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Gravitational Phase Transitions and the Sine-Gordon Model

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We consider the Sine-Gordon model coupled to 2D gravity. We find a nonperturbative expression for the partition function as a function of the cosmological constant, the SG mass and the SG coupling constant. At genus zero, the partition function exhibits singularities which are interpreted as signals of phase transitions. A semiclassical picture of one of these transitions is proposed. According to this picture, a phase in which the Sine-Gordon field and the geometry are frozen melts into another phase in which the fields and geometry become dynamical.

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1. Introduction

2D string theory has thus far been most thoroughly studied in a specific background, defined by the standard gaussian model coupled to c=25 Liouville theory. An important open problem is to obtain an equally complete description of strings moving in other 2D backgrounds.

One approach to this problem is based on perturbing the free action for two uncompactified real fields ϕ, X:

$$S_{\text{Liouville}} + S_{\text{Gaussian}} = \int d^2z\sqrt{\hat{g}}\Big(\frac{1}{8\pi}(\hat{\nabla}\phi)^2 + \frac{Q}{8\pi}\phi R(\hat{g})\Big) + \int d^2z\sqrt{\hat{g}}\frac{1}{8\pi}(\hat{\nabla}X)^2 \quad (1.1)$$

by an operator $\sum e^{\alpha_i\phi}\mathcal{O}_i$, where $\mathcal{O}_i$ are operators in the c=1 gaussian model. In this paper we study the example of the Euclidean Sine-Gordon model coupled to 2D gravity:

$$\begin{aligned} S = \int d^2z\sqrt{\hat{g}}\Big[\frac{1}{8\pi}(\hat{\nabla}\phi)^2 + \frac{\mu}{8\pi\gamma^2}e^{\gamma\phi} + \frac{Q}{8\pi}\phi R(\hat{g})\Big] \\ + \int d^2z\sqrt{\hat{g}}\Big[\frac{1}{8\pi}(\hat{\nabla}X)^2 + me^{\xi\phi}\cos(pX/\sqrt{2})\Big] \end{aligned} \quad (1.2)$$

Here $\hat{g}$ is some background metric. In flat space the Sine-Gordon model is not conformal for $m \neq 0$. Coupling the Sine-Gordon model to gravity produces a nontrivial $c = 26$ conformal field theory. General covariance (and hence conformal invariance) is maintained in the quantum theory for $\gamma = \sqrt{2}$, $Q = \sqrt{8}$, $\xi = \gamma(1 - |p|/2)$. We generally use the notation, conventions, (and insights) of [1]. The $c = 1$ model in 2D gravity is reviewed in [2].

Correlation functions in the theory (1.2) will be defined by "conformal perturbation theory." That is, introducing the vertex operator

$$V_p \equiv \int d^2z\sqrt{\hat{g}}e^{\xi\phi}e^{ipX/\sqrt{2}} \quad (1.3)$$

we define correlation functions at $m \neq 0$ by the series

$$\langle\prod V_{q_i}e^{\frac{1}{2}m(V_p+V_{-p})}\rangle \equiv \sum_{n_1,n_2\geq 0}\frac{m^{n_1+n_2}}{2^{n_1+n_2}n_1!n_2!}\langle\prod V_{q_i}(V_p)^{n_1}(V_{-p})^{n_2}\rangle \quad (1.4)$$

The coefficients in the expansion (1.4) are calculated in the standard background with $m = 0$ but $\mu \neq 0$. Recent results on the c=1 matrix model have yielded a complete set of formulae for c=1 correlators [3]. In this paper we use these formulae to learn about the theory (1.2). Our main result is the phase diagram shown in figs.2 and 3, and described in section four. Some physical interpretations of this diagram are proposed in section five.

2. Flows and phase transitions in $2D$ gravity

In this section we review some (well-known) aspects of coupling constant flows in the $c < 1$ models coupled to 2D gravity. The discussion is meant to put the phase transitions discussed in sections four and five into perspective.

The continuum approach to the $c < 1$ models begins with a "(p, q) theory" which is a tensor product of a Liouville theory with the minimal model $M_{p,q}$. Perturbations around this theory are defined by an action

$$S = \mathcal{S}_{p,q} + \sum_{r,k} \tau_{r,k} \mathcal{O}_{r,k} \tag{2.1}$$

where the $\mathcal{O}_{r,k}$ are KPZ dressed operators in the Kac table, the latter being parametrized as in [1]. One of the couplings $\tau_{r,k}$ must be nonzero to "set the scale," i.e., some operator must provide an infrared cutoff on the functional integral over surfaces. Correlation functions for nearby perturbed theories are defined by conformal perturbation expansions such as (1.4). The nature of such expansions is difficult to analyze in the continuum theory since the coefficients are difficult to compute. Enter the matrix model.

The solution of the continuum limit of the "q-matrix model" indicates the existence of an infinite dimensional space of coupling constants, namely the space of real tuples $\{t_{r,k}\}$, $1 \leq r \leq q$, $0 \leq k$, with all but finitely many t's $= 0$ [4]. There is a certain amount of evidence [5] that the continuum matrix model defined by the tuple $\bar{t}^{(p)}_{r,k} = \delta_{kq+r,p}$ is identical to the theory $\mathcal{S}_{p,q}$. Just as the path integral for (2.1) is ill-defined if $\tau_{r,k} = 0$, the (p, q) string equation is singular for $t_{r,k} - \bar{t}^{(p)}_{r,k} = 0$. The mapping from $t_{r,k}$ to $\tau_{r,k}$ is nontrivial and has only been partially worked out in some special cases [6].

The advantage of the matrix model formulation is that the string equation and KP flow give a complete mathematical description of the crossover phenomena for coupling constant flow between the neighborhoods of two (p, q) "fixed points." Choosing some coupling, say $x = t_{r_0,k_0}$, to set the scale, the solution of the string equation $u(x; t_{r,k})$ will be an analytic power series in the other couplings $t_{r,k}$. After proper identification of the t's with the τ's this power series should correspond to conformal perturbation theory. At genus zero the string equations reduce to algebraic equations for u. Hence, the power series in any coupling will in general have a finite radius of convergence. Therefore, if we make any coupling sufficiently large the physical description of the system must change. Let us consider two examples of this.

Example 1. The flow from Ising to pure gravity in the absence of a magnetic field is described by the equation

$$u^3 + tu^2 = x \tag{2.2}$$

Letting x set the scale, for small t we may identify x with the cosmological constant and t with the coupling of the thermal operator ϵ. The physical solution of (2.2) is determined by the branch $u = x^{1/3}$ for $x \to \infty$. For $t < t_c \equiv (27x/4)^{1/3}$, $u(x,t)$ is a convergent power series in t. In this regime the continuum description of the theory uses the action $S = S_{\text{Liouville}} + S_{\text{Ising}} + t\int e^{\xi\phi}\epsilon$. [1] Thus, for $t < t_c$ we are perturbing the Ising fixed point by a relevant operator. Beyond the radius of convergence the solution to (2.2) must be expanded as a convergent power series in $(x/t^3)^{1/2}$. We are now in the neighborhood of the $(2,3)$ fixed point and in the continuum theory we should describe this power series as a perturbation of the pure $c = 26$ Liouville theory by a certain irrelevant operator. Thus the action is $S = S_{\text{Liouville}} + t^{-3/2}\int \mathcal{O}$ where $\mathcal{O}$ is an operator in the $c = 26$ Liouville theory. Although the solution to (2.2) is analytic in the neighborhood of $t = t_c$ the σ-model description of the physics changes.

Example 2. It is easy to find examples of flows in coupling constants where there must be a true phase transition. Consider the string equation of the $(2, 2m-1)$ theories: $u^m + \sum_{i\geq 1} t_i u^i = x$ where we let $t_0 = x$ set the scale. Consider the graph of the function $f(u) = u^m + \sum_{i\geq 1} t_i u^i$ as a function of u, and denote the value of f at the local minimum with the largest value of u by $h(t_i)$. If, as we change the couplings t_i, $h(t_i)$ crosses through x from below there will be a phase transition. In this case, if we simply analytically continue the specific heat around the branch point in complex t-space u will take complex values.

These examples suggest a general idea, which is borne out by the results of this paper. There is a strong analogy between the coordinates $\tau_{r,k}$ on "theory-space," and weighted projective coordinates of complex manifold theory. First, gravitationally dressed operators depend on the Liouville zero mode only through a single exponential factor, hence the overall normalization of the τ's can be changed by a shift of ϕ. Second, as we have remarked, the theory is singular if all the $\tau_{r,k} = 0$, reminiscent of the fact that the origin of $\mathbb{C}^{n+1}$ is in no sense a point of $\mathbb{C}P^n$. Third, in projective space regions in which a given coordinate can be scaled to one provide coordinate patches for the manifold. In τ-space,

[1] Actually, this is only true for small t, and the results of [6] show that there is more to understand in this example.

different coordinate patches are defined by letting different operators $\mathcal{O}_{r,k}$ set the scale. In example 1 above the change of expansion parameters from $t/x^{1/3}$ to x/t^3 is analogous to the change of coordinates between two coordinate patches of weighted projective space. In general, different coordinate patches correspond to different phases of the same theory.

Let us apply some of these ideas to the Sine-Gordon model. Depending on the relative magnitudes of m and μ either the cosmological constant or the Sine-Gordon interaction will set the scale [1]. If the cosmological constant sets the scale we expect that correlators will be expressed as power series in $m^2\mu^{p-2}$. For example the genus zero partition function is $Z = \frac{1}{2}\mu^2\log\mu + \mu^2 f(m^2\mu^{p-2})$ where $f(z)$ has an analytic expansion around zero. On the other hand, if μ is small m sets the scale, and we expand in $\mu(m^2)^{-1/(2-p)}$. In section five below we will find that these expectations are in accord with matrix model calculations. We find a surprise in that there is a phase transition, possibly analogous to that of example 2 above, for $0 < p < 1$ and for $p > 2$, while the model behaves much more like example 1 above for $1 < p < 2$. In section five we offer a physical picture that describes these phase transitions in terms of semiclassical field theory.

Finally, we conclude with a few remarks on the relation between coupling-constant flow and renormalization group flow. A point where all but one τ vanishes is rather like a fixed point of the renormalization group. At the fixed points we have a well-defined notion of matter central charge c^X and bare matter-field dimensions. As in flat space, the operators perturbing away from the fixed points may be divided into relevant, marginal, and irrelevant. Since the Liouville field ϕ defines the local scale, the Liouville charge ξ of the KPZ dressed operator $e^{\xi\phi}\Phi_X$ is positive for relevant, zero for marginal, and negative for irrelevant operators. Thus, relevant operators grow in the infrared $\phi \to +\infty$, etc. The classification of such operators is exactly the same as in flat space since by the KPZ formula and Seiberg's bound [1]: $\xi = \frac{Q}{2}\left(1 - \sqrt{1 + \frac{8}{Q^2}(\Delta_X - 1)}\right)$ gives $\xi < 0$ for $\Delta^X > 1$ and vice versa. In the Sine-Gordon theory for infinitesimal m the operator $e^{ipX/\sqrt{2}}$ is relevant for $p < 2$, marginal for $p = 2$ and irrelevant for $p > 2$. More precisely, in flat space the renormalization group flow in the neighborhood of $(m,p) = (0,2)$ is given by the flow diagram of the Kosterlitz-Thouless model [7].

Ordinary renormalization group flow is dissipative [8]. In gravity one may note, phenomenologically, that the flows by relevant perturbations of the q-matrix model between two fixed points always *increases* c^X, while $c^X_{eff} \equiv c - 24\Delta^X_{\min}$ always decreases. Conversely, c^X_{eff} always increases under irrelevant perturbations. Thus, if we perturb by an irrelevant operator and we find a phase transition we may expect that c^X_{eff} has increased provided we have a phase transition to another surface theory. We will show in section four that one can perturb the Sine-Gordon model by an irrelevant operator $p > 2$ to obtain a phase transition. The extremely interesting question of whether this is a phase transition to a $c^X_{eff} > 1$ model remains open.

3. The partition function for $m \neq 0$.

In this section we describe the result of a matrix model calculation of a one-point function of the form (1.4). For technical reasons it is convenient to calculate the correlation functions of the vertex operators $\mathcal{T}_{\pm p} = \frac{\Gamma(p)}{\Gamma(-p)} V_{\pm p}$, where $p > 0$ here and hereafter. We define the coupling constant

$$\alpha \equiv \tfrac{1}{2}\frac{\Gamma(-p)}{\Gamma(p)} m \tag{3.1}$$

Using the calculational techniques of [3] and conformal perturbation theory we have found an explicit nonperturbative expression for the one-point function of the cosmological constant $\langle \mathcal{T}_0 \rangle_\alpha \equiv \frac{\partial}{\partial \mu} \mathcal{Z}$, where

$$\mathcal{Z} \equiv \langle e^{\alpha \mathcal{T}_p + \alpha \mathcal{T}_{-p}} \rangle \tag{3.2}$$

is the partition function. The somewhat complicated formula is given in equation $(A.6)$ of appendix A. In particular, defining a certain function, the "bounce factor" R_q by:

$$R_q \equiv \mu^{-|q|} \sqrt{\frac{2}{\pi}} e^{i\pi/4} \cos\left(\frac{\pi}{2}\left(\tfrac{1}{2} + i\mu - |q|\right)\right) \Gamma\left(\tfrac{1}{2} - i\mu + |q|\right) \tag{3.3}$$

we find that the amplitude

$$A_n(\mu, p) \equiv \mu^{-np} \langle \mathcal{T}_0 \mathcal{T}_p^n \mathcal{T}_{-p}^n \rangle \tag{3.4}$$

may be expressed as a polynomial in the R_q evaluated for q's at various integer multiples of p. See eq. $(A.1)$ for more detail.

3.1. String Perturbation Theory

In order to study the partition function on a fixed topology we must expand the nonperturbative answer $(A.1)$ in $1/\mu$. This may be obtained from the asymptotic expansion for the bounce factor:

$$R_p \overset{\mu\to\infty}{\sim} exp\left[p\tilde{\psi}_0 + \sum_{n\geq 1}\frac{i^n p^{n+1}}{(n+1)!}(\frac{d}{d\mu})^n(log\mu + \tilde{\psi}_0)\right] = 1 + \frac{ip^2}{\mu} + \cdots \tag{3.5}$$

where $\tilde{\psi}_0$ denotes the expansion

$$\tilde{\psi}_0 \sim \sum_{k\geq 1}\frac{(-1)^k B_{2k}}{2k}(1-2^{-2k+1})\frac{1}{\mu^{2k}} \sim Re\Psi(\tfrac{1}{2}-i\mu) - log\mu \tag{3.6}$$

Ψ is the digamma function and B_{2k} are Bernoulli numbers.

Substitution of (3.5) into equation (A.1) for $A_n(\mu,p)$ gives an asymptotic expansion of the form:

$$A_n(\mu,p) \sim \sum_{h\geq 0}\frac{1}{\mu^{2n-1+2h}}A_n^h(p) \tag{3.7}$$

The very statement of KPZ scaling, namely, that the above expansion begins at order $\frac{1}{\mu^{2n-1}}$ is somewhat miraculous from the point of view of the matrix model and implies the existence of nontrivial combinatorial identities on Bernoulli numbers. Nevertheless we may extract from (3.5) and (A.1) the following facts about the correlation function:

1. At each order of perturbation theory $A_n^h(p)$ is a polynomial in p of degree $4n-2+4h$.
2. $p=0$ is a zero of $A_n^h(p)$ of order $2n$. Indeed, as $p\to 0$:

$$\begin{aligned} A_n(\mu,p) &\to p^{2n}\left(\frac{\partial}{\partial\mu}\right)^{2n-1}Re\psi(\tfrac{1}{2}-i\mu) \\ &\sim p^{2n}\left[\frac{(2n-2)!}{\mu^{2n-1}} + \frac{(2n)!}{24\mu^{2n+1}} + \cdots\right] \end{aligned} \tag{3.8}$$

3. The value $p=1$ is a root of order n of $A_n^h(p)$.
4. Moreover, the value $p=2$ is a root of order 1 for A_n^h for $h\geq 1$, and in general for m a positive integer, $p=m$ is a root of $A_n^h(p)$ for $h > \frac{1}{2}(1+n(m-2))$.

Statement one is easily proved by examination of (3.5). The expansion of the term in the exponent in powers p^a/μ^b has a maximum value of $a-b$ for the term p^2/μ. In [3] the amplitude $\langle \mathcal{T}_0 \prod_{i=1}^k \mathcal{T}_{p_i}\rangle$ was shown to be a polynomial in bounce factors R_p. It follows that at genus h the amplitude is a polynomial in p_i of degree $2k-2+4h$. [2] A special case of this result is statement one.

Statement two is easily proved from the low energy theorem in [3]. As $p \to 0$ we have $\mathcal{T}_p \to p\mathcal{T}_0 = p\frac{\partial}{\partial\mu}$ (except in the genus zero two-point function). Statement (3.8) immediately follows from the well-known value of the specific heat [10].

The proof of statements three and four is sketched in appendix B.

3.2. The genus zero amplitude

We now focus on the spherical topology. The above remarks show that $A_n^{h=0}(p) = (2n-2)!p^{2n}(1-p)^n Q_n(p)$ where Q_n is a polynomial of order $n-2$ with $Q_n(0)=1$. Finding a formula for this polynomial has proved to be a rather difficult problem. Explicitly expanding the formula (A.1) we find, experimentally, the curious result

$$
\begin{aligned}
A_n^{h=0}(p) &= (2n-2)!p^{2n}(1-p)^n \prod_{i=1}^{n-2}(1-p/r_i) \\
\langle \mathcal{T}_p^n \mathcal{T}_{-p}^n \rangle &= -\mu^{np-2n+2} n! p^{2n}(1-p)^n \frac{\Gamma(n(1-p)+n-2)}{\Gamma(n(1-p)+1)}
\end{aligned}
\tag{3.9}
$$

where $r_i = 1 + i/n$. We emphasize that this is a phenomenological formula, checked for $1 \le n \le 11$. We have made many unsuccessful attempts to prove (3.9) for all n. [3] Two remarks might be useful to anyone else who tries:

1. By explicit calculation, the special roots r_i are *not* roots of the genus one amplitude. [4]
2. The result can be summarized as a "Ward identity" similar to those which have been intensively studied recently in central New Jersey. Specifically, defining $\mathcal{T}_p = p\tilde{\mathcal{T}}_p$ we may use the boundary-operator Ward-identity [12] [13] to restate the result (3.9) as

$$
\langle \tilde{\mathcal{T}}_0 \tilde{\mathcal{T}}_{1+\epsilon}^n \tilde{\mathcal{T}}_{-1-\epsilon}^n \rangle = \langle \tilde{\mathcal{T}}_1^n \tilde{\mathcal{T}}_{-1-\epsilon}^n \tilde{\mathcal{T}}_{n\epsilon} \rangle
\tag{3.10}
$$

for $0 < \epsilon < 1$.

In section four we will simply assume that (3.9) holds for all n, p and explore the physical consequences.

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[2] This confirms the observation of [9] that at large energies the effective string coupling in the $c=1$ model is $g_{eff} \sim \frac{p^2}{\mu}$.

[3] We thank R. Plesser for his participation in several of these efforts.

[4] Thus the existence of these roots is reminiscent of the roots of the chromatic polynomial predicted by the Beraha conjecture [11].

3.3. The special cases $p = 1, 2$

Using discrete tachyon "Ward identities" recently derived in [13] we can give a much more complete description of the partition function and correlation functions for the special cases of a Sine-Gordon background with $p = 1, 2$.

At $p = 1$ the dependence on α is polynomial. Correlation functions at $\alpha \neq 0$ are easily related to correlation functions at $\alpha = 0$. The general formula expressing this relation is somewhat long, so we simply quote a typical result

$$\langle \mathcal{T}_0 \mathcal{T}_q \mathcal{T}_{-q} e^{\alpha \mathcal{T}_1 + \alpha \mathcal{T}_{-1}} \rangle = {}_2F_1(1-q, 1-q; 1; \alpha^2 \mu^{-1}) \langle \mathcal{T}_0 \mathcal{T}_q \mathcal{T}_{-q} \rangle \tag{3.11}$$

for q a positive integer.

At $p = 2$ (when the matter perturbation is formally marginal) the $\mathcal{T}_2$ Ward identity of [13] implies:

$$\lim_{\epsilon \to 0+} \langle \tilde{\mathcal{T}}_{-n\epsilon} \tilde{\mathcal{T}}_2^n \tilde{\mathcal{T}}_{-2+\epsilon}^n \prod_{i=1}^{k} \tilde{\mathcal{T}}_{k_i} \prod_{j=1}^{l} \tilde{\mathcal{T}}_{-q_j} \rangle = n! \frac{\Gamma(Q+n)}{\Gamma(Q)} \langle \tilde{\mathcal{T}}_0 \prod_{i=1}^{k} \tilde{\mathcal{T}}_{k_i} \prod_{j=1}^{l} \tilde{\mathcal{T}}_{-q_j} \rangle \tag{3.12}$$

where $k_i, q_j > 0$, $\sum k_i = \sum q_j \equiv Q$, and $q_i + q_j < 2$ for all pairs i, j. It follows that correlation functions in this kinematic regime are given at $\alpha \neq 0$ by

$$\langle \prod_{i=1}^{k} \mathcal{T}_{k_i} \prod_{j=1}^{l} \mathcal{T}_{-q_j} e^{\alpha \mathcal{T}_2 + \alpha \mathcal{T}_{-2}} \rangle = (1 - 4\alpha^2)^{-Q} \langle \prod_{i=1}^{k} \mathcal{T}_{k_i} \prod_{j=1}^{l} \mathcal{T}_{-q_j} \rangle \tag{3.13}$$

The correlation functions in other kinematic regimes will not be so simply related. For example, one can show

$$\langle \mathcal{T}_q \mathcal{T}_{-q} e^{\alpha \mathcal{T}_2 + \alpha \mathcal{T}_{-2}} \rangle = (1 - 4\alpha^2)^{-q} \left[1 + 4\alpha^2 q(q-2)\right] \langle \mathcal{T}_q \mathcal{T}_{-q} \rangle \tag{3.14}$$

for $2 < q < 4$.

Using the above methods one can directly derive the specific heat:

$$\langle \mathcal{T}_0 \mathcal{T}_0 e^{\alpha \mathcal{T}_2 + \alpha \mathcal{T}_{-2}} \rangle = \log\mu - \log(1 - 4\alpha^2) \tag{3.15}$$

which is, of course, in accord with (3.9). An amusing, and perhaps important, feature of (3.15) is that it is true to *all* orders of perturbation theory, since as noted in section (3.1) $p = 2$ is a root of the correlation functions at genus $h \geq 1$.

We remark that the amplitudes (3.13)(3.15) exhibit an interesting duality between the theory at α and at $\tilde{\alpha} = 1/(4\alpha)$. This duality will be generalized below.

4. Phase Diagram in the α,p plane

According to (3.9) the genus zero specific heat is given by

$$\langle \mathcal{T}_0 \mathcal{T}_0 e^{\alpha \mathcal{T}_p + \alpha \mathcal{T}_{-p}} \rangle - \langle \mathcal{T}_0 \mathcal{T}_0 \rangle = -H(p;z) \tag{4.1}$$

where $z = \mu^{p-2}\alpha^2 p^2(1-p)$ and we have defined an analytic function

$$H(p;z) \equiv \sum_{n\geq 1} \frac{\Gamma(n(2-p))}{n!\Gamma(n(1-p)+1)} z^n \tag{4.2}$$

Some useful mathematical facts about the function $H(p;z)$ are collected and proved in appendix C. In particular, H is a convergent power series in z for all real values of p with radius of convergence $R_c(p)$ given by $(C.3)$ and plotted in fig. 1. On the circle of convergence the series has one or two branch point singularities given by

$$\begin{aligned} z_c(p) &= R_c(p) & p<1 \\ z_c^{\pm}(p) &= -e^{\pm i\pi p} R_c(p) & 1<p<2 \\ z_c(p) &= -R_c(p) & p>2 \end{aligned} \tag{4.3}$$

Moreover, the values of $H(p;z)$ for $|z| > |z_c|$ may be related to values within the circle of convergence by connection formulae similar to those for the hypergeometric function.

Using the above facts we may draw a phase diagram as shown in fig. 2 There are six different regions:

I.) $0 < p < 1$, $0 \leq \mu^{p-2}\alpha^2 < R_c(p)/(p^2(1-p))$

The specific heat is a power series in α^2 with real coefficients. There is a finite radius of convergence with singularity of the form $(1-z/z_c)^{1/2}$.

II.) $1 < p < 2$, $0 \leq \mu^{p-2}\alpha^2 < R_c(p)/(p^2(p-1))$

Again we have a real power series in α^2, but the branch point has moved off into the complex plane, so there is no singularity as we approach the dotted line in fig. 2.

III.) $2 < p < \infty$, $0 \leq \mu^{p-2}\alpha^2 < R_c(p)/(p^2(p-1))$

There is again a finite radius of convergence with singularity $(1-z/z_c)^{1/2}$ since the branch point has moved back to the real axis. Of course, there is no singularity in passing between regions I, II, III.

IV.) $0 < p < 1$, $\mu^{p-2}\alpha^2 > R_c(p)/(p^2(1-p))$

The values in this region are defined by analytic continuation around the branch point. Explicitly, the connection formula ($C.6$) gives

$$\begin{aligned}\langle \mathcal{T}_0 \mathcal{T}_0 e^{\alpha \mathcal{T}_p + \alpha \mathcal{T}_{-p}} \rangle_{IV} = & \frac{1}{2-p}\Big[\log(\alpha^2 p^2 (1-p)) \pm i\pi\Big] \\ & + \frac{1}{p-2} H\Big[p'; -e^{\pm i\pi/(p-2)} \frac{\mu}{(\alpha^2 p^2 (1-p))^{1/(2-p)}}\Big]\end{aligned} \tag{4.4}$$

where $p' = (2p-3)/(p-2)$. We have a power series in μ with *complex* coefficients. The $\pm$ sign depends on the sense in which we analytically continue around the branch point.

V.) $1 < p < 2$, $\mu^{p-2}\alpha^2 > R_c(p)/(p^2(p-1))$

Now again applying the same connection formula,

$$\langle \mathcal{T}_0 \mathcal{T}_0 e^{\alpha \mathcal{T}_p + \alpha \mathcal{T}_{-p}} \rangle_{V} = \frac{1}{2-p}\log(\alpha^2 p^2 (p-1)) + \frac{1}{p-2} H\Big[p'; -\frac{\mu}{(\alpha^2 p^2 (p-1))^{1/(2-p)}}\Big] \tag{4.5}$$

giving a power series in μ with real coefficients.

VI.) $2 \le p < \infty$, $\mu^{p-2}\alpha^2 > R_c(p)/(p^2(p-1))$

In this region we use connection formula ($C.8$). The result is a power series with complex coefficients and expansion parameter

$$(\alpha^2)^{-1/(p-1)} \mu^{-(p-2)/(p-1)} \tag{4.6}$$

the expansion is analytic neither in α nor in μ.

The case $p = 2$ requires special attention as discussed in section 3.3.

Finally, we have used the coupling α which is natural from the matrix model. Changing variables $\alpha \to m$ using (3.1) the phase diagram in the p, m plane looks somewhat different, and is illustrated in fig. 3.

5. Physical interpretation

There are phase transitions when crossing the solid lines in fig. 2. In this section we offer some qualitative physical interpretations of these transitions based on semiclassical analysis. The following considerations are only meant to be heuristic, and it would be interesting to make them more rigorous.

The action (1.2) may be written as

$$\begin{aligned}S = & \frac{1}{4\pi\gamma^2}\Big\{\int d^2 z \sqrt{\hat{g}}\big[\frac{1}{2}(\hat{\nabla}\phi)^2 + e^{\phi} + \frac{Q\gamma}{2}\phi R(\hat{g})\big] \\ & + \int d^2 z \sqrt{\hat{g}}\big[\frac{1}{2}(\hat{\nabla}X)^2 + m e^{\xi\phi}\cos(pX)\big]\Big\}\end{aligned} \tag{5.1}$$

where for convenience we have rescaled and shifted

$$\begin{aligned} m &\to 4\pi m\gamma^2(2/\mu)^{\xi/\gamma} \qquad && \phi \to \gamma\phi + \log(\mu/2) \\ p \to p/(\gamma\sqrt{2}) \qquad X &\to \gamma X && \xi \to \xi/\gamma \end{aligned} \tag{5.2}$$

For $Q = 2/\gamma$, $\xi = 1$ we have a classical conformal field theory. Of course, quantum effects are strong at $c = 1$, but they can be summarized by the usual KPZ/DDK renormalization of parameters, so that $\gamma = \sqrt{2}$, $Q = \sqrt{8}$, $\xi = 1 - p/2$. Working semiclassically, the precise value of Q turns out to be unimportant so we will take the classical value $Q = 2/\gamma$ for simplicity. The physics depends sensitively on ξ so we will leave this as a free parameter with $0 < \xi < 1$. The equations of motion following from (5.1) are then

$$\begin{aligned} R(e^{\phi}\hat{g}) + 1 + m\xi e^{-(1-\xi)\phi}\cos pX &= 0 \\ \hat{\nabla}^2 X + mpe^{\xi\phi}\sin pX &= 0 \end{aligned} \tag{5.3}$$

We restrict attention to the sphere with background metric:

$$\hat{g} = \frac{|dz|^2}{(1+|z|^2)^2} \qquad \hat{R} = 8 \tag{5.4}$$

the constant solutions ("vacua") of (5.3) are given by $(X_n, \bar{\phi})$ where $X_n = \frac{\pi}{p}(2n+1)$, $n \in \mathbb{Z}$, and $\bar{\phi}$ solves

$$8 + e^{\bar{\phi}} - \xi m e^{\xi\bar{\phi}} = 0 \tag{5.5}$$

This equation only has solutions for

$$\log(m\xi) \geq (1-\xi)\log 8 - \Big((1-\xi)\log(1-\xi) + \xi\log\xi\Big) \tag{5.6}$$

in which case the zero-mode potential looks like fig. 4. (When there is no solution one must introduce Lagrange multipliers to fix the area of the surface. See [1].)

The existence of classical solutions for m larger than a critical value explains some features of the phase diagram of the previous section. In a phase where a solution exists we can expand around it and therefore we expect the partition function to be nonsingular for $\mu \to 0$ [1]. This is in accord with the difference between regions I, II and IV, V.

When the condition (5.6) is satisfied there are in fact *two* allowed constant curvatures, i.e., there are two solutions to (5.5). For large values of m these are approximately

$$\begin{aligned} e^{\phi_s} &\sim \left(\frac{8}{m\xi}\right)^{1/\xi} \quad \ll 1 \\ e^{\phi_b} &\sim (m\xi)^{1/(1-\xi)} \gg 1 \end{aligned} \tag{5.7}$$

At $p = 2$ the sum in (6.2) behaves like a power series and remains convergent. More precisely, if we let $p = 2 - is$ for s real we have for large n:

$$\frac{|R_{np}|}{n!^2} \leq \frac{1}{\pi} e^{-\mu\theta(s)+\frac{1}{2}\pi\mu} \frac{1}{n} \mu^{-2n} (f(s))^n \tag{6.3}$$

where

$$f(s) = 4(1 + s^2/4) e^{\frac{1}{2}\pi|s| - s\theta(s)}$$

and $\theta(s) = \tan^{-1}(s/2)$. Thus if $4C\alpha^2 < 1$ there is a finite interval $-s_c < s < s_c$ along the $Re(p) = 2$ axis where the series is absolutely convergent. Thus there exists an analytic continuation of the determinant to the domain $Re(p) > 2$, where we are perturbing the Sine-Gordon theory by an irrelevant operator.

We have just shown that, nonperturbatively, there are analogs of the regions I, II, III of section four. It is more difficult to see if the radius of convergence in $|\alpha|$ will be finite. We expect that at fixed p there is a finite radius of convergence in $|\alpha|$. As we increase $|\alpha|$, $1 + \Sigma$ probably develops a left or right zero mode and the determinant has a logarithmic singularity, although we have not proven this. It is easy to show that for sufficiently large $|\alpha|$, $\| \Sigma \| > 1$ so there is no reason for the determinant to be nonsingular. It would be interesting to understand the singularities better and to have a physical picture of how the phase transitions are modified by topology-change.

7. Future Directions

There are several projects which would extend the present work:

1. Of course, it is important to *prove* (3.9)! The recent results of [13] are an important step in this direction.
2. We also skipped over some hard analysis in section five, regarding the existence of instanton solutions.
3. In section four we used the connection formulae $(C.4), (C.6), (C.8)$. These relate different backgrounds via the action of the discrete group S_3 and are thus reminiscent of target space duality and mirror symmetry. It would be very interesting to see if these symmetries survive in other correlation functions.
4. We would like to have an equally complete understanding of the amplitudes at genus one which are implicitly contained in $(A.1)$. These would be most useful for understanding better the nature of the transition $III \rightarrow VI$. If the transition is due to tachyon dominance

that should become apparent in the behavior of the genus one amplitudes. Unfortunately, we have not managed to recognize any special pattern in the first few amplitudes.

5. The simple result (3.15) deserves to be understood better. Naively, at $p = 2$ we have a tensor product of Liouville and Sine-Gordon theories, but for $m \neq 0$, $\cos(\sqrt{2}X)$ is not exactly marginal so this is an illusion [7]. Indeed (3.15) is not a product, but a sum of functions of μ and α^2.

6. It would be interesting to interpret the minimal models as restricted Sine-Gordon theories and relate the above results more directly to the $c < 1$ models.

The present paper also touches on some deeper issues. The matrix model defines finite integrated correlators even for the irrelevant operators. How does it choose the finite parts? A natural guess for the underlying principle is the W_∞ symmetry of the theory. This implies a corresponding W_∞ symmetry of the continuum Liouville $\times$ matter system. Perhaps the finite parts are chosen based on the principle that the W_∞ Ward identities must be maintained.

A second issue is the spacetime interpretation of the Euclidean theory. If, for example, we compactify X when $\alpha = 0$ then we calculate the free energy of a string at temperature $1/R$. What happens when $\alpha \neq 0$? The Euclidean Hamiltonian has now nontrivial X-dependence. Should we interpret the calculations in terms of nonequilibrium statistical mechanics?

One may also ask about the Minkowskian analog of the above results and the corresponding Minkowskian spacetime interpretation. Some of the relevant issues are discussed in [13].

Finally, we may ask the evident question: Do analogous phase transitions exist in more realistic theories of gravity?

Acknowledgements

We would like to thank R. Plesser and N. Seiberg for very important conversations. We further thank N. Seiberg for useful criticism and comments on an earlier draft of the paper. We have also benefitted from helpful discussions with and suggestions from T. Banks, M. Douglas, E. Martinec, S. Ramgoolam, N.Read, H. Saleur, R. Shankar, S. Shatashvili, S. Shenker, and A.B. Zamolodchikov. This work is supported by DOE grant DE-AC02-76ER03075 and by a Presidential Young Investigator Award.

Appendix A. Derivation of the formula for the correlation functions

We derive the formula for $\langle \mathcal{T}_0(\mathcal{T}_p\mathcal{T}_{-p})^n\rangle$ by considering the $\epsilon \to 0+$ limit of the correlator $\langle \mathcal{T}_\epsilon(\mathcal{T}_{p-\epsilon/n})^n(\mathcal{T}_{-p})^n\rangle$ using the graphical rules of [3]. A short calculation yields

$$
\begin{aligned}
A_n(\mu,p) &\equiv \mu^{-np}\langle \mathcal{T}_0\mathcal{T}_p^n\mathcal{T}_{-p}^n\rangle \\
&= i(-1)^n(n!)^2\sum_{k=1}^{n}\frac{(-1)^k}{k}\sum_{a_i,b_i}\Big(\prod_{i=1}^{k} b_i^2-\prod_{i=1}^{k} a_i^2\Big)\mathcal{C}(a_1,\ldots,b_k)\prod_{i=1}^{k}\frac{R_{a_ip}}{(a_i)!^2}\frac{R^*_{b_ip}}{(b_i!)^2}
\end{aligned}
\tag{A.1}
$$

where the sum runs over all partitions $n = a_1+b_1+\cdots+a_k+b_k$ with $a_i, b_i \geq 0$ such that the denominator of

$$
\mathcal{C}(a_1,b_1,a_2,b_2,\ldots,a_k,b_k) \equiv \frac{1}{(a_1+b_1)(b_1+a_2)(a_2+b_2)\cdots(a_k+b_k)(b_k+a_1)}
\tag{A.2}
$$

is nonzero. R_p is the "bounce factor" of [3] given by

$$
R_p = \mu^{-p}\sqrt{\frac{2}{\pi}}e^{i\pi/4}\cos\Big(\frac{\pi}{2}(\tfrac{1}{2}+i\mu-p)\Big)\Gamma(\tfrac{1}{2}-i\mu+p)
\tag{A.3}
$$

for $p > 0$.

The expression (A.1) can be written more succinctly by introducing an algebra of 1-dimensional projection operators $\mathcal{P}_{a,b}$ for $a, b \in \mathbb{Z}_+$, not both zero, satisfying the relations

$$
\mathcal{P}_{a_1,b_1}\mathcal{P}_{a_2,b_2} = \frac{(a_1+b_1)(a_2+b_2)}{(b_1+a_2)(b_2+a_1)}\mathcal{P}_{a_1,b_2}
\tag{A.4}
$$

For such an algebra we may write the factor $\mathcal{C}$ as

$$
\frac{1}{\prod_i(a_i+b_i)^2}Tr\Big(\mathcal{P}_{a_1,b_1}\cdots\mathcal{P}_{a_k,b_k}\Big) = \mathcal{C}(a_1,\ldots b_k)
\tag{A.5}
$$

Such projection operators may be explicitly constructed as operators on a Hilbert space. Let $|z_a\rangle$, $a = 0,1,\ldots$ be an ON basis. Define $|w_b\rangle = \sum_a(a+b)^{-1}|z_a\rangle$ so that $\langle z_a|w_b\rangle = (a+b)^{-1}$. Then $\mathcal{P}_{a,b} \equiv (a+b)|z_a\rangle\langle w_b|$.

Using the projection operators $\mathcal{P}_{a,b}$ the full partition function can be nicely expressed as a determinant of an operator Σ:

$$
\langle \mathcal{T}_0 e^{\alpha\mathcal{T}_p+\alpha\mathcal{T}_{-p}}\rangle - \langle\mathcal{T}_0\rangle = i\log Det\Big[(1+\Sigma)(1+\Sigma^*)^{-1}\Big]
\tag{A.6}
$$

where

$$
\Sigma \equiv \sum_{n\geq 1}(-\mu^p\alpha^2)^n\sum_{a+b=n}\frac{R_{ap}}{(a)!^2}\frac{R^*_{bp}}{(b!)^2}\frac{b^2}{n^2}\mathcal{P}_{a,b}
\tag{A.7}
$$

Remarks:

1. It might be an interesting exercise to obtain the above determinant directly from the fermion determinant in the original free-fermion formulation of the theory.

2. Nonperturbatively the series vanishes identically at $\mu = 0$. This follows immediately since $R_p R_q^*$ is real for $\mu = 0$ and any real momenta p, q.

3. In a similar way one can write slightly more complicated formulae for the three point function $\langle \mathcal{T}_0 \mathcal{T}_q \mathcal{T}_{-q} exp(\alpha \mathcal{T}_p + \alpha \mathcal{T}_{-p}) \rangle$.

Appendix B. Integer roots of the amplitudes

The basic idea of the proof is very simple. To all orders of perturbation theory the bounce factor can be replaced by

$$\begin{aligned} R_p &= \mu^{-p} e^{i\pi p/2} \frac{\Gamma(\frac{1}{2} - i\mu + p)}{\Gamma(\frac{1}{2} - i\mu)} \\ &\sim 1 + \sum_{k=1}^{\infty} \frac{Q_k(p)}{\mu^k} \end{aligned} \tag{B.1}$$

For $p = n \in \mathbb{Z}_+$, R_p becomes a polynomial in $1/\mu$ so that $Q_k(p = n) = 0$ for $k > n$. On the other hand, by KPZ scaling, we know the power of $1/\mu$ for the leading term in any amplitude. If this power exceeds the order of the relevant polynomials then we may prove vanishing theorems. For example, if we put $p = m$ in (A.1), then the expression must be of the form $1/\mu^{nm}$ times a polynomial in μ. By KPZ scaling the genus h contribution goes like $\sim 1/\mu^{2n-1+2h}$ so that the appropriate contribution must vanish for $h > \frac{1}{2} + n(m-2)/2$.

The proof that $p = 1$ is an n^{th} order zero is much more tedious but uses the same idea. Having proved $p = 1$ is a root we take a derivative with respect to p of (A.1). Plugging in $p = 1$ and using properties of gamma and polygamma functions we show that $\frac{\partial}{\partial p} A_n$ has an expansion in $1/\mu$ terminating at $1/\mu^n$. The proof then proceeds inductively and the inductive step fails when we consider $(\frac{\partial}{\partial p})^n A_n$.

We may note parenthetically that by the above reasoning *any* $c = 1$ amplitude with integral external momenta vanishes at sufficiently large orders of perturbation theory. This supports the general idea that special tachyons are associated with topological field theory.

Appendix C. Properties of the function $H(p;z)$

In this appendix we prove some useful facts about the function

$$\begin{aligned} H(p;z) &\equiv \sum_{n\geq 1} \frac{\Gamma(n(2-p))}{n!\Gamma(n(1-p)+1)} z^n \\ &= \frac{1}{\pi}\sum_{n\geq 1} \frac{\Gamma(n(2-p))\Gamma(n(p-1))}{n!} \sin(n\pi p)(-z)^n \\ &= -\sum_{n\geq 1} \frac{\Gamma(n(p-1))}{n!\Gamma(n(p-2)+1)}(-z)^n \end{aligned} \tag{C.1}$$

The ratio of gamma functions behaves at large n like

$$\begin{aligned} &n^{-3/2}(exp\left[(2-p)log(2-p)-(1-p)log(1-p)\right])^n && 0<p<1 \\ &n^{-3/2}(-1)^n sin(n\pi p)(exp\left[(2-p)log(2-p)+(p-1)log(p-1)\right])^n && 1<p<2 \\ &n^{-3/2}(-1)^n(exp\left[(p-2)log(p-2)-(p-1)log(p-1)\right])^n && 2<p<\infty \end{aligned} \tag{C.2}$$

showing that the series defining $H(p;z)$ converges absolutely for

$$|z| < R_c(p) = exp\Big[(p-2)\log|p-2| - (p-1)\log|p-1|\Big] \tag{C.3}$$

Note that from comparing the first and third lines we have the first connection formula:

$$H(p;z) = -H(3-p;-z) \tag{C.4}$$

We can define analytic continuations of the function $H(p;z)$ using various integral representations.

Our first integral representation is the Mellin-Barnes representation

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(s(2-p))}{\Gamma(s(1-p)+1)}\Gamma(-s)(-z)^s ds \tag{C.5}$$

where $z \notin \mathbb{R}^+$ and we use the standard branch of the logarithm. The integral over s converges absolutely for all such z if $p<1$ and converges for $|arg(z)| \geq \pi(p-1)$ for $1<p<2$. For $|z|<|z_c(p)|$ we can close the s integral in the right half-plane to obtain the series in the first line of (C.1). If $|z|>|z_c(p)|$ then we can close in the left half-plane, thus proving the connection formula

$$H(p;z) = \frac{1}{p-2}\log(-z) + \frac{1}{2-p}H(p';-(-z)^{-1/(2-p)}) \tag{C.6}$$

for $p' - 2 = \frac{1}{p-2}$.

Our second integral representation is the Mellin-Barnes integral

$$\frac{-1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(s(p-1))}{\Gamma(s(p-2)+1)}\Gamma(-s)z^s ds \tag{C.7}$$

where $z \notin \mathbb{R}^-$. For $p > 2$ this converges absolutely for all such z and defines an analytic continuation of H. For $1 < p < 2$ the integral converges absolutely for $|arg(z)| \leq \pi(p-1)$. By closing into the right half-plane we obtain the series in the third line of (C.1) and by closing in the left half-plane we obtain the third connection formula:

$$H(p; z) = \frac{-1}{1-p}\log z + \frac{1}{p-1}H(p'; z^{-1/(p-1)}) \tag{C.8}$$

where $p' - 1 = \frac{1}{p-1}$.

Our third integral representation is derived from the middle series in (C.1) using the integral representation of the Beta function. The result is

$$z\frac{\partial}{\partial z}H(p; z) = -\frac{z \sin \pi p}{\pi}\int_0^1 dt \frac{t^{1-p}(1-t)^{p-2}}{\left(1 + ze^{i\pi p}t^{2-p}(1-t)^{p-1}\right)\left(1 + ze^{-i\pi p}t^{2-p}(1-t)^{p-1}\right)} \tag{C.9}$$

This integral always converges at the endpoints $t = 0, 1$ for $1 < p < 2$ and defines an analytic continuation in z for these values of p. The existence of singularities can be examined by looking for pinching of the contour. In this way it is easy to check that there are no singularities as z increases from zero to infinity through real values.

In general $H(p; z)$ does not seem to be expressible in terms of standard special functions, although at some special values we can write H more explicitly:

$$\begin{aligned}
H(0; z) &= -\log\left(\frac{1+\sqrt{1-4z}}{2}\right) \\
H(\tfrac{1}{2}; z) &= \pi\int_0^z \left[{}_2F_1(\frac{5}{6}, \frac{7}{6}; \frac{3}{2}; \frac{27t^2}{4}) - 1\right] dt \\
&\quad + \frac{2}{3}\pi\int_0^z \left[{}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{27t^2}{4}) - 1\right]\frac{dt}{t} \\
H(1; z) &= -\log(1-z) \\
H(\frac{3}{2}; z) &= 2\log\left[\sqrt{1+\frac{z^2}{4}} + \frac{z}{2}\right] \\
H(2; z) &= \log(1+z)
\end{aligned} \tag{C.10}$$

When p is rational, $H(p;z)$ can be written in terms of generalised hypergeometric functions ${}_aF_b$. In general for fixed p the branch point singularity in z is a square root singularity, except at $p = 1, 2$ where we have a logarithmic singularity.

Finally we note that the set of transformations of the (p, z) plane defined in (C.4)(C.6)(C.8) defines an action of the permutation group S_3 on this plane.

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Figure Captions

Fig. 1. Radius of convergence as a function of p.

Fig. 2. The phase diagram in the α^2 vs p plane. The solid lines indicate lines across which there are phase transitions.

Fig. 3. The phase diagram using the normalization standard for vertex operators. There are now infinitely many separated regions where we have a transition of the type $III \rightarrow VI$.

Fig. 4. The zero mode potential $V(\phi, X) = 8\phi + e^{\phi} + me^{\xi\phi} \cos pX$ in the case when there are constant postive curvature solutions to the equations of motion.

Fig. 5. Particle motion of $(X(r), \phi(r))$ for a proposed field configuration connecting large and small geometries.

Fig. 6. A multi-instanton configuration. Many small instantons of scale size $r^2 \sim e^{\phi_s}$ join onto a large sphere of scale size $r^2 \sim e^{\phi_b}$.

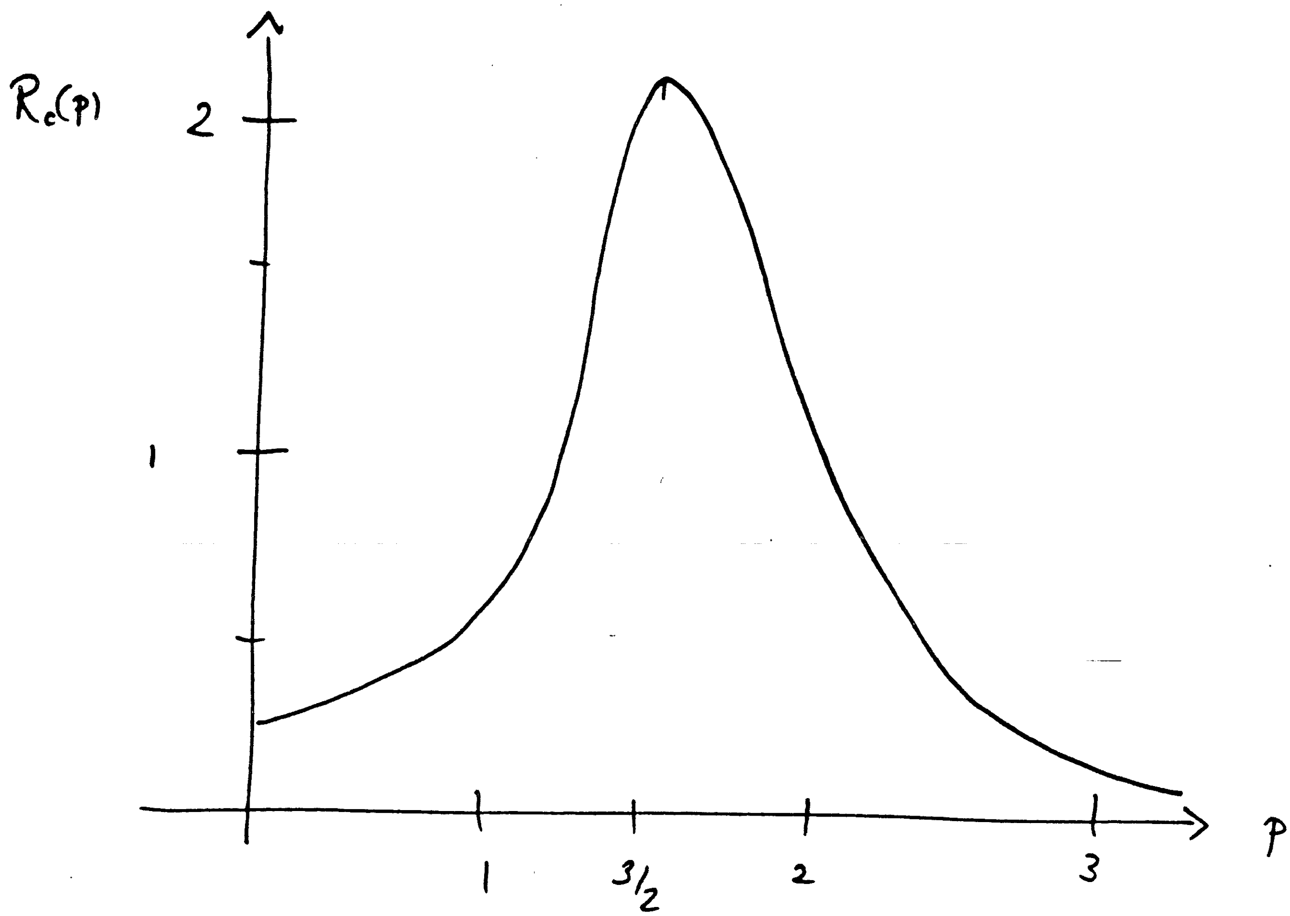

Fig. 1

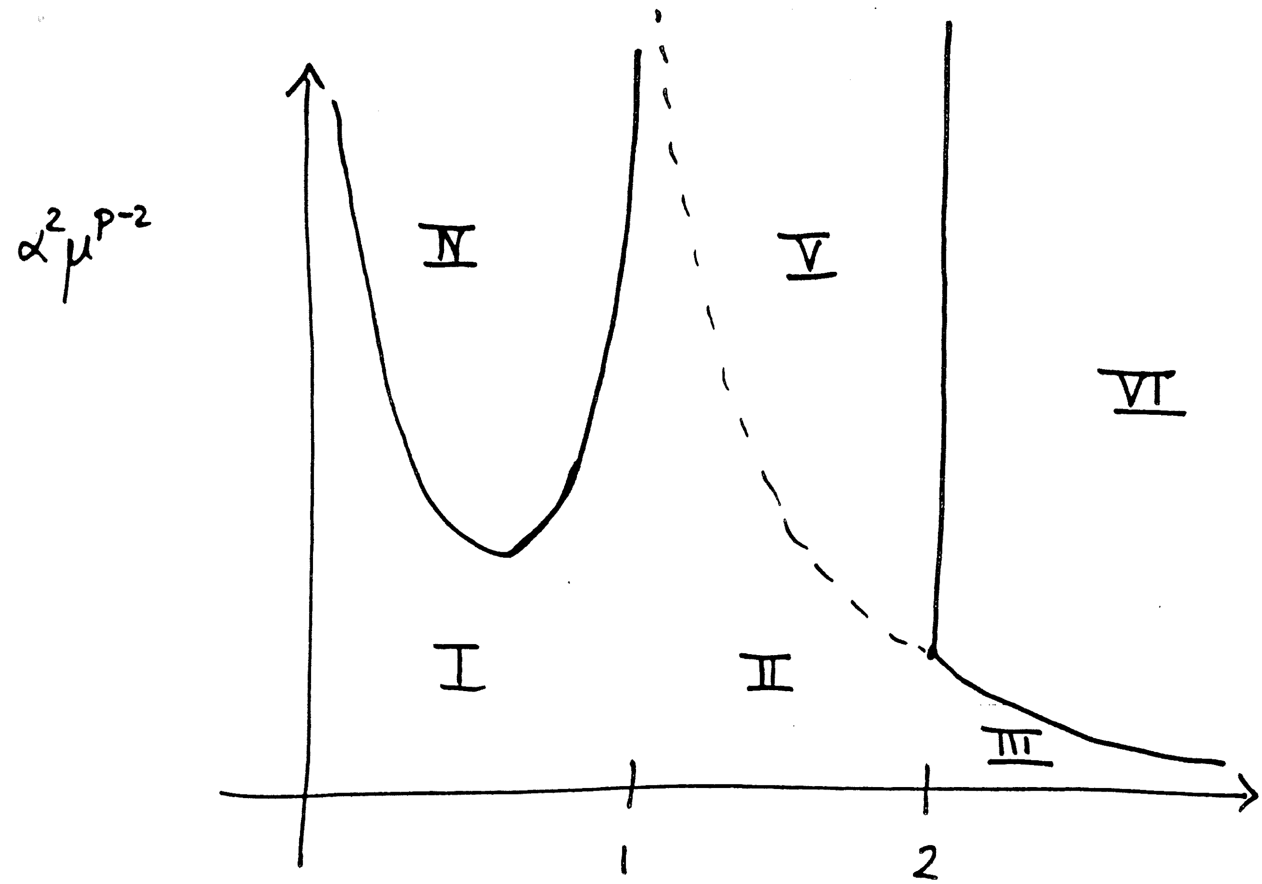

Fig. 2

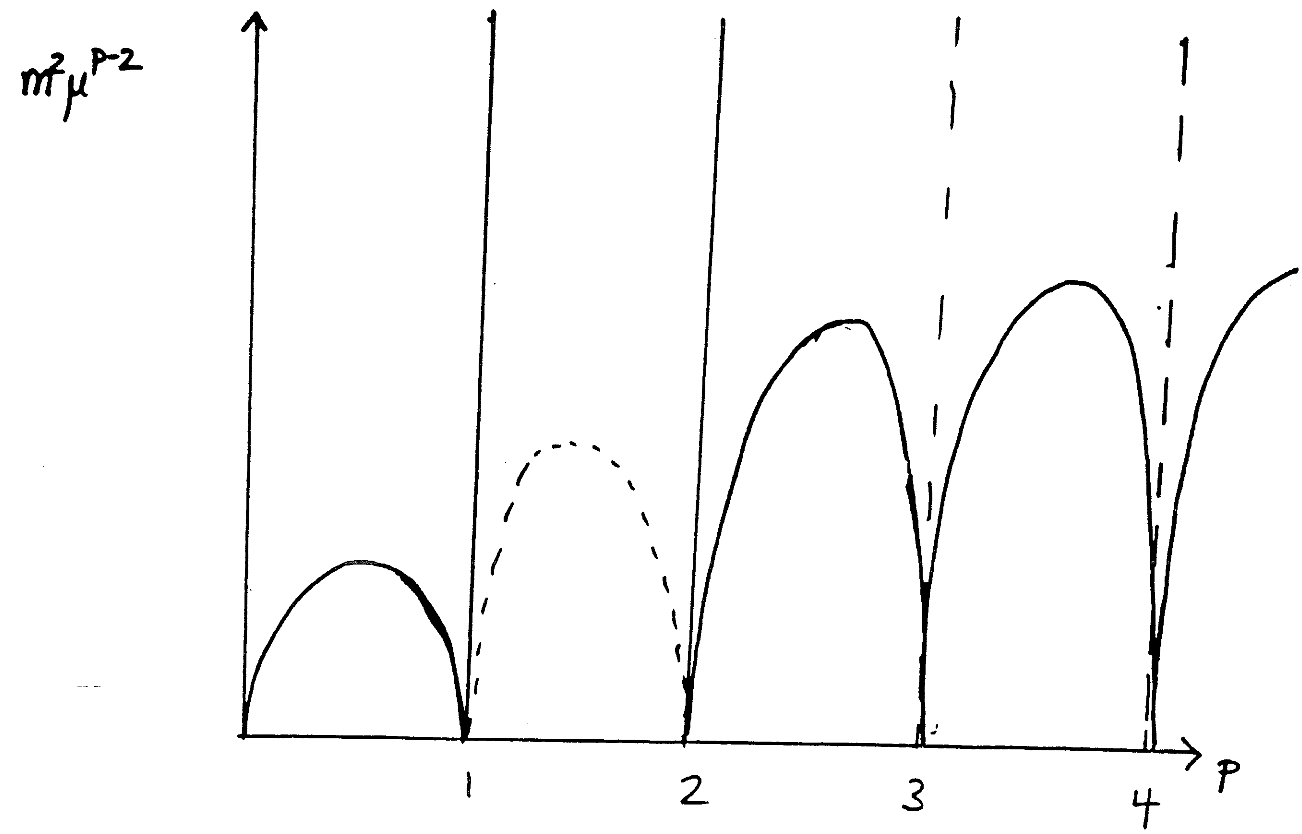

Fig.3

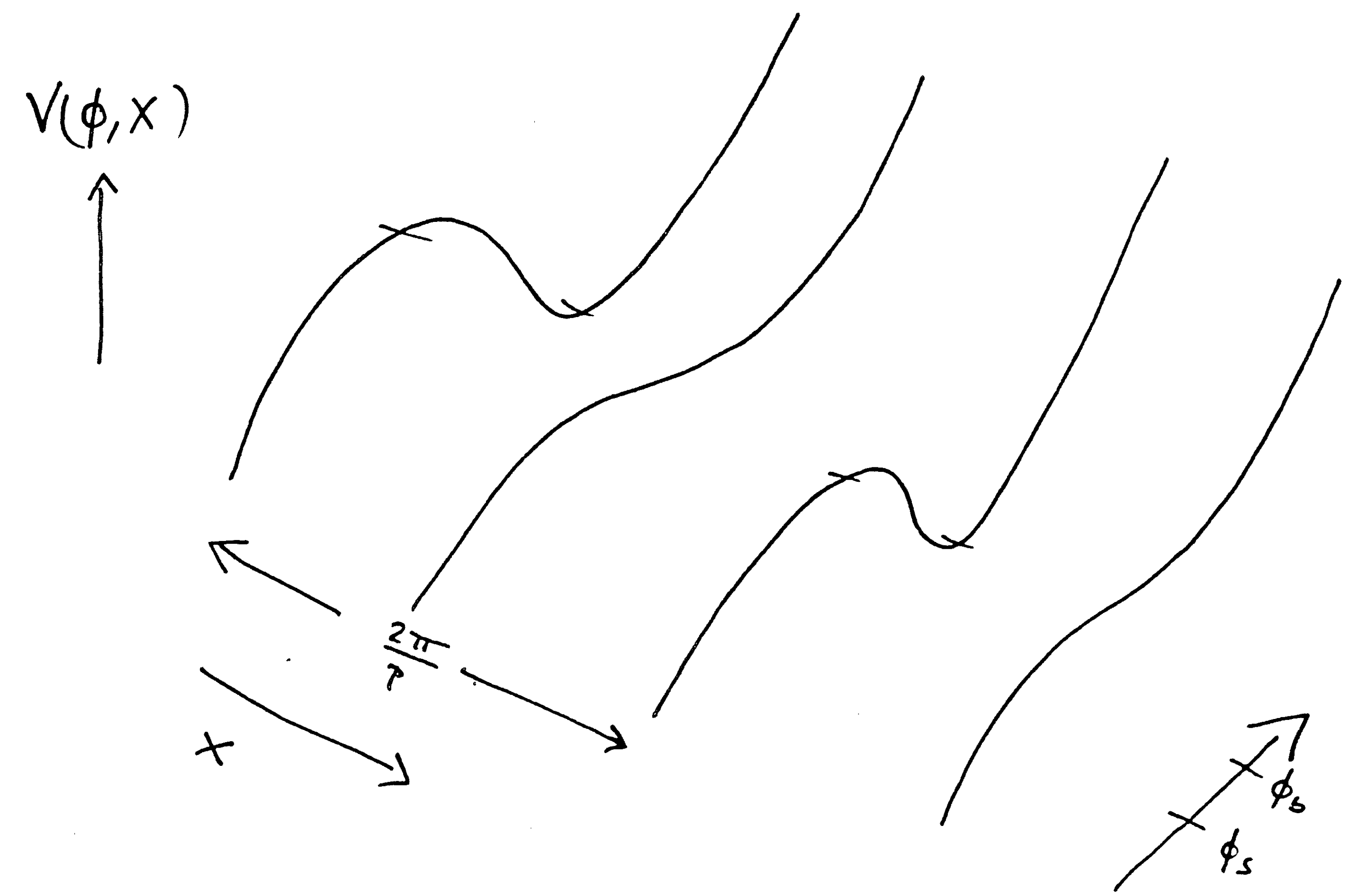

Fig. 4

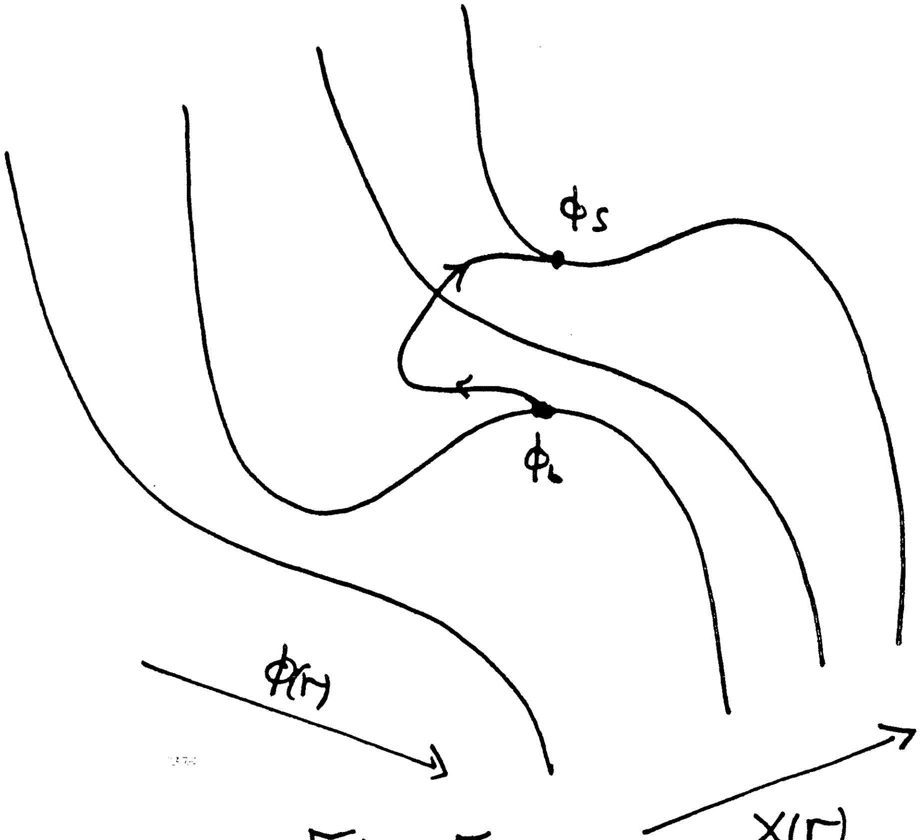

Fig 5

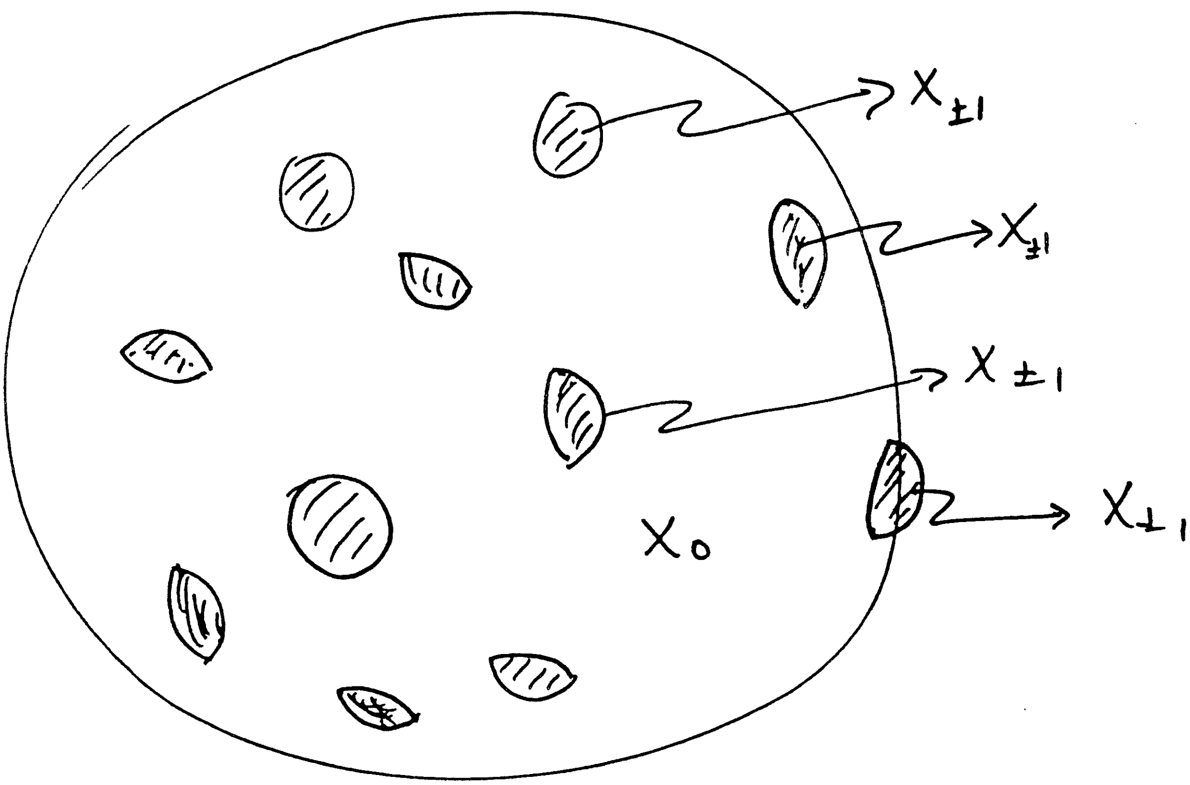

Fig.6