Homework #3

Due February 20, 2026

Problem I

(I) Consider a long string $q(\tau)$, $\tau \in [0, L]$, subject to the boundary condition

$$q(0) = q_0 \ .$$

(a) Show that in the limit $L \to \infty$

$$Z_{\rm conf} \to {\rm Const}\, {\rm e}^{-FL}\, \Psi_0(q_0)\ , \qquad (\star)$$

where F is the specific (per unit length) free energy of the string, and $\Psi_0(q)$ is the ground–state wave function of the Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q})\,. \qquad (\star\star)$$

Here *Const* is independent of q_0. Observe that the boundary condition at $\tau = L$ is not important for this conclusion.

(b) If L is large but finite, determine the form of the leading correction to $(\star)$. Assume that the spectrum of $\hat{H}$ is discrete.

(II) For an infinite string, $\tau \in (-\infty, \infty)$, write down the correlation function $\langle q(\tau_1)q(\tau_2)\rangle$ in terms of the eigenvalues E_n and the associated wave functions $\Psi_n(q)$ of the Hamiltonian $(\star\star)$.

Problem II

Consider 4-dimensional hyper-cubic lattice $\mathbb{Z}^4$, with the lattice spacing Δ. Compute the lattice propagator as the sum over all lattice path from 0 to x (both being the lattice points $\in \mathbb{Z}^4$, weighted with ${\rm e}^{-m_0 L}$, where L is the length of the path, $L = \Delta \times$ (# of the lattice links in the path). Consider the limit $\Delta \to 0$. Observe that at fixed m_0 the sum either diverges, or is dominated by the shortest path from 0 to x. Determine how $m_0 = m_0(\Delta)$ should depend on Δ for the long ($L \gg |x|$) path to remain relevant in the limit $\Delta \to 0$.

Hint: Easy way to evaluate the sum is to write it as

$$\sum_{n=0}^{\infty} {\rm e}^{-m_0\,\Delta\, n} \sum_{\{\nu_k\}} \prod_{\mu=1}^{4} \delta\Big(\sum_{k=1}^{n} \nu_k^\mu - x^\mu/\Delta \Big)\,,$$

where now ν_k^μ are unit vectors pointing along the lattice links, and $\delta(n)$ is the Kronecker's symbol. Use the representation

$$\delta(n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, {\rm e}^{{\rm i}kn}\,.$$

Problem III

(a) The Hamiltonian of a free two-dimensional quantum rotator is

$$\hat{H}_0 = -\frac{1}{2I}\frac{d^2}{d\phi^2} \qquad (\hbar = 1)\ ,$$

where $\phi \sim \phi + 2\pi$ is an angular variable. Consider the Euclidean-time Heisenberg operators

$$\mathrm{e}^{\mathrm{i}\sigma\hat{\phi}(\tau)} = \mathrm{e}^{\tau \hat{H}_0}\, \mathrm{e}^{\mathrm{i}\sigma\hat{\phi}}\, \mathrm{e}^{-\tau\hat{H}_0} \qquad (\sigma = \pm 1)\ .$$

Compute the two-point thermal correlation function

$$\langle \mathrm{e}^{\mathrm{i}\sigma_2\phi(\tau_2)}\mathrm{e}^{\mathrm{i}\sigma_1\phi(\tau_1)}\rangle_\beta \equiv \frac{1}{Z_0}\,\mathrm{Tr}\Big[\mathrm{e}^{-\beta\hat{H}_0}\, \mathrm{e}^{\mathrm{i}\sigma_2\hat{\phi}(\tau_2)}\, \mathrm{e}^{\mathrm{i}\sigma_1\hat{\phi}(\tau_1)}\Big],$$

where

$$Z_0 = \mathrm{Tr}\Big[\mathrm{e}^{-\beta\hat{H}_0}\Big]\ .$$

Hint. Use the eigenbasis of $\hat{H}_0$ (angular momentum states). Note that $\mathrm{e}^{\pm\mathrm{i}\phi}$ act as ladder operators shifting the quantum number by ± 1.

(b) Show that the result obtained in part (a) can also be derived using the Euclidean path-integral representation

$$\left\langle \prod_{k=1}^{2} \mathrm{e}^{\mathrm{i}\sigma_k\phi(\tau_k)} \right\rangle_\beta = \frac{\displaystyle\sum_{m=-\infty}^{\infty}\int_{\phi(\tau+\beta)=\phi(\tau)+2\pi m} \mathcal{D}\phi\ \mathrm{e}^{-\mathcal{A}_{\mathrm{E}}[\phi]} \prod_{k=1}^{2}\mathrm{e}^{\mathrm{i}\sigma_k\phi(\tau_k)}}{\displaystyle\sum_{m=-\infty}^{\infty}\int_{\phi(\tau+\beta)=\phi(\tau)+2\pi m} \mathcal{D}\phi\ \mathrm{e}^{-\mathcal{A}_{\mathrm{E}}[\phi]}}\ ,$$

where

$$\mathcal{A}_{\mathrm{E}}[\phi] = \int_0^\beta d\tau\,\frac{I}{2}\left(\partial_\tau\phi\right)^2\ .$$

Generalize this result and compute the n-point correlation function

$$\left\langle \prod_{k=1}^{n} \mathrm{e}^{\mathrm{i}\sigma_k\phi(\tau_k)} \right\rangle_\beta\ .$$

Hint. Decompose the field as $\phi(\tau) = 2\pi m\,\tau/\beta + X(\tau)$, where $X(\tau)$ is periodic. The sum over winding numbers m plays the role of the discrete angular momentum spectrum in the operator formalism. Charge neutrality, $\sum_k \sigma_k = 0$, is essential. Consider how this condition arises in the path-integral calculation.

(c) Consider the rotator in the presence of a constant external electric field $\boldsymbol{\mathcal{E}}$. Assume that the rotator has a dipole moment $\boldsymbol{d}$, so that the Hamiltonian becomes

$$\hat{H} = -\frac{1}{2I}\frac{d^2}{d\phi^2} - \mathcal{E}d\,\cos\phi\ .$$

Using the path-integral representation, develop a weak-field perturbative expansion for the partition function

$$Z = \mathrm{Tr}\Big[\mathrm{e}^{-\beta \hat{H}}\Big] \ .$$

Hint. Expand $\exp\Big(2\kappa \int_0^\beta d\tau\ \cos\phi\Big)$ in powers of $\kappa \equiv \frac{1}{2}\mathcal{E}d$. Each order generates insertions of $\mathrm{e}^{\pm \mathrm{i}\phi(\tau)}$, whose correlators were computed in parts (b).

(d) Recall that the Coulomb potential of a particle with charge e in d-dimensional Euclidean space is defined as the solution of the Laplace equation

$$\triangle \Phi(\mathbf{x}) = -\, e\, S_d\, \delta^{(d)}(\mathbf{x}) \ , \qquad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \ .$$

It then follows that, in the case $d = 1$, the Coulomb potential energy between two particles with charges e_1 and e_2 is

$$U(x) = -\, e_1 e_2\, |x| \ .$$

Consider a one-dimensional plasma consisting of a mixture of particles with charges $\pm e$. Write down the expression for the grand partition function $\mathcal{Q} = \mathcal{Q}(T, V, \mu)$ of this classical statistical system in the grand canonical ensemble. Using the result above, show that the equation of state has the form

$$\frac{P}{T\rho} = F\bigg(\frac{T\rho}{P_0}\bigg) \qquad\qquad (P_0 \equiv e^2) \ .$$

Here P, T, and ρ denote the pressure, temperature, and particle density, respectively. Determine the function $F(z)$.[1]

Hint. Identify the fugacity with the electric-field expansion parameter. Relate $\mathcal{Q}$ to the partition function of the quantum rotator and extract the thermodynamic quantities from the grand potential $\mathcal{G} = -T\, \log(\mathcal{Q})$.

You will need some facts from the theory of the Mathieu equation,

$$\frac{d^2 y}{dx^2} + \big(a - 2q\cos(2x)\big)\, y = 0 \ .$$

In some contexts, the term *Mathieu function* refers to solutions of the Mathieu differential equation for arbitrary values of a and q. When no confusion can arise, some authors use this term more narrowly to denote π- or 2π-periodic solutions, which exist only for special values of a and q.

More precisely, for a given (real) value of q, such periodic solutions exist for an infinite set of values of a, called *characteristic values*. These are conventionally organized into two separate sequences, $a_m(q)$ and $b_{m+1}(q)$, with $m = 0, 1, 2, \ldots$. The corresponding functions are denoted by $\mathrm{ce}_m(x, q)$ and $\mathrm{se}_{m+1}(x, q)$, respectively. They are sometimes also referred to as *cosine–elliptic* and *sine–elliptic* functions, or as Mathieu functions of the first kind.

[1] The equation of state of the one-dimensional plasma was originally obtained in A. Lenard, *Exact Statistical Mechanics of a One-Dimensional System with Coulomb Forces*, J. Math. Phys. **2**, 682 (1961).

For further details, see, for example, the Wikipedia article on Mathieu functions and sec. 20 in Abramowitz & Stegun). The functions $\mathrm{ce}_m(x,q)$ and $\mathrm{se}_{m+1}(x,q)$, as well as the characteristic values $a_m(q)$ and $b_{m+1}(q)$, are implemented in *Mathematica*. In particular,

$$a_0(q) := \text{MathieuCharacteristicA}[0,q] \; .$$

(e) Study the low- and high-temperature limits of the one-dimensional plasma. In particular, show that at high temperatures the system is in a plasma phase, while at low temperatures it can be interpreted as a gas of neutral molecules. Verify that there is no phase transition between these two regimes.

Hint. Analyze the limiting values of the function $F(z)$ appearing in the equation of state and show that

$$F(z) \to \begin{cases} \frac{1}{2} \; , & z \to 0 \; , \\ 1 \; , & z \to \infty \; . \end{cases}$$

This implies that at high temperatures interactions are weak, while at low temperatures opposite charges bind into neutral pairs. Plot $F(z)$ using *Mathematica* and verify that $F(z)$ interpolates smoothly between the two limits.