ASPEN TALK Aug. 9, 2005

↳ Electric + Magnetic Fluxes Don't Commute

1. Introduction

Fluxes and branes have been playing a key role in string theory and M-theory for about 10 years now, and continue to be important for example in recent claims of moduli stabilization. As we heard from Liam McAllister that's crucial to attempts to understand stringy cosmology.

Nevertheless - I feel that the mathematical formulation of these fluxes is not yet in a satisfactory state. So I continue to think about the general theory of fluxes.
Let me list some of the issues which are still open and might have an impact on string cosmology and/or flux stabilization.

1. Dirac Quantization:

The formulation in II/M/het is different. Until we understand how these views are naturally compatible we're missing something deep and fundamental

2. RR Action:
Has not been properly written in the literature.
Work w/ D.F. and D.B. → we now know it for IIA/IIB.

3. Tadpole conditions : torsion corrections

Torsion flux or charge : $N \times$ flux $= 0$
Not exotic: Discrete Wilson lines, SSB,

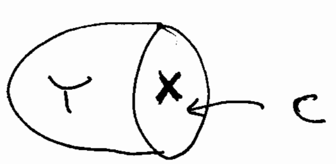
any CY $\pi_1 \neq 0 \longrightarrow \mathrm{Tors}\, K \neq 0$.

M-theory : $\left[\frac{1}{2} G \wedge G - I_8\right] = 0$ $\exists$ integral refinement

4. Anomaly Cancellation : effects of fluxes, orientifolds :
In attempts to implement KKLT scenario people wrap D-branes on nonspin manifolds - raises many issues.
W/ D. Freed + B. Florea

* 5. Hamiltonian Formulation - important for string cosmology

HH wavefunction for C-field, RR field.



$\Psi(C) \in \mathcal{H}$.

long distance / weak coupling $\Psi \sim \Theta$ - function

subtle phases from point 2.

Last year at Strings 2004 I reported on some aspects ~~for M-theory~~ of the Ham. formulation for M-Theory.
Today I want to address the case of type II string theory.

Summarize the result:

Consider type II / $X_9 \times \mathbb{R}$. Usually assumed that the superselection sectors – the flux sectors – are given

$$\mathcal{H} = \bigoplus_{x \in K(X)} \mathcal{H}_x$$

$$K(X) = K_B^{\epsilon}(X) \qquad \epsilon = 0/1 \quad \text{IIA/IIB}$$

w/ D. Freed + G. Segal – Not true!

Basic observation even applies to 3+1 dim'l Maxwell theory.

Real story torsion fluxes don't commute.
Generate a Heisenberg group.
$\mathcal{H}$ = rep$^{\underline{n}}$ of that Heisenberg group.

For me this is conceptually important because it means that our standard picture of a Dbrane as a submanifold w/ v.b. and Connection needs to be modified.

Before we begin with technicalities — one general remark:

What I will say applies to a broad class of theories called "GAGT's."

Old subject of topology: Generalized Cohomology theory.

H^* characterized by axioms — essentially naturality and gluing (Meyer-Vietoris) together with the "dimension axiom"

$$H^k(pt;G) = G\,\delta_{k,0} \qquad G = \text{abelian group}$$

(I will write $H^k(M) = H^k(M,\mathbb{Z})$ unless explicitly said otherwise.)

Generalized Coho: Drop the dimension axiom $\longrightarrow$ bordism, K-theory, elliptic cohomology...

To do physics we need <u>local</u> fields — Constrained by topological considerations — This is the new subject of "differential generalized Cohomology"

Physicists should learn it

$\check{H}^\ell(M)$ — B-fields, M5 3-form, M-theory 3-form

$\check{K}^\epsilon(M)$ type II RR

$\check{K}O$ type I, orientifolds

<u>Def:</u> A GAGT is a field theory whose space of <u>gauge invt</u> fields is a gen. diff'l Coho. theory.

2. Generalized Maxwell Theory

Maxwell theory = theory of a connection 1-form on a line bundle

For a fixed $\mathcal{L} \longrightarrow M$, the gauge invariant field space is

$$\text{Conn's} \longrightarrow \mathcal{A}(\mathcal{L})/\mathcal{G} \longleftarrow \text{gauge group.}$$

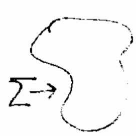
Considering all line bundles together

$$\{\text{gauge inequiv. fields}\} = \bigcup_{c_1 \in H^2(M)} \mathcal{A}(\mathcal{L}_{c_1})/\mathcal{G}$$

- We'd like to generalize to arbitrary form degree
- Above is actually an abelian group.

Right point of view: The gauge inv't info in a gauge field is encoded by the holonomy function

$$Z_1(M) \longrightarrow U(1) = \exp(2\pi i \mathbb{R}/\mathbb{Z})$$



$$\Sigma \longrightarrow \exp\left(2\pi i \oint_\Sigma A\right)$$

The space of all homomorphisms $Z_1(M) \to U(1)$ is an abelian group: $= \check{H}^2(M) = \bigcup_{c_1} \left(\mathcal{A}(\mathcal{L}_{c_1})/\mathcal{G}\right)$

Def: The Deligne–Cheeger–Simons coho.

$$\check{H}^{\ell}(M) = \{ \text{Homomorphisms } \chi : Z_{\ell-1}(M) \to U(1) \}$$

We can view this as a group of gauge inequivalent ℓ-form gauge fields.

By "generalized Maxwell" I mean a field theory such that $\{\text{gauge inequiv. fields}\} = \check{H}^{\ell}(M)$

Sometimes I'll denote $[\check{A}] \in \check{H}^{\ell}(M)$

So that

$$\chi(\Sigma) = \exp\left(2\pi i \int A \right)$$

But it's important to stress that A is NOT a globally well-defined $(\ell-1)$-form.

Now I'll need to spend some time explaining some properties of $\check{H}^{\ell}(M)$ - so we digress for a little math lesson

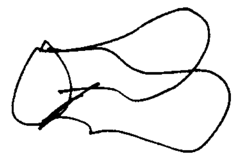
Fieldstrength To a given χ $\exists$ $F \in \Omega^{\ell}(M)$ s.t. if $\Sigma = \partial B$ then

$$\chi(\Sigma) = \exp\left(2\pi i \int_B F \right)$$

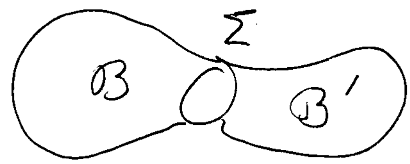
as opposed to A, F is a globally well-defined form

Now different $\mathcal{B}$'s bound the same Σ

Small changes in $\mathcal{B}$



$\Rightarrow \quad dF = 0$



Large changes in $\mathcal{B}$

$\Rightarrow \quad F \in \Omega^{\ell}_{\mathbb{Z}}(M)$

So F has *integral periods*.

This is where physicists often stop - but you can't.
Some information is missing.

Suppose $k\Sigma = \partial \mathcal{B}$ then

$$\left(\exp 2\pi i \int_\Sigma A \right)^k = \exp\left(2\pi i \int_{\mathcal{B}} F\right)$$

But knowing F alone does not tell you how to take k^{th} root.
That extra info is encoded in the "torsion part" of the <u>characteristic class</u>

$$a([\check{A}]) \in H^k(M;\mathbb{Z})$$

$$\begin{tikzcd}
\check{H}^\ell(M) \arrow[r] \arrow[d] & \Omega^\ell_{\mathbb{Z}}(M) \ni F \arrow[d] \\
H^\ell(M;\mathbb{Z}) \arrow[r] & H^\ell(M;\mathbb{R})
\end{tikzcd}
\qquad \bar{a} = [F]_{DR}$$

$$\underset{\check{a}}{\overset{\in}{H^\ell(M,\mathbb{Z})}}$$

$$\begin{tikzcd}
\check{H}^\ell(M) \arrow[r] \arrow[d] & \Omega^\ell_{\mathbb{Z}}(M) \ni F \arrow[d] \\
H^\ell(M,\mathbb{Z}) \arrow[r] & H^\ell(M,\mathbb{R}) \cong H^\ell_{DR}(M) \ni [F] \\
a \arrow[r] & \bar{a}
\end{tikzcd}$$

There is still more information: the topologically trivial flat fields.
There are two ways to summarize the structure of this group:

These live in

$$H^{\ell-1}(M) \otimes \mathbb{R}/\mathbb{Z} \cong \mathcal{H}^{\ell-1}/\mathcal{H}^{\ell-1}_{\mathbb{Z}}$$

flat Wilson lines continuously tuneable to zero.

All this info is summarized in the two exact sequences:

$$\left.\begin{array}{l} 0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \longrightarrow \overset{F}{\underset{\cap}{}}\,\Omega^{\ell}_{\mathbb{Z}}(M) \to 0 \\[3ex] 0 \to \underbrace{\Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}}}_{\text{topol. trivial}} \longrightarrow \check{H}^{\ell}(M) \longrightarrow \underset{\underset{a}{\cup}}{H^{\ell}(M,\mathbb{Z})} \to 0 \end{array}\right\}\ \text{keep}$$

Notice

1. Flat fields need not be topologically trivial

$$H^{\ell-1}(M;\mathbb{R}/\mathbb{Z}) = \text{compact abelian group} = \underbrace{\underset{1}{\bigcirc} + \bigcirc \cdots + \bigcirc}_{\substack{\text{group of cpts} \\ = H^{\ell}_{T}(M)}}$$

$$\text{Component of the identity} = H^{\ell-1}(M)\otimes\mathbb{R}/\mathbb{Z} \cong \mathcal{H}^{\ell-1}/\mathcal{H}^{\ell-1}_{\mathbb{Z}}$$

on the other hand

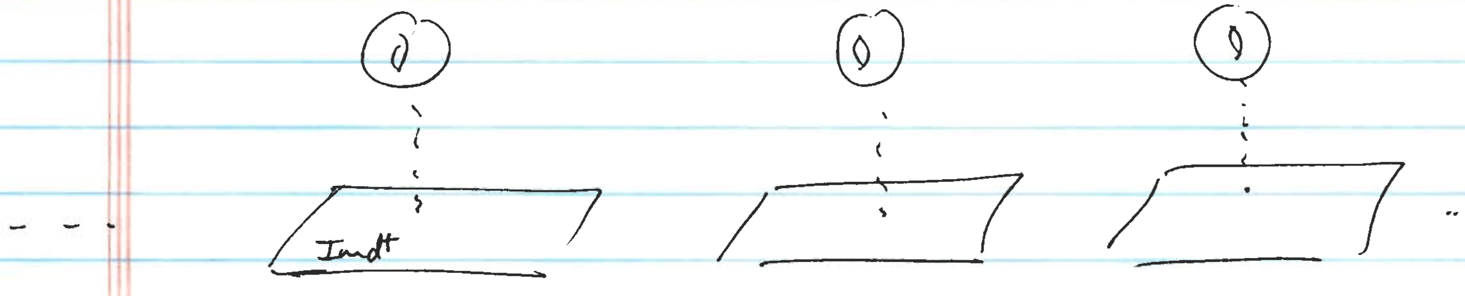
2. Topologically trivial fields do not just form a v.s.

$$\underbrace{\mathcal{H}^{\ell-1}/\mathcal{H}^{\ell-1}_{\mathbb{Z}} \cong \Omega^{\ell-1}_{d}/\Omega^{\ell-1}_{\mathbb{Z}}}_{\text{torus}} \longrightarrow \Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}}$$

$$\Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}} \Big\downarrow$$

$$\Omega^{\ell-1}/\Omega^{\ell-1}_{d} \cong \mathrm{Im}\, d^{\dagger} = \text{vector space}$$

So the picture of $\check{H}^\ell$ is



$$\longrightarrow H^\ell(M)$$

Examples:

1.) $\check{H}^1(M) = \{ f : M \to U(1) \}$

$$a = f^*[d\theta] \qquad F = \frac{1}{2\pi i} d(\log f)$$

2.) $\check{H}^2(M) = \{\text{line bdles w/ connection}\}/\sim$

Finally we'll need a crucial fact

If M is compact and oriented ($\dim M = n$) then we have Poincaré duality — there is a perfect pairing

$$\check{H}^\ell(M) \times \check{H}^{n+1-\ell}(M) \longrightarrow U(1)$$

$$0 \to H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}) \to \check{H}^\ell(M) \to \Omega^\ell_{\mathbb{Z}}(M) \to 0 \qquad (F \in \Omega^\ell_{\mathbb{Z}}(M))$$

$$0 \to \Omega^{n-\ell}/\Omega^{n-\ell}_{\mathbb{Z}} \to \check{H}^{n+1-\ell}(M) \to H^{n+1-\ell}(M) \to 0 \qquad (A_D \in \Omega^{n-\ell})$$

Pairing is: $\exp\left(2\pi i \int_M A_D \wedge F\right)$

3/1 Hamiltonian Formulation of Generalized Maxwell

Now $M = X \times \mathbb{R}$

$$[\check{A}] \in \check{H}^{\ell}(M) \qquad S = \int \tfrac{1}{2}\lambda^{-1} F * F$$

Completely straightforward $\mathcal{H} = L^2(\check{H}^{\ell}(X)) \ni \psi(\check{A})$

$\hookrightarrow$ grading by magnetic flux

keep leave space $\rightarrow$
$$\mathcal{H} = \bigoplus_m \mathcal{H}_m \qquad m \in H^{\ell}(X, \mathbb{Z})$$

because m labels the components of config. space

There is an electro-magnetic dual formulation by

$[\check{A}_D] \in \check{H}^{n-\ell}(M)$ so

Keep leave space $\rightarrow$
$$\mathcal{H} = \bigoplus_e \mathcal{H}_e \qquad e \in H^{n-\ell}(X, \mathbb{Z})$$

But - can we simultaneously measure electric and magnetic flux?

$$\mathcal{H} = \bigoplus_{e,m} \mathcal{H}_{e,m}$$

Your intuition should be "yes!"

$$\left[\int_{\Sigma_1} F , \int_{\Sigma_2} * F \right] = \left[\int_{\Sigma_1} F , \int_{\Sigma_2} \Pi \right]$$

$$= \left[\int_X \omega_1 F , \int_X \omega_2 \Pi \right]$$

$$= \int \omega_1 d\omega_2 = 0 .$$

But these period integrals only measure flux / torsion.

To see what happens at the torsion level we need to understand the grading by electric flux in the $\check{A}$ - formulation

Diagonalizing $*F$ = Diagonalizing Π
= <u>translation</u> eigenstate

We are only interested in the topological class of electric flux and therefore we define

<u>Def$^{\underline{n}}$</u>: A state of definite (topological) electric flux satisfies

$$\forall \check{\phi}_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \quad \psi(\check{A} + \check{\phi}_f) = \exp\left(2\pi i \int_X \phi_f e\right) \psi(\check{A})$$

$$e \in H^{n-\ell}(X, \mathbb{Z})$$

So

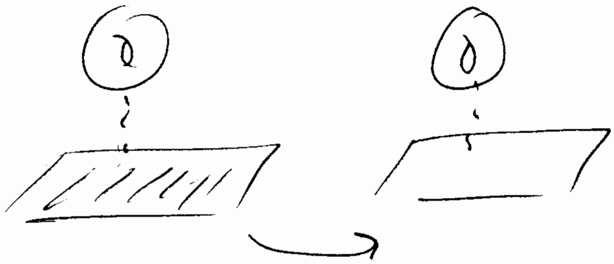
$$\mathcal{H} = \bigoplus_e \mathcal{H}_e \qquad e \in H^{n-\ell}(X,\mathbb{Z}) \quad \text{Diagonalizing } H^{\ell-1}(X,\mathbb{R}/\mathbb{Z}) \subset \check{H}^{\ell}(X)$$

Dually:

$$\mathcal{H} = \bigoplus_m \mathcal{H}_m \qquad m \in H^{\ell}(X,\mathbb{Z}) \quad \text{Diagonalizing } H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z}) \subset \check{H}^{n-\ell}(X)$$

Now, our main claim is that these translation operators don't commute.

Actually this follows immediately since translation by a topologically nontrivial, but flat field changes the magnetic flux sector



~~This actually follows immediately since translation by a topologically nontrivial flat field changes the magnetic flux sector.~~

But, to understand this more systematically, let's consider the following remark

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Let $S =$ any abelian group. w/ measure $\Rightarrow L^2(S) = \mathcal{H}$

$\mathcal{H}$ is a repn of S : $s_0 \in S$ $(L_{s_0}\psi)(s) := \psi(s+s_0)$

" " " " $\check{S}$ $\chi \in \check{S}$ $(m_\chi \psi)(s) := \chi(s)\psi(s)$

Pontryagin dual

Not a repn of $S \times \check{S}$ $\quad L_{s_0} m_\chi = \chi(s_0)\, m_\chi L_{s_0}$

Is a repn of $1 \to U(1) \to \mathrm{Heis}(S \times \check{S}) \to S \times \check{S} \to 1$

In fact: Thm (S-vN) : $\mathrm{Heis}(S\times\check{S})$ has a unique irrep. $= L^2(S)$

Apply this to $S = \check{H}^\ell(X)$

Poincaré duality $\Rightarrow$ Pontryagin dual $\check{S} = \check{H}^{n-\ell}(X)$

$\Rightarrow \mathcal{H} =$ Unique irrep of $\mathrm{Heis}\left(\check{H}^\ell(X) \times \check{H}^{n-\ell}(X)\right)$

Now recall

$$H^{\ell-1}(X,\mathbb{R}/\mathbb{Z}) = \left(H^{\ell-1}(X)\otimes\mathbb{R}/\mathbb{Z}\right)\times H^{\ell}_T(X)$$

$$H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z}) = \left(H^{n-\ell-1}(X)\otimes\mathbb{R}/\mathbb{Z}\right)\times H^{n-\ell}_T(X)$$

trivial pairing

perfect pairing

Therefore

$$\mathcal{H} = \bigoplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}} \quad \circlearrowleft \quad \mathrm{Heis}\left(H^{\ell}_T \times H^{n-\ell}_T\right)$$

<u>Example</u> Maxwell on $S^3/\mathbb{Z}_k \times \mathbb{R}$

$$H^2(S^3/\mathbb{Z}_k) = \mathbb{Z}_k$$

$$\mathrm{Heis} = \langle P, Q \mid PQ = \omega QP \rangle$$

~~Cannot simultaneously measure elec, mag flux~~

In a basis we magnetic fluxes

$$\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{k-1} \end{pmatrix} \quad \text{electric fluxes} \quad \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{pmatrix}$$

Close with 2 Remarks

(1.) A closely related observation in the special case of Maxwell on Lens spaces was made by Gukov-Rangamani-Witten: Noncommuting Wilson line operators on a Lens space.

$D3/S^3/\mathbb{Z}_k$ cannot simultaneously measure the F1 and D1 strings.

Present discussion places that remark in a wider context.

(2.) Notice that our formulation of the Hilbert Space is <u>manifestly</u> electric-magnetic dual.

This gives a very crisp formulation of a selfdual field so $F = *F$ for:

$\dim X = 2p-1$ $\quad [\check{A}] \in \check{H}^p(M)$ selfdual

$$\mathcal{H} := \mathrm{Heis}\left(\check{H}^p(X)\right)$$

4// RR Fields

Now let's apply these ideas to type II strings

First we have $[\check{B}] \in \check{H}^3(M_{10})$

RR fields $[\check{C}] \in \check{K}^{\epsilon}_{\check{B}}(M_{10})$

We have a picture of differential K-theory similar to what we had before

Characteristic class: $x \in K^{\epsilon}_{B}(M)$

Fieldstrength: $\tilde{F} = \tilde{F}_0 + u^{-1}\tilde{F}_2 + \cdots + u^{-5}\tilde{F}_{10}$

$$d_H \tilde{F} = 0 \qquad d_H = d - H$$

Let $R = \mathbb{R}[u, u^{-1}]$ $\deg u = 2$.

$$\Omega^k_{d_H}(M;R) = \{ d_H\text{-closed forms valued in } R \}$$

degree $= k$

$\exists$ Chern character $ch_B : K^{\epsilon}_B(M) \longrightarrow H^{\epsilon}_{d_H}(M)$

Quantization of periods: $[\tilde{F}]_{d_H} = ch_B(x)\sqrt{\hat{A}}$

defines $\Omega^k_{d_H,\mathbb{Z}}(M;R)$

The two exact sequences from before have precise analogs: (take IIA for definiteness)

$$0 \to \overbrace{K_B^{-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \longrightarrow \check{K}^{0}_{\check{B}}(M) \longrightarrow \overbrace{\Omega^{0}_{d_H,\mathbb{Z}}(M;\mathbb{R})}^{\text{fieldstrength}} \to 0$$

$$0 \to \underbrace{\Omega^{odd}(M)/\Omega^{1}_{d_H,\mathbb{Z}}(M)}_{\text{top. trivial}} \to \check{K}^{0}_{\check{B}}(M) \longrightarrow \underbrace{K^{0}_{B}(M)}_{\text{characteristic class}} \to 0$$

But now there is a new ingredient: the RR field is self-dual, both in IIA and in IIB.

$\Longrightarrow$ subtleties in formulating the action – but recent progress

$\Longrightarrow$ Hilbert space should be formulated along the above lines

Thm Let X be odd-dim'l, (cpt, K-oriented)

Then there is a perfect pairing

$$\check{K}^{\epsilon}_{\check{B}}(X) \times \check{K}^{\epsilon}_{\check{B}}(X) \longrightarrow \mathbb{R}/\mathbb{Z}$$

Therefore $\mathrm{Heis}\left(\check{K}^{\epsilon}_{\check{B}}(X)\right)$ has a unique irrep, and we define this to be the Hilbert space $\mathcal{H}_{RR}$.

Now we can return to our main result mentioned at the beginning.

Is there a grading $\mathcal{H}_{RR} = \bigoplus_{x \in K_B^{\epsilon}(X)} \mathcal{H}_{RR,x}$?

<u>No!</u> A K-theory class encodes both electric <u>and</u> magnetic flux, and these do not commute.

More mathematically: such a grading would be diagonalizing translation by the flat fields $= K_B^{-1}(X; \mathbb{R}/\mathbb{Z})$

But the Heisenberg pairing is nontrivial on these

What we do have

$$\mathcal{H}_{RR} = \bigoplus_{\bar{x} \in K/\text{tors.}} \mathcal{H}_{RR,\bar{x}} \circlearrowleft \text{Heis}\left(\text{Tors}\, K_B^0(X)\right)$$

<u>Conclusion:</u> Quantum states of the RR field cannot be in a definite K-theory class!

fig. 1

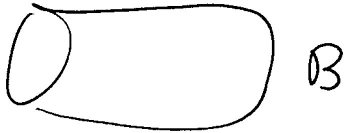


fig. 2

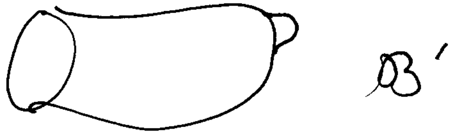


fig. 3

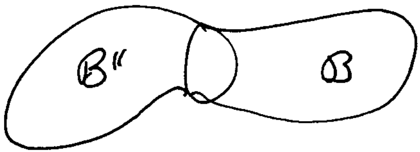
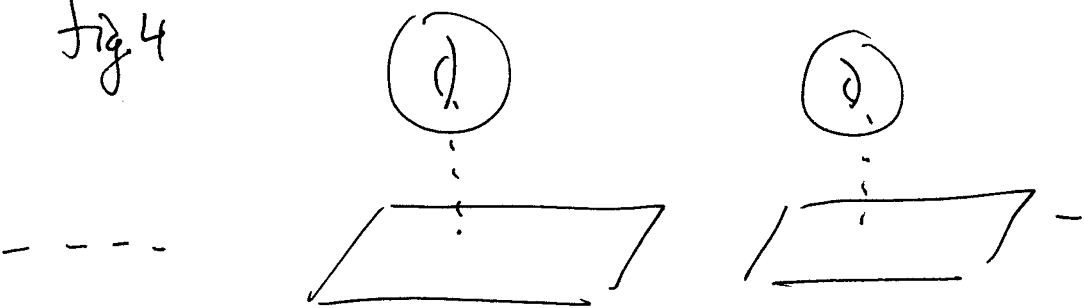


fig. 4



$\longrightarrow H^{k}(M;\mathbb{Z})$

10 : 48 : 30

1. Introduction.

A. Dirac Quantization $\quad$ II / M / het different!

B. RR Action: $\quad$ D. Belov + D. Freed -

C. Tadpole cond's: $\quad$ torsion corrections?

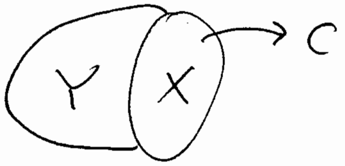
any CY w/ $\pi_1 \neq 0 \rightarrow \mathrm{Tors} K \neq 0$.

$\left[\frac{1}{2} G \wedge G - I_8\right] = 0 \quad \exists$ integral refinement

D. Anomaly cancellation: $\quad$ fluxes, orientifolds.

* E. Hamiltonian Formulation – stringy cosmology

HH wavefunction for C, RR



$\Psi(C) \in \mathcal{H}$.

long distance / wk coupling $\quad \Psi \sim \underline{\underline{\Theta}}$.

Summary: $\quad$ Type II / $X \times \mathbb{R}$

$$\mathcal{H} \neq \bigoplus_{x \in K(X)} \mathcal{H}_x \quad !$$

$K(X) = K_B^{\epsilon}(X) \quad \epsilon = 0/1 \quad$ IIA/IIB

D. Freed + G. Segal.

torsion fluxes don't commute
generate a Heisenberg group
$\mathcal{H}$ = rep^n of that H.G.

Applies to GAGT's
Generalized cohomology theory: drops dimension axiom

$H^k(M)$ – axioms ....
dimension: $H^k(pt; G) = \delta_{k,0}\, G$

Differential G.C.T.

$\check{H}^k(M)$ B fields, M-thry C
$\check{K}^{\epsilon}(M)$
$\check{K}O$

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2. Generalized Maxwell Theory.

$$\mathcal{L} \longrightarrow M.$$

$$\check{H}^2(M) = \bigcup_{c_1 \in H^2(M)} \mathcal{A}(\mathcal{L}_{c_1}) / \mathcal{G} = \{\text{gauge inequiv. fields}\}.$$

- this is an abelian group
- generalize to higher degree.

gauge invt info $\Longleftrightarrow$ holonomy function

$$\Sigma \longrightarrow \exp\left(2\pi i \oint_\Sigma A\right)$$

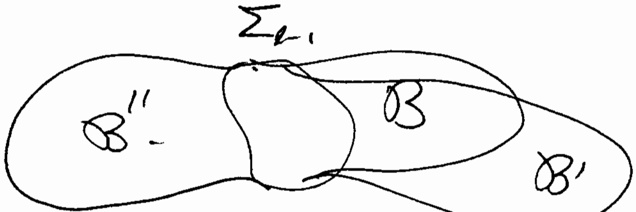
$$Z_1(M) \longrightarrow U(1) = \exp\left(2\pi i \, \mathbb{R}/\mathbb{Z}\right)$$

<u>Def:</u>

$$\check{H}^\ell(M) = \{\text{Hom's } \chi : Z_{\ell-1}(M) \longrightarrow U(1)\}.$$

$= \{$gauge inequiv. fields for "$\ell-1$ -form potential"$\}$
"Generalized Maxwell"

<u>Properties</u>:



For fixed χ $\exists$ $F_\chi \in \Omega^\ell(M)$ s.t.

$$\chi(\Sigma) = \exp\left(2\pi i \int_{\mathcal{B}} F_\chi\right)$$

$$\longrightarrow \quad dF = 0. \quad \longrightarrow [F] \in H^\ell_{DR}(M) \cong H^\ell(M;\mathbb{R})$$

$$\longrightarrow \quad F_{(\chi)} \in \Omega^\ell_{\mathbb{Z}}$$

$$k\Sigma = \partial \mathcal{B}.$$

$$(\chi(\Sigma))^k = \left(\exp 2\pi i \int_\Sigma A\right)^k = \exp 2\pi i \int_{\mathcal{B}} F.$$

Characteristic class $a_\chi \in H^\ell(M;\mathbb{Z})$

$$
\begin{tikzcd}
{[\check{A}]} \arrow[d, phantom, "\in" rotate=90] & & F \arrow[d, phantom, "\in" rotate=90] & \\
\check{H}^{\ell}(M) \arrow[rr] \arrow[d] & & \Omega^{\ell}_{\mathbb{Z}}(M) \arrow[dr] & \\
a \in H^{\ell}(M;\mathbb{Z}) \arrow[rr] & & H^{\ell}(M;\mathbb{R}) \arrow[r, phantom, "\cong"] & H^{\ell}_{DR}(M) \ni [F] \\
 & & \bar{a} \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}
$$

$$\bar{a} = [F]$$

$$
0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \to \Omega^{\ell}_{\mathbb{Z}}(M) \ni F \to 0
$$

$$
0 \to A_D \in \underbrace{\Omega^{n-\ell}/\Omega^{n-\ell}_{\mathbb{Z}}}_{\text{top. triv.}} \to \check{H}^{n+1-\ell}(M) \to H^{n+1-\ell}(M,\mathbb{Z}) \ni a \to 0
$$

$$\exp\left(2\pi i \int_M A_D F\right) \qquad H^n(M;\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$$

1.) Flat need not be top. trivial

$$H^{\ell-1}(M;\mathbb{R}/\mathbb{Z}) = \mathbb{1} + \underbrace{\mathbb{0} + \cdots + \mathbb{0}}_{H^{\ell}_{T}(M)}$$

$$H^{\ell-1}(M) \otimes \mathbb{R}/\mathbb{Z} = \mathcal{H}^{\ell-1}/\mathcal{H}^{\ell-1}_{\mathbb{Z}}$$

2.) Top. triv. fields not just a v.s.

$$A \in \Omega^{\ell-1} \qquad \chi_A(\Sigma) = \exp\left(2\pi i \int_\Sigma A\right)$$

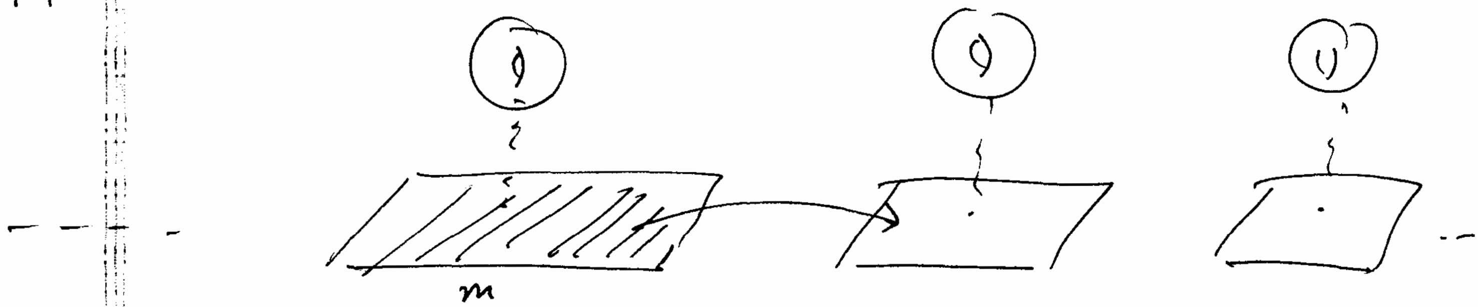
$$A \to A + \omega.$$

$$\mathcal{H}^{\ell-1}/\mathcal{H}^{\ell-1}_{\mathbb{Z}} = \Omega^{\ell-1}_d / \Omega^{\ell-1}_{\mathbb{Z}} \longrightarrow \Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}}$$

$$\downarrow$$

$$\underline{\Omega^{\ell-1}/\Omega^{\ell-1}_d} = \mathrm{Im}\, d^{+} = \text{vectorspace}$$

$\check{H}^{\ell}$



$$\longrightarrow H^{\ell}(M;\mathbb{Z})$$

Exples: 1) $\check{H}^{1}(M) = \{f : M \to U(1)\}$

$$a = f^*[d\theta] \qquad F = \frac{1}{2\pi i}\, d(\log f)$$

2.) $\check{H}^{2}(M)$

<u>Poincaré duality</u> If M cpt, oriented $\dim M = n$

$$\check{H}^{\ell}(M) \times \check{H}^{n+1-\ell}(M) \to U(1)$$

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3// Hamiltonian Formulation

$$M = X \times \mathbb{R}$$

$[\check{A}_D] \in \check{H}^{n-\ell}(M) \longleftrightarrow [\check{A}] \in \check{H}^{\ell}(M) \qquad S = \int \frac{1}{2}\lambda^{-1} F * F$

$$\mathcal{H} = L^2(\check{H}^{\ell}(X)) \qquad \psi([\check{A}])$$

$$\mathcal{H} = \bigoplus_m \mathcal{H}_m \qquad m \in H^{\ell}(X, \mathbb{Z}) \quad \underline{H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z}) \subset \check{H}^{n-\ell}(X)} \text{ Diag.}$$

$$= \bigoplus_e \mathcal{H}_e \qquad e \in H^{n-\ell}(X, \mathbb{Z}) \xleftarrow{\text{Diag.}} \underline{H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \subset \check{H}^{\ell}(X)}$$

$$\mathcal{H} \stackrel{?}{=} \bigoplus_{e,m} \mathcal{H}_{e,m}$$

$$\left[\underline{\int_{\Sigma_1} F}, \underline{\int_{\Sigma_2} *F} \right] = \left[\int_{\Sigma_1} F, \int_{\Sigma_2} \Pi \right]$$

$$= \left[\int_X \omega_1 F, \int_X \omega_2 \Pi \right]$$

$$= \int_X \omega_1 \, d\omega_2 = 0.$$

__Def:__ A state of definite (topological) electric flux

$$\forall \check{\phi}_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \qquad \psi(\check{A} + \check{\phi}_f) = \exp\left(2\pi i \int_X \phi_f e\right) \psi(\check{A})$$

$$e \in H^{n-\ell}(X, \mathbb{Z})$$

S - any abelian group w/ measure $\mathcal{H} = L^2(S)$

$\mathcal{H}$ rep$^{\underline{n}}$ of S $\quad s_0 \in S \quad (L_{s_0}\psi)(s) = \psi(s+s_0)$

rep^{n} of $\check{S}$ $\quad \chi \in \check{S} \quad (m_\chi \psi)(s) = \chi(s)\psi(s)$

<u>Not</u> rep$^{\underline{n}}$ of $S \times \check{S}$ $\qquad \underline{L_{s_0} m_\chi = \chi(s_0)\, m_\chi L_{s_0}}$

$$1 \to U(1) \to \mathrm{Heis}(S \times \check{S}) \to S \times \check{S} \to 1$$

<u>Thm</u> (S-vN) $\quad \mathrm{Heis}(S\times\check{S})$ has <u>unique</u> irrep $= L^2(S)$

$S = \check{H}^\ell(X)$ $\qquad$ Poincaré dual $\Rightarrow$ Pontryagin dual group $\check{S} = \check{H}^{n-\ell}(X)$

$\Rightarrow$ $\mathcal{H}$ = Unique irrep of $\mathrm{Heis}\left(\check{H}^\ell(X) \times \check{H}^{n-\ell}(X)\right)$

$$H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) = \left(H^{\ell-1}(X) \otimes \mathbb{R}/\mathbb{Z}\right) \times H^\ell_T$$

$$H^{n-\ell-1} = (\qquad) \times H^{n-\ell}_T$$

trivial pairing $\qquad\qquad$ perfect pairing

$$\mathcal{H} = \bigoplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}} \quad \circlearrowleft \quad \mathrm{Heis}\left(H^\ell_T \times H^{n-\ell}_T\right)$$

Expl: Maxwell $S^3/\mathbb{Z}_k \times \mathbb{R}$

$$H^2(S^3/\mathbb{Z}_k) = \mathbb{Z}_k$$

$$\mathrm{Heis} = \langle P, Q \mid PQ = \omega QP \rangle$$

$$\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{k-1} \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{pmatrix}$$

(1.) Gukov–Rangamani–Witten '98

$AdS^5 \times S^5/\mathbb{Z}_k$ $\qquad$ $D3/\ S^3/\mathbb{Z}_k$.

F1 + D1 string

(2.) $\dim X = 2p-1$ $\qquad$ $[\check{A}] \in \check{H}^p(M)$

$$\boxed{\mathcal{H}_{S.D.} = \mathrm{Heis}(\check{H}^p(X))}$$

4// RR fields type II

$$[\check{B}] \in \check{H}^3(M_{10})$$

RR $\qquad [\check{C}] \in \check{K}^{\epsilon}_{\check{B}}(M_{10})$

Characteristic class $\qquad x \in K^{\epsilon}_{B}(M)$

Fieldstrength $\qquad \tilde{F} = \tilde{F}_0 + u^{-1}\tilde{F}_2 + \cdots u^{-5}\tilde{F}_{10}$

$$d_H \tilde{F} = 0 \qquad d_H = d - H.$$

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$$R = \mathbb{R}[u, u^{-1}] \qquad \deg u = 2.$$

$$\Omega^k_{d_H}(M;R) = \{ d_H\text{-closed forms in } R \text{ degree} = k \}.$$

~~Dirac quant: $ch_B : K_B^\epsilon(M) \longrightarrow H^\epsilon_{d_H}(M)$~~

~~$\tilde{F}$~~

Fieldstrength $\in \Omega^0_{d_H, \mathbb{Z}}(M;R)$

Dirac Quant. $\quad ch_B : K_B^\epsilon(M) \longrightarrow H^\epsilon_{d_H}(M)$

$$[\tilde{F}]_{d_H} = ch_B(x) \sqrt{\hat{A}}$$

$$0 \to \overset{\text{flat}}{K_B^{-1}}(M; \mathbb{R}/\mathbb{Z}) \to \check{K}^0_{\check{B}}(M) \longrightarrow \Omega^0_{d_H,\mathbb{Z}}(M;R) \to$$

$$0 \to \Omega^{odd}(M) / \Omega^1_{d_H,\mathbb{Z}}(M) \to \check{K}^0_{\check{B}}(M) \longrightarrow K^0_B(M) \longrightarrow 0$$

$$C = C_1 + C_3 + C_5 + \cdots$$

Selfdual : $\longrightarrow$ action. . . .

$\longrightarrow$ Hilbert space.

Thm Let X be odd-dim'l (cpt, K-oriented). $\exists$ perfect pairing

$$\check{K}^{\epsilon}_{\check{\beta}}(X) \times \check{K}^{\epsilon}_{\check{\beta}}(X) \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$\Longrightarrow \quad \mathrm{Heis}\left(\check{K}^{\epsilon}_{\check{\beta}}(X)\right) \text{ has ! irrep.} := \mathcal{H}_{RR}$$

$\mathrm{Heis}(\mathrm{Tors}\, K^{0}_{\beta}(X))$

$$\mathcal{H}_{RR} = \bigoplus_{\bar{x} \in K^{\epsilon}_{\beta}(X)/\text{torsion}} \mathcal{H}_{RR,\bar{x}}$$

~~Not~~

$\check{K}^{-1}_{B}(X;\mathbb{R}/\mathbb{Z})$ has nontrivial pairing

Conclusion: Quantum states of the RR field cannot be in a definite K-theory class!

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