Homework #1

Due January 30, 2026

Problem I

Theorem (Noether): Consider a mechanical system governed by the Lagrangian $L = L(q, \dot{q}, t)$. Suppose we perform a transformation of the co-ordinates q and time t:

$$\begin{aligned} t' &= t + \epsilon\, X(q,t) \\ q^i(t') &= q^i(t) + \epsilon\, \Psi^i(q,t) \end{aligned}$$

with ϵ an infinitesimal parameter. If

$$\mathrm{d}t\, L(q,\dot{q};t) = \mathrm{d}t' \left(L(q', \dot{q}'; t') + \epsilon \frac{d}{dt'} F(q', t') \right),$$

then the quantity

$$I = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}^i} \left(\dot{q}^i X - \Psi^i \right) - LX - F$$

is conserved $\dot{I} = 0$.

This problem illustrates the theorem stated above.

(i) Imagine a classical mechanics particle moving in a gravitational field, which is generated by a homogeneous mass distribution. What are the symmetries and conserved quantities for the following cases?

a) The mass is uniformly distributed in the plane $z = 0$

b) The mass is uniformly distributed in the half plane $z = 0$, $y > 0$

c) The mass is uniformly distributed along a circular cylinder of infinite length, whose axis extends in the vertical direction

d) As in c) but now the cylinder is of finite length

e) The mass is uniformly distributed along a cylinder of infinite length, extending in the z direction, whose cross-section is an ellipse

f) The mass is in the form of a uniform, infinite wire, which has the geometry of a helical solenoid wound about the z axis

(ii) A one-dimensional mechanical system is governed by the action

$$S = -\int \mathrm{d}t \, \sqrt{1 - \dot{q}^2} \; .$$

Show that the transformation

$$\begin{aligned} q' &= q \cosh(\theta) + t \sinh(\theta) \\ t' &= q \sinh(\theta) + t \cosh(\theta) \end{aligned}$$

with θ an arbitrary parameter is a symmetry of the action. Using Noether's theorem for a classical mechanics system, find the conserved quantity corresponding to this symmetry.

Problem II

The Klein–Gordon action

$$S = \frac{1}{2} \int \mathrm{d}^4x \, \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right)$$

is invariant with respect to Lorentz transformations of space–time:

$$x \mapsto \tilde{x}, \qquad \tilde{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu .$$

(i) Using Noether's theorem, find the three conserved charges corresponding to the invariance of the model under spatial rotations about the x^1, x^2, and x^3 axes.

(ii) Do the same as in (i) for the three infinitesimal boosts in the x^1, x^2, and x^3 directions. Compare the conserved charges obtained in (ii) with the ones you found in Problem 1, part (ii).

Problem III

Consider the Lagrangian density for two real scalar fields φ_j with equal masses,

$$\mathcal{L} = \frac{1}{2} \sum_{s=1}^{2} \partial_\mu \varphi_s \partial^\mu \varphi_s - \frac{m^2}{2} \sum_{s=1}^{2} \varphi_s^2 \; .$$

(i) Define the complex field

$$\phi = \frac{1}{\sqrt{2}} \left(\varphi_1 + \mathrm{i} \, \varphi_2 \right) \qquad \left(\mathrm{i} \equiv \sqrt{-1} \right)$$

and rewrite the Lagrangian density in terms of ϕ and its complex conjugate ϕ^*. Find the equations of motion for φ_s, and show that the same results follow if one treats $\mathcal{L}(\phi, \phi^*)$ as if ϕ and ϕ^* were independent dynamical variables.

(ii) Consider the transformation

$$\phi \to \phi\, \mathrm{e}^{\mathrm{i}\alpha}\,, \qquad \phi^* \to \phi^* \mathrm{e}^{-\mathrm{i}\alpha}\,.$$

Show that this is a symmetry, and find the associated Noether current j^μ and the conserved charge Q.

Problem IV

Consider a classical field theory defined in the semi-infinite region

$$x^1, x^2 \in (-\infty, +\infty), \qquad x^3 \equiv z \in [0, +\infty),$$

with a boundary at $z = 0$. Suppose the Lagrangian is

$$L = \int_0^\infty dz \int d^2\mathbf{x}^\parallel \left(\frac{1}{2}\,\dot{\varphi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 - \frac{m^2}{2}\,\varphi^2 \right) + \int d^2\mathbf{x}^\parallel\, V\big(\varphi_B(t, \mathbf{x}^\parallel)\big)\,,$$

where $\mathbf{x}^\parallel = (x^1, x^2)$ labels points along the boundary, and

$$\varphi_B(\mathbf{x}^\parallel, t) = \varphi(\mathbf{x}, t)\big|_{z=0}$$

is the boundary value of φ.

(i) Find the field equations of motion, including the boundary condition at $z = 0$.

(ii) Determine which of the quantities E, P^1, P^2 and P^3 are conserved.

(iii) Assuming $V(\varphi_B) = M\,\varphi_B^2$, find the general solution of the field equations.