Problem I (10 pt)

An electron moves in the electric field $\vec{E}(\vec{r}) = (E \cos(kz), 0, 0)$, where $k = const.$ Find $\vec{r}(t)$ if $\vec{r}(0) = 0$ and $\vec{v}(0) = (0, 0, v_0)$ (Ignore the gravity)
Solution

Using Newton’s equation ($e > 0$)

\[ m \ddot{x} = -e E \cos(kz) , \quad \ddot{y} = 0, \quad \ddot{z} = 0. \]

one finds

\[ y(t) = 0 \quad z(t) = v_0 t \]

and

\[ x(t) = - \frac{2e E}{mk^2 v_0^2} \sin^2 \left( \frac{kv_0 t}{2} \right). \]
**Problem II (10 pt)**

An uniform beam of particles with an incident velocity $\vec{v}_{in} \parallel OZ$ scatters on a surface of revolution defined by the equation,

$$r = b \sin \left( \frac{z}{a} \right), \quad 0 \leq z \leq \pi a.$$  

Here $r, z$ are cylindrical coordinates, and $a, b$ are positive parameters. Determine the differential cross section for the case of completely elastic scattering.
Solution

The scattering angle

\[ \cos(\Theta) = \frac{\vec{v}_{in} \cdot \vec{v}_{out}}{v_{in}^2} \]

can be found from the relation (see Fig.),

\[ \tan \left( \frac{\Theta}{2} \right) = \frac{dr}{dz} = \frac{b}{a} \cos \left( \frac{z}{a} \right). \]

The impact parameter \( s \) coincides with \( r \), so

\[ s^2 = r^2 = b^2 - b^2 \cos^2 \left( \frac{z}{a} \right) = b^2 - a^2 \tan^2 \left( \frac{\Theta}{2} \right), \]

and

\[ \sigma(\Theta) = \frac{s}{\sin(\Theta)} \left| \frac{d}{d\Theta} \right| = \frac{1}{2 \sin(\Theta)} \left| \frac{d(s^2)}{d\Theta} \right| = a^2 \tan \left( \frac{\Theta}{2} \right) \frac{\tan \left( \frac{\Theta}{2} \right)}{2 \sin(\Theta) \cos^2 \left( \frac{\Theta}{2} \right)} \cdot \]

Using the relation

\[ \sin(\Theta) = 2 \sin \left( \frac{\Theta}{2} \right) \cos \left( \frac{\Theta}{2} \right), \]

one obtains

\[ \sigma(\Theta) = \frac{a^2}{4 \cos^4 \left( \frac{\Theta}{2} \right)}. \]

The admissible scattering angles belong to the interval

\[ 0 \leq \Theta \leq \Theta_{max} = 2 \arctan(b/a). \]

The scattering angle \( \Theta = 0 \) corresponds to \( s \to b \) and the maximum scattering angle corresponds to \( s \to 0. \)
Problem III (10 pt)

A mechanical system with two degrees of freedom characterized by the kinetic

\[ T = \frac{1}{2} \frac{\dot{\alpha}^2}{A + B \beta^2} + \frac{1}{2} \dot{\beta}^2 \]

and potential

\[ U = C + D \beta^2 \]

energies. Here \((\alpha, \beta)\) are generalized coordinates, and \(A, B, C, D\) are real constants. Determine the motion of the system for all possible values of the constants.
Solution

The Lagrangian reads,

\[ L = T - U = \frac{1}{2} \frac{\dot{\alpha}^2}{A + B\beta^2} + \frac{1}{2} \beta^2 - D\beta^2 - C. \]

Notice that \( \alpha \) is a cyclic coordinate thus the conjugated generalized momentum,

\[ p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = \frac{\dot{\alpha}}{A + B\beta^2} = \text{const} \]

is an integral of motion. The Routhian has a form,

\[ R = \left( L - \dot{\alpha}p_\alpha \right) \bigg|_{\dot{\alpha}=p_\alpha(A+B\beta^2)} = \frac{1}{2} \beta^2 - D\beta^2 - C - \frac{p_\alpha^2}{2} (A + B\beta^2). \]

Let us define

\[ \omega^2 = 2D + B p_\alpha^2 , \]

then

\[ R = \frac{\dot{\beta}^2}{2} - \frac{\omega^2 \beta^2}{2} + \cdots . \]

Here dots \( \cdots \) mean the constant which does not effect on the equation of motion,

\[ \frac{d}{dt} \left( \frac{\partial R}{\partial \beta} \right) - \frac{\partial R}{\partial \beta} = 0 , \]

or

\[ \ddot{\beta} + \omega^2 \beta = 0 . \]

A general solution of this equation depends on two integration constants \( \beta_1, \beta_2 \) and reads explicitly,

\[ \beta = \beta_1 \cos(\omega t) + \beta_2 \sin(\omega t), \quad \text{if} \quad \omega^2 > 0 \]

\[ \beta = \beta_1 \cosh(|\omega|t) + \beta_2 \sinh(|\omega|t), \quad \text{if} \quad \omega^2 < 0 \]

\[ \beta = \beta_1 + \beta_2 t, \quad \text{if} \quad \omega^2 = 0 . \]

The function \( \alpha = \alpha(t) \) can be found from the equation,

\[ \alpha = p_\alpha \int dt(A + B\beta^2) \]

which gives,

\[ \alpha = \alpha_0 + p_\alpha A + B\frac{(\beta_1^2 + \beta_2^2)}{2} t + \frac{p_\alpha B(\beta_1^2 - \beta_2^2)}{4\omega} \sin(2\omega t) - \frac{p_\alpha B\beta_1 \beta_2}{2\omega} \cos(2\omega t) , \]

for \( \omega^2 > 0 \). Here \( \alpha_0 \) is an integration constant as well as \( p_\alpha, \beta_1 \) and \( \beta_2 \). For \( \omega^2 < 0 \) one has

\[ \alpha = \alpha_0 + p_\alpha A + B\frac{(\beta_1^2 - \beta_2^2)}{2} t + \frac{p_\alpha B(\beta_1^2 + \beta_2^2)}{4|\omega|} \sinh(2|\omega|t) + \frac{p_\alpha B\beta_1 \beta_2}{2|\omega|} \cosh(2|\omega|t) . \]

In the case \( \omega^2 = 0 \) the solution reads,

\[ \alpha = \alpha_0 + p_\alpha (A + B\beta_1^2) t + p_\alpha B\beta_1 \beta_2 t^2 + \frac{p_\alpha B\beta_2^2}{3} t^3 . \]
Problem IV (10pt)

In a certain non-linear medium, the kinetic energy of particle is no longer a quadratic function of its velocity. The Lagrangian for the motion of one dimensional oscillator immersed in this medium is given by,

\[ L = \frac{m}{2} \left\{ \lambda (q^2)^{1+\alpha} - \omega_0^2 q^2 \right\}, \]

where \( q \) is the displacement, \( m \) is the mass of the particle, \( \lambda, \omega \) and \( \alpha \) are positive constants.

\( a. \) Write down an expressions for the generalized momentum and the total mechanical energy of the particle.

\( b. \) What is the power law dependence of the period \( T \) of the oscillator on the total energy? Compare it with a standard harmonic oscillator (\( \alpha = 0 \)).
Solution

The generalized momentum and the total mechanical energy read,

\[ p = \frac{\partial L}{\partial \dot{q}} = m \lambda (1 + \alpha) (q^2)^{\alpha} \dot{q}, \]
\[ E = q \frac{\partial L}{\partial \dot{q}} - L = \frac{m}{2} \left\{ \lambda (1 + 2\alpha) (q^2)^{1+\alpha} + \omega_0^2 q^2 \right\}. \]

To find the period of oscillation we can use the energy conservation law. Then,

\[ q^2 = \left( \frac{2E - m\omega_0^2 q^2}{m\lambda(1 + 2\alpha)} \right)^{\frac{1}{1+\alpha}} \implies dt = \pm \left( \frac{m\lambda(1 + 2\alpha)}{2E} \right)^{\frac{1}{1+\alpha}} \frac{dq}{(1 - \frac{m\omega_0^2 q^2}{2E})^{\frac{1}{1+\alpha}}}.
\]

The period of oscillation is defined by the relation,

\[ T = 4 \left( \frac{m\lambda(1 + 2\alpha)}{2E} \right)^{\frac{1}{1+\alpha}} \int_0^{\frac{2E}{m\omega_0^2}} \frac{dq}{(1 - \frac{m\omega_0^2 q^2}{2E})^{\frac{1}{1+\alpha}}} = \left( \frac{m\lambda(1 + 2\alpha)}{2E} \right)^{\frac{1}{1+2\alpha}} \frac{1}{\omega_0} \sqrt{\frac{2E}{m}} \int_0^1 \frac{d\xi}{(1 - \xi^2)^{\frac{1}{1+2\alpha}}}.
\]

One can simplify this expression as follows \((z = \xi^2)\),

\[ T = \frac{2}{\omega_0} \left( \frac{2E}{m} \right)^{\frac{1}{1+2\alpha}} \int_0^1 dz z^{-\frac{1}{2}} (1 - z)^{\frac{1}{2+2\alpha}}.
\]

Notice that

\[ B(x, y) = \int_0^1 dz z^{x-1} (1 - z)^{y-1} \]

is so called the Euler beta-function. It can be expressed in term of the Gamma-function,

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad \Gamma\left( \frac{1}{2} \right) = \sqrt{\pi}.
\]

Hence,

\[ T = \frac{2}{\omega_0} \left( \frac{2E}{m} \right)^{\frac{1}{1+2\alpha}} \frac{\alpha}{\omega_0} \Gamma\left( \frac{1}{2} \right) B\left( \frac{1}{2}, \frac{1 + 2\alpha}{2 + 2\alpha} \right).
\]

The period depends on \(E\) as follows:

\[ T \sim E^{\frac{\alpha}{2+2\alpha}}.
\]

For \(\alpha = 0\), the period does not depend on the energy:

\[ T = \frac{2\sqrt{\lambda}}{\omega_0} B\left( \frac{1}{2}, \frac{1}{2} \right) = \frac{2\pi\sqrt{\lambda}}{\omega_0}.
\]