Physics 507

MIDTERM EXAM
November 2, 2006

Problem I (10 pt)

An electron moves in the electric field $\vec{E}(\vec{r}) = (E\cos(kz), 0, 0)$, where $k = const$. Find $\vec{r}(t)$ if $\vec{r}(0) = 0$ and $\vec{v}(0) = (0, 0, v_0)$ (Ignore the gravity)

Solution

Using Newton's equation $(e > 0)$

$$m\ddot{x} = -e\,E\ \cos(kz)\ , \quad \ddot{y} = 0, \quad \ddot{z} = 0\ .$$

one finds

$$y(t) = 0 \qquad z(t) = v_0\ t$$

and

$$x(t) = -\frac{2eE}{mk^2v_0^2}\ \sin^2\left(\frac{kv_0t}{2}\right)\ .$$

Problem II (10 pt)

An uniform beam of particles with an incident velocity $\vec{v}_{in} \parallel OZ$ scatters on a surface of revolution defined by the equation,

$$r = b\ \sin\left(\frac{z}{a}\right), \quad 0 \leq z \leq \pi a\ .$$

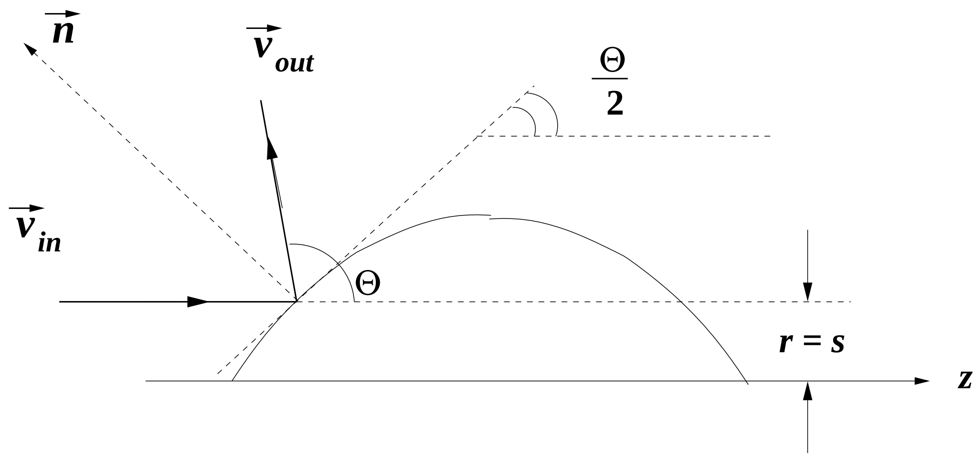
Here r, z are cylindrical coordinates, and a, b are positive parameters. Determine the differential cross section for the case of completely elastic scattering.

Solution

The scattering angle

$$\cos(\Theta) = \frac{\vec{v}_{in} \cdot \vec{v}_{out}}{\vec{v}_{in}^2}$$

can be found from the relation (see Fig.),



$$\tan\left(\frac{\Theta}{2}\right) = \frac{dr}{dz} = \frac{b}{a} \cos\left(\frac{z}{a}\right) .$$

The impact parameter s coincides with r, so

$$s^2 = r^2 = b^2 - b^2 \cos^2\left(\frac{z}{a}\right) = b^2 - a^2 \tan^2\left(\frac{\Theta}{2}\right) ,$$

and

$$\sigma(\Theta) = \frac{s}{\sin(\Theta)} \left|\frac{ds}{d\Theta}\right| = \frac{1}{2\sin(\Theta)} \left|\frac{d(s^2)}{d\Theta}\right| = a^2 \frac{\tan(\frac{\Theta}{2})}{2\sin(\Theta)\cos^2(\frac{\Theta}{2})} .$$

Using the relation

$$\sin(\Theta) = 2 \sin(\frac{\Theta}{2}) \cos(\frac{\Theta}{2}) ,$$

one obtains

$$\sigma(\Theta) = \frac{a^2}{4 \cos^4(\frac{\Theta}{2})} .$$

The admissible scattering angles belong to the interval

$$0 \leq \Theta \leq \Theta_{max} = 2 \arctan(b/a) .$$

The scattering angle $\Theta = 0$ corresponds to $s \to b$ and the maximum scattering angle corresponds to $s \to 0$.

Problem III (10 pt)

A mechanical system with two degrees of freedom characterized by the kinetic

$$T = \frac{1}{2}\,\frac{\dot{\alpha}^2}{A + B\,\beta^2} + \frac{1}{2}\,\dot{\beta}^2$$

and potential

$$U = C + D\,\beta^2$$

energies. Here $(\alpha,\, \beta)$ are generalized coordinates, and A, B, C, D are real constants. Determine the motion of the system for all possible values of the constants.

Solution

The Lagrangian reads,

$$L = T - U = \frac{1}{2}\ \frac{\dot{\alpha}^2}{A + B\,\beta^2} + \frac{1}{2}\,\dot{\beta}^2 - D\,\beta^2 - C\ .$$

Notice that α is a cyclic coordinate thus the conjugated generalized momentum,

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = \frac{\dot{\alpha}}{A + B\,\beta^2} = const$$

is an integral of motion. The Routhian has a form,

$$R = \Big(L - \dot{\alpha} p_\alpha\Big)\Big|_{\dot{\alpha} = p_\alpha (A + B\,\beta^2)} = \frac{1}{2}\,\dot{\beta}^2 - D\,\beta^2 - C - \frac{p_\alpha^2}{2}\ (A + B\,\beta^2)\ .$$

Let us define

$$\omega^2 = 2D + B\,p_\alpha^2\ ,$$

then

$$R = \frac{\dot{\beta}^2}{2} - \frac{\omega^2 \beta^2}{2} + \cdots\ .$$

Here dots $\cdots$ mean the constant which does not effect on the equation of motion,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\beta}}\right) - \frac{\partial R}{\partial \beta} = 0\ ,$$

or

$$\ddot{\beta} + \omega^2\,\beta = 0\ .$$

A general solution of this equation depends on two integration constants $\beta_1,\ \beta_2$ and reads explicitly,

$$\beta = \beta_1\ \cos(\omega t) + \beta_2\ \sin(\omega t)\,, \qquad \text{if}\ \ \omega^2 > 0$$

$$\beta = \beta_1\ \cosh(|\omega| t) + \beta_2\ \sinh(|\omega| t)\,, \qquad \text{if}\ \ \omega^2 < 0$$

$$\beta = \beta_1 + \beta_2\, t\,, \qquad \text{if}\ \ \omega^2 = 0\ .$$

The function $\alpha = \alpha(t)$ can be found from the equation,

$$\alpha = p_\alpha\ \int dt (A + B\,\beta^2)$$

which gives,

$$\alpha = \alpha_0 + p_\alpha\left(A + \frac{B(\beta_1^2 + \beta_2^2)}{2}\right) t + \frac{p_\alpha B(\beta_1^2 - \beta_2^2)}{4\omega}\ \sin(2\omega t) - \frac{p_\alpha B\beta_1\,\beta_2}{2\omega}\ \cos(2\omega t)\ ,$$

for $\omega^2 > 0$. Here α_0 is an integration constant as well as p_α, β_1 and β_2. For $\omega^2 < 0$ one has

$$\alpha = \alpha_0 + p_\alpha\left(A + \frac{B(\beta_1^2 - \beta_2^2)}{2}\right) t + \frac{p_\alpha B(\beta_1^2 + \beta_2^2)}{4|\omega|}\ \sinh(2|\omega| t) + \frac{p_\alpha B\beta_1\,\beta_2}{2|\omega|}\ \cosh(2|\omega| t)\ .$$

In the case $\omega^2 = 0$ the solution reads,

$$\alpha = \alpha_0 + p_\alpha\left(A + B\beta_1^2\right) t + p_\alpha B\beta_1\beta_2\, t^2 + \frac{p_\alpha B\beta_2^2}{3}\ t^3\ .$$

Problem IV (10pt)

In a certain non-linear medium, the kinetic energy of particle is no longer a quadratic function of its velocity. The Lagrangian for the motion of one dimensional oscillator immersed in this medium is given by,

$$L = \frac{m}{2} \left\{ \lambda \left(\dot{q}^2\right)^{1+\alpha} - \omega_0^2 \, q^2 \right\},$$

where q is the displacement, m is the mass of the particle, λ, ω and α are positive constants.

a. Write down an expressions for the generalized momentum and the total mechanical energy of the particle.

b. What is the power law dependence of the period T of the oscillator on the total energy? Compare it with a standard harmonic oscillator ($\alpha = 0$).

Solution

The generalized momentum and the total mechanical energy read,

$$p = \frac{\partial L}{\partial \dot{q}} = m\lambda\ (1+\alpha)\ (\dot{q}^2)^{\alpha}\ \dot{q}\,,$$
$$E = \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L = \frac{m}{2}\ \Big\{\, \lambda\,(1+2\alpha)\,(\dot{q}^2)^{1+\alpha} + \omega_0^2\, q^2 \,\Big\}\ .$$

To find the period of oscilation we can use the energy conservation law. Then,

$$\dot{q}^2 = \Big(\frac{2E - m\omega_0^2\, q^2}{m\lambda(1+2\alpha)}\Big)^{\frac{1}{1+\alpha}} \quad\Longrightarrow\quad dt = \pm\ \Big(\frac{m\lambda(1+2\alpha)}{2E}\Big)^{\frac{1}{2+2\alpha}}\ \frac{dq}{\big(1-\frac{m\omega_0^2\, q^2}{2E}\big)^{\frac{1}{2+2\alpha}}}\ .$$

The period of oscilation is defined by the relation,

$$\frac{T}{4} = \Big(\frac{m\lambda(1+2\alpha)}{2E}\Big)^{\frac{1}{2+2\alpha}}\ \int_0^{\sqrt{\frac{2E}{m\omega_0^2}}} \frac{dq}{\big(1-\frac{m\omega_0^2\, q^2}{2E}\big)^{\frac{1}{2+2\alpha}}} =$$
$$\Big(\frac{m\lambda(1+2\alpha)}{2E}\Big)^{\frac{1}{2+2\alpha}}\ \frac{1}{\omega_0}\ \sqrt{\frac{2E}{m}}\ \int_0^1 \frac{d\xi}{(1-\xi^2)^{\frac{1}{2+2\alpha}}}\ .$$

One can simplify this expression as follows ($z=\xi^2$),

$$T = \frac{2}{\omega_0}\ \big(\lambda(1+2\alpha)\big)^{\frac{1}{2+2\alpha}}\ \Big(\frac{2E}{m}\Big)^{\frac{\alpha}{2+2\alpha}}\ \int_0^1 dz\, z^{-\frac{1}{2}}\ (1-z)^{-\frac{1}{2+2\alpha}}\ .$$

Notice that

$$B(x,y) = \int_0^1 dz\, z^{x-1}\ (1-z)^{y-1}$$

is so called the Euler beta-function. It can be expressed in term of the Gamma-function,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \qquad \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}\ .$$

Hence,

$$T = \frac{2}{\omega_0}\ \big(\lambda(1+2\alpha)\big)^{\frac{1}{2+2\alpha}}\ \Big(\frac{2E}{m}\Big)^{\frac{\alpha}{2+2\alpha}}\ B\Big(\frac{1}{2}\,,\,\frac{1+2\alpha}{2+2\alpha}\Big)\ .$$

The period depends on E as follows:

$$T \sim E^{\frac{\alpha}{2+2\alpha}}\ .$$

For $\alpha = 0$, the period does not depend on the energy:

$$T = \frac{2\sqrt{\lambda}}{\omega_0}\ B\Big(\frac{1}{2}\,,\,\frac{1}{2}\Big) = \frac{2\pi\sqrt{\lambda}}{\omega_0}\ .$$