# Homework #4

## Due March 3, 2026

# Problem I

(i) List all the (inequivalent) contractions appearing in

$$\frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4z\, d^4w\, \langle \varphi(x)\, \varphi(y)\, \varphi^4(z)\, \varphi^4(w) \rangle$$

that give a non-vanishing contribution to the expectation value. Draw the corresponding diagrams. Write down the mathematical expression corresponding to each diagram in the form of integrals over the Euclidean propagator $D(x)$.

(ii) Consider the perturbative expansion to second order ($\lambda^2$) of the correlation functions

$$\langle \varphi(x_1)\varphi(x_2)\rangle \qquad \text{and} \qquad \langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

in $\varphi^4$ theory. Draw all the corresponding diagrams that contribute (i.e., connected diagrams). Indicate their symmetry factors. In the case of $\langle \varphi(x_1)\varphi(x_2)\rangle$ write down the expression corresponding to each Feynman diagram in momentum space. You **do not** need to give the momentum space expression for the diagrams corresponding to the terms in the expansion of $\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$ as this would be too much work.

# Problem II

The creation of Klein-Gordon particles by a classical source can be modeled by the Hamiltonian:

$$H = H_0 + \int d^3\mathbf{x} \left( - j(\mathbf{x}, t)\, \phi(x) \right),$$

where $H_0$ is the free Klein-Gordon Hamiltonian, i.e.,

$$H_0 = \frac{1}{2} \int d^3\mathbf{x} \left( \pi^2(\mathbf{x}) + (\boldsymbol{\nabla}\varphi)^2(\mathbf{x}) + m^2\varphi^2(\mathbf{x}) \right),$$

$\varphi(\mathbf{x})$ is the Klein-Gordon field and $j(\mathbf{x}, t\,)$ is some given real function of the space-time variables.

(i) Argue that the probability that the source creates no particles is given by:

$$P(0) = \left|\mathcal{M}\right|^2 \qquad \text{with} \qquad \mathcal{M} = \langle 0|\, \mathrm{T}\Big\{ \exp\left( \mathrm{i} \int d^4x\, j(x)\, \hat{\varphi}(x) \right) \Big\}\, |0\rangle\ .$$

Here $\hat{\varphi}(x)$ is the quantum Klein-Gordon field operator.

(ii) Evaluate $P(0)$ up to order $j^2$. Specifically, show that

$$P(0) = 1 - \lambda + O(j^4),$$

and determine $\lambda$.

(iii) Represent the term computed in part ii) as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates so that it can be summed exactly:

$$P(0) = \exp(-\lambda)\,.$$

(iv) Compute the probability that the source creates one particle of momentum $\mathbf{k}$. Perform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of part iii) to sum the series.

(v) Show that the probability of producing $n$ particles is given by

$$P(n) = \frac{1}{n!}\,\lambda^n\,\exp(-\lambda)\ .$$

This is known as a Poisson distribution. Prove that for a Poisson distribution

$$\sum_{n=0}^{\infty} P(n) = 1\,, \qquad \langle N \rangle = \sum_{n=0}^{\infty} n\,P(n) = \lambda\,.$$

Compute the mean square fluctuation $\left\langle \big(N - \langle N \rangle\big)^2 \right\rangle$.

# Problem III

In the $\varphi^3$ theory

$$\mathcal{A}_I = \frac{\lambda_3}{3!}\int d^4x\,\varphi^3(x)$$

carry out the Legendre transform of $W[J]$ to the order 4 in $J$ and $\phi$. Express $W^{(3)}$ and $W^{(4)}$ in terms of $\Gamma^{(4)}$, $\Gamma^{(3)}$, $\Gamma^{(2)}$. Give diagramatic representation of your result.