Quantum Field Theory And Invariants Of Smooth Four-Dimensional Manifolds

Gregory Moore Rutgers University

British Math Colloquium, June 6, 2022

1. Physical Mathematics
2. Math & Physics Problems
3. Instantons & Donaldson Invariants
4. Topological Field Theory
5. Low Energy Effective Field Theory
6. New Results On Z_u
7. Directions For Future Research

Phys-i-cal Math-e-ma-tics, n.

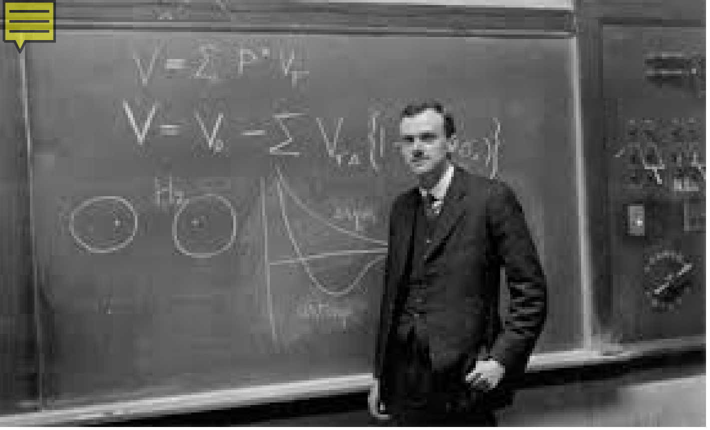
Pronunciation: Brit. /ˈfɪzɨkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, *but equally central*, goals of

1. Elucidating the laws of nature at their most fundamental level,

together with

2. Discovering deep mathematical truths.



1931: Dirac's Paper on Monopoles

Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

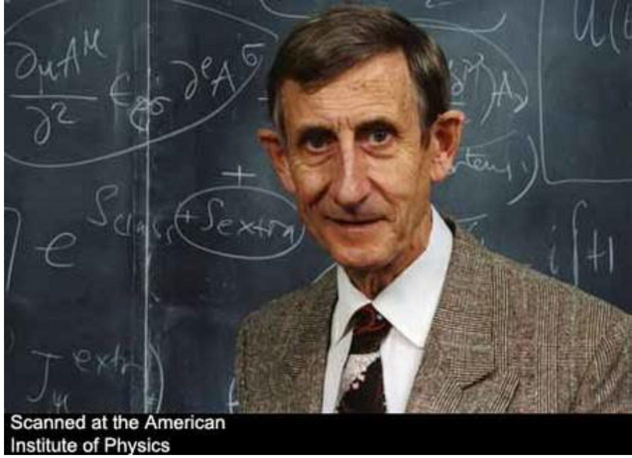
Received May 29, 1931

§ *1. Introduction*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

1972: Dyson's Announcement



MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

> It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
> JACQUES HADAMARD

1. **Introduction.** The purpose of the Gibbs lectures is officially defined as "to enable the public and the academic community to become aware of the contribution that mathematics is making to present-day thinking and to modern civilization." This puts me in a difficult position. I happen to be a physicist who started life as a mathematician. As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce. Discussing this divorce, the

Well, I am happy to report that Mathematics and Physics have remarried!

Change began in the 1970's …..

Some great mathematicians got interested in aspects of fundamental physics .....

While some great physicists started producing results requiring ever increasing mathematical sophistication, ...

Physical Mathematics

In the past few decades a new field has emerged
with its own distinctive character, its own aims and values,
its own standards of proof.

One of the guiding principles is certainly the discovery
of the ultimate foundations of physics.

This quest has led to ever more sophisticated mathematics…

A second guiding principle is that physical insights can lead to
surprising and new results in mathematics

Such insights are a great success - just as profound and
notable as an experimental confirmation
of a theoretical prediction.

Today:

I will explain just one emblematic example of a remarkable convergence of physical and mathematical ideas.

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6 New Results On Z_u

7 Directions For Future Research

Four Dimensional Differentiable Topology

X: Four-dimensional, compact, oriented, simply connected, smooth manifold without boundary.

We do not know anything even close to a complete topological invariant.

One thing we know for sure is that the world of such four-dimensional manifolds is really wild.

Clay Mathematics Proceedings
Volume 5, 2006

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

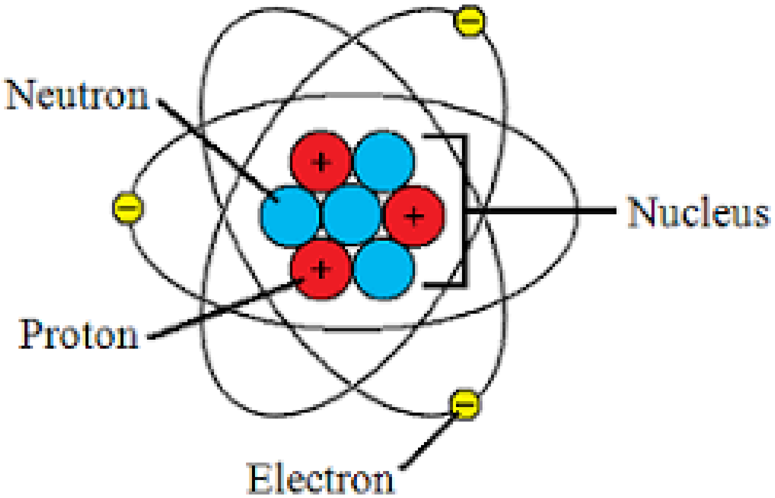
Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction

Nuclear Force

It's cozy in there:

$r = 10^{-15} m.$



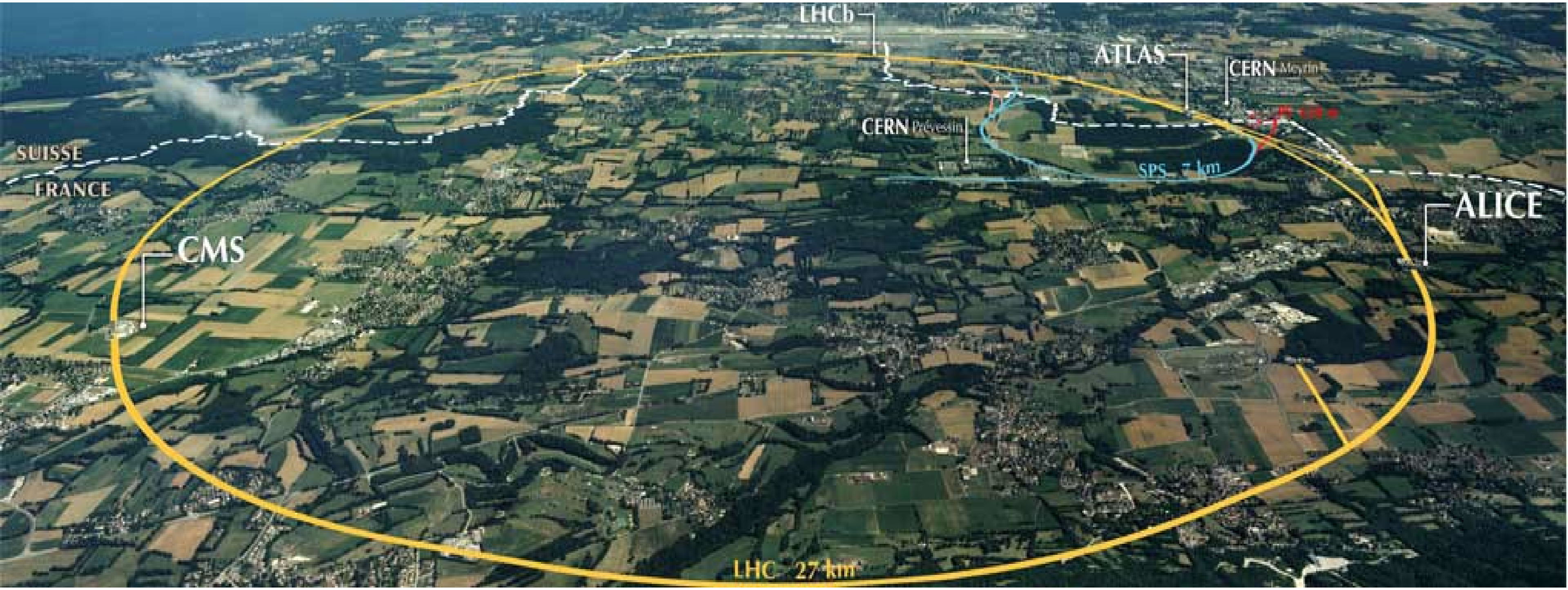
Protons have positive electric charge

Electrical force between two protons at this distance produces an acceleration .... $\sim 10^{28}\ g$

Fastest roller coaster $\sim 6\ g$

Fighter pilots $\sim 9\ g$

The strong force is very subtle – it has been studied with particle accelerators for decades - up to the present day…



Large Hadron Collider at CERN

Mathematical Formulation Of The Strong Force: Yang-Mills Theory

G: Compact finite dimensional Lie group

Lie algebra $\mathfrak{g}$ equipped with invariant bilinear form tr

$\mathcal{A}$: Space of connections on principal G-bundles over X

$\mathcal{A}/\mathcal{G}$: Connections up to isomorphism (gauge equivalence classes)

Given ∇_P : connection on $P \to X$ ***AND*** a Riemannian metric on X

Yang-Mills action $A_{YM} = \int_X tr(F \wedge * F)$

$*$: Hodge star: Depends on the metric $g_{\mu\nu}$ on X

A_{YM} descends to a function on $\mathcal{A}/\mathcal{G}$

Formally, $\exists$ natural translation invariant measure on $\mathcal{A}$ pushes forward to a measure $d\mu$ on $\mathcal{A}/\mathcal{G}$

Physicists want: $d\mu\, e^{-A_{YM}}$ as a probability measure on $\mathcal{A}/\mathcal{G}$

Expectation values: ``path integral''

Overwhelming evidence suggests it makes mathematical sense

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Reminder On Self-Duality

$*^2 = 1$. For any 2-form ω define (anti-)self-dual projections:

$$\omega_{\pm} := \frac{1 \pm *}{2}\omega = \frac{1}{2}(\omega \pm * \omega)$$

$\mathcal{H}^2(X)$ splits into orthogonal sum of (anti-)self-dual forms dimensions $b_2^{\pm}$

$$b_2^+ + b_2^- = \dim H^2(X)$$

$\pm$ subspaces for intersection form on $H^2(X)$

Most important connections: The minima of A_{YM}

In general we cannot set $F = 0$.

$$k := -\int_X tr\, F^2 \in \mathbb{Z}$$

Locally constant on $\mathcal{A}$

$$\mathcal{A} = \coprod_{k\in\mathbb{Z}} \mathcal{A}_k$$

$$2\int_X tr(F \ * F\ ) = \int_X tr[(F + * F) * (F + * F)]$$

$$-2\int_X tr\, F^2$$

Instantons -1.1

$$2\int_X tr(F \; * F) = \int_X tr(F + * F)^2 - 2\int_X tr\, F^2$$

A solution of the anti-self-dual YM equation:

$$F_+ := \frac{1}{2}(F + * F) = 0$$

will mimimize the action.

Such solutions only exist for $k \geq 0$

Instantons – 1.2

Solutions to $F_+ = 0$ are called *instantons*.

Ever since their discovery in 1975 by Belavin, Polyakov, Schwartz, and Tyupkin they have been intensively studied by physicists interested in Yang-Mills theory.

Meanwhile, back at the ranch

…. mathematicians such as Atiyah, Bott, Drinfeld, Hitchin, Manin, Singer, Uhlenbeck…. were producing remarkable mathematical results about these instantons.

There are continuous families of instanton solutions:

$\mathcal{M} \subset \mathcal{A}/\mathcal{G}$: Instanton moduli space: $\mathcal{M} = \coprod_{k\in\mathbb{Z}} \mathcal{M}_k$

G: Simple Lie group:

$$\dim_{\mathbb{R}} \mathcal{M}_k = 4hk - \dim G \frac{(\chi+\sigma)}{2}$$

[Atiyah-Hitchin-Singer 1977]

$$\dim_{\mathbb{R}} \mathcal{M}_k = 8k - \frac{3}{2}(\chi+\sigma) \qquad G = SU(2), SO(3)$$

$\mathcal{M}$ depends on the Riemannian metric on X

Donaldson Invariants

$S \subset X$: smooth surface.

$\mathcal{M}(S)$: subspace where the Dirac equation on S coupled to ∇_P has a solution

Poincare dual to $\mathcal{M}(S)$ defines $\mu(S) \in H^2(\mathcal{M};\mathbb{Z})$

$$Z_D(S) := \int_{\mathcal{M}} e^{\mu(S)} = \sum_k \int_{\mathcal{M}_k} \frac{\mu(S)^r}{r!}$$

$\mu(S)$: a linear function of $S \in H_2(X)$

$Z_D(S)$ is a function on $H_2(X)$

$Z_D(S)$ is metric independent

$\Rightarrow$ Topological Invariant of X !

The discovery of these and related topological invariants led to major advances in the differential topology of four-manifolds

Donaldson invariants are extremely hard to compute

It took a lot of effort to compute a few special examples….

Mathematicians hit a wall…

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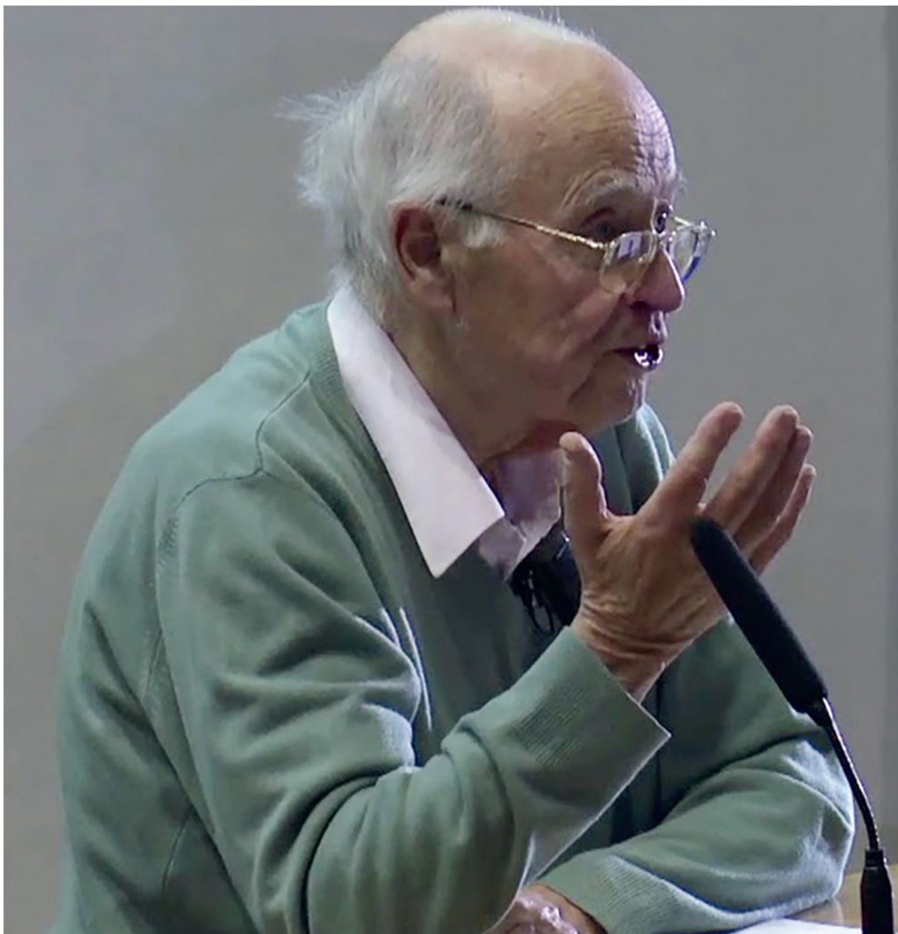
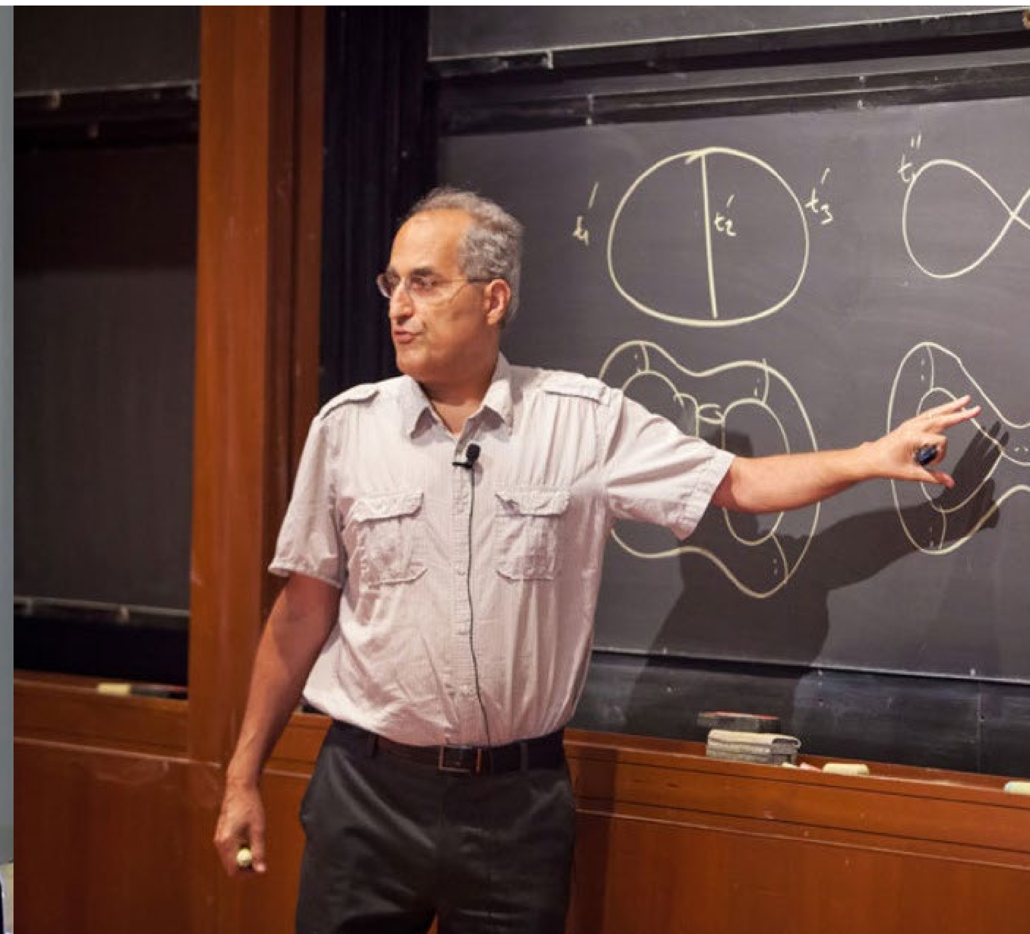
A Turning Point: Atiyah’s Question:

What is ***<u>the physical interpretation</u>*** of the Donaldson invariants?





Witten's Answer

Donaldson invariants can be computed within the framework of a Yang-Mills field theory (with gauge group $SU(2)$)

Commun. Math. Phys. 117, 353–386 (1988)

Communications in
Mathematical
Physics

Topological Quantum Field Theory

Edward Witten*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

N=$\mathbb{2}$ Super-Yang-Mills Theory

Fields: $\nabla_P \in \mathcal{A}$ $\phi \in \Gamma(adP \otimes \mathbb{C})$

Fermions/anticommuting: $\psi \in \Gamma(adP \otimes S^+ \otimes E)$

S^+: Chiral spin bundle over X

$E \to X$: Rank $\mathbb{2}$ complex vector bundle with connection ∇_{E} and structure group $SU(2)_R$

N.B. If X is not spin we can still define ψ making a proper choice of E

Topological Twisting

In SYM we replace $\mathcal{A}/\mathcal{G}$ by a super-manifold $\mathcal{F}$ fibered over $\mathcal{A}/\mathcal{G}$

There is still a formal probability measure (Berezinian) $d\mu\, e^{-A_{SYM}}$ on $\mathcal{F}$ with A_{SYM}:

$$\int_X \frac{1}{g^2} tr\,(F * F + D\phi * D\phi^* + [\phi, \phi^*]^2 + \bar{\psi} D_E \psi + \cdots)$$

For a special choice of E, ∇_E the variation of the measure wrt metric $g_{\mu\nu}$ on X is a *total derivative*

Observables In Witten's Theory

$S \subset X$: smooth surface.

$$\mathcal{O}(S) := \int_S tr(\phi F + \psi^2)$$

$$Z_W(S) := \langle e^{\mathcal{O}(S)} \rangle$$

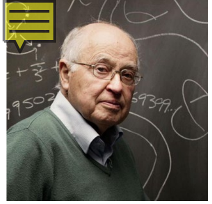
With Top. Twist, i.e., special E, ∇_E ,

$\mathcal{O}(S)$ only depends on $S \in H_2(X)$

So $Z_W(S)$ is a function on $H_2(X)$

$Z_W(S)$ independent of metric

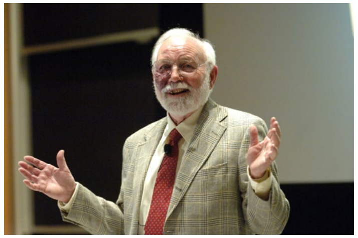
Localization



M. Atiyah & L. Jeffrey: The measure can be interpreted as a representative of a Thom class for the normal bundle of $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$

$\Rightarrow$ Path integral over $\mathcal{F}$ (assumed to exist) localizes to an integral over $\mathcal{M}$

L. Baulieu & I. Singer:



Under localization $\mathcal{O}(S)$ pulls back to $\mu(S)$

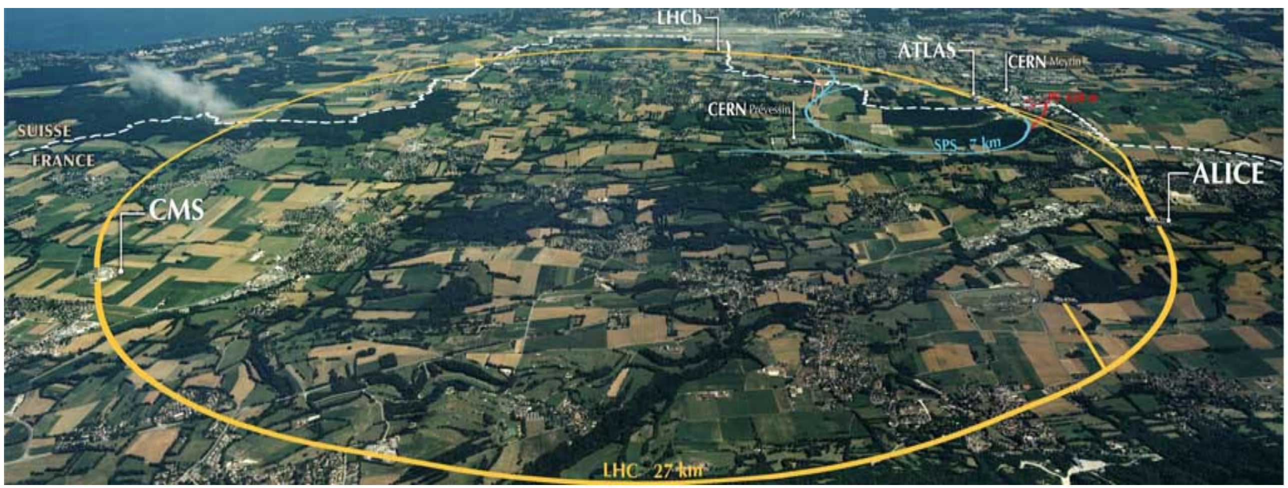
Claim: $Z_D(S) = Z_W(S)$

Quantum Field Theory

Computing Donaldson invariants a la Witten requires computing expectation values in a Yang-Mills theory

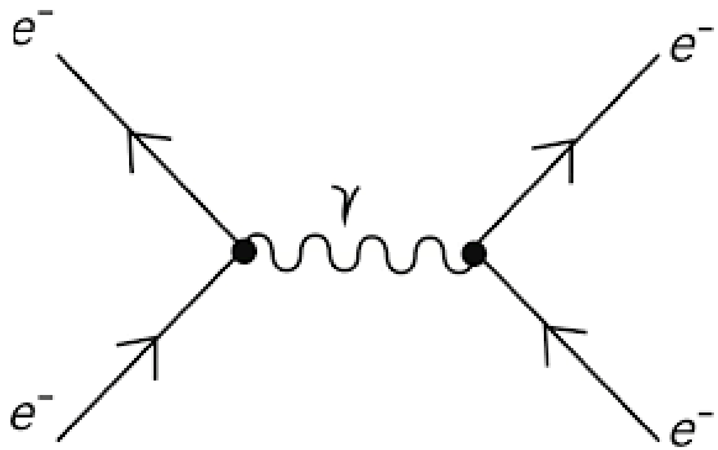
How hard can that be?

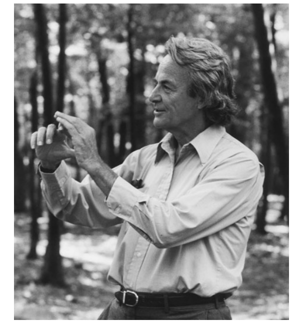
Abelian field theory

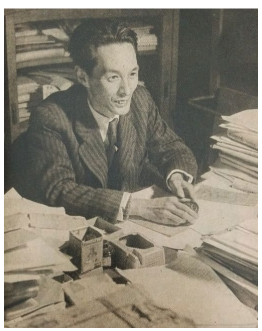


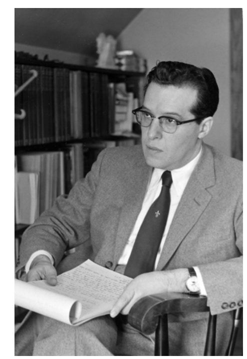
Nonabelian field theory

Abelian: Maxwell's theory: Hard, but solvable - Dyson, Feynman, Schwinger, Tomonaga (1946-1949)









Nonabelian: $MUCH^n$ harder:

Not solved yet

So, computing correlation functions of operators in *nonabelian* Yang-Mills theory is extremely difficult

So we seem to have exchanged one hard problem for another…..

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Low Energy Effective Theory – 1.1

$Z_{DW}(S)$ independent of $g_{\mu\nu}$ on X

Consider metric $L^2 g_{\mu\nu}$ in the limit $L \to \infty$

Quantum theory: Length ~ 1/Energy

Minima of energy (or action): "Vacua"

Small deviations from a vacuum can be described by a DIFFERENT QFT: The LEET

Low Energy Effective Theory – 1.2

We ***ask*** a question Q of a QFT, with action A,

We ***answer*** using a *different* QFT
with action A_{LEET}
(perhaps with a totally new set of fields)

The answer to Q is the ***SAME*** as the answer to an analogous question Q_{LEET}.

Moduli Spaces Of Vacua On $\mathbb{R}^4$

$$A_{SYM} = \int_{\mathbb{R}^4} tr(F * F + |D\phi|^2 + [\phi, \phi^*]^2)$$

Vacua: $\phi(\mathrm{x})$ is constant: $\phi(x) = \phi_{vac} \in \mathfrak{g}$

$[\phi, \phi^*] = 0 \Rightarrow \phi_{vac}$ is semisimple.

Any constant ϕ_{vac} will do!

There is not a unique
Poincare - invariant vacuum.

There is a ***moduli space of vacua***:

Moduli Spaces Of Vacua - 2/2

Persists in the quantum theory:

$\exists\ |\Omega(u)\rangle$: Poincare invariant quantum vacua.

For $SU(2)$: labeled by $u \in \mathbb{C}$

$$\langle\Omega(u)|tr(\phi^2)|\Omega(u)\rangle = u$$

There is not one probability measure on $\mathcal{F}$ but (for $G = SU(2)$) a continuous family labled by ``vacua'' $u \in \mathbb{C}$

Spontaneous Symmetry Breaking

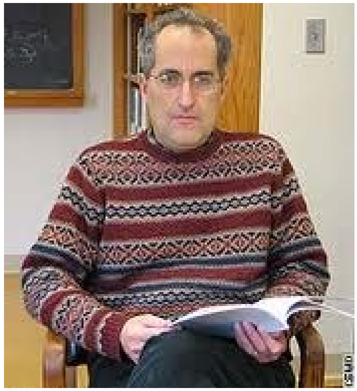
For $G = SU(2)$: gauge ϕ_{vac} to the form

$$\phi_{vac} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

Automorphisms of the vacuum are U(1) gauge transformations.

$\Rightarrow$ The LEET of small field deviations around a vacuum u is an ABELIAN GAUGE THEORY!

Seiberg-Witten Paper



Seiberg & Witten (1994)

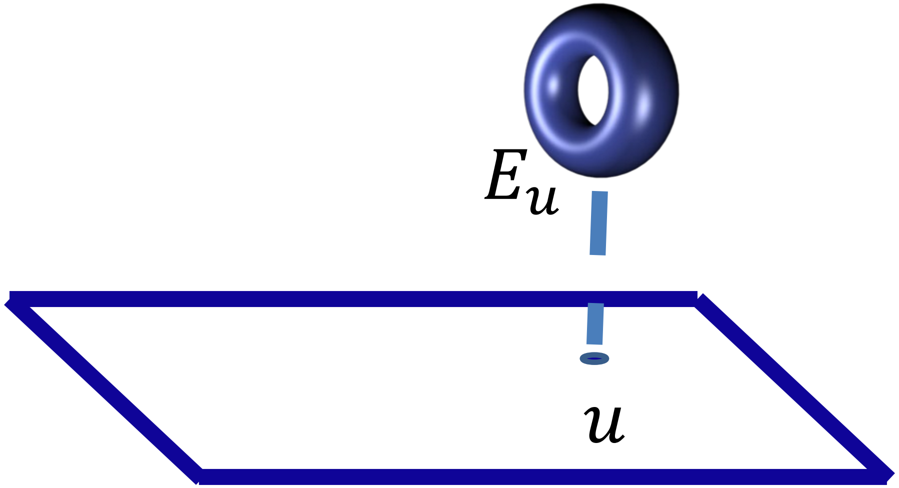
The LEET has action A^{u}_{LEET} that depends on u

Seiberg-Witten: A^{u}_{LEET} can be computed from the periods of a meromorphic 1-form λ_u on a family of Riemann surfaces E_u

$$G = SU(2):$$

$$E_u: \quad y^2 = x^2(x-u) + \frac{1}{4}x$$

$$\lambda = \frac{dx}{y}(x-u)$$



Local System Of Charges

Electro-mag. charge lattice: $\Gamma_u = H_1(E_u; \mathbb{Z})$

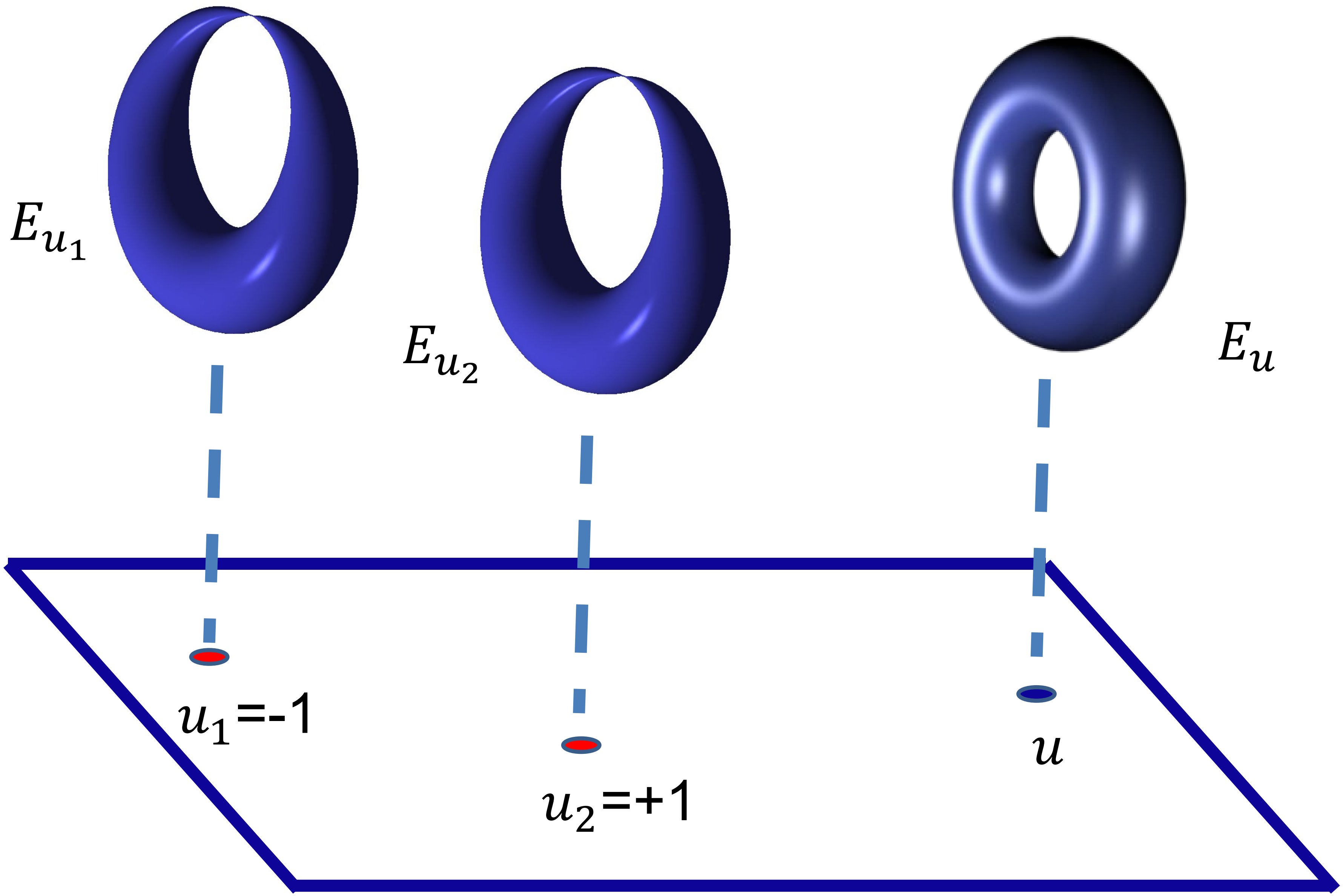
LEET only depends on (E_u, λ_u).

But! to write an action we must *choose* a splitting: $\Gamma_u \cong \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m \Rightarrow \tau(u)$

$$A^u_{LEET} \sim \int_X \overline{\tau(u)} F_+^2 + \tau(u) F_-^2 + \cdots$$

LEETs with different splittings are related by ``electromagnetic duality''

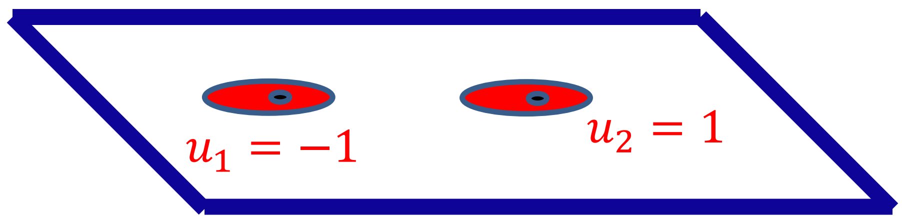
Γ_u has nontrivial monodromy around discriminant locus: $\Delta(u_j) = 0$



Seiberg-Witten Theory - II

Cannot use the same gauge field for all u:
Transitions by electro-magnetic duality

LEET breaks down near discriminant locus:



BUT!

Near $\mathrm{u_j}$ we can use a *different* LEET – there are new fields (``monopole fields'') which need to be included.

It is *also* an Abelian gauge theory and has an action $A_{LEET}^{\mathrm{u_j}}$

X: SUMMING OVER VACUA

Quantum effects $\Rightarrow$ the path integral for *compact* X will be given by a sum over ALL the vacua on $\mathbb{R}^4$.

Two kinds of vacua

Continuous moduli space of vacua parametrized by $\mathrm{u} := \langle tr\phi^2 \rangle_u \in \mathbb{C}$

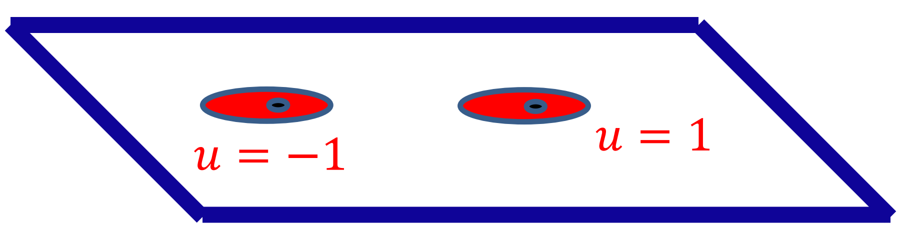
ALSO: The LEET $A_{LEET}^{u_j}$ near the discriminant loci $u_j \in \{ -1, +1 \}$ each has its own set of vacua.

Vacua of $A_{LEET}^{u_j}$ are enumerated by the solutions to the renowned Seiberg-Witten equations

Evaluate $Z_{DW}(S)$ Using LEET

$$Z_D(S) = \int_{\mathcal{M}} e^{\mu(S)} = Z_W(S) = \langle e^{\mathcal{O}(S)} \rangle$$

$$Z_{DW} = Z_u + \sum_{u_j} Z_j^{SW}$$



Preliminary Comments On Z_u

Z_u an explicit integral over complex u −plane computed using QFT techniques

We will return to it and discuss it in detail.

But first let's finish writing down the full answer for the path integral.

Contributions From u_j

Path integral for the LEET with action $A_{LEET}^{u_j}$ takes the general form of a sum over vacua (soln's to SWE):

$\mathfrak{spin}c$ structure:

$w_2(TX) = \lambda \ mod\ 2$

$$\sum_{\lambda \in \mathfrak{Spin}c} SW(\lambda)\, R_\lambda(S)$$

$$R_\lambda(S) := Res_{u_j}\left[\left(\frac{du}{u}\right) e^{S^2 T(u) + i\frac{S\cdot\lambda}{\omega(u)}} C(u)^{\lambda^2} P(u)^\sigma E(u)^\chi \right]$$

$\omega(u)$: Nonvanishing period in the neighborhood of u_j

$$T(u) \sim \omega(u)^{-2} E_2(\tau(u))$$

Everything is already known ***EXCEPT***:

$$\boldsymbol{C(u)}^{\lambda^2} P(u)^{\sigma} E(u)^{\chi}$$

Functions C, P, E : independent of X but difficult to derive from first principles.

Using a phenomenon known as ``wall crossing'' one can derive C, P, E, directly from Z_u

$\Rightarrow \quad Z_u$ is of fundamental importance.

Witten Conjecture

$Z_u = 0$ when $\mathrm{b}_2^+(X) > 1 \Rightarrow$

If X has $b_2^+(X) > 1$ then:

$$Z_{DW}(S) = 2^{c^2-\chi_h} \left(e^{\frac{1}{2}S^2} \sum_{\lambda} SW(\lambda)\, e^{S\cdot\lambda} + \right.$$

$$\left. + \, i^{\chi_h} e^{-\frac{1}{2}S^2} \sum_{\lambda} SW(\lambda) e^{-iS\cdot\lambda} \right)$$

$$\chi_h = \frac{\chi + \sigma}{4}$$

$$c^2 = 2\chi + 3\,\sigma$$

Example: $X = K3$:

$$Z_{DW}(S) = \sinh\left(\frac{1}{2}S^2\right)$$

The Donaldson invariants can be written in terms of the Seiberg-Witten invariants: They carry the same information about the four-dimensional space

Much easier to compute!


MIDNIGHT
MADNESS
All Your Holiday Shopping Done Before The Sun Comes Up

Comments On The Witten Conjecture

Agrees with structure theorem of P. Kronheimer and T. Mrowka – whose work provided important background for Witten's conjecture.

Physical derivation sketched above was given by G. Moore and E. Witten in 1997

Complex surfaces: L. Gottsche, H. Nakajima, & K. Yoshioka 2006

P. Feehan and T. Leness have outlined a rigorous proof of this conjecture using standard techniques of differential geometry.

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u-Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$.   When $b_2^+ = 1$:

$$Z_u = \int du\, d\bar{u}\ \ \mathcal{H}\ \ \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: Sum over principal $U(1)$ bundles for gauge field of the Abelian LEET:

$$\Psi \sim \sum_{v=c_1(L)} e^{-i\,\pi\bar{\tau}(\bar{u})v_+^2 - i\,\pi\tau(u)v_-^2}$$

NOT holomorphic and ***metric- DEPENDENT***

u-Plane: Recent Progress 1.1

The u-plane integral turns out to be closely related to the theory of mock modular (and Jacobi) forms

Special examples: Moore-Witten (1997) and Malmendier-Ono (2008)

New point: The relation holds for ALL 4-manifolds with $b_2^+ = 1$

G. Korpas, J. Manschot, G. Moore, I. Nidaiev (2019)

u-Plane: Recent Progress 1.2

$$E_u: \quad y^2 = x^2(x-u) + \frac{1}{4}x$$

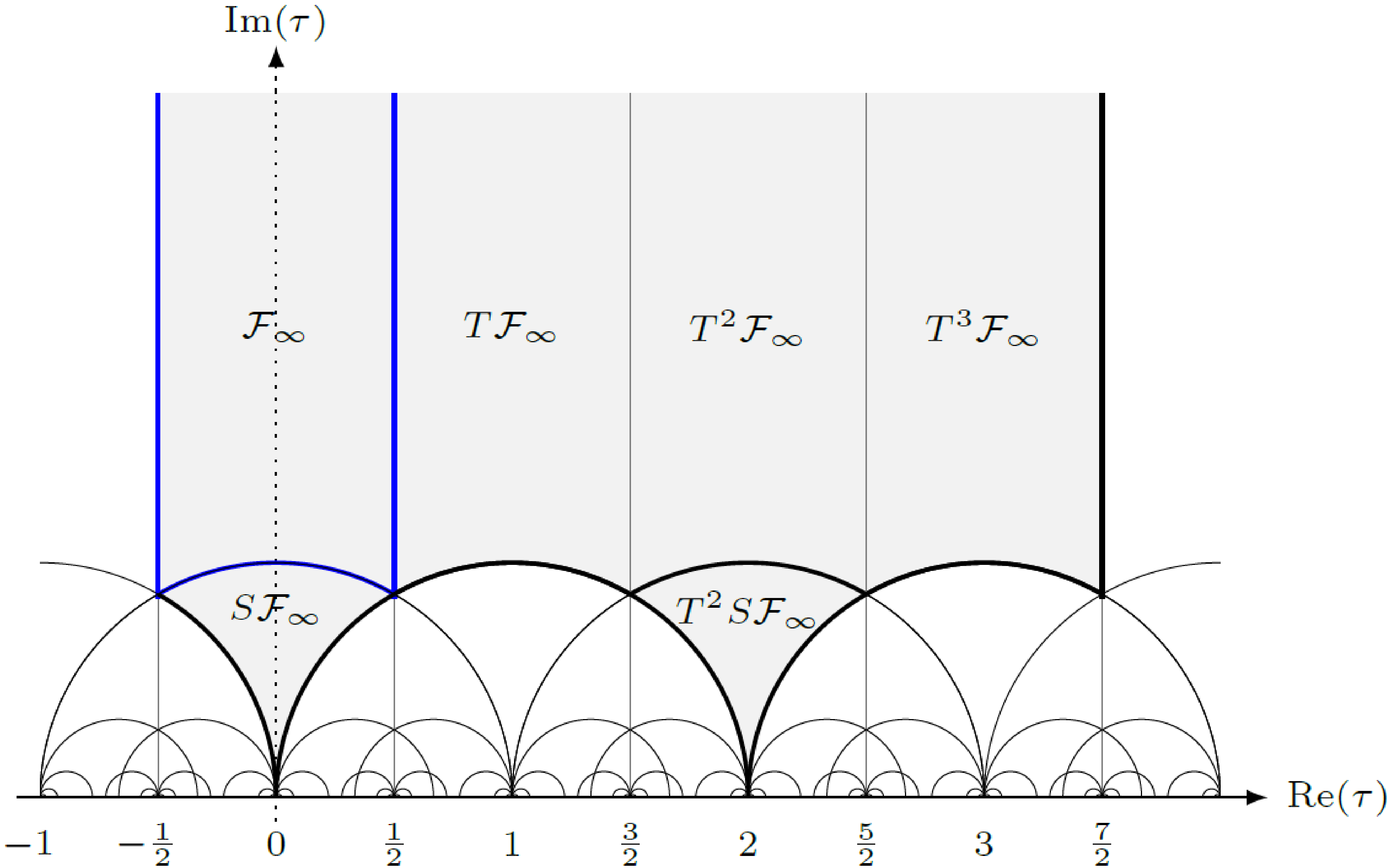
There is a unique A-cycle in $H_1(E_u;\mathbb{Z})$ invariant under monodromy $u \to e^{2\pi i} u$ (for $|u| > 1$)

$$\omega := \oint_A \frac{dx}{y}$$

Choose B-cycle:⇒

$$\tau(u) = \oint_B \frac{dx}{y} \Big/ \oint_A \frac{dx}{y}$$

$$u = \frac{\vartheta_2^4 + \vartheta_4^4}{2\,\vartheta_2^2\vartheta_4^2} = \frac{1}{8}q^{-\frac{1}{4}} + \frac{5}{2}q^{\frac{1}{4}} + \cdots \qquad q = e^{2\pi i\tau}$$



$$\Gamma^0(4) = \{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} mod4\} \subset SL(2,\mathbb{Z})$$

Relation To Mock Modular Forms – 1.1

Z_u : A sum of integrals of the form :

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, (Im\,\tau)^{-s}\, f(\tau,\bar{\tau})$$

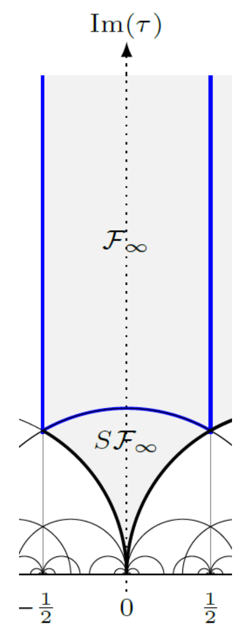
Support of c is bounded below

$$f(\tau,\bar{\tau}) = \sum_{m-n\in\mathbb{Z}} c(m,n) q^m\, \bar{q}^n$$

Strategy: Find $\hat{h}(\tau,\bar{\tau})$ such that

$$\partial_{\bar{\tau}} \hat{h} = (Im\,\tau)^{-s}\, f(\tau,\bar{\tau})$$

$\hat{h}\,(\tau,\bar{\tau})$ is modular of weight (2,0)



Relation To Mock Modular Forms – 1.2

$$\hat{h}(\tau,\bar{\tau}) = h(\tau) + R$$

We choose an explicit solution

$$\partial_{\bar{\tau}} R \; = (Im\tau)^{-s} f(\tau,\bar{\tau})$$

vanishing exponentially fast at $Im\tau \to \infty$

$h(\tau)$: mock modular form

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m \qquad q = e^{2\pi i\tau}$$

$$h\left(-\frac{1}{\tau}\right) = \tau^2 h(\tau) \; + \tau^2 \int_{-i\infty}^{0} \frac{f(\tau,\bar{v})}{(\bar{v}-\tau)^s} d\bar{v}$$

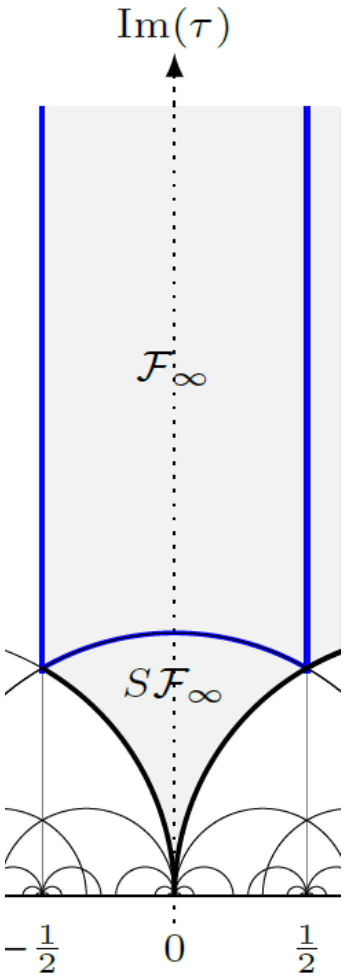
Doing The Integral

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, y^{-s}\, f(\tau,\bar{\tau})$$

$$\partial_{\bar{\tau}} \hat{h} = y^{-s}\, f(\tau,\bar{\tau})$$

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m$$

$$I_f = d(0)$$



Note: $d(0)$ undetermined by diffeq but fixed by the modular properties: Subtle!

∃ Long history of the definition & evaluation of such integrals with singular modular forms

Some references for evaluation of modular integrals:

1. H. Peterson, Math. Ann. 127, 33 (1954)
2. Dixon-Kaplunovsky-Louis, Nucl. Phys. B329 (1990) 27
3. Harvey-Moore, Nucl. Phys. B463 (1996) 315
4. Moore-Witten, Adv. Theor. Math. Phys. 1 (1997) 298
5. Borcherds, Inv. Math. 132 (1998) 491
6. Brunnier-Funke, Duke Math J. 125.1 (2004)
7. Bringmann-Diamantis-Ehlen, Int. Math. Res. Not. (2017)
8. Korpas-Manschot-Moore-Nidaiev, e-print arXiv:1910.13410

1. Physical Mathematics
2. Math & Physics Problems
3. Instantons & Donaldson Invariants
4. Topological Field Theory
5. Low Energy Effective Field Theory
6. New Results On Z_u
7. **Directions For Future Research**

FUTURE DIRECTIONS

1. Other N=2 Theories

2. Family Invariants

3. $b_2^+ = 0$

4. Manifolds with boundary and Floer theory

5. Physical Interpretation of Bauer-Furuta Invariant?

6. K-theoretic & Elliptic Generalizations

What About Other N=2 Theories?

There are infinitely many other four-dimensional N=2 supersymmetric quantum field theories.

Topological twisting should sense for any $\mathcal{N} = 2$ theory $\mathcal{T}$.

(but $\mathcal{T}$-dependent details remain to be worked out)

Natural Question: Given the successful application of $\mathcal{N} = 2$ SYM for $G = SU(2)$ to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N} = 2$ field theories?

How Lagrangian Theories Generalize Donaldson Invariants

$$Z(S) = \langle e^{\mathcal{O}(S)} \rangle_{\mathcal{T}} \;\; = \int_{\mathcal{M}} e^{\mu(S)} \mathcal{E}(\mathcal{V})$$

But now $\mathcal{M}$: is the moduli space of:

$$F^{+} = \mathcal{D}(M, \bar{M}) \qquad \gamma \cdot D\, M = 0$$

$$M \in \Gamma(W^{+} \otimes V)$$

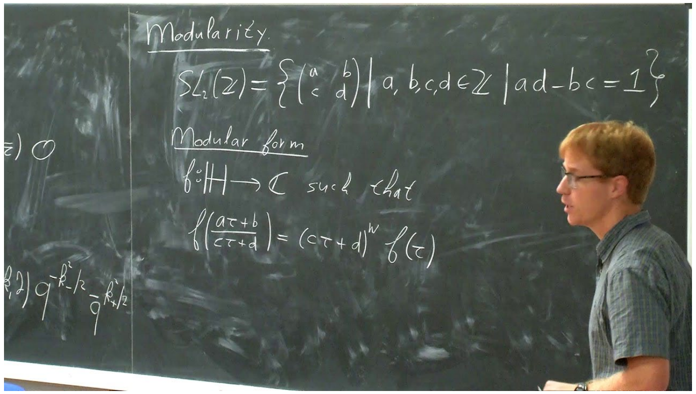
W^{+}: Rank 2 bundle associated to a $\mathfrak{spin}^c$ structure

``Generalized monopole equations''

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

U(1) case: Seiberg-Witten equations.

Example: SU(2) $\mathcal{N} = 2^*$



arXiv:2104.06492

$$M \in \Gamma(W^+ \otimes ad\, P \otimes \mathbb{C})$$

(UV) Spin-c structure $\mathfrak{s}_{uv}$,

$$c_{uv} := c(\mathfrak{s}_{uv}) \in H^2(X, \mathbb{Z})$$

$$\tau_{uv} \sim \theta + \frac{i}{g_{uv}^2} \in \mathcal{H} \qquad q_{uv} := e^{2\pi i\, \tau_{uv}}$$

$$t := m/\Lambda$$

$$Z(S;\tau_{uv},c_{uv},t) := \langle\, e^{\mathcal{O}(S)} \rangle_{\mathcal{N}=2^*}$$

$$= \sum_{k\geq 0} q_{uv}^{k} \int_{\mathcal{M}_{Q,k}} e^{\mu(S)} Eul(\mathcal{E};t)$$

Equivariant Euler class for U(1) symmetry rotating M

$\mathcal{E}$: Obstruction bundle for elliptic complex

Interpolates between Donaldson-Witten ($t \to \infty$)
and Vafa-Witten ($t \to 0$) partition functions

Suitably S-duality covariant under
$SL(2,\mathbb{Z})$ transformations of τ_{uv}

But nonholomorphic in τ_{uv} for $b_2^+ = 1$

∃ exact expressions in terms of Jacobi-Maass forms

∃ explicit analog of the ``Witten conjecture''
for $b_2^+ > 1$

Class S: Conjecture A

Another class of theories is derived from *six-dimensional* QFT by compactification on a Riemann surface C. [Lerche et. al.; Witten; Gaiotto; Gaiotto,Moore,Neitzke]

Not always described by an action principle – somewhat more mysterious.

Conjecture A: For all Lagrangian and all class S theories the four-manifold invariants can be written in terms of Seiberg-Witten invariants, using formulae generalizing the Witten conjecture.

Nontrivial checks: Pure SU(N) [M. Marino & G. Moore]
& simplest non-Lagrangian [Moore-Nidaiev]

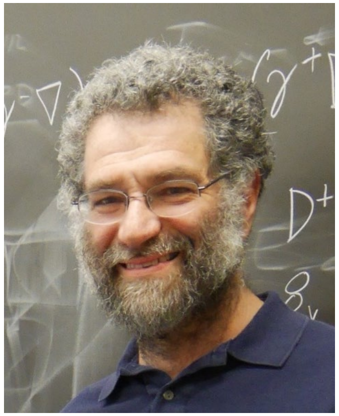
Family Donaldson Invariants

There is an interesting generalization to invariants for families of four-manifolds.

Mentioned by Donaldson long ago.
A modest amount of work has been done in the math literature .

$$Z_D \in H^*(BDiff^+(X))$$





Donaldson-Witten a la Baulieu-Singer

$$P \to \mathbb{X} \qquad \mathcal{G} := Aut(P)$$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P)$

$$\left(\Omega^*(\mathcal{A}(P)) \otimes S^*(Lie\mathcal{G}) \right)^{\mathcal{G}}$$

$$Q\, A_\mu = \psi_\mu \qquad Q\, \psi_\mu = -D_\mu \phi \qquad Q\, \phi = 0$$

Atiyah & Jeffrey + Baulieu & Singer

Z_{DW} : Pushforward in $\mathcal{G}$ −equivariant cohomology.

$$\mathcal{G}_d := Diff^+(\mathbb{X})$$

$\mathcal{G}_d$ −equivariant cohomology of $MET(\mathbb{X})$

$$Q\, g_{\mu\nu} = \Psi_{\mu\nu} \qquad Q\, \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \qquad Q \Phi^\mu = 0$$

Action e^{-S} is a closed equivariant class in the $\mathcal{G} \rtimes \mathcal{G}_d$ − equivariant cohomology of $MET(\mathbb{X}) \times \mathcal{A}(P)$

Push-forward in $\mathcal{G}$ −equivariant cohomology is a $\mathcal{G}_d$ −equivariant class on $MET(\mathbb{X})$

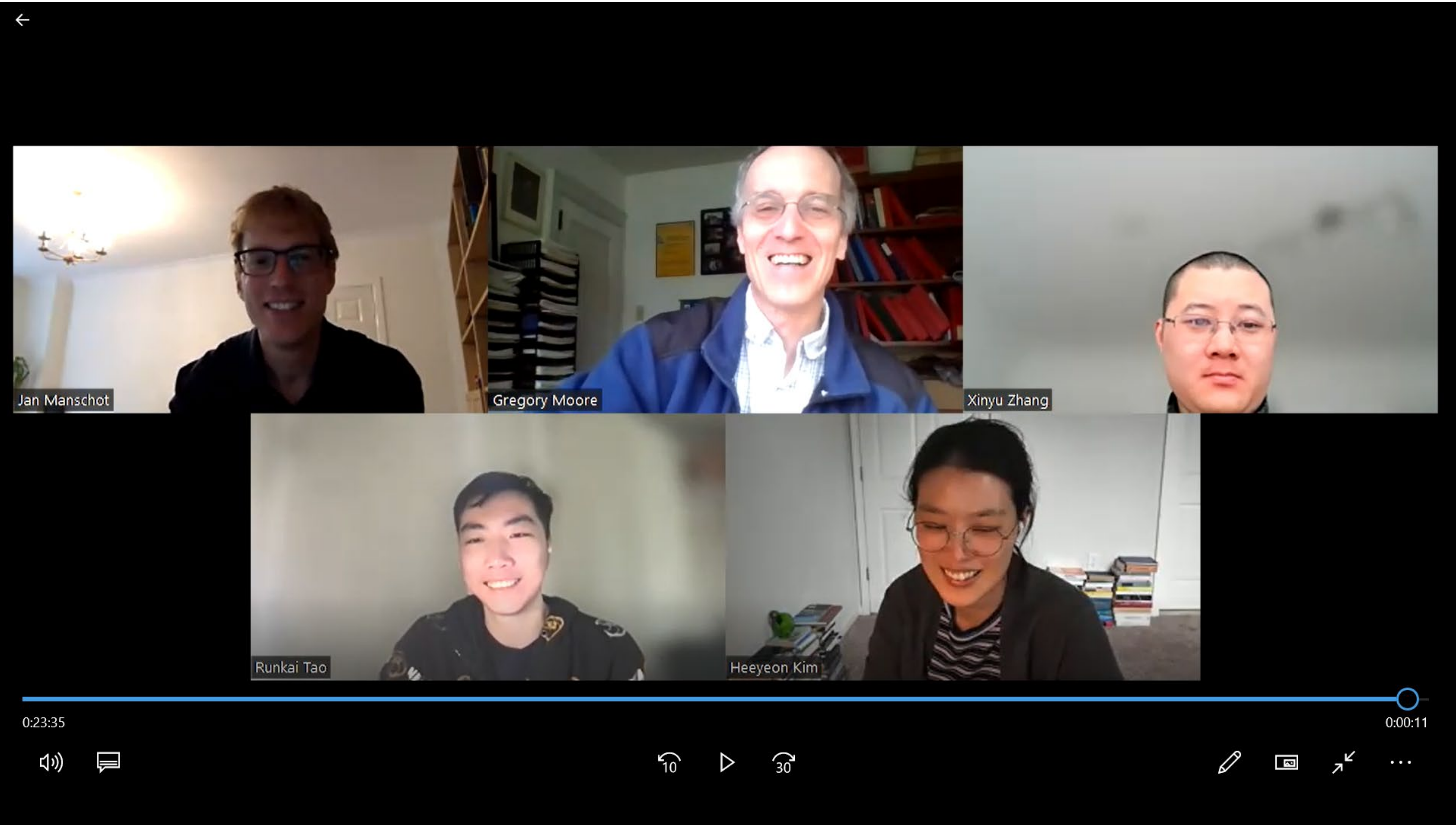
Families Of Four-Manifolds - 2/3

Thanks to heroic computations by JC and VS we have explicit actions e^{-S} obtained by coupling to truncated & twisted $N = 2$ conformal supergravity

$$[Z[g_{\mu\nu}, \Psi_{\mu\nu}, \Phi^{\mu}]] \in H^*(BDiff^+(\mathbb{X}))$$

Conjecture B: These will produce the family invariants envisioned by Donaldson, and generalizations thereof

``K-Theoretic Donaldson Invariants''



Five Dimensions

Partial Topological Twist of 5d SYM on $\mathbb{X} \times S^1$

Reduces to SQM on the moduli space of instantons:

(Requires that $\mathcal{M}$ be Spin-c)

$$\mathcal{R} := R\,\Lambda$$

$$Z[\mathcal{R}] = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} \hat{A}(T\,\mathcal{M}_k)$$

[Nekrasov (1996); Losev, Nekrasov, Shatashvili; Gottsche et. al. ....]

+ interesting story including observables…

Chern-Simons Observables

$U(1)_{inst}$ symmetry with current $J = Tr(f \wedge f\ )$

Couple to background gauge field A:

$$n := \left[\frac{F(A)}{2\pi}\right] \in H^2(\mathbb{X},\mathbb{Z})$$

$$\mathcal{O}(n) = \int_{\Sigma(n)\times S^1} Tr\left(a\, da + \frac{2}{3}a^3\right)$$

$$= \int_{\mathbb{X}\times S^1} F(A) \wedge Tr\left(a\, da + \frac{2}{3}a^3\right)$$

$$Z(\mathcal{R}, n) := \langle\, e^{\mathcal{O}(n)} \rangle$$

Five Dimensions

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} e^{c_1(L(n))} \hat{A}(\mathcal{M}_k)$$

Using both the U-plane integral, and, independently, localization techniques, we reproduce & generalize

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

ABSTRACT. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

ABSTRACT. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

This should generalize to 6d SYM on $\mathbb{X} \times \mathbb{E}$

$$\hat{A}(\mathcal{M}_k) \to Ell(\mathcal{M}_k, q)$$

Conjecture C:

Integrals in elliptic cohomology of distinguished classes defined by the susy sigma model with target space $\mathcal{M}_k$ define smooth invariants of four-manifolds


NOT
That's all Folks!