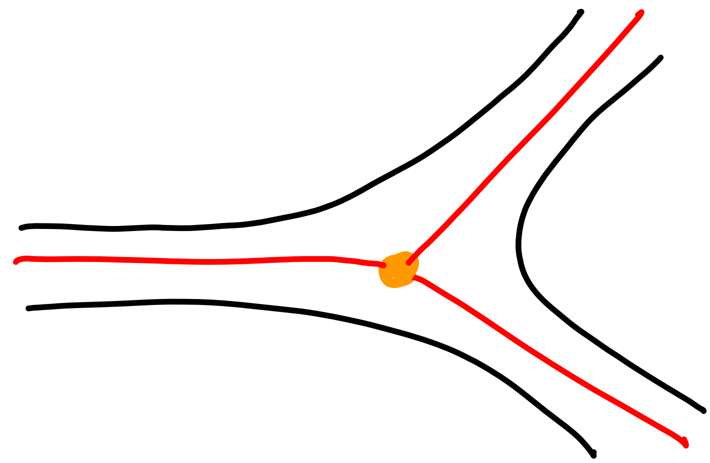
IMPORTANT FACT: FOR GENERIC VALUES OF ϑ THERE ARE NO FINITE WKB CURVES. BUT AT SPECIAL CRITICAL VALUES OF ϑ THERE ARE FINITE WKB CURVES.

RECALL THAT FOR FINITE WKB CURVES $\langle \lambda, \partial_t \rangle = e^{i\vartheta_*}$ WITH $\vartheta_* = \arg Z$.

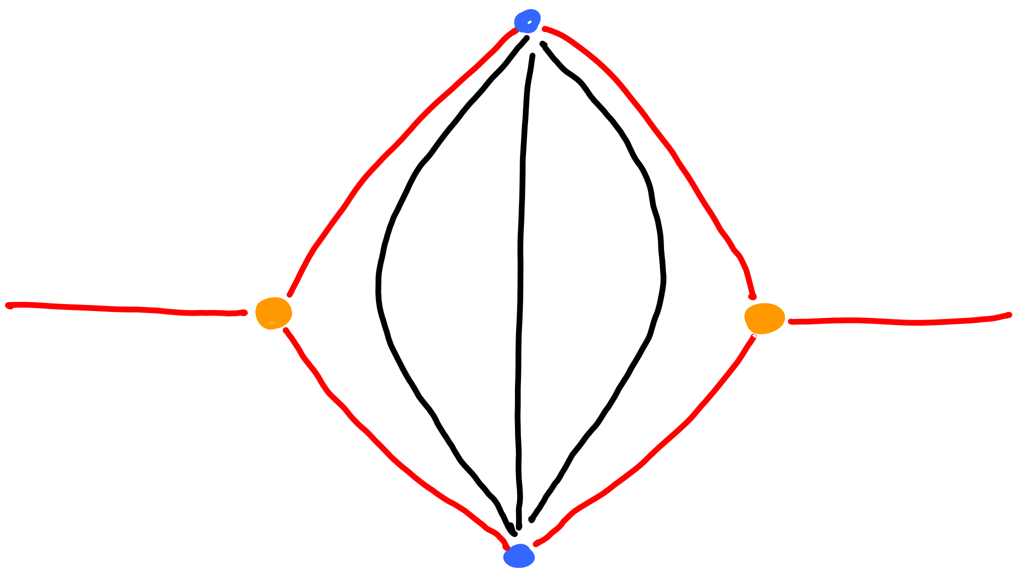
SO: THE CRITICAL VALUES ARE THE PHASES ϑ_* OF BPS STATES.

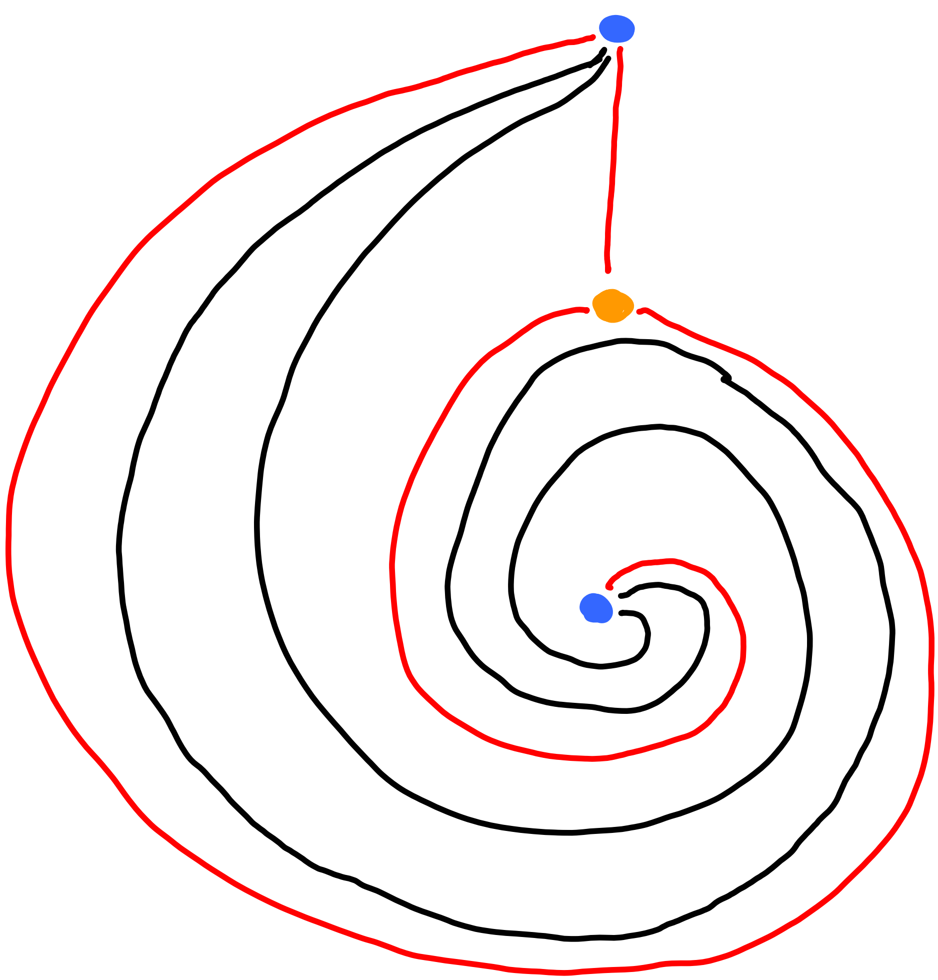
TO DEFINE OUR TRIANGULATION WE FIRST USE THE <u>SEPARATING CURVES</u> TO SPLIT C INTO <u>WKB CELLS</u>

LOCALLY:

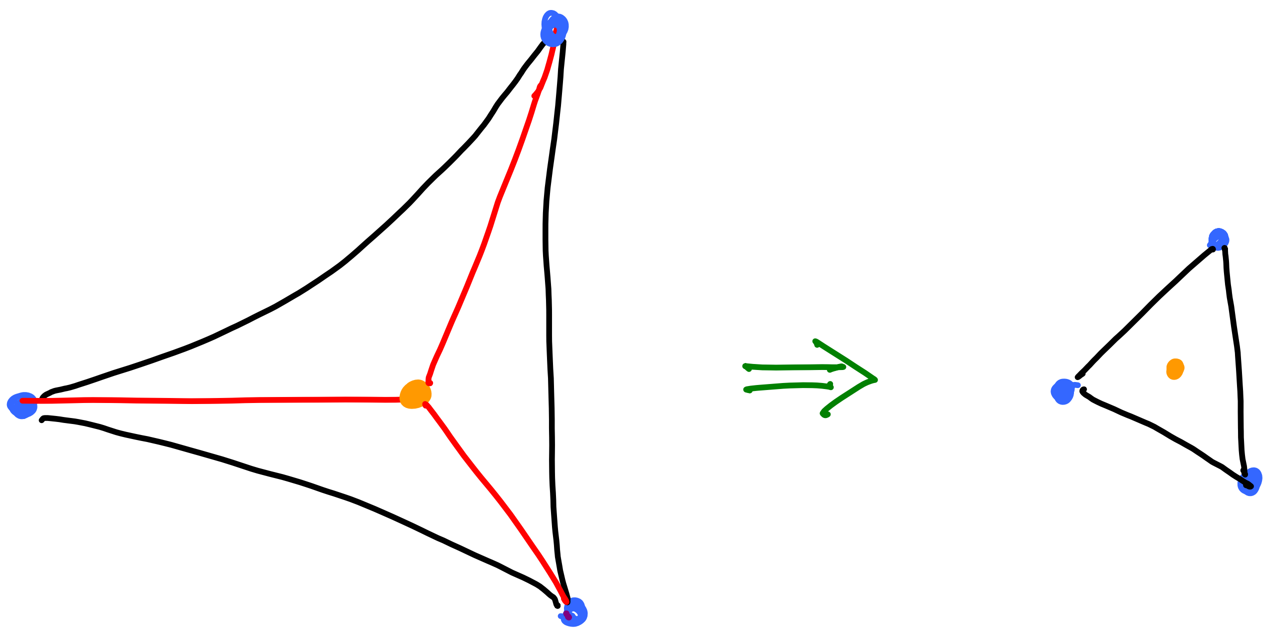


FOR <u>GENERIC</u> λ, ϑ IT TURNS OUT THERE ARE ONLY TWO KINDS OF CELLS:

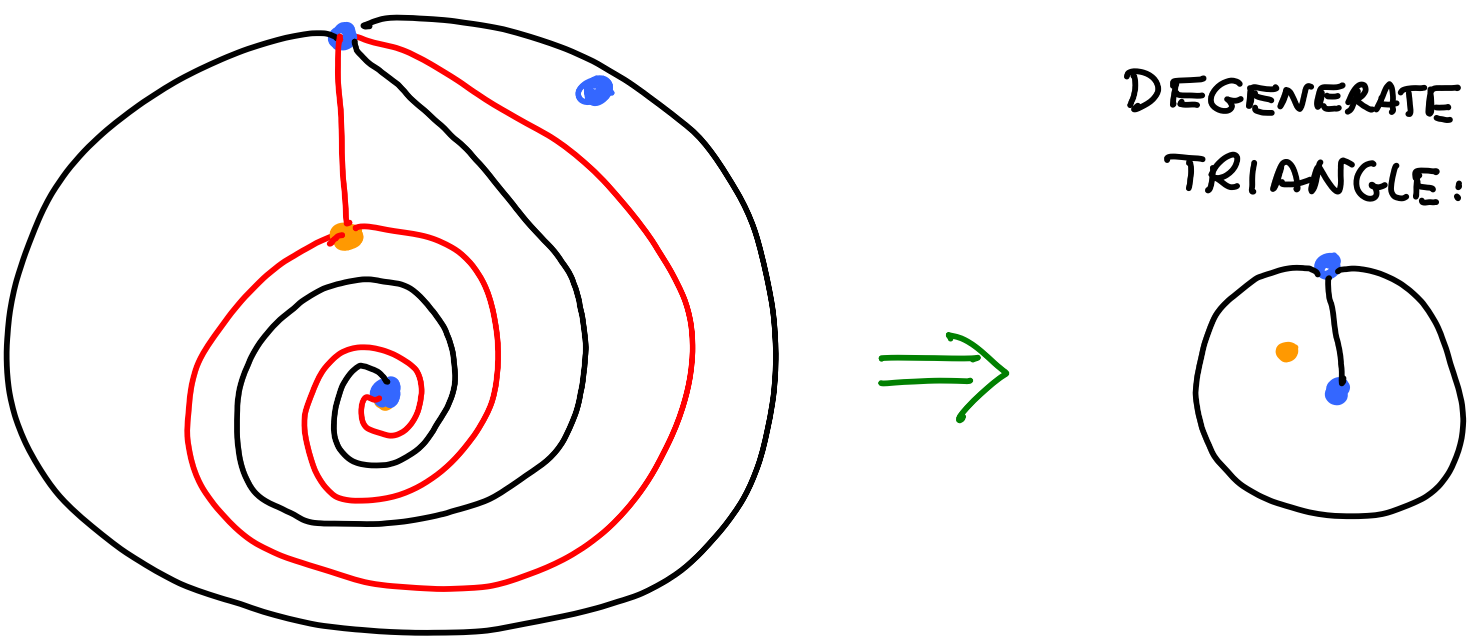




FOR THE WKB TRIANGULATION WE CHOOSE A GENERIC WKB CURVE IN EACH CELL:



DEGENERATE TRIANGLE:



CHOICE OF μ_i

RECALL THAT WE MUST DEFINE A "DECORATED TRIANGULATION."

$(\lambda, \vartheta) \Rightarrow$ DISTINGUISHED EIGENVALUE OF M_i

"SMALL FLAT SECTION": THE FLAT SECTION WHICH DECAYS ALONG THE WKB CURVE GOING INTO THE SINGULARITY

THESE ARE THE SECTIONS FOR WHICH WE HAVE GOOD CONTROL IN WKB APPXT.

DENOTE THE RESULTING DECORATED TRIANGULATION $T(\vartheta, \lambda)$

MORPHISMS OF WKB TRIANG'S

VARY ϑ $\Rightarrow$ (HOMOTOPY CLASS OF) $T(\vartheta, \lambda)$ IS UNCHANGED

EXCEPT AT CRITICAL VALUES ϑ_c WHERE FINITE WKB CURVES DEVELOP.

WHEN VARYING ϑ, $T(\vartheta, \lambda)$ JUMPS PRECISELY AT THE VALUES OF PHASES OF BPS STATES!

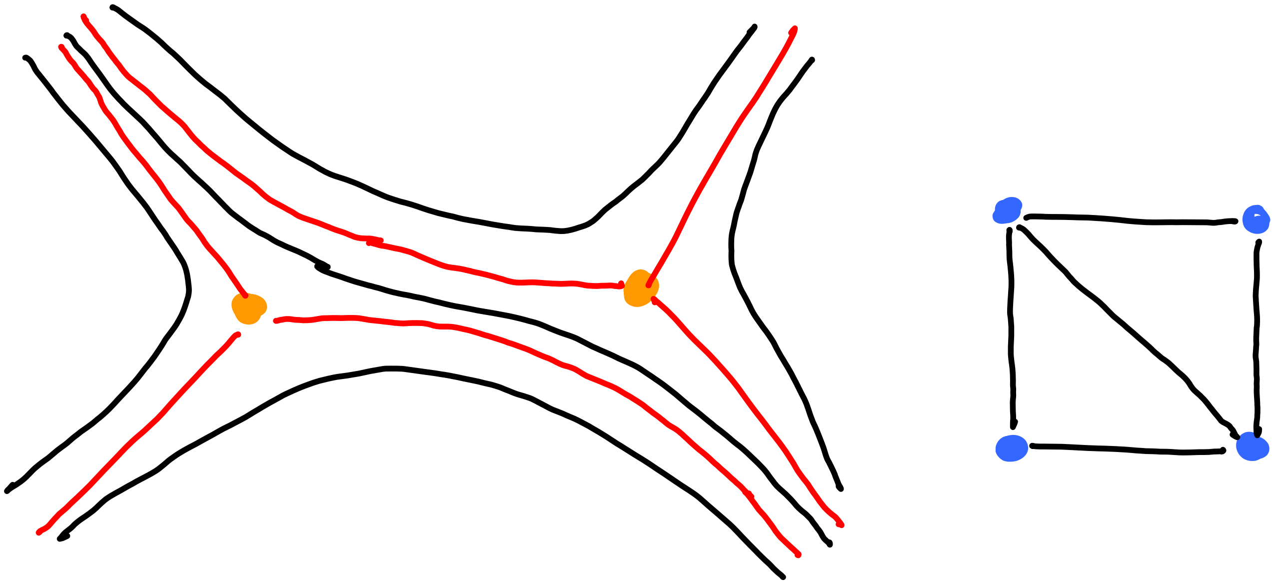
FOR GENERIC λ A JUMP IN $T(\vartheta, \lambda)$ ONLY HAPPENS WHEN A SEPARATING CURVE DEGENERATES TO A FINITE WKB CURVE JOINING TURNING POINTS w_a, w_b.

THUS, FOR GENERIC λ THERE ARE ONLY TWO KINDS OF JUMPS:

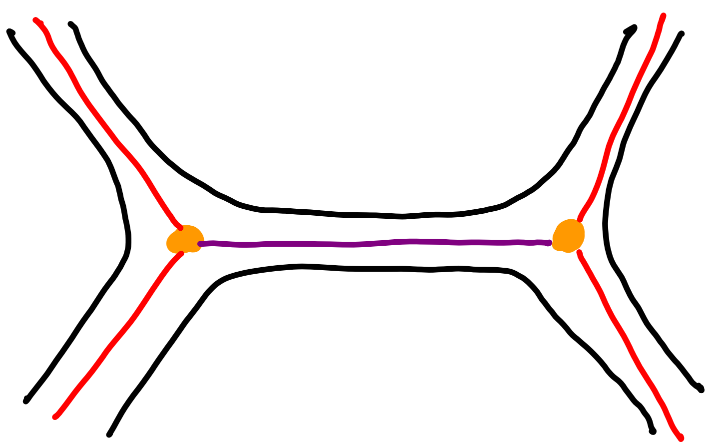
- EITHER $w_a \neq w_b$
- OR $w_a = w_b$

HYPERMULTIPLET JUMP: $W_a \neq W_b$

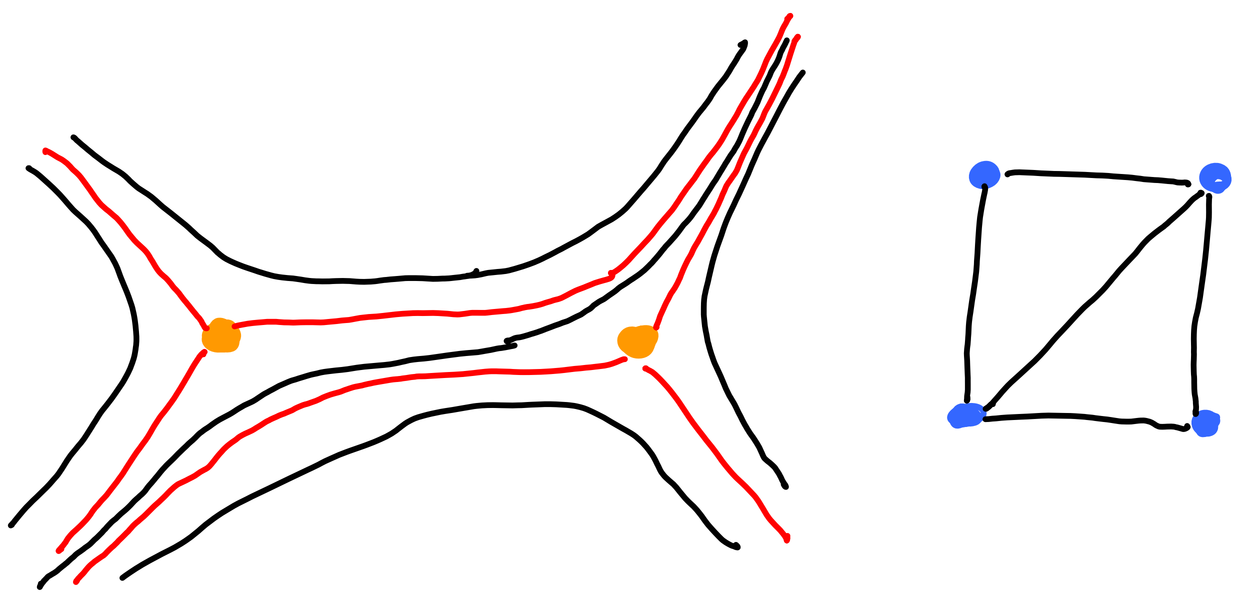
$\vartheta < \vartheta_c$



$\vartheta = \vartheta_c$

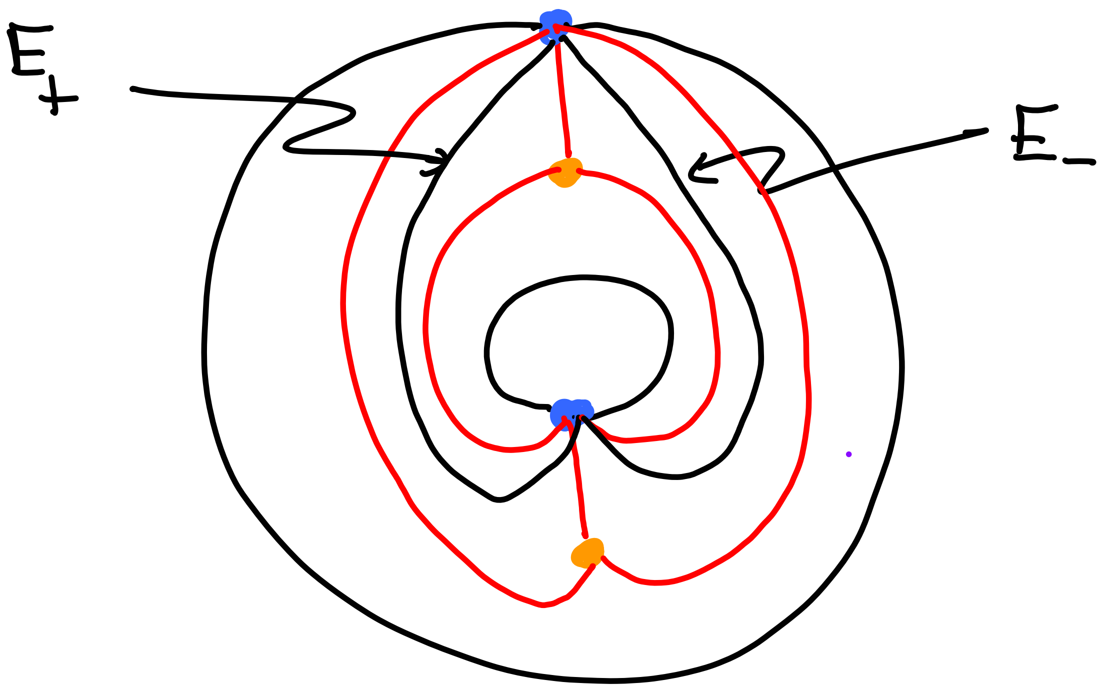


$\vartheta > \vartheta_c$



THIS IS JUST A FLIP

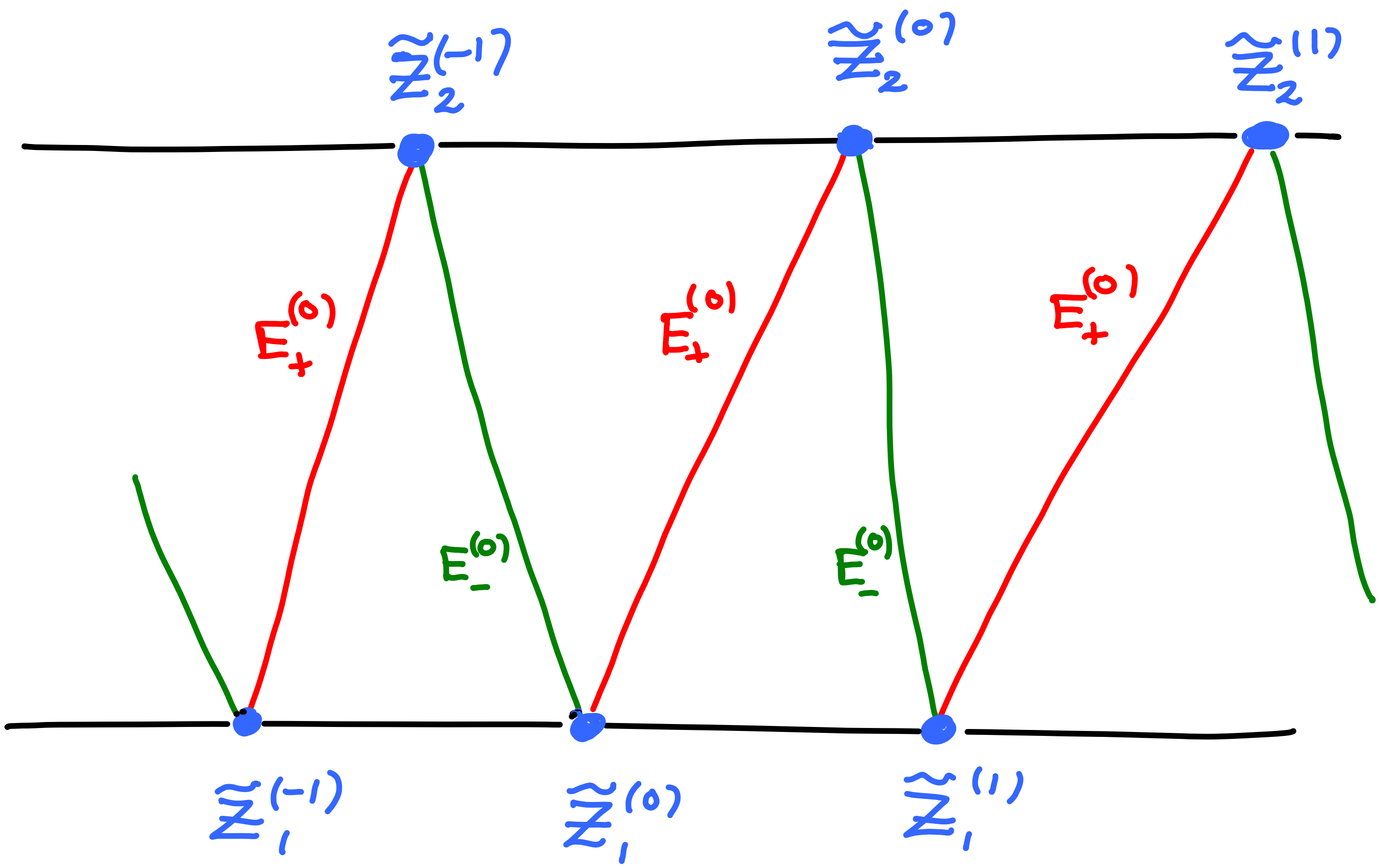
VECTORMULTIPLET JUMP: $W_a = W_b$



$\vartheta < \vartheta_c$

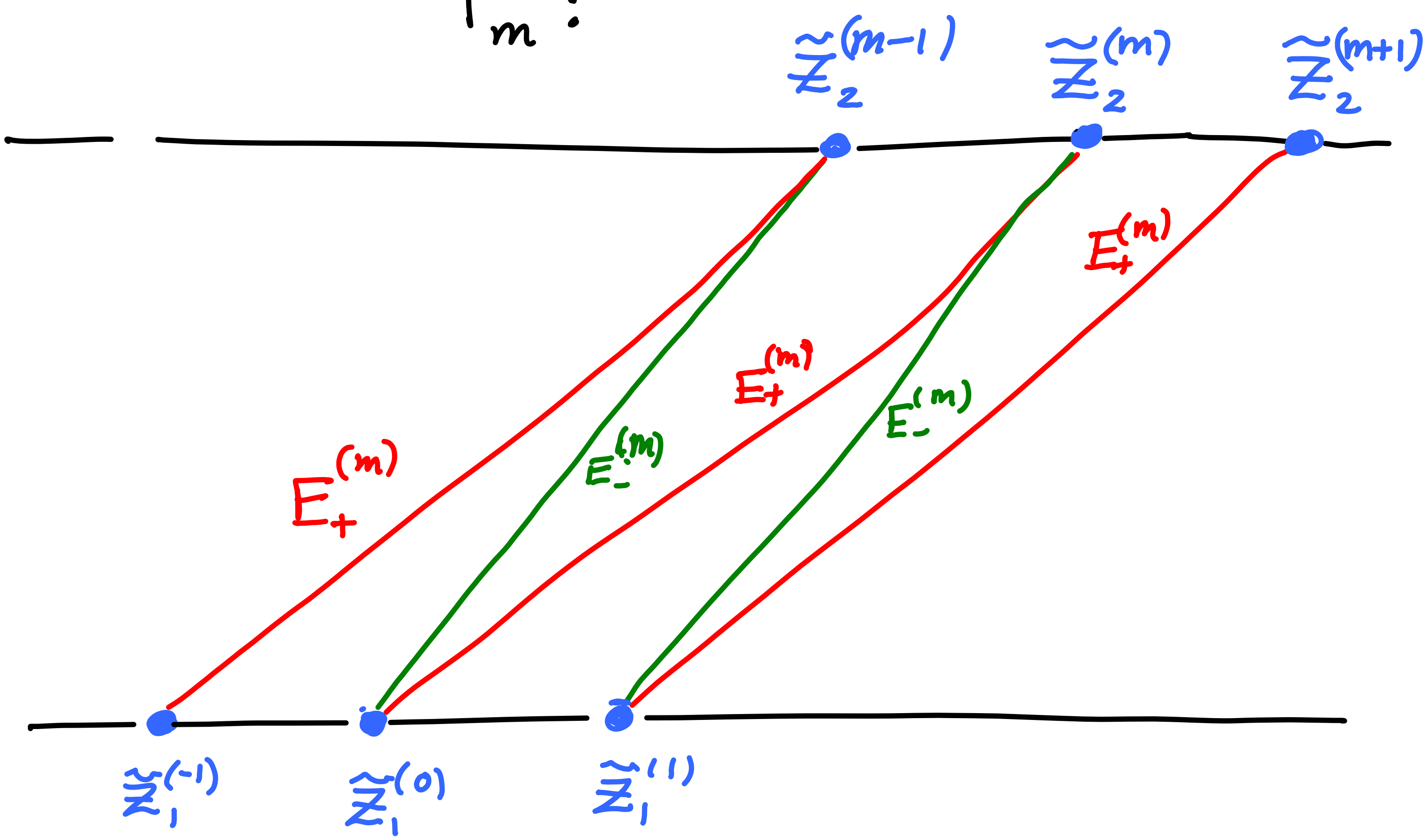
AS $\vartheta \to \vartheta_c^-$ $T(\vartheta, \lambda)$ HAS AN INFINITE SEQUENCE OF FLIPS

$E_+ E_- E_+ E_- \cdots\cdots$



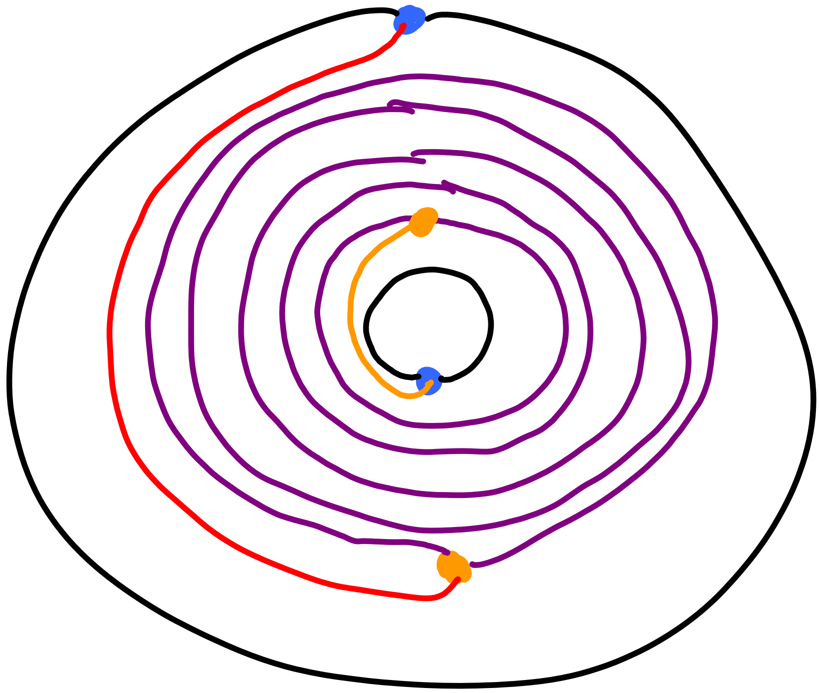
$\tilde{Z}_2^{(-1)}$
$\tilde{Z}_2^{(0)}$
$\tilde{Z}_2^{(1)}$
$E_+^{(0)}$
$E_+^{(0)}$
$E_+^{(0)}$
$E_-^{(0)}$
$E_-^{(0)}$
$\tilde{Z}_1^{(-1)}$
$\tilde{Z}_1^{(0)}$
$\tilde{Z}_1^{(1)}$

T_m :



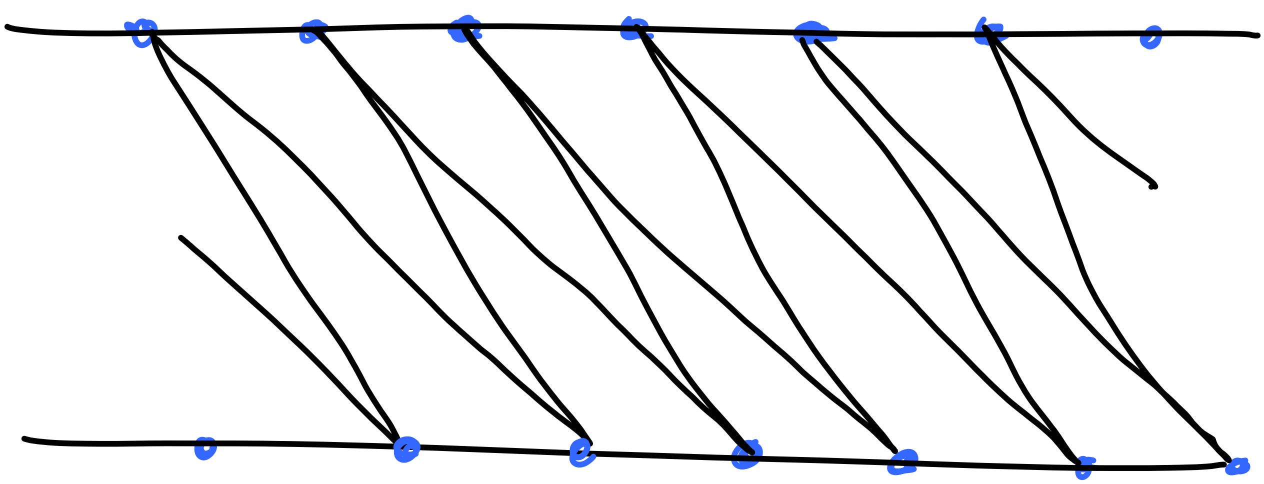
$\tilde{Z}_2^{(m-1)}$
$\tilde{Z}_2^{(m)}$
$\tilde{Z}_2^{(m+1)}$
$E_+^{(m)}$
$E_+^{(m)}$
$E_-^{(m)}$
$E_+^{(m)}$
$E_-^{(m)}$
$\tilde{Z}_1^{(-1)}$
$\tilde{Z}_1^{(0)}$
$\tilde{Z}_1^{(1)}$

AT $\vartheta = \vartheta_c$



$\vartheta = \vartheta_c$

AT $\vartheta > \vartheta_c$



T_{-m}

SUITABLE COMBINATIONS OF

$\chi_{E_+}^{T_m}$, $\chi_{E_-}^{T_m}$ HAVE LIMITS

FOR $m \to \infty$:

$$\chi_A^{T_{+\infty}} = \lim_{m \to \infty} \chi_{E_+}^{T_m} \chi_{E_-}^{T_m}$$

$$\chi_A^{T_{-\infty}} = \lim_{m \to -\infty} \chi_{E_+}^{T_m} \chi_{E_-}^{T_m}$$

$\Rightarrow$ ADD NEW OBJECTS $T_{\pm\infty}$ TO THE GROUPOID, CALL THEM "LIMIT TRIANGULATIONS"

NEW MORPHISMS:

$$T_{+\infty} \longrightarrow T_{-\infty}$$

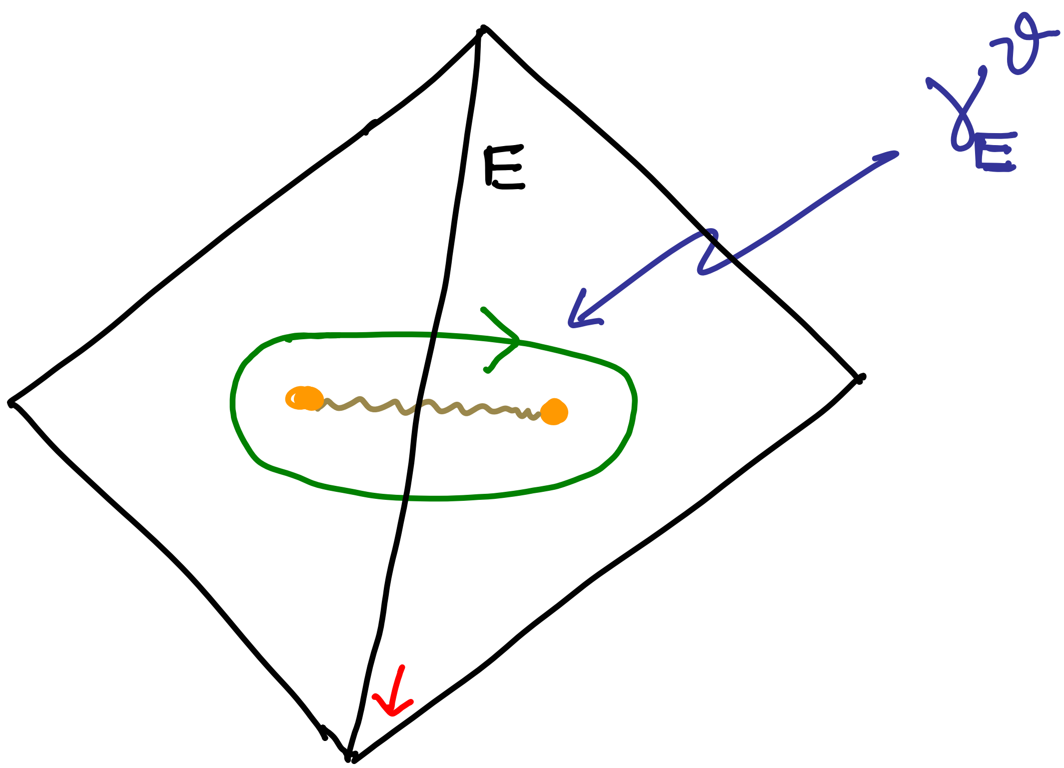
CALLED "TWISTS"

8. DEFINING THE TWISTOR COORD'S

FINALLY, TO DEFINE $\mathcal{X}^{\vartheta}_{\gamma}(\cdot,\zeta)$

WE ASSOCIATE TO $E \in \mathcal{E}(T(\vartheta,\lambda))$

CERTAIN CYCLES $\gamma^{\vartheta}_{E} \in H_1(\Sigma, \mathbb{Z})$



RULE: ORIENT THE LIFTS $\hat{E}$ SO THAT $e^{-i\vartheta}\langle \lambda, \partial_t \rangle > 0$: +VE

DEMAND $\langle \gamma^{\vartheta}_{E}, \hat{E} \rangle = +1$

THE $\{\gamma_E^{\vartheta}\}_{E \in \mathcal{E}(T)}$ FORM A (POSITIVE) BASIS FOR Γ.

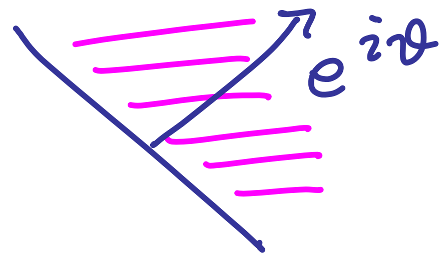
NOW DEFINE :

$$\chi^{\vartheta}_{\gamma_E^{\vartheta}} := \chi_E^{T(\vartheta, \lambda)}$$

$$\chi^{\vartheta}_{\gamma + \gamma'} := \chi^{\vartheta}_{\gamma} \, \chi^{\vartheta}_{\gamma'}$$

THEOREM 1: IF $R \to \infty$ AND

ζ IS IN $\mathbb{H}_\vartheta$:

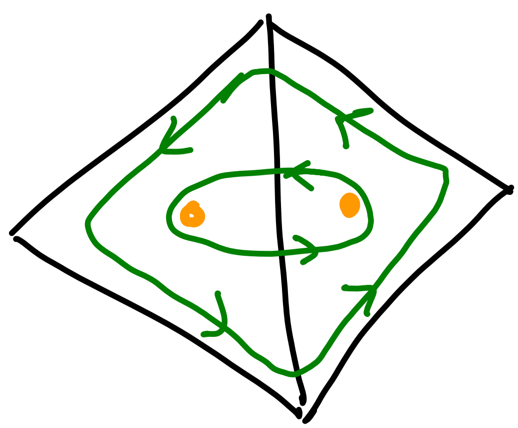


THEN

$$\mathcal{X}^{\vartheta}_{\gamma}(\cdot,\zeta) \underset{R\to\infty}{\sim} \exp\left(\frac{\pi R}{\zeta} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right)$$

RECOVERS NEITZKE-PIOLINE SEMIFLAT TWISTOR COORDINATES.

PROOFS:



$$s_i \sim \exp\left(\pm \frac{R}{\zeta}\int_{z_i}^{z} \lambda\right)\cdot\left\{\begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

{ USE RELATION TO 2D sinh-Gordon $\blacksquare$

THEOREM 2:

WITH RESPECT TO SYMPLECTIC STRUCTURE:

$$\omega_\zeta = \int_C \mathrm{Tr}\, \delta A\, \delta A$$

$$\{ \mathcal{X}^{\vartheta}_{\gamma}, \mathcal{X}^{\vartheta}_{\gamma'} \} = \langle \gamma, \gamma' \rangle \mathcal{X}^{\vartheta}_{\gamma+\gamma'}$$

THEOREM 3:

AT SUFFICIENTLY LARGE R

$$\mathcal{X}_\gamma(\cdot, \zeta) = \mathcal{X}^{\vartheta = \arg \zeta}_{\gamma}(\cdot, \zeta)$$

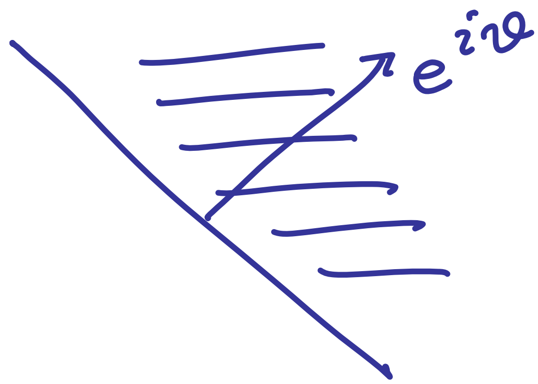
SATISFY THE 5 DEFINING PROPERTIES.

PROOF:

(1) $\mathcal{X}_\gamma(\cdot, \zeta)$ HOLOMORPHIC ON $\mathcal{M}^\zeta$:
FOCK & GONCHAROV

(2),(3) FOLLOW EASILY FROM THE DEFINITION

(4B) FOR $\zeta \to 0$ IN THE HALF-PLANE

$H_\vartheta =$ 

$$\lim_{\zeta \to 0} \mathcal{X}_\gamma^\vartheta(\zeta) \exp\left(\frac{-\pi R}{\zeta} Z_\gamma\right) \quad \text{EXISTS}$$

FOLLOWS FROM WKB ASYMPTOTICS AS WITH $R \to \infty$

(5) IF $\vartheta = \vartheta_c$ IS THE PHASE OF A BPS STATE OF CHARGE γ_0 THEN, DEFINING

$$\mathcal{X}^{\pm}_{\gamma} = \lim_{\vartheta \to \vartheta_c^{\pm}} \mathcal{X}^{\vartheta}_{\gamma}$$

$$\boxed{\mathcal{X}^{+}_{\gamma} = \mathcal{X}^{-}_{\gamma} \left(1 - \sigma(\gamma_0)\mathcal{X}^{-}_{\gamma_0}\right)^{\Omega(\gamma_0)\langle\gamma,\gamma_0\rangle}}$$

NOTE: $\sigma(\gamma_0) = +1, \ \Omega(\gamma_0) = -2$ VM

$\sigma(\gamma_0) = -1, \ \Omega(\gamma_0) = +1$ HM

1. FOR HM: CLUSTER TRMN.

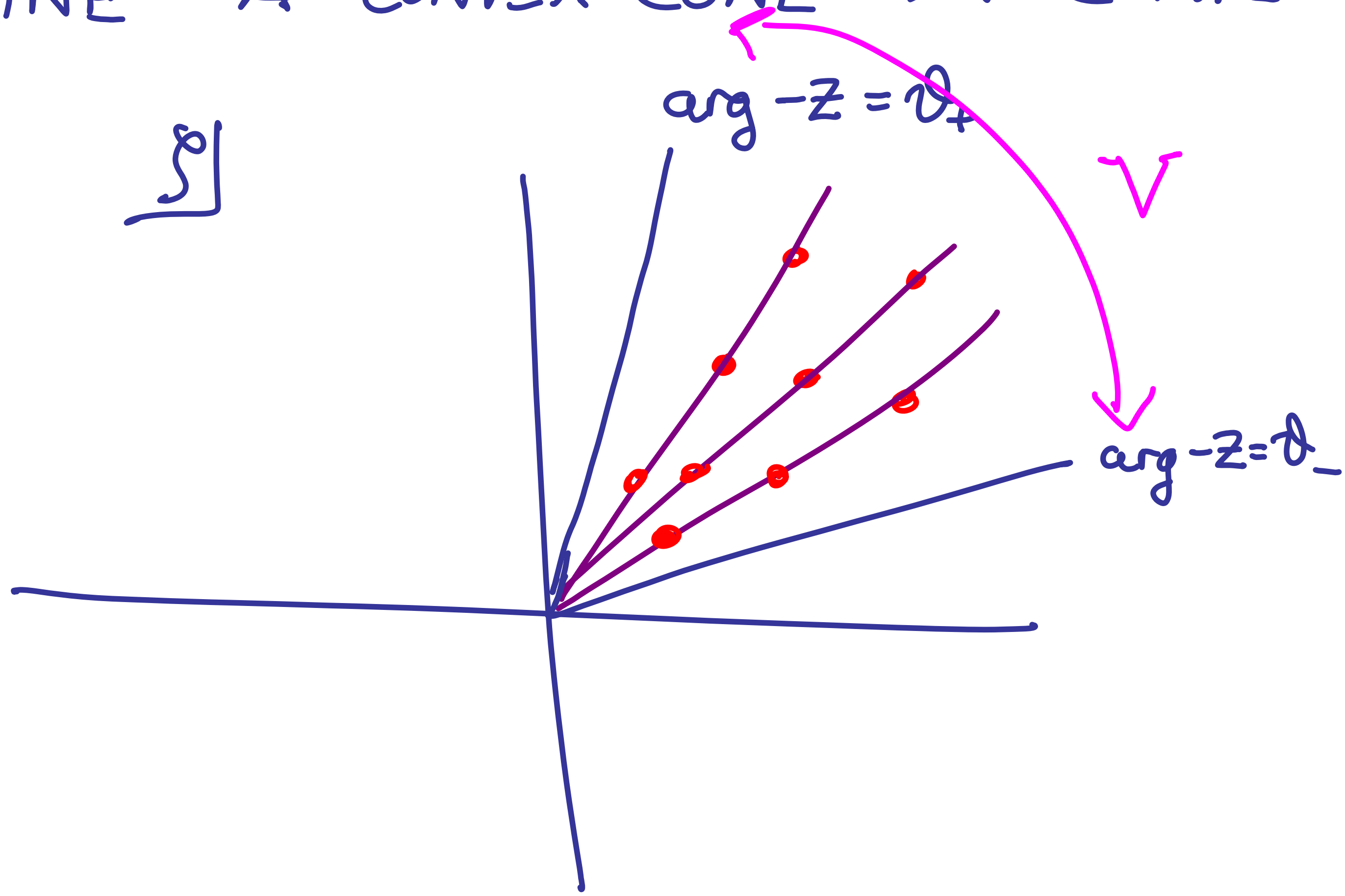
2. FOR VM: EXPLICIT COMPUTATION OF TWIST TMN:

$$\mathcal{X}^{T_{+\infty}} \longrightarrow \mathcal{X}^{T_{-\infty}}$$

∎

9. WALL CROSSING

CHOOSE $\vartheta_- < \vartheta_+$ TO DEFINE A CONVEX CONE IN COMPLEX $\mathcal{S}$



SUPPOSE WE FOLLOW A PATH u_- TO u_+ SO THAT NO BPS RAY CROSSES $\arg(-Z) = \vartheta_\pm$.

THEN $T(\vartheta_\pm, \lambda_-)$ SMOOTHLY EVOLVES TO $T(\vartheta_\pm, \lambda_+)$

ON THE OTHER HAND, EVOLVING ϑ_- TO ϑ_+ AT FIXED λ PRODUCES A SEQUENCE OF FLIPS, TWISTS, AND POPS.

<u>FACT</u>: ALL POPS OCCUR IN DEGENERATE TRIANGLES, AND THE INDUCED TRANSFORMATION IS 1 FOR SUCH POPS.

THEREFORE χ^{ϑ_+} IS RELATED TO χ^{ϑ_-} VIA THE IMAGE OF

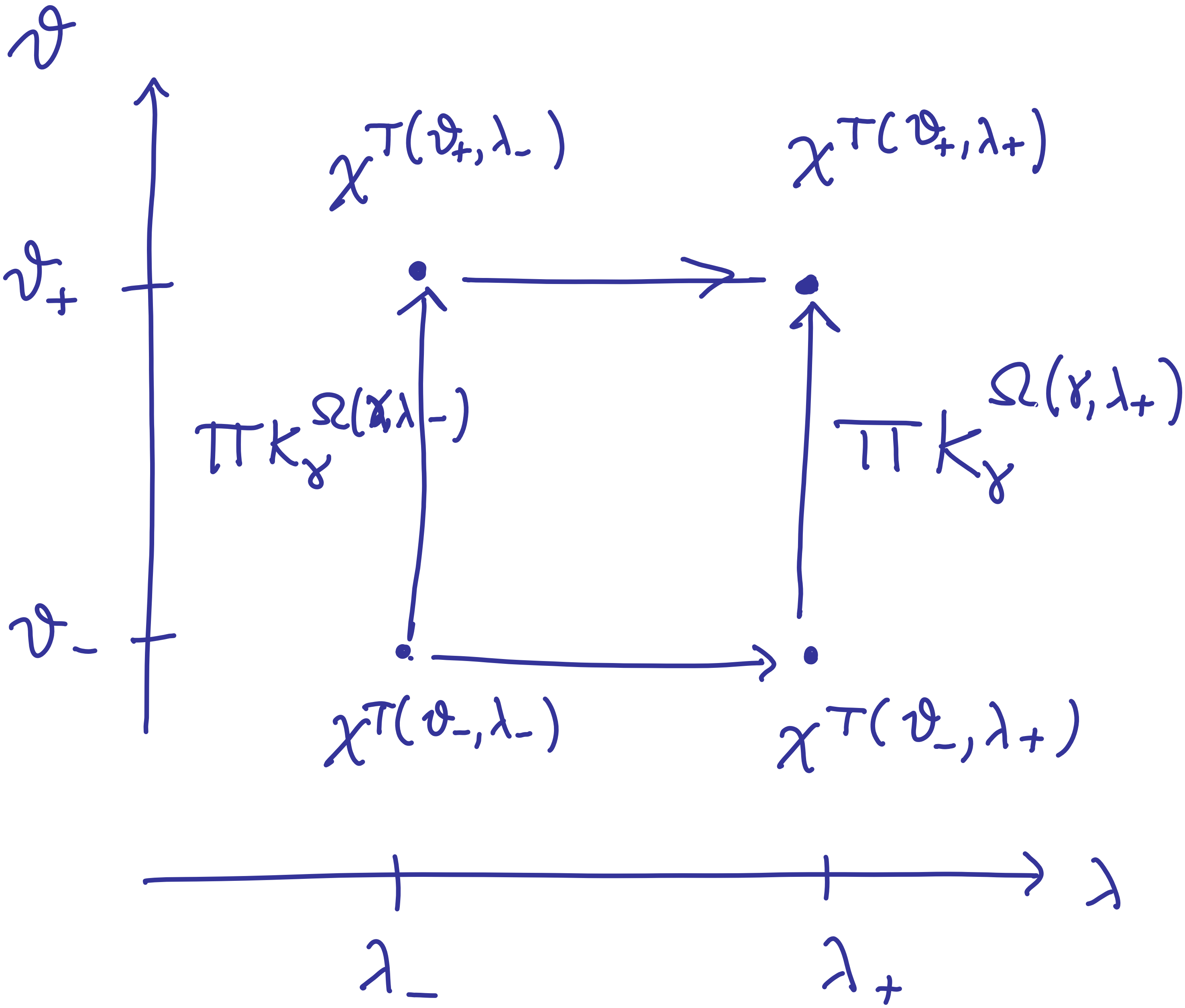
$$\prod_{\vartheta_- < \vartheta_c < \vartheta_+} \sigma_{\gamma^{\vartheta_c}_{E_c}}$$

i.e.

$$\prod_{\vartheta_- < \arg(-Z_\gamma) < \vartheta_+} K_\gamma^{\Omega(\gamma,\lambda)} = A_V$$

BUT THERE IS NO DISCONTINUITY

IN $\chi^{T}(\vartheta, \lambda_-) \longrightarrow \chi^{T}(\vartheta, \lambda_+)$



$$\boxed{\prod K_\gamma^{\Omega(\gamma, \lambda_-)} = \prod K_\gamma^{\Omega(\gamma, \lambda_+)}}$$

MOVIES

10. DETERMINING THE BPS SPECTRUM

NOW LET US VARY ϑ TO $\vartheta+\pi$.

WE CAPTURE ALL THE BPS STATES

$$\prod_{\vartheta_c} \sigma_{\gamma^{\vartheta_c}_{E_c}} \longrightarrow \prod_{\vartheta < -\arg Z < \vartheta+\pi} K_\gamma^{\Omega(\gamma,\lambda)}$$

ON THE OTHER HAND,

$T(\vartheta, \lambda)$ AND $T(\vartheta+\pi, \lambda)$

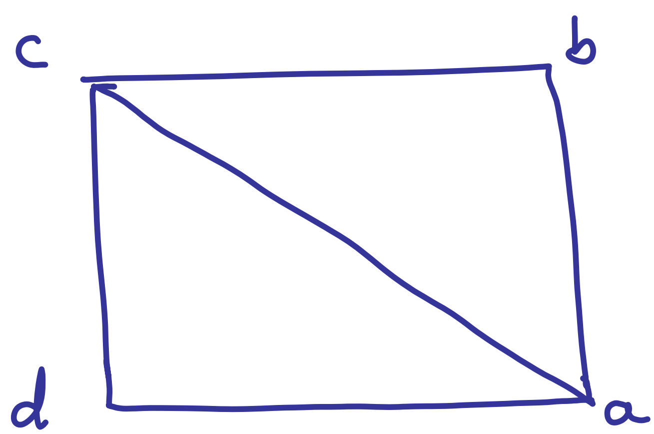
<u>ONLY</u> DIFFER BY

SIMULTANEOUSLY POPPING

ALL THE VERTICES!!

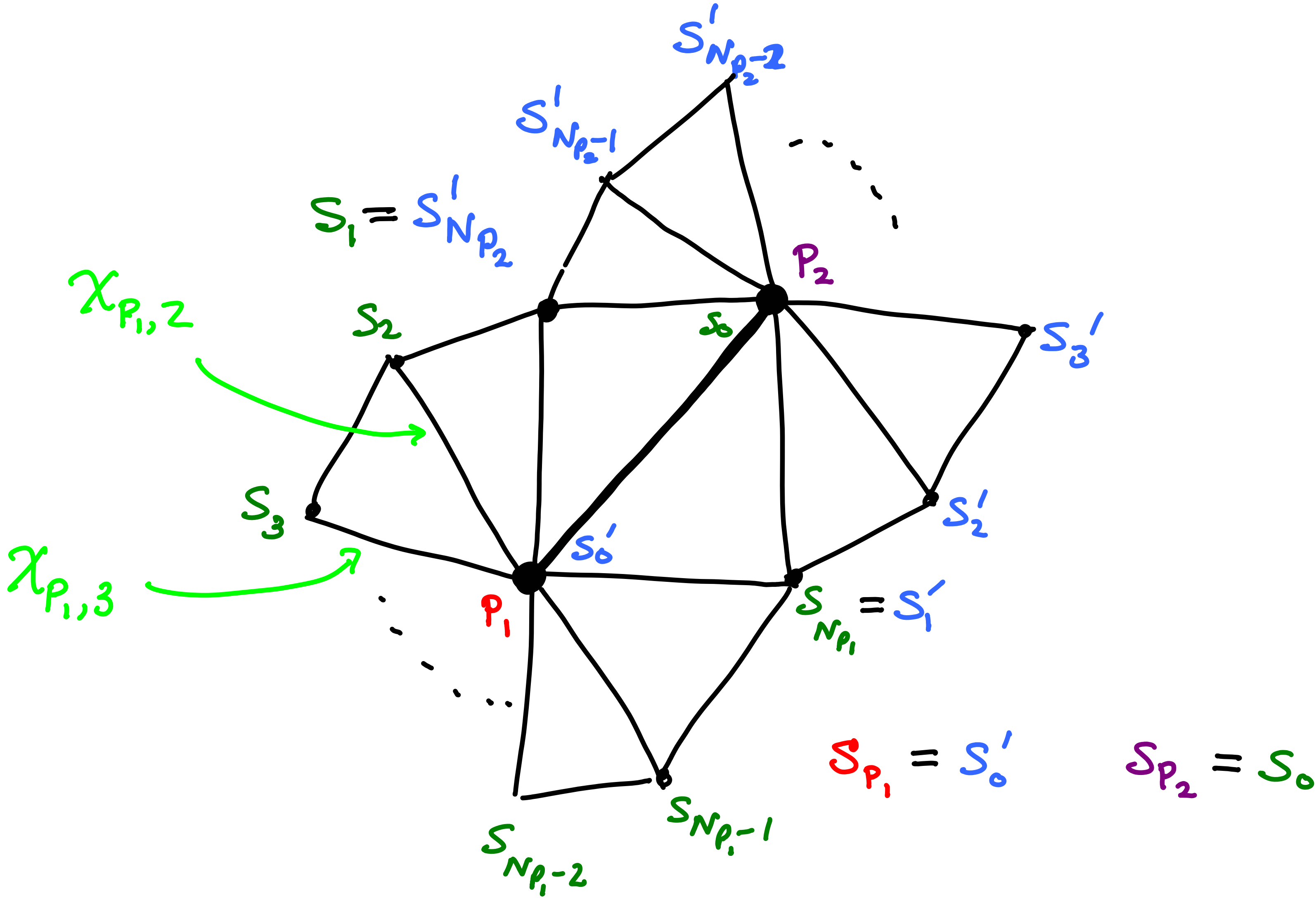
UNIVERSAL STOKES MATRIX

WHILE THE CHANGE χ^T_E FOR POPPING ONE VERTEX IS COMPLICATED, IT TURNS OUT THAT POPPING ALL VERTICES LEADS TO A RATHER SIMPLE FORMULA!



$$\tilde{\chi}^{\tilde{T}}_{ac}\,\chi^{T}_{ac} = \frac{(1+A_{ab})(1+A_{cd})}{(1+A_{bc})(1+A_{da})}$$

TO GIVE A FORMULA FOR $A_{P_1P_2}$:



$$A_{p_1 p_2} = \frac{1}{1-\mu_{p_1}^2} \frac{1}{1-\mu_{p_2}^2} \cdot \chi_{p_1 p_2} \cdot \left(1 + \sum_{k=1}^{N_{p_1}-1} \prod_{j=1}^{k} \chi_{p_1,j}\right) \cdot \left(1 + \sum_{k=1}^{N_{p_2}-1} \prod_{j=1}^{k} \chi_{p_2,j}\right)$$

TO FIND THE BPS SPECTRUM

THE TRANSFORMATION

$$S : \mathcal{X}_i \longrightarrow \tilde{\mathcal{X}}_i$$

$$\tilde{\mathcal{X}}_i = \mathcal{X}_i \frac{(1+A_{ab}(i))(1+A_{cd}(i))}{(1+A_{bc}(i))(1+A_{da}(i))}$$

HAS A UNIQUE DECOMPOSITION

OF THE FORM:

$$S = \prod_{\vartheta < -\arg Z < \vartheta+\pi} K_\gamma^{\Omega(\gamma,\lambda)}$$

$$\boxed{\text{THIS DETERMINES THE } \Omega(\gamma,u)}$$

CONCLUSION: FUTURE DIRECTIONS

1. WE HAVE SOME IDEAS ABOUT HOW TO GO TO RANK $k > 2$.

2. RELATION TO INTEGRABLE SYSTEMS (e.g. THE INTEGRAL EQUATION FOR $\mathcal{X}_\gamma$ IS A VERSION OF THE TBA.)

3. SUPERGRAVITY

4. NEW MODULAR FUNCTORS