Chapter 5

Small Vibrations

5.1 Overview

In chapter 3, we learned how to linearize problems with one degree of freedom about fixed points, stable and unstable equilibria. That is a very sensible thing to do if we only want to know about the motion in the vicinity of the fixed point. Beyond that it gave some qualitative information. For systems with many degrees of freedom, there are the same arguments for doing this. Motion close to an equilibrium is often of considerable interest. But there are other compelling reasons for the linearization program with many degrees of freedom. With only one degree of freedom it is possible to reduce even a fully nonlinear problem to a straightforward (though possibly very messy, numerically evaluated) integration. This was discussed in chapter 4. With many degrees of freedom, such a reduction is impossible in general. About the best tools one has are various sorts of perturbation theory which start with a linearized problem and then seek to put the rest back in bit-by-bit. There are exceptions which can be handled much more thoroughly and we shall discuss some of them, but in general that's the sort of shameful state we're reduced to.

Of course, the motivation is not completely negative. There are many systems which are nearly harmonic in character, whose small oscillations are of considerable interest. A prominent example is molecules. Continuous elastic media (infinitely many degrees of freedom!) are another, but we shall only be able to have the briefest peek at that.

This chapter is therefore devoted to the study of higher-dimensional linearized problems. This calls for the use of some linear algebra by its nature, so some of the requisite tools will be hastily assembled. Then we'll see how the general program works and take a detailed look at a pair of coupled oscillators. Going much beyond two degrees of freedom quickly becomes an algebraic nightmare. It's in

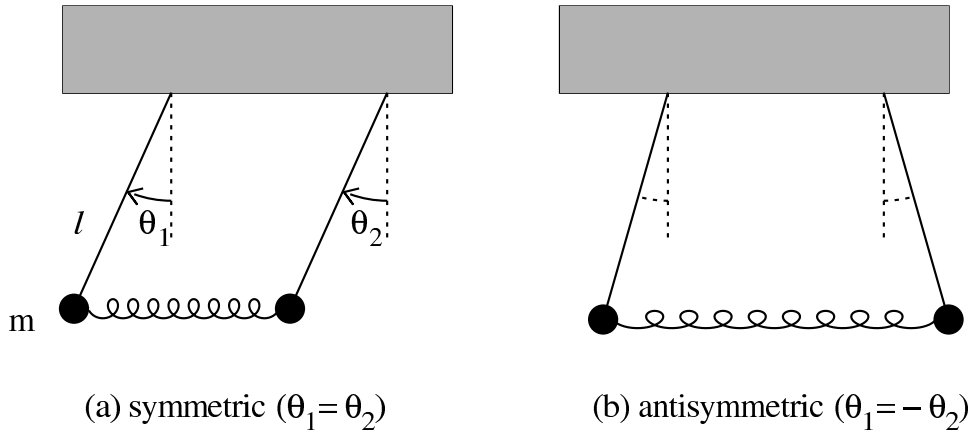

Figure 5.1: Two coupled pendula

principle straightforward, but not the sort of thing you want to do by hand. In many important problems to which the theory is applied, symmetries have a lot to say about the sorts of vibrations which can occur and their frequencies. Unfortunately, that is beyond our scope.

5.2 An Example: two coupled pendula

Because we will soon plunge into some mathematics which may be quite unfamiliar, it's a good idea to start with a motivational example. Then at least there will be some excuse. Let's consider a pair of identical pendulums as in figure 5.1 which are connected by a weak spring. The spring obeys Hooke's law, so the potential associated with it is

$$U_{\text{spring}} = \frac{k\ell^2}{2}(\sin\theta_1 - \sin\theta_2)^2.$$

Having dealt with such things before, we know how to write down the kinetic energies and the gravitational potential energies for the pendula, so just add in the spring potential to get the Lagrangian for this system, which is

$$L = \frac{m\ell^2}{2}(\dot\theta_1^2 + \dot\theta_2^2) - mg\ell(2 - \cos\theta_1 - \cos\theta_2) - \frac{k\ell^2}{2}(\sin\theta_1 - \sin\theta_2)^2. \tag{2.1}$$

Computing the equations of motion from this Lagrangian is straightforward, again a slight variation on a familiar theme. Here they are:

$$\begin{aligned} m\ell^2\ddot\theta_1 + mg\ell\sin\theta_1 + k\ell^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1 &= 0, \\ m\ell^2\ddot\theta_2 + mg\ell\sin\theta_2 - k\ell^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2 &= 0. \end{aligned} \tag{2.2}$$

Exercise Work those out in the margin.

To tidy things up a bit, clear out all the constants in favor of a frequency ω_0 and a dimensionless parameter ϵ which measures the strength of the coupling between the pendula,

$$\frac{g}{\ell} = \omega_0^2, \qquad \frac{k}{m} = \epsilon\omega_0^2, \qquad \frac{k\ell}{mg} = \epsilon. \tag{2.3}$$

Then the equation of motion for θ_1 becomes

$$\ddot\theta_1 + \omega_0^2\sin\theta_1 + \epsilon\omega_0^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1 = 0.$$

Now, if there were no θ_2 here, we could write down a solution for this in the form of an integral as was done in the previous chapter. The presence of θ_2 means that life is very difficult, however. There is a stable equilibrium at $\theta_1 = \theta_2 = 0$, and our way of making some progress is to linearize the equations of motion about this fixed point. That is done by expanding the sines and cosines and keeping terms containing at most one power of θ_1 or θ_2:

$$[\sin\theta_1 - \sin\theta_2]\cos\theta_1 = [(\theta_1 - \cdots) - (\theta_2 - \cdots)](1 - \theta_1^2/2 + \cdots) = \theta_1 - \theta_2 + \mathcal{O}(\theta^3). \tag{2.4}$$

So throw away all the higher order stuff. The linear parts are

$$\begin{aligned} \ddot\theta_1 + \omega_0^2[(1+\epsilon)\theta_1 - \epsilon\theta_2] &= 0 \\ \ddot\theta_2 + \omega_0^2[(1+\epsilon)\theta_2 - \epsilon\theta_1] &= 0. \end{aligned} \tag{2.5}$$

This is much nicer. It is a pair of coupled linear differential equations. We could perhaps tackle it head on. But, instead we take a detour to develop some linear algebra tools. These are absolutely essential for dealing with linearized problems with more than two degrees of freedom. After setting up the general framework, we'll come back and finish solving the equations (2.5).

5.3 Linear Algebra, the short course

This section is a brief introduction to linear algebra. Alongside calculus, linear algebra occupies a very central position in modern mathematics and finds uses in an incredible variety of applications.

5.3.1 vector spaces

Linear algebra is an abstraction from the familiar properties of ordinary, physical vectors. The basic structure is a **vector space**, which is a collection of objects, called **vectors**, together with two operations defined on them. First, there is a commutative and associative addition, so that vectors can be added and the order in which we do so is completely immaterial. For example, $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{w}+\mathbf{v})$. We require a zero element $\mathbf{0}$, so that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for any vector $\mathbf{u}$. Secondly, there is a multiplication by real numbers or complex numbers, often referred to as **scalars** in this context. Depending upon which set of scalars are in use, one refers to a **real vector space** or a **complex vector space**. Multiplication by scalars is required to distribute across addition,

$$\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v},$$

and to satisfy

$$\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u},$$

and

$$(1)\mathbf{u} = \mathbf{u}.$$

These are all so natural that it may very well have been unnecessary to even bring them up. Too late.

ordinary physical vectors Since freshman physics, you have been using displacement vectors which obeyed exactly these kinds of rules. It made sense to add two displacements and to multiply one by a real number, so this was a real vector space.

function spaces Consider the set V of continuous real-valued functions on the unit interval $0 \leq x \leq 1$. Given two functions $f(x)$ and $g(x)$ in V, the sum $f+g$ is defined to be the function which takes the value $f(x)+g(x)$ at x. This is so natural it may even seem strange that I spelled it out. It's also quite reasonable to multiply by a real number α, with αf being the function with value $\alpha f(x)$ at x. So this is a real vector space. By making other sorts of restrictions on the allowed functions, other function spaces can be made. A very simple variation is to take continuous complex-valued functions. Then we would have a complex function space. This illustrates just how far the idea of a vector space goes. And function spaces of this sort do arise in physics: quantum mechanics is built out of them.

5.3.2 bases

Decomposing vectors into components along x, y and z is an old familiar process. Any vector can be written down as a combination

$$\mathbf{v} = v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y + v_z\hat{\mathbf{e}}_z,$$

and we need all three of $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ in order to be able to write down all vectors this way. We could not, for instance, write down a vector in the z direction as a combination of just $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$.

The generalization of this notion is the idea of a **basis.** A basis is a set of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in terms of which we can express every vector as a combination of the sort

$$\mathbf{v} = \alpha_1\mathbf{e}_1 + \cdots + \alpha_n\mathbf{e}_n,$$

for some collection of scalars $\alpha_1, \ldots, \alpha_n$ and such that if we threw out even one of the $\mathbf{e}_i$ there would be some vector which we couldn't expand in that way. The expansion coefficients α_i in the equation above are called the **components** of $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

There are many ways to choose a basis. We could have rotated our axes for example, or used $\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y$ in place of $\hat{\mathbf{e}}_y$. This you can see from

$$v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y + v_z\hat{\mathbf{e}}_z = (v_x - v_y)\hat{\mathbf{e}}_x + v_y(\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y) + v_z\hat{\mathbf{e}}_z.$$

Although there is a wide range of choices, all bases have the same size. It is not difficult to prove that, but I won't do so here because I suspect you're willing to believe it. In the case we're discussing, that size is three. Since the number of elements in a basis doesn't depend upon the basis, it is a property of the vector space itself and is called the **dimension** of the vector space.

The function space mentioned above is an infinite-dimensional vector space. As you may well imagine, there are subtleties involved in working with such things. Fortunately, we won't be needing them here.

5.3.3 linear operators

Consider the following two operations defined on ordinary three-dimensional vectors $\mathbf{v}$, ($\boldsymbol{\omega}$ is some fixed vector):

$$\mathbf{v} \mapsto \mathbf{A}\mathbf{v} \stackrel{def}{=} \boldsymbol{\omega} \times \mathbf{v}, \tag{3.6}$$

and

$$\mathbf{v} = v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y + v_z\hat{\mathbf{e}}_z \mapsto \mathbf{B}\mathbf{v} \stackrel{def}{=} v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y. \tag{3.7}$$

These are linear operators. A **linear operator** is a machine which takes a vector in, gives a vector back and completely respects all the basic vector space operations. What this last requirement means is that you get the same answer if you add two vectors and then put them through the machine as you do if you put them through and add them afterwards. Similarly, it makes no difference whether you multiply by a scalar before or after. For the first example, this is spelled out by

$$\begin{aligned} \boldsymbol{\omega} \times (\mathbf{v} + \mathbf{w}) &= (\boldsymbol{\omega} \times \mathbf{v}) + (\boldsymbol{\omega} \times \mathbf{w}). \\ \boldsymbol{\omega} \times (\alpha \mathbf{v}) &= \alpha(\boldsymbol{\omega} \times \mathbf{v}). \end{aligned}$$

An example of something which is not a linear operator is $\mathbf{v} \mapsto (\omega \times \mathbf{v}) \times \mathbf{v}$.

An important question to ask about a linear operator is whether it has an inverse. A linear operator $\mathbf{A}$, considered as simply a mapping of a vector space V into itself, has an inverse if it matches elements of V up in a one-to-one fashion, in other words, for each $\mathbf{w} \in V$, there is precisely one $\mathbf{v}$ such that $\mathbf{w} = \mathbf{A}\mathbf{v}$. A fairly remarkable fact is that if $\mathbf{A}$ *does not map any nonzero vector to zero*, then it is invertible in this sense, and the inverse is also a linear operator, which is denoted $\mathbf{A}^{-1}$. A linear operator which sends some non-zero vector to zero is called **singular**, so another way of stating the criterion for invertibility is that the operator needs to be nonsingular.

5.3.4 columns, rows and matrices

Everything you need to know about a linear operator is contained in its effect on a basis. If $\mathbf{A}$ is a linear operator on ordinary three-dimensional vectors,

$$\mathbf{A}(v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z) = v_x(\mathbf{A}\hat{\mathbf{e}}_x) + v_y(\mathbf{A}\hat{\mathbf{e}}_y) + v_z(\mathbf{A}\hat{\mathbf{e}}_z).$$

This simple observation is at the heart of the representation of linear algebra in terms of columns, rows and arrays of numbers.

Suppose you have chosen a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in your vector space. Then the vector

$$\mathbf{v} = v_1 \mathbf{e}_1 + \cdots + v_n \mathbf{e}_n$$

can be represented by the column of components

$$\begin{pmatrix} v_1 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

As long as you know what basis this is being used, this **column vector** is a perfectly unambiguous way of referring to the vector $\mathbf{v}$. Now, let's consider the vector which

results from applying $\mathbf{A}$ to the basis vector $\mathbf{e}_i$, and give its components names according to

$$\mathbf{A}\mathbf{e}_i = A_{i1}\mathbf{e}_1 + \cdots + A_{in}\mathbf{e}_n.$$

We can collect all these together into a square array, lining up the $\mathbf{A}\mathbf{e}_i$ columns next to each other thus:

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

No arguing that this contains everything there is to know about $\mathbf{A}$, but it also gives us a convenient calculational algorithm. To illustrate we'll take $n = 3$, for the sake of simplicity. The application of a linear operator on a vector produces another vector, so we need that to be reflected in the multiplication of a matrix and a column giving another column. Here's how it works:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} A_{11}v_1 + A_{12}v_2 + A_{13}v_3 \\ A_{21}v_1 + A_{22}v_2 + A_{23}v_3 \\ A_{31}v_1 + A_{32}v_2 + A_{33}v_3 \end{pmatrix}. \tag{3.8}$$

This is bristling with indices, but the way it works is quite simple. To get the top component of the answer $\mathbf{A}\mathbf{v}$, skim of the top row of the matrix for $\mathbf{A}$, turn it sideways and multiply it into the column for $\mathbf{v}$ component by component, just like a dot product. To get the second component of $\mathbf{A}\mathbf{v}$, do the same with the second row of the matrix and so forth. The proof that this recipe actually does what is advertised is very simple. It works for basis vectors. For example, for $\mathbf{e}_1$ we get

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix}.$$

This picks off the first column of the matrix for $\mathbf{A}$, which was the column representation of $\mathbf{A}\mathbf{e}_1$ by definition. It works in much the same way with the other two basis vectors. Looking back at equation (3.8), you can rewrite the right-hand side as

$$v_1 \begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix} + v_2 \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix} + v_3 \begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \end{pmatrix},$$

in which we recognize the images of the basis vectors multiplied by the corresponding components of $\mathbf{v}$, just as required, since it should be $v_1(\mathbf{Ae}_1) + v_2(\mathbf{Ae}_2) + v_3(\mathbf{Ae}_3)$.

There is yet another way to write these things, which is sometimes convenient and uses less paper. The i-th component of the vector $\mathbf{Av}$ is given by

$$[\mathbf{Av}]_i = \sum_j A_{ij} v_j. \tag{3.9}$$

This is completely in agreement with

Sometimes we want to apply a linear operator $\mathbf{B}$ to a vector which itself is the result of the action of another linear operator $\mathbf{A}$. This is the composition of $\mathbf{B}$ with $\mathbf{A}$, usually called the **product** and written simply as $\mathbf{BA}$:

$$(\mathbf{BA})\mathbf{v} \stackrel{def}{=} \mathbf{B}(\mathbf{Av}),$$

where the parentheses on the right-hand side serve to indicate the order of operations, and on the left are just for emphasis. What this equation means is that parentheses are never necessary in such a concatenation. Let's compute the matrix of the product $\mathbf{BA}$. Writing down the i-th component of the result of applying it to some vector $\mathbf{v}$,

$$\begin{aligned} [\mathbf{BAv}]_i &= \sum_j B_{ij} [\mathbf{Av}]_j = \sum_j B_{ij} \sum_k A_{jk} v_k \\ &= \sum_{jk} (B_{ij} A_{jk}) v_k = \sum_k \left(\sum_j B_{ij} A_{jk} \right) v_k. \end{aligned} \tag{3.10}$$

Since $\mathbf{v}$ could be anything, this says that the ik component of the product is given by

$$[\mathbf{BA}]_{ik} = \sum_j B_{ij} A_{jk}. \tag{3.11}$$

What does this tell us to do in terms of matrix manipulations? It says that the k-th column of the product matrix is obtained by taking the k-th column of $[\mathbf{A}]$ and acting on it with the matrix for $\mathbf{B}$ just as if it were a vector. Here is an example that should make it clear:

$$\begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} (2)(-5) + (-3)(7) & 0 \\ (0)(-5) + (4)(7) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (2)(1) + (-3)(6) \\ 0 & (0)(1) + (4)(6) \end{pmatrix} = \begin{pmatrix} -31 & -16 \\ 28 & 24 \end{pmatrix}.$$

Most vector spaces with which we deal are not given to us *a priori* in the form of columns of numbers. The connection to column vectors only comes once a basis

has been chosen. However, it is frequently convenient to confuse a vector with the column vector representing it, or a linear operator with the matrix representing it. This cuts down on the verbiage of the "...matrix which represents..." sort. But we must always keep in the backs of our minds that the columns and matrices are contingent upon a choice of basis.

5.3.5 eigenvectors and eigenvalues

In general, of course, the result $\mathbf{Av}$ of applying a linear operator to a vector can be any old thing. However it may happen (and it is of great interest when it does), that the effect of $\mathbf{A}$ on $\mathbf{v}$ is to simply multiply it by a scalar:

$$\mathbf{Av} = \lambda\mathbf{v}. \tag{3.12}$$

When this happens, $\mathbf{v}$ is said to be an **eigenvector** of $\mathbf{A}$, and λ is the associated **eigenvalue**. Another terminology is *characteristic vector* and *characteristic value*. Notice that if $\mathbf{v}$ is an eigenvector, then any scalar multiple $\alpha\mathbf{v}$ will be too, with the same eigenvalue, because α will simply factor through everything. So we can always choose one component of an eigenvector to our liking. When we count eigenvectors, we generally don't call such things different.

A linear operator picked out of a hat may not have any eigenvectors, or only a few. For example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

does not have any real eigenvectors at all. Let's see. Suppose it had one so that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

This means

$$\begin{aligned} y &= \lambda x \\ x &= -\lambda y = -\lambda^2 x \end{aligned}$$

so that $\lambda = \pm i$. This certainly cannot happen if x and y are real numbers! But it can if they are allowed to be complex. We can always choose $x = 1$, so let's do that. Then we get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} = i \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has one eigenvector, which is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with an eigenvalue of 0. There are no others. Using complex numbers will not help, either.

There is a class of matrices which are guaranteed to have enough eigenvectors that an entire basis can be made from them. If a matrix is **symmetric**,

$$A_{ij} = A_{ji} \quad \text{for all } i, j,$$

then there is a basis consisting of eigenvectors. They may be complex, however. This symmetry property means the matrix doesn't change if we reflect all the entries across the upper-left to lower-right diagonal. If we write the matrix for the operator **A** in that basis, it will be diagonal:

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

5.3.6 determinants

The way to find out whether a linear operator is singular is to compute the determinant of its matrix. If the determinant is zero, then it is singular and therefore has a zero eigenvalue.

$$\det[\mathbf{A}] = 0 \Leftrightarrow \mathbf{A} \text{ is singular.} \tag{3.13}$$

So how do you compute the determinant? For two-by-two matrices it is very simple:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{21}A_{12}. \tag{3.14}$$

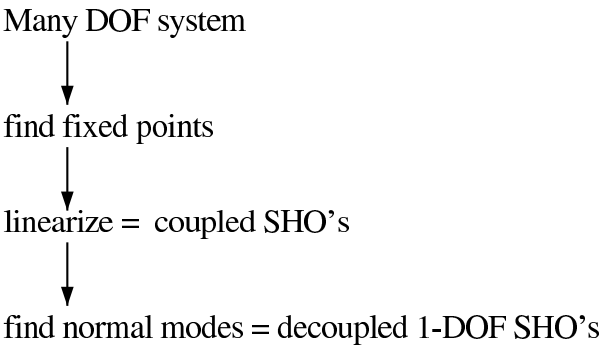

Figure 5.2: The general program

In other words, the northwest to souteast diagonal minus the southwest to northeast diagonal. For bigger matrices, the best thing is a technique called expansion by minors. Here's how it works:

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} = \sum_{m=1}^{m=n} (-1)^{m-1} A_{1m} \det[C]_{1m} \tag{3.15}$$

where $[C]_{1m}$ denotes the matrix obtained by removing the 1st row and m-th column from A. This reduces the computation of $\det A$ to a sum of determinants of $n-1 \times n-1$ determinants. Now apply the same rule to those until you're down to 2×2. There's also a useful special recipe for 3×3 matrices that you probably know.

5.4 Linearized Dynamics in the Vicinity of a Fixed Point

The idea of linearizing the dynamics in the vicinity of an equilibrium works the same way with many degrees of freedom as with one. First, adjust your coordinates if necessary so that the equilibrium is at $q_1 = q_2 = \cdots = q_n = 0$ (and, of course $\dot{q}_1 = \dot{q}_2 = \cdots = \dot{q}_n = 0$). Then expand the kinetic and potential energies to second

order in q's and $\dot{q}$'s to get

$$T = \frac{1}{2}\sum_{i,j} M_{ij}\dot{q}_i\dot{q}_j, \quad \text{and} \quad V = \frac{1}{2}\sum_{i,j} V_{ij}\dot{q}_i\dot{q}_j. \tag{4.16}$$

The M_{ij} and V_{ij} are constants. In doing this we may have thrown away higher order terms like $q_1\dot{q}_1\dot{q}_2$ or $q_1q_2q_3$, etc. The reason there are no terms with only one q in the potential is that we're expanding about an equilibrium. We must have $\partial V/\partial q_i = 0$ for every i.

It's important to choose the constant symmetrically. If the terms in V involving q_1 and q_2 were aq_1q_2, you could take $V_{12} = a$, $V_{21} = 0$, or the other way around, or any number of other choices. We can choose them symmetrically, as

$$M_{ij} = M_{ji}, \quad V_{ij} = V_{ji}. \tag{4.17}$$

Since that simplifies matters, we will do that. Putting them together, we find a Lagrangian

$$L = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} V_{ij}q_iq_j. \tag{4.18}$$

And, following the usual recipe, the equations of motion are

$$\sum_j [M_{ij}\ddot{q}_j + V_{ij}q_j] = 0, \quad \text{for each } i. \tag{4.19}$$

To make any further progress with these equations of motion, it's very helpful to use techniques of linear algebra. If we conceive of $q_1, \ldots, q_n$ as being the components of a vector, we can rewrite the equations of motion as a vector equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{V}\mathbf{q} = 0. \tag{4.20}$$

Exercise Verify those equations of motion.

Now, we are going to look for motions of the system which are oscillatory in the sense that there is a sinusoidal variation of the positions with time, but all positions oscillating with the same frequency. A motion of this sort is called a **normal mode**. Since we want to encompass the possibility of unstable equilibria, we will broaden the definition of normal modes to include solutions which expand or contract exponentially, just as we had for the one degree of freedom case. We will show that any motion of the system can be decomposed into a sum of normal

modes of various amplitudes. In terms of the way we've expressed things here, a normal mode $\boldsymbol{\eta}$ satisfies

$$\alpha \mathbf{M}\boldsymbol{\eta} + \mathbf{V}\boldsymbol{\eta} = 0,$$

so that

$$\mathbf{q} = \boldsymbol{\eta} e^{\pm i\omega t}, \quad \omega = \sqrt{|\alpha|}$$

satisfies the equation of motion if $\alpha < 0$, or

$$\mathbf{q} = \boldsymbol{\eta} e^{\pm \lambda t}, \quad \lambda = \sqrt{\alpha}$$

satisfies the equation of motion if $\alpha > 0$.

The existence of a complete set of normal modes means that the general solution to the equations of motion can be expressed as a linear combination of them:

$$\mathbf{q}(t) = \sum a_\mu \boldsymbol{\eta}_\mu e^{i\omega_\mu t} + \sum \boldsymbol{\eta}_\nu (b_\nu e^{\lambda_\nu t} + c_\nu e^{-\lambda_\nu t}), \tag{4.21}$$

where the first sum runs over the oscillatory modes and the second over the exponentially growing and shrinking ones. In total, there are n different η_μ.

Here is the argument for the existence of that complete set. You may want to skip it until after seeing something more concrete. On purely physical grounds, we know that $\mathbf{M}$ must be nonsingular. If there were any vector $\mathbf{v}$ which was mapped to zero: $\mathbf{M}\mathbf{v} = 0$, that would mean there was absolutely no kinetic energy associated with motion in the 'direction' $\mathbf{v}$. That makes no sense. We also know that $\mathbf{M}$ is symmetric (we made it that way!), so there is a basis consisting of eigenvectors. If we use these eigenvectors for our basis, the matrix of $\mathbf{M}$ becomes diagonal. This means we are using new coordinates y_i with

$$T = \frac{1}{2} \sum_i m_i \dot{y}_i^2.$$

Finally, we can rescale our coordinates with $x_i = \sqrt{m_i}\, y_i$, so that

$$T = \frac{1}{2} \sum_i \dot{x}_i^2.$$

Now the expression for V in terms of the x_i is different than in terms of the q_i, but it still has the same form:

$$V = \frac{1}{2} \sum_{ij} \tilde{V}_{ij} x_i x_j,$$

with $\tilde{V}_{ij}$ symmetric. Working out the equations of motion,

$$\ddot{\mathbf{x}} + \tilde{\mathbf{V}}\mathbf{x} = 0.$$

Since $\tilde{\mathbf{V}}$ is symmetric, it has a complete set of eigenvectors $\boldsymbol{\zeta}_\alpha$, with $\tilde{\mathbf{V}}\boldsymbol{\zeta} = \alpha\boldsymbol{\zeta}$. These are precisely the normal modes we're after. In fact, doing things in stages like this, first choosing a basis which renders the kinetic energy very simple and only then tackling the potential is quite possibly a good general strategy.

But knowing the normal modes are there isn't the same as having them.

5.5 Normal Modes: back to coupled pendula

The condition for the existence of a vector η with

$$(\alpha\mathbf{M} + \mathbf{V})\,\boldsymbol{\eta} = 0,$$

is

$$\det|\alpha\mathbf{M} - \mathbf{V}| = 0. \tag{5.22}$$

In our pendulum problem (remember that?), the equation we want to solve for α and $\boldsymbol{\eta}$ is

$$\left\{\alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \omega_0^2\begin{pmatrix}1+\epsilon & -\epsilon\\ -\epsilon & 1+\epsilon\end{pmatrix}\right\}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = 0. \tag{5.23}$$

First, find the eigenvalues by setting the determinant of the matrix in braces equal to zero. Since we know that we will find only oscillatory motion, we will just replace α by $-\omega^2$ at this point

$$\begin{vmatrix}(1+\epsilon)\omega_0^2 - \omega^2 & -\epsilon\omega_0^2\\ -\epsilon\omega_0^2 & (1+\epsilon)\omega_0^2 - \omega^2\end{vmatrix} = (1+\epsilon)^2\omega_0^4 - 2(1+\epsilon)\omega_0^2\omega^2 + \omega^4 - \epsilon^2\omega_0^4.$$

Setting this to zero gives the so-called *secular equation*

$$\omega^4 - 2(1+\epsilon)\omega_0^2\omega^2 + (1+2\epsilon)\omega_0^4 = 0.$$

Since this is a quadratic equation for ω^2, the solutions are

$$\begin{aligned}\omega^2 &= (1+\epsilon)\omega_0^2 \pm \omega_0^2\left[(1+\epsilon)^2 - (1+2\epsilon)\right]^{1/2}\\ &= (1+\epsilon\pm\epsilon)\omega_0^2\end{aligned}$$

Finally then, the two normal mode frequencies are

$$\omega_1 = \omega_0, \quad \omega_2 = (1+2\epsilon)^{1/2}\omega_0. \tag{5.24}$$

That's only half the fun. Now we need the normal modes themselves. Putting the first eigenvalue $\alpha = -\omega_0^2$ into equation (5.23),

$$\epsilon\omega_0^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \eta_{11} \\ \eta_{12} \end{pmatrix} = 0,$$

so that

$$\eta_{11} - \eta_{12} = 0.$$

And for the other, $\alpha = (1 + 2\epsilon)\omega_0^2$,

$$\epsilon\omega_0^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_{21} \\ \eta_{22} \end{pmatrix} = 0,$$

which implies

$$\eta_{21} + \eta_{22} = 0.$$

The general solution to the equation of motion is a linear combination of the normal modes,

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}. \tag{5.25}$$

The first normal mode, with $\omega_1 = \omega_0$ has $\theta_1(t) = \theta_2(t)$ for all time. It is the symmetric mode shown in figure 5.1. This frequency makes sense because the spring is never stretched, so that both pendula continue to move with their uncoupled frequency. The second normal mode has $\theta_1(t) = -\theta_2(t)$ so that the two pendula are always moving in opposite directions. The spring pulls them back together and then pushes them apart and this raises the frequency above that of a lone pendulum.

5.5.1 the role of symmetry

You will notice that both these modes have a very well defined symmetry to them. That may make you suspect that we should have seen it coming and could have cut short some of this calculation. Indeed that is true. For very complicated vibration problems, as with polyatomic molecules for example, the exploitation of symmetries is very important. Taking advantage of that relies on group representation theory, which is well beyond our scope.

The simple lesson though is that whenever a physical situation displays a symmetry we should try to take advantage of it. We should have thought to try one or the other of these simple modes. It doesn't matter how you find the normal modes if you can demonstrate that they are such.

5.6 A pair of coupled harmonic oscillators: trends

In this section we want to understand some general trends of the normal mode frequencies for a pair of coupled oscillators as various parameters are changed. We can consider the concrete example of a pair of masses on springs coupled together by yet another spring, as in figure 5.6. The masses are both equal to m, but the spring constants will be allowed to be different.

The equations of motion for x_1 and x_2 are

$$\begin{aligned} m\ddot{x}_1 &= -k_1 x_1 - k_3(x_1 + x_2) \\ m\ddot{x}_2 &= -k_2 x_2 - k_3(x_1 + x_2). \end{aligned} \tag{6.26}$$

Things are going to become messy, so we'll make some shorthand definitions at this point. Specifically,

$$\begin{aligned} \kappa_i &= \kappa_i/m, \quad i = 1, 2, 3 \\ \kappa_1' &= \kappa_1 + \kappa_3 \quad \kappa_2' = \kappa_2 + \kappa_3 \\ \omega_1 &= \sqrt{\kappa_1}, \quad \omega_2 = \sqrt{\kappa_2}. \end{aligned} \tag{6.27}$$

Now, the 'eigenequation' we need to solve for the normal modes and their frequencies is

$$\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\kappa_1' & -\kappa_3 \\ -\kappa_3 & -\kappa_2' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6.28}$$

The eigenvalues are determined by setting the determinant of the matrix equal to zero:

$$(\omega^2 - \kappa_1')(\omega^2 - \kappa_2') - \kappa_3^2 = 0,$$

which is rearranged to give

$$\omega^4 - (\kappa_1' + \kappa_2')\omega^2 - \kappa_3^2 = 0,$$

and finally

$$\omega^4 - (\kappa_1' + \kappa_2')\omega^2 + \kappa_1'\kappa_2' - \kappa_3^2 = a\omega^4 + b\omega^2 + c = 0.$$

I've indicated the usual a b and c of a quadratic equation because we need to work on those a little. Nothing for it but to compute directly:

$$c = \kappa_1'\kappa_2' - \kappa_3^2 = (\kappa_1 + \kappa_3)(\kappa_2 + \kappa_3) - \kappa_3^2 = \kappa_1\kappa_2 + \kappa_3(\kappa_1 + \kappa_2),$$

$$b^2 = (\kappa_1 + \kappa_2 + 2\kappa_3)^2 = (\kappa_1 + \kappa_2)^2 + 4\kappa_3(\kappa_1 + \kappa_2) + 4\kappa_3^2$$

and

$$\begin{aligned} b^2 - 4ac &= (\kappa_1 + \kappa_2)^2 + 4\kappa_3(\kappa_1 + \kappa_2) + 4\kappa_3^2 - 4\kappa_1\kappa_2 - 4\kappa_3(\kappa_1 + \kappa_2) \\ &= (\kappa_1 - \kappa_2)^2 + 4\kappa_3^2. \end{aligned}$$

In the end, at least we see a little simplification. The roots are

$$\omega^2 = \overline{\kappa}\left\{1 + \frac{\kappa_3}{\overline{\kappa}} \pm \left[\left(\frac{\Delta\kappa}{2\overline{\kappa}}\right)^2 + \left(\frac{\kappa_3}{\overline{\kappa}}\right)^2\right]^{1/2}\right\}, \tag{6.29}$$

where I've introduced yet more shorthand:

$$\begin{aligned} \overline{\kappa} &= \frac{\kappa_1 + \kappa_2}{2} = \text{ average spring constant over m} \\ \Delta\kappa &= \kappa_1 - \kappa_2 = \text{ difference of spring constants over m} \end{aligned} \tag{6.30}$$

Now we'll look at some limiting ranges for the parameters $\overline{\kappa}$, $\Delta\kappa$ and κ_3.

1. $\underline{\kappa_1 = \kappa_2 = \overline{\kappa}}$

In this case,

$$\omega^2 = \kappa_0 \pm \kappa_3, \tag{6.31}$$

where

$$\kappa_0 = \overline{\kappa} + \kappa_3$$

is the spring constant that would be effective if one mass were nailed down to $x = 0$ and only the other allowed to oscillate. Coupling the two oscillators together produces a symmetrical splitting about this value. This is a ubiquitous phenomenon when two identical systems are brought together and interact weakly. It is exactly what happens in the formation of bonding and antibonding orbitals in the H_2 molecule.

From your experience with the pair of pendula, you know what modes these frequencies correspond to. $\omega^2 = \overline{\kappa}$ has the two masses moving back and forth

together so that the central spring is never even stretched. The higher frequency mode is an in-and-out breathing mode.

2. $\kappa_1 = \kappa_2$ and $\kappa_3 \ll \overline{\kappa}$: beats

Now let's suppose that the coupling spring κ_3 is very weak. An interesting phenomenon known as **beats** arises in this case. We can make an approximation for the higher frequency as follows,

$$\omega_1 = \sqrt{\overline{\kappa}}, \quad \omega_2 = \sqrt{\overline{\kappa} + 2\kappa_3} = \omega_1 \left[1 + \kappa_3/\overline{\kappa} + \cdots\right].$$

Now consider pushing mass 1 toward the center a distance d while holding mass 2 fixed. Then release everything from rest. Incorporating those initial conditions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{d}{2} \cos\omega_1 t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{d}{2} \cos\omega_2 t.$$

$$\cos\omega_2 t \approx \cos\omega_1 t \cos\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right) - \sin\omega_1 t \sin\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)$$

$$\begin{aligned} x_1 &= \frac{d}{2}\left(1 + \cos\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\right)\cos\omega_1 t - \frac{d}{2}\sin\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\sin\omega_1 t \\ x_2 &= -\frac{d}{2}\left(1 - \cos\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\right)\cos\omega_1 t - \frac{d}{2}\sin\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\sin\omega_1 t \end{aligned}$$

It may be clear already from these expressions that mass 2 will very slowly pick up energy until $(\omega_1\kappa_3/\overline{\kappa})t$ is near π. At that point, mass 2 will be moving with its maximum amplitude and mass 1 will be nearly stationary. This very low frequency transfer of the energy back and forth between masses 1 and 2 is known as beats.

We can quantify that a bit better, however. Since $\omega_1\kappa_3/\overline{\kappa}$ is much smaller than ω_1, everything is very nearly oscillating with frequency ω_1. If we average over one period of that motion in an approximate way, by taking the sine and cosine of $(\omega_1\kappa_3/\overline{\kappa})t$ to be constant over that period, we can easily compute the mean square amplitude of the motion. The mean square amplitude of mass 1 is x_1^2 averaged over $2\pi/\omega_1$ and similarly for mass 2. The result is

$$\begin{aligned} \text{mean square amplitude 1} &\approx \frac{d^2}{2}\left[1 + \cos\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\right] \\ \text{mean square amplitude 2} &\approx \frac{d^2}{2}\left[1 - \cos\left(\omega_1 \frac{\kappa_3}{\overline{\kappa}} t\right)\right]. \end{aligned} \tag{6.32}$$

Here's how that's done in more detail. Writing ϵ for $\omega_1\kappa_3/\overline{\kappa}$ so we don't drown,

$$x_1^2 = \frac{d^2}{4}\left\{(1+\cos\epsilon t)^2\cos^2\omega_1 t + \sin^2\epsilon t\,\sin^2\omega_1 t - 2\,(1+\cos\epsilon t)\sin\epsilon t\sin\omega_1 t\cos\omega_1 t\right\}.$$

Over one complete period, $2\pi/\omega_1$, $\sin^2\omega_1 t$ and $\cos^2\omega_1 t$ average to $1/2$, but $\sin\omega_1 t\cos\omega_1 t$ averages to zero.

Exercise Carry out those averages, in order to obtain the mean square amplitudes.

Exercise sketch graphs of the positions of both masses as a function of time.

3. $\Delta\kappa \neq 0$, $\kappa_3 \ll \Delta\kappa$

Finally we consider the case that the frequencies of the two separate oscillators are different, and that the coupling is weak compared to this splitting. In that case, the square root in equation (6.29) can be expanded to get

$$\omega^2 \approx \overline{\kappa} + \kappa_3 \pm \left(\frac{\Delta\kappa}{2} + \frac{\kappa_3^2}{\Delta\kappa}\right). \tag{6.33}$$

For definiteness, let's say that $\kappa_1 > \kappa_2$, it really makes no difference. Then, setting $\kappa_3 = 0$,

$$\begin{aligned} \overline{\kappa} + \frac{\Delta\kappa}{2} &= \kappa_1 \\ \overline{\kappa} - \frac{\Delta\kappa}{2} &= \kappa_2. \end{aligned}$$

This is as it should be. If $\kappa_3 = 0$ we must have the frequencies of the two separate oscillators. Now notice what happens when κ_3 is present. First, there is a shift of both frequencies squared by κ_3 as we saw for the $\Delta\kappa = 0$ case. But there is an additional shift by $\pm\kappa_3^2/\Delta\kappa$. The higher frequency goes up and the lower goes down. In other words, the coupling drives the two frequencies apart. This too is a ubiquitous phenomenom, seen in quantum mechanical contexts as well as classical ones. Notice however, that the additional splitting is small as the original $\Delta\kappa$ is large. Those who have studied second order perturbation theory in quantum mechanics may enjoy comparing this to the formula derived there for the energy levels.

Exercise Carry out the expansion of the square root to obtain equation (6.33.

5.7 Lissajous figures and quasi-periodic motion

For a system with quadratic kinetic and potential energies, every motion can be decomposed into a sum of normal modes. The normal modes are periodic, but this does not mean that a superposition of normal modes is. In general it will not be.

For the simplest example, consider a mass m moving in two dimensions in the potential

$$U(x,y) = \frac{k_x}{2}x^2 + \frac{k_y}{2}y^2.$$

Since x and y are separated here, the equations of motion are completely decoupled:

$$\begin{aligned} \ddot{x} &= -\frac{k_x}{m}x \\ \ddot{y} &= -\frac{k_y}{m}y. \end{aligned}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_x \cos(\omega_x t + \delta_x) \\ A_y \cos(\omega_y t + \delta_y) \end{pmatrix},$$

where $\omega_x^2 = k_x/m$, $\omega_y^2 = k_y/m$. If $\omega_x = \omega_y$, then the trajectories of this system are ellipses, as you showed on a homework problem. If they are not however, matters are quite different. To understand that, introduce some new variables u and v by means of

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_x \cos u \\ A_y \cos v \end{pmatrix}.$$

So,

$$\dot{u} = \omega_x, \quad \dot{v} = \omega_y.$$

These variables are much more convenient that x and y because they change uniformly in time. Notice that x does not determine u uniquely – there are two values of u which give each x. One of them corresponds to $\dot{x} > 0$ and the other to $\dot{x} < 0$. Since u and v change uniformly in time, a plot of $(u(t), v(t))$ as in figure 5.3 (a) is a straight line. The slope of this line is precisely

$$\frac{\omega_y}{\omega_x}.$$

Since $u = 2\pi$ is really no different from $u = 0$, it makes sense to fold the whole picture back into the $[0, 2\pi] \times [0, 2\pi]$ square at the bottom left of the figure. If we do that (and then blow it up) we get figure 5.3 (b).

If ω_y/ω_x is a rational number, this curve will eventually close on itself. In fact, if

$$\frac{\omega_y}{\omega_x} = \frac{p}{q}$$

with p and q reduced to lowest terms, then when u has changed by $q(2\pi)$, v has changed by $p(2\pi)$. This means that they're really back to their original values, so

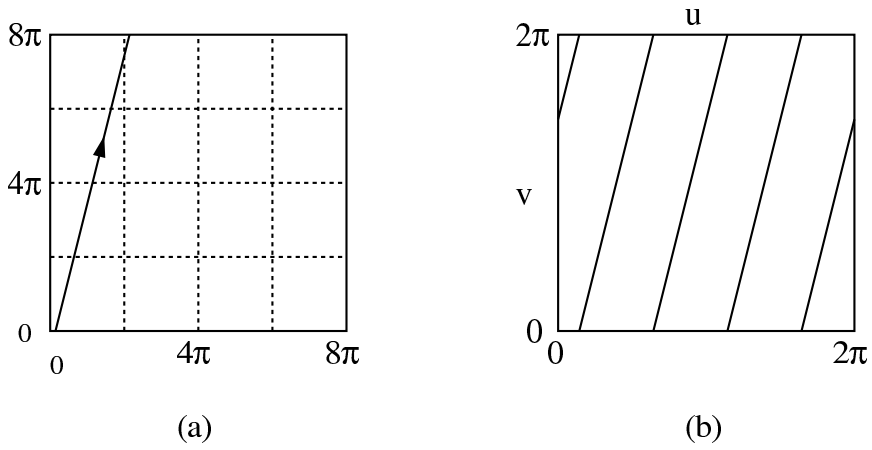

Figure 5.3: A trajectory in u and v for $\omega_y/\omega_x = 4$. v goes through four full cycles while u goes through one.

the curve closes at that point. Since u and v determine x and y, this means that the x-y trace of the trajectory also closes.

On the other hand, if ω_y/ω_x is irrational, the curve can never close. If it did, that would mean that at some instant of time, both u and v have changed by integral multiples of 2π, which immediately translates to $\omega_x t$ and $\omega_y t$ having changed by integral multiples of 2π, so that their ratio is rational.

In fact, more can be said. If the frequency ratio is irrational, the curve will pass arbitrarily near any point in the u-v square. One says that the orbit is **dense** in the square. Motion of this sort is called **quasi-periodic**. Although the trajectory may seem very complex from watching its trace in x and y, that complexity is illusory since it falls into two very simple independent pieces.

5.8 more things we can do

5.8.1 first order vector ODE's and exponentiation

In this section, we are going to study first order differential equations like this:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}. \tag{8.34}$$

This provides some useful practice in the manipulative techniques, and it's also important for its own sake, since equations of this sort arise in a variety of contexts in physics. The basic equation of time evolution in quantum mechanics has exactly this form, for example.

The equation of motion for oscillators can be recast in this form. For the one degree-of-freedom dampled harmonic oscillator we studied in chapter 3, the system of equations

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} \\ \frac{d\dot{x}}{dt} &= -\omega_0^2 x - \Gamma \dot{x} \end{aligned} \tag{8.35}$$

can be written in that way as

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\Gamma \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}. \tag{8.36}$$

This is a little different than what we were doing earlier because the vector now contains a velocity. That's the price we pay for making a first order equation. We will come back to this in a bit, but first let's work on a slightly different problem.

Suppose the point $\mathbf{r}$ in the x-y plane is rotating around the z-axis with angular velocity ω. Its velocity is perpendicular to $\mathbf{r}$ and $\hat{\mathbf{e}}_z$ (after all, it's going in a circle), and has magnitude $\omega|\mathbf{r}|$, so it is easy to see that the equation describing this motion is

$$\frac{d\mathbf{r}}{dt} = \omega \hat{\mathbf{e}}_z \times \mathbf{r}. \tag{8.37}$$

(Apparently positive ω corresponds to rotating counter-clockwise when viewed from above) Breaking $\mathbf{r}$ into components,

$$\hat{\mathbf{e}}_z \times \mathbf{r} = \hat{\mathbf{e}}_z \times (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y) = -y\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_y.$$

Written in terms of column vectors, this operation is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

So then, the operator $\omega\hat{\mathbf{e}}_z\times$ (the $\mathbf{A}$ of equation (8.34) for this case) becomes

$$\mathbf{A} \stackrel{def}{=} \omega\hat{\mathbf{e}}_z \times \text{“}=\text{”} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Now, if equation (8.34) were a scalar equation

$$\frac{dx}{dt} = ax,$$

you would immediately have the solution $x(t) = e^{at}x(0)$. The solution of the vector equation is no harder. It is

$$\mathbf{r}(t) = \exp(\mathbf{A}t)\,\mathbf{r}(0), \tag{8.38}$$

where the exponential of a matrix or linear operator is defined by the power series for e^x:

$$\exp(\mathbf{A}t) \stackrel{def}{=} \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n. \tag{8.39}$$

Proving equation (8.38) is a piece of cake. Differentiating the definition of the exponential,

$$\frac{d}{dt}e^{\mathbf{A}t} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{A}^n = \sum_{n=0}^{\infty}\frac{nt^{n-1}}{n!}\mathbf{A}^n = \sum_{n=0}^{\infty}\frac{t^{n-1}}{(n-1)!}\mathbf{A}^n$$

and setting $m = n-1$

$$\cdots = \mathbf{A}\left\{\sum_{m=0}^{\infty}\frac{t^m}{(m)!}\mathbf{A}^m\right\} = \mathbf{A}\,e^{\mathbf{A}t}. \tag{8.40}$$

So,

$$\frac{d}{dt}e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\,e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\mathbf{x}(t),$$

as promised.

Now, for the $\mathbf{A}$ in equation (5.8.1),

$$\mathbf{A}^2 = \omega^2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \omega^2\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -\omega^2\,\mathbf{1}. \tag{8.41}$$

So,

$$\mathbf{A}^{2n} = (-1)^n\omega^{2n}\mathbf{1}, \quad \mathbf{A}^{2n+1} = (-1)^n\omega^{2n}\mathbf{A}.$$

Packing these results into the exponential,

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{n=0}^{\infty}(-1)^n\frac{(\omega t)^{2n}}{(2n)!}\mathbf{1} + \sum_{n=0}^{\infty}(-1)^n\frac{(\omega t)^{2n+1}}{(2n+1)!}\,\mathbf{A}/\omega \\ &= \cos(\omega t)\mathbf{1} + \sin(\omega t)\mathbf{A}/\omega. \\ &= \begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}. \end{aligned} \tag{8.42}$$

Knowing all along that after time t the point has rotated through an angle ωt, you were probably expecting that result. But it's nice to see it emerge all by itself.

There is another way to work this out. That is to find eigenvectors for the operator $\mathbf{A}$. The eigenvalues of $\mathbf{A}$ are found from

$$0 = \det[\lambda \mathbf{1} - \mathbf{A}] = \begin{vmatrix} \lambda & -\omega \\ \omega & \lambda \end{vmatrix} = \lambda^2 + \omega^2,$$

which implies

$$\lambda = \pm i\omega.$$

The associated eigenvectors are easily worked out to be

$$\mathbf{v}_1 = \hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y = \begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ and } \quad \mathbf{v}_2 = \hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Exercise Pretend I hadn't told you what the eigenvectors were, and find them for yourself.

From either the original differential equation (8.34) or the solution (8.38), you can see that

$$\begin{aligned} e^{t\mathbf{A}}\mathbf{v}_1 &= e^{t\lambda_1}\mathbf{v}_1 = e^{i\omega t}\mathbf{v}_1 \\ e^{t\mathbf{A}}\mathbf{v}_2 &= e^{t\lambda_2}\mathbf{v}_2 = e^{-i\omega t}\mathbf{v}_2. \end{aligned} \tag{8.43}$$

We probably want to write down $\exp(\mathbf{A}t)\mathbf{r}$ in terms of the x and y components of $\mathbf{r}$. To get that, we should decompose $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ into linear combinations of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{aligned} \hat{\mathbf{e}}_x &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) \\ \hat{\mathbf{e}}_y &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2i}(\mathbf{v}_1 - \mathbf{v}_2). \end{aligned} \tag{8.44}$$

From these expressions, you easily get

$$\begin{aligned} e^{t\mathbf{A}}\hat{\mathbf{e}}_x &= \frac{e^{i\omega t}}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{e^{-i\omega t}}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos\omega t \\ -\sin\omega t \end{pmatrix} \\ e^{t\mathbf{A}}\hat{\mathbf{e}}_y &= \frac{e^{i\omega t}}{2i}\begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{e^{-i\omega t}}{2i}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix}. \end{aligned} \tag{8.45}$$

5.8.2 Green functions revisited

We have just solved the homogeneous equation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x} \quad \Rightarrow \quad \mathbf{x}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{x}(t_0).$$

Now it would be interesting to add a driving term to get

$$\frac{d}{dt}\mathbf{x}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{f}(t).$$

To solve this, we take our cue from an analoguos scalar equation again. Consider

$$\frac{dx}{dt} - ax = f(t),$$

where a is a constant. This is solved by use of an integrating factor. Rewrite it as

$$e^{at}\frac{d}{dt}\left(e^{-at}x\right) = f(t),$$

multiply through by the exponential out front and relabel to get

$$\frac{d}{ds}\left(e^{-as}x(s)\right) = e^{-as}f(s) \quad \Rightarrow \quad e^{-as}x(s)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-as}f(s)\,ds.$$

Finally,

$$\begin{aligned} x(t) &= e^{at}\left[e^{-t_0}x(t_0) + \int_{t_0}^{t} e^{-as}f(s)\,ds\right] \\ &= e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-s)}f(s)\,ds. \end{aligned}$$

Solving the vector equation

$$\frac{d}{dt}\mathbf{x}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{f}(t) \tag{8.46}$$

amounts to little more than using a different typeface because all the operations employed in solving the scalar equation are still valid. The solution is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-s)}\mathbf{f}(s)\,ds. \tag{8.47}$$

This can also be verified by substituting back into the differential equation, of course.

The equation of motion for the undamped ($\Gamma = 0$) harmonic oscillator (see equation 8.36) can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x}/\omega_0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x}/\omega_0 \end{pmatrix}. \tag{8.48}$$

Dividing $\dot{x}$ by ω makes it look just like the rotation problem we just considered, so the solution is immediate.

$$\begin{pmatrix} x(t) \\ \dot{x}(t)/\omega_0 \end{pmatrix} = \begin{pmatrix} x(t_0) \\ \dot{x}(t_0)/\omega_0 \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \cos\omega_0(t-s) & \sin\omega_0(t-s) \\ -\sin\omega_0(t-s) & \cos\omega_0(t-s) \end{pmatrix} \mathbf{f}(s)\, ds. \tag{8.49}$$

The sort of driving term used in chapter 3 was

$$\mathbf{f}(s) = \begin{pmatrix} 0 \\ f(s)/\omega_0 \end{pmatrix}.$$

If you put this into the integral, take the top component to find $x(t)$ and then let $t_0 \to -\infty$, you will find exactly the result we obtained earlier.

Exercise Fill in the small gaps in this last part. Why is $f(s)$ divided by ω_0?