Noether's theorem.

Theorem. Let

$$t' = t + \epsilon\ \Phi(q,t)$$
$$Q^k(t') = q^k(t) + \epsilon\ \Psi^k(q,t)$$

be an infinitesimal ($\epsilon \to 0$) transformation of time and generalized coordinates q^k ($k = 1, \dots f$), and the action is invariant under such transformation[1] :

$$\int_{t_1}^{t_2} dt\ L(q,\dot{q},t) = \int_{t_1'}^{t_2'} dt'\ L(Q,\dot{Q},t')\ . \tag{1}$$

Then the quantity,

$$I = \sum_{k=1}^{f} \frac{\partial L}{\partial \dot{q}^k}\ \left(\dot{q}^k\,\Phi - \Psi^k\right) - L\,\Phi\ , \tag{2}$$

is an integral of motion:

$$\frac{dI}{dt} = 0\ .$$

Proof: First of all let us introduce the notations,

$$\delta t = t' - t = \epsilon\,\Phi$$
$$\delta^* q^k = Q^k(t') - q^k(t) = \epsilon\ \Psi^k$$
$$\delta q^k = Q^k(t) - q^k(t)\ .$$

Notice that,

$$\delta^* q^k = Q^k(t') - q^k(t) = Q^k(t') - Q^k(t) + Q^k(t) - q^k(t) = \dot{q}^k\,\delta t + \delta q^k + O(\epsilon^2)\ .$$

We also define,

$$\delta^* L = L\big[Q(t'),\dot{Q}(t'),t'\big] - L\big[q(t),\dot{q}(t),t\big]$$
$$\delta L = L\big[Q(t),\dot{Q}(t),t\big] - L\big[q(t),\dot{q}(t),t\big]\ .$$

It is clear that

$$\delta^* L = \delta L + \delta t\ \frac{dL}{dt} + O(\epsilon^2)\ .$$

[1] Such transformation is said to be the *infinitesimal symmetry* of the mechanical system.

One has

$$\delta^* S = \int_{t_1'}^{t_2'} dt'\, L\big[Q(t'), \dot{Q}(t'), t'\big] - \int_{t_1}^{t_2} dt\, L\big[q(t), \dot{q}(t), t\big] = \\ \int_{t_1'}^{t_2'} dt' \Big(L\big[q(t), \dot{q}(t), t\big] + \delta^* L\Big) - \int_{t_1}^{t_2} dt\, L\big[q(t), \dot{q}(t), t\big]\ . \tag{3}$$

Using the relations

$$t' = t + \delta t\,, \qquad dt' = dt\, \left(1 + \frac{d\delta t}{dt}\right), \qquad \delta^* L = \delta L + \delta t\, \frac{dL}{dt} + O(\epsilon^2)$$

we can change the variables in the first integral in (3) and obtain,

$$\delta^* S = \int_{t_1}^{t_2} dt \left\{ \left(1 + \frac{d\delta t}{dt}\right)\left(L + \delta L + \delta t\, \frac{dL}{dt}\right) - L \right\} + O(\epsilon^2) = \\ \int_{t_1}^{t_2} dt \left(\delta L + L \frac{d\delta t}{dt} + \delta t \frac{dL}{dt}\right) + O(\epsilon^2) = \\ \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt}\,(\delta t\, L) + \sum_{k=1}^{f} \left(\frac{\partial L}{\partial q^k}\, \delta q^k + \frac{\partial L}{\partial \dot{q}^k}\, \delta \dot{q}^k \right) \right\} + O(\epsilon^2)\ . \tag{4}$$

Now let us apply the relation $\delta \dot{q}^k = \frac{d}{dt}\, \delta q^k$, and integrate the last term in (4) by parts,

$$\delta^* S = \int_{t_1}^{t_2} dt\, \frac{d}{dt}\Big(\delta t\, L + \sum_{k=1}^{f} \delta q^k\, \frac{\partial L}{\partial \dot{q}^k}\Big) + \int_{t_1}^{t_2} dt\, \sum_{k=1}^{f} \Big(\frac{\partial L}{\partial q^k} - \frac{d}{dt}\, \frac{\partial L}{\partial \dot{q}^k}\Big) + O(\epsilon^2)\ .$$

Consider this equation for $q^k = q^k(t)$ being the solution of equations of motion,

$$\frac{\partial L}{\partial q^k} = \frac{d}{dt}\, \frac{\partial L}{\partial \dot{q}^k} \qquad (k = 1, \ldots, f)\ .$$

Then,

$$\delta^* S = \int_{t_1}^{t_2} dt\, \frac{d}{dt}\Big(\delta t\, L + \sum_{k=1}^{f} \delta q^k\, \frac{\partial L}{\partial \dot{q}^k}\Big) + O(\epsilon^2) = \Big(\delta t\, L + \sum_{k=1}^{f} \delta q^k\, \frac{\partial L}{\partial \dot{q}^k}\Big)\Big|_{t_1}^{t_2} + O(\epsilon^2)\ . \tag{5}$$

Combining Eq.(5) with,

$$\delta t = \epsilon\ \Phi\,, \qquad \delta q^k = \delta^* q^k - \dot{q}^k\, \delta t = \epsilon\ (\Psi^k - \dot{q}^k\, \Phi)\ ,$$

one obtains,

$$\delta^* S = \epsilon \left\{ \Phi\left(L - \sum_{k=1}^{f} \dot{q}^k\, \frac{\partial L}{\partial \dot{q}^k}\right) + \sum_{k=1}^{f} \Psi^k\, \frac{\partial L}{\partial \dot{q}^k} \right\}\Bigg|_{t_1}^{t_2} + O(\epsilon^2)\ .$$

If we introduce the quantity (2), then

$$\delta^* S = \epsilon\, \Big(I(t_2) - I(t_1)\Big) + O(\epsilon^2)\ .$$

The invariance of the action (1) means that $\delta^* S = 0$, hence

$$I(t_2) = I(t_1)\ .$$

The last relation is valid for arbitrary t_1 and t_2, therefore $I(t)$ is an integral of motion.