Chapter 3

Linear Oscillators – One Degree of Freedom

3.1 Overview

What do these things have in common: propagation of seismic waves generated by earthquakes, remote detection of complex molecules via infrared spectra, the electric and magnetic fields in a microwave cavity, and grandfather clocks ? The humble **simple harmonic oscillator** plays a starring rôle in the understanding of all of these. Not just for springs anymore, Hooke's law even reaches into modern quantum field theory!

It is hard to overstate the importance of the harmonic oscillator as a basic concept. Many physical systems, such as the examples just mentioned, are usefully treated as one or more harmonic oscillators. The equations of motion of a simple harmonic oscillator (often abbreviated to SHO) are linear, making the mathematical analysis very easy. So easy in fact, that external driving forces and damping can be included, and the problem is still completely soluble. Best of all, with these ingredients a pretty good approximate treatment of many phenomena is possible.

Later, we will find out how to handle small oscillations in systems with many degrees of freedom. In this chapter, however, we will stick to one degree of freedom. First, we examine the idea of constructing linearized equations of motion near static equilibria. That's how the simple harmonic oscillator dynamics arise in the first place. Then, the solutions with damping are studied in detail in section 3.4. After a break to have a good look at things in phase space in section 3.5, the final ingredient, external driving forces, are added in section 3.6. Simple sinusoidal driving is considered first, revealing the phenomenon of resonance. Then arbitrary driving, using an important tool known as a Green function.

3.2 Static Equilibria: Stable and Unstable

Figure 3.1(a) depicts a more-or-less random potential in one dimension. The force on a particle in this potential is the derivative of $V(x)$, so the particle experiences no force at the

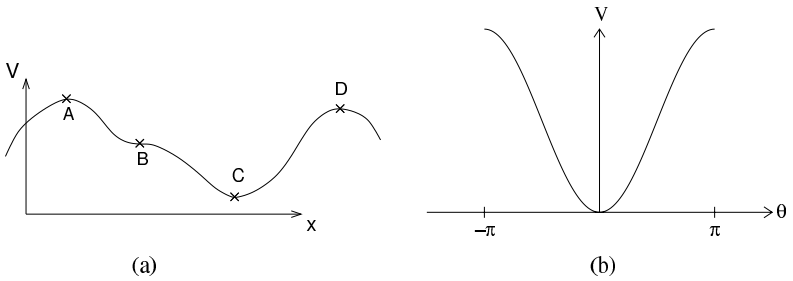

Figure 3.1: (a) A generic one-dimensional potential showing some static equilibria. (b) The potential for a plane pendulum. $\theta = 0$ is a stable equilibrium, $\theta = \pi$ and unstable one ($\theta = -\pi$ is exactly the same thing).

positions where the graph of V is flat. Such a position is called a **static equilibrium**, and the particle can remain at rest indefinitely at such a point. The labelled points in the figure are all static equilibria.

The fate of a particle placed near a static equilibrium, but not precisely at it, is at least as interesting. After all, it is surely impossible to get it precisely on the equilibrium (and at rest, to boot), and even if it were, tiny environmental disturbances would surely shift it off. So the question arises: if it's close to the equilibrium, does it stay close or does it move ever further away? There are several possibilities. If the equilibrium is a local minimum of the potential, as Point C in the figure, the resulting force will tend to push it back if it is displaced slightly. This is a **stable equilibrium**. If the equilibrium is a local maximum, as are points A and D, it is an **unstable equilibria**, because a slight displacement leads to a force which tends to push the particle yet further from the equilibrium. It is also possible to have points such as B, which are neither local minima nor maxima. Pushed one way, the particle experiences a force back toward equilibrium; pushed the other way, a force away from it. It is really unstable, though, as any displacement will result in its ending up on the downhill side.

3.3 Linearized Dynamics Near Static Equilibrium

Quite often, the potential near an equilibrium x_0 is smooth. In that case, a truncated Taylor series expansion can be made. We write

$$V(x_0 + \delta) \approx V(x_0) + V'(x_0)\,\delta + \frac{1}{2}V''(x_0)\,\delta^2. \tag{3.1}$$

The mere fact that x_0 is a static equilibrium means that the first derivative vanishes there, so this approximates the potential as a parabola, either opening up or opening down. It is possible that the second derivative of V also vanishes at x_0, buth that would be rather unusual, and we do not consider it further in this chapter.

For a body of mass m moving along a line, the equation of motion is found from the

force,

$$F(x_0+\delta) = -\frac{dV}{dx} \approx V''(x_0)\delta.$$

The result is

$$m\ddot{\delta} = V''(x_0)\,\delta.$$

This differential equation is linear and therefore very easy to solve. We'll return to it shortly.

This process of **linearization** can be carried out near a static equilibrium of a system with an arbitrary number of degrees of freedom, but for now only one degree of freedom is considered. However, we will deal with more general systems than particles moving in one Cartesian direction. For instance, consider a plane pendulum, with the potential shown in Figure 3.1(b). Clearly, $\theta = 0$ is a stable equilibrium and $\theta = \pi$ is unstable. Although the motion is one-dimensional, it is not along a line. We could use Newtonian methods, but since we now have that tool, we'll use the Lagrangian

$$L = \frac{m\ell^2}{2}\dot{\theta}^2 - mg\ell(1-\cos(\theta)).$$

Using

$$\begin{aligned} \cos\theta &= 1 - \frac{1}{2}\theta^2 + \cdots \\ \cos(\pi-\theta) &= -cos(\theta) \end{aligned}$$

the potential can be expanded around these two values of θ. Dropping a constant (a different one for each case),

$$L \approx \begin{cases} \frac{m\ell^2}{2}\dot{\theta}^2 - \frac{mg\ell}{2}\theta^2, & \theta \text{ near } 0, \\ \frac{m\ell^2}{2}\dot{\theta}^2 + \frac{mg\ell}{2}(\theta-\pi)^2, & \theta \text{ near } \pi. \end{cases} \tag{3.2}$$

For other one degree-of-freedom systems, the potential can generally be expanded in the same way, if it is smooth. Near an equilibrium of such a general system, which may as well be taken at $q = 0$,

$$V(q) \approx V(0) + \frac{1}{2}V''(0)q^2.$$

But, the kinetic energy might very well depend on generalized position as well as generalized velocity:

$$T(q,\dot{q}) = \frac{a(q)}{2}\dot{q}^2,$$

with $a(q) > 0$ a function of q. This calls for another Taylor series expansion. The constant term in the potential is essentially arbitrary and the first derivative vanishes at an equilibrium, so we must expand V to at least second order in q. For the kinetic energy, it should be adequate to take

$$a(q) \approx a(0).$$

The general idea is that both q and $\dot{q}$ are going to be 'small' so that the terms with the fewest powers of these quantities are the most important (unless the coefficient is zero!). So we keep the first nonvanishing terms in each case.

The approximation leaves us with a Lagrangian

$$L \approx \frac{a}{2}\dot{q}^2 + \frac{b}{2}q^2, \tag{3.3}$$

where $a > 0$ and b are constants. The resulting equation of motion is

$$\ddot{q} + \omega_0^2 q = 0, \quad \text{with } \omega_0^2 = \frac{|b|}{a} > 0 \quad \text{if } b < 0 \tag{3.4}$$

or

$$\ddot{q} - \lambda_0^2 q = 0, \quad \text{with } \lambda_0^2 = \frac{b}{a} > 0, \quad \text{if } b > 0. \tag{3.5}$$

These could be combined into a single equation, but it makes sense to deal with the stable and unstable cases separately. By making an approximation to the Lagrangian which is quadratic in coordinates and velocities, a linear equation of motion is obtained. And the reason that is nice is that nothing is easier to solve than a linear differential equation.

Question Which of these corresponds to a stable, and which to an unstable, equilibrium?

Exercise If you constructed a generic Lagrangian by making an expansion in powers of q and $\dot{q}$ without splitting into T and V, you might write

$$L = A + Bq + C\dot{q} + Dq^2 + Eq\dot{q} + F\dot{q}^2.$$

The C and E terms are a bit odd looking and not usual, but they can occur sometimes. Compute the equation of motion resulting from this.

You are probably used to writing the solution to Equation (3.4) in the form

$$q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

or

$$q(t) = A\cos(\omega_0 t - \delta),$$

where the angular frequency ω_0 (radians per second), is related to the ordinary frequency f (cycles per second) by

$$\omega_0 = 2\pi f.$$

You have to get used to the fact that both of these are called simply 'frequency.' The shift δ of the argument of the cosine is often referred to as the phase lag of the oscillator and accounts for the possibility that q may not reach its maximum value at $t = 0$.

On the other hand, you would write the general solution of Equation (3.4) as

$$q(t) = a_+ e^{\lambda_0 t} + a_- e^{-\lambda_0 t}, \tag{3.6}$$

for some constants a_+ and a_-. This looks (and is) very different, despite the fact that the differential equations only differ by a sign in one term.

Instead of the more familiar sines and cosines, we are going to use complex exponentials to solve Equation (3.4). Even if you've never seen it, you can verify in a few seconds that the general solution is

$$q(t) = a_+ e^{i\omega_0 t} + a_- e^{-i\omega_0 t}. \tag{3.7}$$

This looks much more like Equation (3.6), but don't be fooled. The exponential of an imaginary number always has modulus one, whereas the solution in equation (3.6) diverges either in the distant past or the distant future (or both).

Since $q(t)$ must be real, it is necessary that the two coefficients be complex conjugates,

$$a_- = a_+^*.$$

Since for a complex number z, $z + z^* = 2\mathrm{Re}z$, we could write the solution as

$$q(t) = 2\mathrm{Re}(a_+ e^{i\omega_0 t}).$$

For a_+ and a_- real, we recover the solution $\cos\omega_0 t$. In general $a_+ = re^{i\phi}$, where r is the modulus $|a_+|$. Then

$$a_+ e^{i\omega_0 t} = re^{i(\omega_0 t + \phi)},$$

and adding the complex conjugate yields

$$\cos(\omega_0 t + \phi).$$

That's how a phase shift of the oscillator appears in this way of writing things. Using the complex exponentials will be especially helpful when we discuss resonance and, later on, systems with many degrees of freedom.

3.4 The Damped Simple Harmonic Oscillator

As mentioned in the overview, damping is important in many real oscillators. Fortunately for us, damping can be reasoably modelled without spoiling the linearity of the equation of motion. The sort of damping force we will consider is

$$F_{\mathrm{damp}} = -\frac{\omega_0}{Q}\dot{q}, \quad 0 < Q. \tag{4.8}$$

This form is *not* dictated by any fundamental physical laws, but rather by mathematical convenience. It is a pretty good approximation to the dissipative forces suffered by a body moving through a fluid, however (look back at section 1.7). In many non-mechanical applications it is also a good representation. In some other cases it is only qualitatively correct.

Exercise Consider the block attached to a harmonic spring in Figure 3.2. Using the usual formula for sliding friction, make a plot of the frictional force versus velocity, and then versus time over a cycle or two. (Ignore the times when the block is momentarily stationary.) How is this even remotely like the damping force we are using?

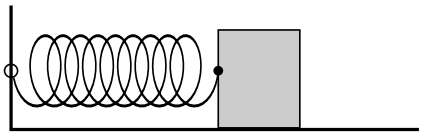

Figure 3.2: An oscillator experiencing a dissipative force.

With the addition of the damping force, our EOM becomes

$$\ddot{q} + \frac{\omega_0}{Q}\dot{q} + \omega_0^2 q = 0. \tag{4.9}$$

Question Do you think this EOM can be derived from a Lagrangian? If you try to reverse engineer one, the exercise in the previous section may be helpful.

Since the linearity of the equation of motion equation has not been destroyed by our tinkering, the solutions are still exponentials in the time, just as for the undamped case:

$$q(t) = e^{i\beta t}.$$

The difference is that the "frequency" β is now complex. For, substituting into Eq. (4.9), one gets the quadratic equation

$$-\beta^2 + \frac{i}{Q}\beta\omega_0 + \omega_0^2 = 0.$$

This has two roots,

$$\beta_\pm = \left[\pm\left(1 - \frac{1}{4Q^2}\right)^{1/2} + \frac{i}{2Q}\right]\omega_0. \tag{4.10}$$

Depending upon the value of Q, three very different kinds of behavior can arise because the square root changes from being real to imaginary at $Q = 1/2$. For $1/2 < Q$, The solutions are more-or-less oscillatory, but decay to zero at large times; this is called **underdamped.** For $Q < 1/2$, there is no oscillation, just an exponential decay; this is called **overdamped.** The borderline case, $Q = 1/2$ is referred to as **critically damped.** We are going to discuss all three of these cases individually, but you may want to have a look at the figures in MT Ch. 3 or Figure 3.3 to get a feel for the solutions first.

Exercise Make a picture showing the values of $\beta_\pm$ in the complex plane for these three regimes. Why is the imaginary part always positive?

3.4.1 underdamped: $Q > 1/2$

In the underdamped case, the general solution of Eq. (4.9) is

$$q(t) = \left(a_+ e^{i\omega' t} + a_- e^{-i\omega' t}\right) e^{-\omega_0 t/2Q}, \qquad \omega' = \omega_0\sqrt{1 - \frac{1}{4Q^2}}. \tag{4.11}$$

It's convenient to split $\beta_\pm$ into real and imaginary parts. The factor in parentheses is an oscillatory function just like that for the undamped oscillator, except that the frequency is shifted down by the damping, reaching zero at critical damping. This oscillatory part is multiplied by an envelope function $e^{-t/2Q}$ decaying to zero exponentially.

Exercise Sketch a rough plot of this solution.

3.4.2 overdamped: $Q < 1/2$

In the overdamped case, we take

$$\beta_\pm = i\lambda_\pm$$

and write the solution as

$$q(t) = a_+ e^{-\lambda_+ t} + a_- e^{-\lambda_- t} \tag{4.12}$$

with

$$\lambda_\pm = \frac{\omega_0}{2Q}\left[1 \pm \sqrt{1 - 4Q^2}\right] < 0. \tag{4.13}$$

There's no sign of any oscillation here. It might seem that $q(t)$ is destined to slide monotonically in to zero. Surprisingly, that is not true. $q(t)$ can have a turning point (but only one).

3.4.3 critically damped: $Q = 1/2$

Finally, if $Q = 1/2$, the roots $\beta_\pm$ of the quadratic equation are both equal to -1. That will yield a solution

$$q(t) = Ae^{-\omega_0 t}.$$

Since the EOM is a second order equation, we know there must be another solution. In fact, $te^{-\omega_0 t}$ also solves it.

Exercise Verify that last statement by explicit computation.

Thus, the general solution can be written as

$$q(t) = (C + D\omega_0 t)e^{-\omega_0 t}. \tag{4.14}$$

I've factored an ω_0 out of the second constant simply so that C and D have the same dimensions.

3.4.4 The Meaning of Q

Obviously enough, from the expression for the dissipative force, Q is concerned with energy dissipation, and the smaller Q the larger the force, so (one expects) the greater the rate of energy dissipation. We will make that relationship much more precise for the underdamped case.

Going back to the discussion just preceeding the linearized Lagrangian in Eq. (3.3), we can identify the kinetic and potential energies of our oscillator as

$$T = \frac{a}{2}\dot{q}^2 \qquad \text{and} \qquad V = \frac{b}{2}q^2.$$

Now adding them to get the total energy, instead of subtracting as we did in forming L,

$$E = \frac{a}{2}\dot{q}^2 + \frac{b}{2}q^2 = \frac{a}{2}\left(\dot{q}^2 + \omega_0^2 q^2\right).$$

For an underdamped oscillator we may write

$$q(t) = Ae^{-\omega_0 t/2Q}\cos(\omega' t - \delta).$$

There no point in carrying the phase shift δ around; we set it to zero by choosing $t = 0$ appropriately. The potential energy is then

$$V = \frac{a}{2}\omega_0^2 \cos^2(\omega' t)\, e^{-\omega_0 t/Q},$$

and the kinetic energy, after a bit of algebra and the double angle formula $\sin 2x = \sin x \cos x$, turns out to be

$$T = \frac{a}{2}\left[(\omega')^2 \sin^2(\omega' t) + \frac{\omega_0^2}{4Q^2}\cos^2(\omega' t) + \frac{\omega_0 \omega'}{2Q}\sin(2\omega' t)\right] e^{-\omega_0 t/Q}.$$

Now we are going to average these over one period of oscillation, but to do that, we'll have to assume that Q is large enough that the exponential factor doesn't change appreciably over one period. That will allow us to leave it out of the averaging. Since $\sin^2 \omega t$ and $\cos^2 \omega t$ average to $1/2$ over an entire period,

$$T_{\text{ave}} \approx V_{\text{ave}} \approx \frac{a}{2}\omega_0^2\, e^{-\omega_0 t/Q}.$$

To get T_{ave} to come out, it was necessary to substitute $(\omega')^2 = \omega_0^2(1 - 1/4Q^2)$. Both the kinetic and potential energies, averaged over a period are decaying exponentially with time, hence so is the total energy,

$$E \approx E(0)e^{-\omega_0 t/Q}.$$

Now the meaning of Q is clear:

$$\frac{dE}{dt} = -\frac{E}{Q} \qquad Q \gg 0. \tag{4.15}$$

Put into words, $1/Q$ is the fraction of energy lost in unit time. The following table gives approximate values of Q for a variety of oscillators. As you can see, values of Q in the hundreds are not so hard to come by.

Some reprentative Q values*

system	Q
50 g mass hanging on coil spring	25
Earth (seismic oscillations)	200
FM radio receiver	5000
tuning fork	10^4
Na atom (yellow spectral line)	5×10^7
superconducting rf cavity	10^10
Fe nucleus (gamma ray transition)	3×10^12

* from *Newtonian Dynamics* by R. Beirlein.

3.5 Phase Portraits

It's hard to exaggerate the value of visualizing dynamics not only in configuration space, but also in phase space. Unfortunately, it's also hard to exaggerate the difficulty of doing that with more than one degree of freedom! Fortunately that's all we have right now. Later we'll investigate some ways to picture more complicated systems.

The configuration and velocity of a system at any given time associates a point in phase space with the system. This point is sometimes called a **representative point**, and as time goes on it moves in phase space.[1] This evolution can be visualized with a **phase portrait.** Figure 3.3 shows phase portraits for an undamped an underdamped and an overdamped SHO. The idea of a phase portrait is to show enough of the orbits to see what's going on. The arrows indicate the direction along which the representative points move, but they aren't really needed since you could put them back in if they accidentally got erased.

Question How? I never got around to putting arrows on the overdamped phase portrait. Fill them in for me.

Let us temporarily agree to distinguish the velocity as a coordinate in phase space from the velocity as the actual time derivative of position along a dynamical trajectory by denoting the former as $\dot{q}$ and the latter as dq/dt. Then we write

$$\begin{aligned} \frac{dq}{dt} &= \dot{q}, \\ \frac{d\dot{q}}{dt} &= f(q,\dot{q}). \end{aligned} \tag{5.16}$$

The first equation here is a little bit silly, in a sense, but it's helpful to make the point. The left hand sides are the components of the *velocity in phase space* of a point moving along a dynamical trajectory (*not* the ordinary velocity). Since every point in the phase space can be a representative point, and the phase space velocity (Eq. 5.16) varies smoothly, it is as if the phase space were filled with a moving fluid. This is a so-called static flow because the fluid velocity at a given point never changes. After a time t, the flow carries point

[1] Personally I like to confuse the representative point with the system itself, but some people seem to think there is value in keeping the distinction.

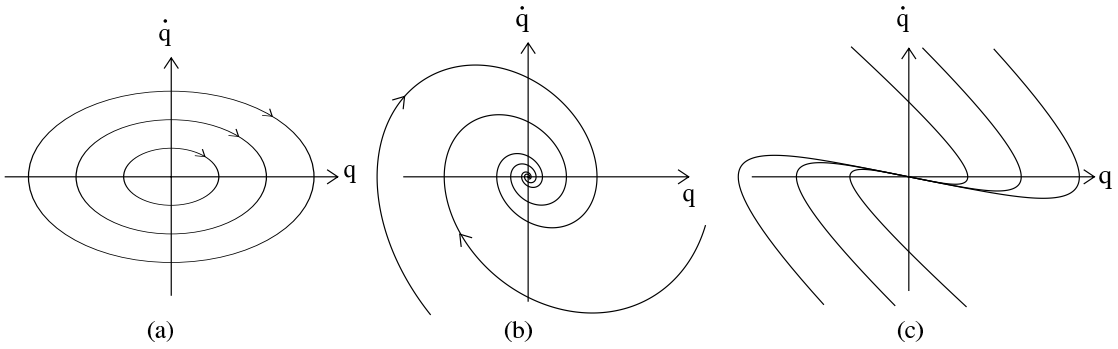

Figure 3.3: Phase portraits for (a) an undamped SHO, (b) an underdamped SHO with $Q = 2.0$, and (c) an overdamped SHO with $Q = 0.4$.

$x(0) = (q(0), \dot{q}(0))$ into $x(t) = (q(t), \dot{q}(t))$. But no matter what time the system starts at $x(0)$, it will reach $x(t)$ a time t later. The set of mappings which describes where all the phase space points go after any interval of time is called a **phase flow**. We formalize that in the following way. The point in phase space into which x is carried after time t is denoted by[2] $\phi(x,t)$. The dynamical trajectory which passes through x at time $t = 0$ is precisely $\phi(x,t)$ and

$$\phi(x(t), t') = \phi(x, t + t') \tag{5.17}$$

is the expression of the invariance property mentioned above. We will investigate the fluid analogy to phase space flow more in chapter 6.

Exercise The phase portrait of the undamped oscillator can be drawn without explicitly solving the equations of motion. How is that done? Why doesn't it work for the damped oscillator?

Question By introducing an extra independent variable, $\dot{q}$, we were able to rewrite a second order differential equation as a system of first order equations (5.16). Can you see how to write even higher order differential equations as systems of first-order equations?

At the beginning of this chapter, we studied the notion of stable and unstable static equilibria. When we look at things in phase space, there are a couple of new concepts of stability which naturally suggest themselves. For all three phase portraits in figure 3.3, $q = \dot{q} = 0$ is a **fixed point**, that is a point which does not move under the time evolution. It is an orbit consisting of a single point. Points which are close to that one move toward it for both the damped cases. This is **asymptotic stability** of the fixed point. A fixed point x_0 is asymptotically stable if it has some neighborhood U such that all points in U eventually tend to x_0

$$x \in U \Rightarrow F(x,t) \to x_0 \quad \text{as } t \to \infty.$$

[2]The letter F is overworked, so we'll write 'phi' for 'phlow'.

A **neighborhood** of a point x in phase space is any set which contains all points y within δ in both position and velocity, for some $\delta > 0$. In fact, the fixed point for both damped cases is better than asymptotically stable because not just a neighborhood of it goes to the fixed point, but the entire phase space.

The fixed point for the undamped flow is not asymptotically stable. Other points in the phase plane are carried eternally round and round on ellipses and never fall in toward the center. However, they don't get any further away either. This is a kind of stability, too, but a somewhat weaker one. It is called **Liapunov stability.** A fixed point x_0 is Liapunov stable if given any neighborhood U of x_0, there is another neighborhood V, possibly smaller, such that points which start in V never escape from U:

$$x \in V \Rightarrow F(x,t) \in U, \quad \text{for all } t > 0.$$

For the simple case of the undamped oscillator, the neighborhoods U and V can be taken to be identical.

3.6 The Periodically Driven SHO

Physics, like life, is made interesting by interactions. As the phase flow in Figure 3.3(b) illustrates, a damped but undriven SHO is actually a pretty boring thing. All the motion quickly damps out and then it just sits there. Since most oscillators encountered in the world are not only damped, but also influenced by external forces, we'd do well to look into that. The simplest sort of interaction between an oscillator and the rest of the world is a pre-determined **driving force** $F(t)$, which enters the equation of motion as

$$\ddot{q} + \frac{\omega}{Q}\dot{q} + \omega_0^2 q = F(t). \tag{6.18}$$

The oscillator responds to the external world, but any influence going the other way is ignored. Since $F(t)$ doesn't actually have the dimensions of a force, I will try to refer to this as a **driving term** instead, so that we don't get confused. (The other terminology is quite prevalent however.) In §3.6.1 we will consider a special, but important, kind of driving, namely a sinusoidally varying force. Then in §3.7 we'll take up the question of arbitrary driving forces.

Exercise How would the Lagrangian need to be modified in order to get this EOM? (You'd better do the undamped case, since that's the only one we have a Lagrangian for!)

3.6.1 Resonance

A sinusoidal driving force has the form

$$F(t) = F_0 \cos(\omega t + \delta) = 2\mathrm{Re}(Fe^{\omega t}), \tag{6.19}$$

where

$$F = |F|e^{i\delta}. \tag{6.20}$$

Since the equation of motion is linear and all the constants are real, solving it with a complex driving term and then taking the real part gives the same solution as results from taking the real part of the driving term. This follows from taking the equation

$$\ddot{q} + \frac{\omega_0}{Q}\dot{q} + \omega_0^2 q = g(t),$$

and adding or subtracting its complex conjugate. Since it is rather simpler, we are therefore going to actually solve the problem with

$$F(t) = Fe^{\omega t},$$

and only take the real part at the very end.

With that understanding, the equation of motion of the oscillator becomes

$$\ddot{q} + \frac{\omega}{Q}\dot{q} + \omega_0^2 q = Fe^{i\omega t}. \tag{6.21}$$

To solve this, you might guess

$$q_0(t) = Ae^{i\omega t},$$

reasoning that the oscillator has got to move with the same frequency as the driving. Insering it into Eq. (6.21), an algebraic equation results, which you can solve for A:

$$A = \frac{F}{\omega_0^2 - \omega^2 + i\omega_0\omega/Q}. \tag{6.22}$$

Although this does yield a solution, it is a **particular solution** and not the general solution. We can always add a solution of the undriven equation to $q_0(t)$ and it will still be a solution. Generally, what should be added to the particular solution is determined by the initial conditions. If there is any damping, this is anyway not so interesting because the undriven solution quickly decays away, leaving just the driven response $q_0(t)$.

Unless $1/Q = 0$, which is the case of no damping, the factor

$$R \stackrel{def}{=} \frac{A}{F}$$

which relates the amplitude of the oscillations to the amplitude of the driving is a complex number. What is the significance of R being complex in general? It means that the oscillator is not necessarily in phase with the driving force. When the force is at its maximum, the oscillator may be at its maximum displacement, or maybe its at zero, or maybe somewhere else. So let's write R as

$$R = |R|e^{i\phi}. \tag{6.23}$$

Then, the (real) response of the oscillator is

$$q(t) = \mathrm{Re}(RFe^{i\omega t}) = |R||F|\cos(\omega t + \delta + \phi). \tag{6.24}$$

Clearly, the phase of R as a complex number is equal to the relative phase between the driving and the oscillator's response. Graphs of the amplitude and phase lag of the oscillator

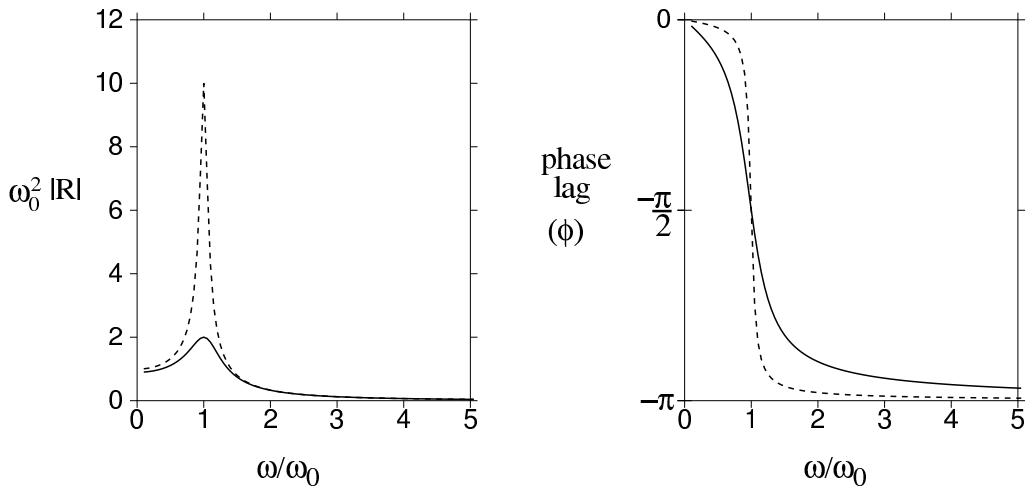

Figure 3.4: The amplitude (left) and phase lag (right) of an oscillator subjected to a sinusoidal driving force. ('R' stands for 'response') The solid curves are for $Q = 2$ and the dashed, for $Q = 10$.

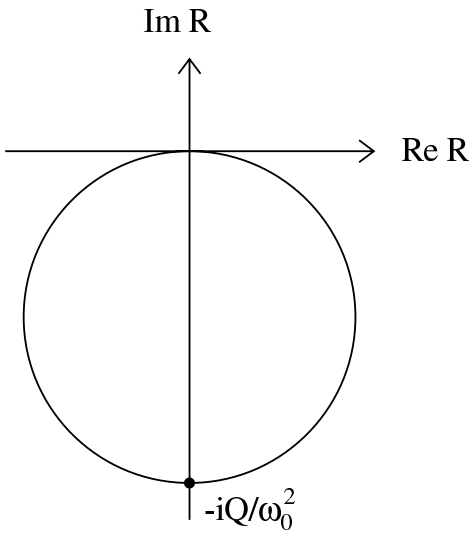

Figure 3.5: The complex response R of a damped oscillator.

response are shown in Fig. 3.4.

Exercise Extend the curves to $\omega < 0$.

The striking thing about the plots is that as Q becomes very large, there is an ever sharper peak in the amplitude of the response near $\omega = \omega_0$ accompanied by an ever more sudden drop of the phase shift from 0 to $-\pi$. From Eq. (6.21), the response of an oscillator driven precisely at its natural frequency is

$$R = -i\frac{Q}{\omega_0^2}, \qquad \omega = \omega_0,$$

and this diverges as $Q \to \infty$.

Question Does this mean that if we start driving a nearly undamped oscillator at its natural frequency, large amplitude motion will appear instantly ? If not, how is that reconciled with the last equation? (Hint: check the time dependence of our driving force.)

This divergence as the damping disappears is probably not terribly surpising to you; what might be is that the response is finite for ω away from ω_0 even in the absence of damping. When $\omega = \omega_0$, we say that the oscillator is being driven *at resonance*, and the sharp features of the response for large Q and $\omega \approx \omega_0$ are known as **resonance**. You are surely familiar with the phenomenon from pushing somebody on a swing. If all the pushes come at the right time, which is to say in phase with the natural oscillations, then they can reinforce each other. The amplitude of the motion builds until it is halted by the damping.

Question How does the damping make the amplitude stop growing? Does it become stronger, and if so why?

Exercise The amplitude is not maximized precisely at resonance, but slightly off. The shift of the maximum goes to zero as $1/Q \to 0$. Compute the driving frequency which results in maximum $|R|$.

Clearly, things are most interesting very close to resonance, particularly when Q is large. So we will suppose that ω/ω_0 does not differ much from one, and make some appropriate approximations. Let us write

$$\omega = \omega_0 + \delta\omega.$$

Keeping only the first non-zero term in each of the real and imaginary parts, the denominator of the complex response amplitude Eq. (6.22) becomes

$$\begin{aligned} \omega_0^2 - \omega^2 + i\omega_0\omega/Q &= \omega_0^2 - (\omega_0 + \delta\omega)^2 + i\omega_0(\omega_0 + \delta\omega)/Q \\ &\approx 2\omega_0\delta\omega + i\omega_0^2/Q \\ &= 2(\omega_0 - \omega) + i\omega_0^2/Q. \end{aligned}$$

Putting this expression back into Eq. (6.22) and multiplying through by the complex conjugate of the denominator,

$$R \approx \frac{1}{2\omega_0^2}\left[\frac{1 - \omega/\omega_0 - i/2Q}{(1 - \omega/\omega_0)^2 + 1/4Q^2}\right]. \tag{6.25}$$

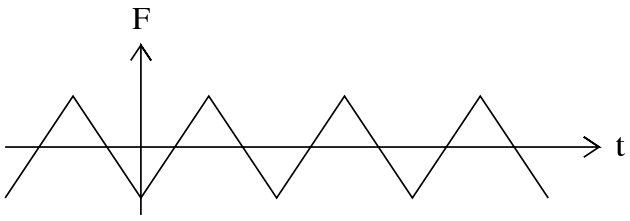

Figure 3.6: A periodic forcing function

This approximate complex response amplitude is plotted in Figure 3.5. It traces out a circle of radius $Q/2\omega_0^2$, but it is very nearly zero except for ω very near to ω_0.

Exercise Try to visualize a concrete forced oscillator as the driving frequency goes through the natural frequency. See, in your mind's eye what the plot for R shows.

3.6.2 general periodic driving: Fourier series

Periodic driving of an oscillator can take many shapes. It certainly need not be perfectly sinusoidal. For instance, one may have a sawtooth force as in Figure 3.6. Once we have solved the problem with a sinusoidal driving force, however, the solution with any periodic driving is not far behind. This hinges on two facts.

The first is the linearity of the equation of motion. Suppose that $q_1(t)$ and $q_2(t)$ are solutions under the driving forces $F_1(t)$ and $F_2(t)$, respectively:

$$\ddot{q}_1 + \frac{\omega}{Q}\dot{q}_1 + \omega_0^2 q_1 = F_1(t),$$
$$\ddot{q}_2 + \frac{\omega}{Q}\dot{q}_2 + \omega_0^2 q_2 = F_2(t).$$

Then, by adding the two equations together,

$$(\ddot{q}_1 + \ddot{q}_2) + \frac{\omega}{Q}(\dot{q}_1 + \dot{q}_2) + \omega_0^2(q_1 + q_2) = F_1(t) + F_2(t),$$

you can see that $q_1(t) + q_2(t)$ is a solution with driving force $F_1(t) + F_2(t)$. As a result, if we know the response to some building block forces, the response to any force which is a superposition of those can be immediately computed.

The second fact is that any periodic driving can be written as a sum of sines and cosines, or alternatively complex exponentials. This representation is known as a **Fourier series.** Without proof, here is the way such things are handled. Let $f(t)$ be any function of period

T, so that $f(t+T) = f(t)$. Then,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \tag{6.26}$$

where

$$\omega = 2\pi/T,$$

and the **Fourier coefficients** are

$$c_n = \frac{1}{T}\int_0^T e^{-in\omega t} f(t)\, dt. \tag{6.27}$$

Since the driving term is certainly real, the complex conjugate of this equation shows that

$$c_{-n} = c_n^*.$$

The complex exponentials we are using here exhaust all the sinusoidally varying functions which have an integral number of periods over the interval T. The expansion provides a decomposition of any periodic force into shorter and shorter wavelength components. Roughly speaking, the very short wavelength pieces correspond to sharp features of the original driving force. Insofar as it has no sharp features, we can reasonable expect to be able to neglect all but a few terms of the series. The sawtooth curve has a sharp feature: the places where it turns around. The Fourier coefficients for the sawtooth in the figure is

$$c_n = \begin{cases} A/n^2, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Exercise If you are familiar with Fourier series written in terms of sines and cosines (as in eqs. 3.102 of MT), convince yourself that what's written here is the same thing by combining Equations (6.26-6.27) with their complex conjugates.

3.6.3 Extended Phase Space

The undriven harmonic oscillator, damped or not, has a certain symmetry with respect to time translations. If it has a given position and velocity at time t, its position at time $t+T$ does not depend upon what particular time t was. This property manifests itself in the differential equation governing the system by the absence of any explicit time dependence. A differential equation of that sort, and the dynamical system described by such an equation, is called **autonomous.** If there is an explicit time dependence, the equation (or system) is called **nonautonomous.** The latter is the case for the driven oscillator. It is no longer enough to know, for instance, that $q = \dot{q} = 0$ now in order to determine the state of the system ten minutes from now. We also need to know at what point in its cycle the driving force is. But which cycle it is in doesn't matter. If the driving is specified as a function with a period of one hour, we only need to be able to see the minute hand of the clock (as well as check the position and velocity) in order to predict the future evolution of the system.

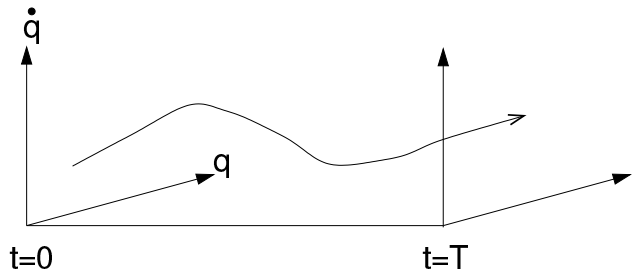

Figure 3.7: extended phase space

For a nonautonomous system, the phase space built on just q and $\dot{q}$ is no longer adequate for one of its major purposes. For an autonomous system, a point x in phase space is carried by the flow into point $\phi(x,t)$ after a time t, regardless of when it started at x. There is precisely one orbit through each point of phase space. In order to regain it that property, we enlarge the phase space, to what is called **extended phase space.** The idea here is illustrated in Figure 3.7. We add an extra dimension corresponding to time, and it extends through one full cycle of the driving force. Now we have what we want. A point in this extended phase space corresponds to a position, a velocity and a point in the cycle of the driving force. Thus, there will be only one orbit through that point. Notice that the $q - \dot{q}$ plane at $t = T$ is to be identified with the plane at $t = 0$. The extended phase space is therefore bounded in the time direction and is more like a circle than a line segment. However, that's hard to draw, so we just have to remember the fact. The phase space of an autonomous system has twice as many dimensions as degrees of freedom, since each degree of freedom brings a position and velocity (sometimes there are constraints which make this not quite true). Since the extended phase space for a one degree of freedom system has three dimensions, it is sometimes said to have 'one and a half' degrees of freedom.

An extended phase space can be made for the undriven system, too, if we like. For the undamped oscillator, the orbits look like helices; in case T is the natural period of the oscillator, they are closed, going through the same point at $t = T$ as at $t = 0$. For a driven system, closed orbits are the exception, not the rule.

Exercise Try to imagine and/or sketch the orbits in extended phase space for an undamped SHO both undriven and driven at resonance.

3.7 Arbitrary Driving Force: Green Functions

Quite often, we are interested in a force $F(t)$ which is zero up until some time t_0, and the solution that is wanted is one with $q(t) = 0$ for $t < t_0$. Such initial conditions (and the Green function associated with them, see below) are generally called **causal,** because

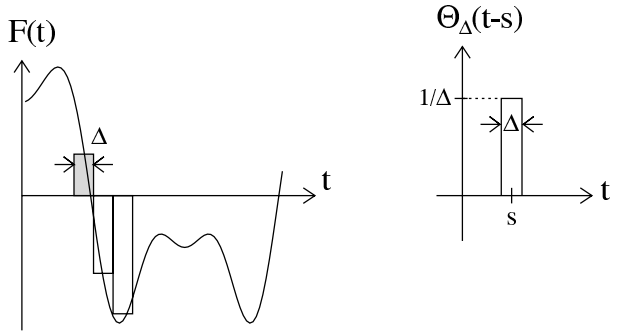

Figure 3.8:

nothing happens until the cause kicks in.

Remarkably, linearity of the driven oscillator equation allows fairly straightforward solution for an *arbitrary* driving force! The Green function technique we will develop can be applied to any inhomogeneous linear differential equation. The key is illustrated in Fig. 3.8. Slice time up into bins of duration Δ centered on $t_n = n\Delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ and approximate $F(t)$ by flattening it out over each of those bins. To refine this idea, define the square pulse of duration Δ and height $1/\Delta$:

$$\Theta_\Delta(s) = \begin{cases} 1/\Delta & \text{if } \Delta/2 \le s < \Delta/2, \\ 0 & \text{otherwise,} \end{cases}$$

Then you can approximate $F(t)$ by $F_\Delta(t)$, defined as

$$F_\Delta(t) = \sum_n \overline{F}_n \Theta_\Delta(t - t_n)\,\Delta = \text{a sum of square pulses,} \tag{7.28}$$

where

$$\overline{F}_n = \frac{1}{\Delta} \int_{\text{n-th bin}} F(t)\,dt$$

is the average of $F(t)$ over the n^{th} bin $(n-1/2)\Delta \le t < (n+1/2)\Delta$. As Δ becomes smaller, the approximation becomes better. There are peculiarities to the limit $\Delta \to 0$, so a slightly roundabout way of getting there is needed.

Denote the causal response to $\Theta_\Delta(t)$ by $G_\Delta(t)$, so that

$$\ddot{G}_\Delta(t) + \frac{\omega}{Q}\dot{G}_\Delta(t) + \omega_0^2 G_\Delta(t) = \Theta_\Delta(t). \tag{7.29}$$

This does not in any way depend upon $F(t)$. The response to F_Δ is exactly

$$q_\Delta(t) = \sum_n \overline{F}_n\, G_\Delta(t - t_n)\,\Delta. \tag{7.30}$$

The limit can be taken safely here, with the result

$$q(t) = \int_{-\infty}^{\infty} F(s)G(t-s)\,ds. \tag{7.31}$$

$G(t)$ is called a **Green function.**[3] In this case, it's a *causal* Green function. The formula looks nice, but it's not much use until $G(t)$ is actually calculated. We will do that, but it involves a digression, so to to relieve the suspense, here is the formula:

$$G(t) = \frac{\sin(\omega' t)}{\omega'} e^{-\omega_0 t/2Q}. \tag{7.32}$$

Question What are the dimensions of G according to this formula? Is that what it should be?

Now, to find G, we apply the differential operator $d^2/dt^2 + (\omega_0/Q)d/dt + \omega_0^2$ to both sides of Equation (7.31). On the left-hand side, you get simply $F(t)$, so that we ought to have

$$F(t) = \int_{-\infty}^{\infty} \left(\frac{d^2}{dt^2} + \frac{\omega_0}{Q}\frac{d}{dt} + \omega_0^2\right) G(t-s)F(s)\,ds.$$

In other words, we want

$$\left(\frac{d^2}{dt^2} + \frac{\omega_0}{Q}\frac{d}{dt} + \omega_0^2\right) G(t-s) = \delta(t-s), \tag{7.33}$$

where $\delta(t-s)$ is a function such that

$$\int_{-\infty}^{\infty} \delta(t-s)F(s)\,ds = F(t)$$

no matter what $F(t)$ is. But that's crazy; there's no function with that property.

3.7.1 Detour: The Dirac Delta "Function" and other Distributions

Obviously, $\delta(t-s)$ must be zero for $t - s \neq 0$ else the answer would surely depend on $F(s)$ for values other than t. But a function which is zero everywhere but one point has a zero integral. Actually, there really is such a thing, it is called a Dirac delta function (though it isn't really a function), and it is given by

$$\delta(t) = \frac{dH}{dt} \tag{7.34}$$

where

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases} \tag{7.35}$$

[3] After the Irish mathematician George Green of Green's Theorem fame.

is the Heaviside step function. The meaning of the differentiation here requires some elucidation.

To see what it's all about, compute

$$J = -\int_{-\infty}^{\infty} H(t)\frac{df(t)}{dt}\,dt. \tag{7.36}$$

In order that this make sense, we insist that f be once differentiable and also that it be identically zero outside of some bounded region. (This property is referred to as f having *bounded support*) Then,

$$J = f(0),$$

as you can check for yourself. We can integrate by parts — almost. The fact that f is zero for large $|t|$ means that there are no surface terms and we would have

$$J = \int_{-\infty}^{\infty} \frac{dH(t)}{dt} f(t)\,dt,$$

except for the fact that the derivative of the step function does not exist at $t = 0$. If we promise to use it only inside integrals like that however, we may perfectly well *define* the derivative dH/dt by the right-hand side of Eq. (7.36). That is exactly what we shall do, with the aim of being able to use our old rules of differentiation on a wider class of objects. For any function h whatever, whether smooth or with a discontinuity like the step function, we will define dh/dt "in the sense of distributions" by

$$\int \frac{dh(t)}{dt} f(t)\,dt = -\int h(t)\frac{df(t)}{dt}\,dt, \tag{7.37}$$

where $f(t)$ is any differentiable function of bounded support. This was already true for differentiable functions h, so it does not conflict with the original notion of derivative, but extends it. As stated, this does not tell us what dh/dt means, standing alone, but only if it is inside an integral. It turns out to be the case that if h is actually a function with continuous derivative, this equation does determine dh/dt completely. But it is probably not obvious, and I won't prove it. But it also allows us to assign a meaning to the derivatives of things which didn't have them under the old, narrow definition.

Now, by a change of variables (in the first and last steps) we may compute

$$\frac{d}{dt}\int H(t-s)f(s)\,ds = -\frac{d}{dt}\int H(s)f(t-s)\,ds = \int H(s)\frac{df}{dt}(t-s)\,ds = -\int H(s)\frac{df}{ds}(t-s)\,ds$$

by the old rules of the game. The new definition makes this last integral equal to

$$\int \frac{dH(s)}{ds} f(t-s)\,ds,$$

and with a final change of variable, the beginning and end of this chain of equations reads

$$\frac{d}{dt}\int H(t-s)f(s)\,ds = \int \frac{dH}{dt}(t-s)f(s)\,ds. \tag{7.38}$$

So, we can take the derivative under the integral sign just as if H were a well-behaved function.

That is how the formula

$$\delta(t) = \frac{dH}{dt}$$

is to be understood. At an intuitive level, it makes some sense. Clearly the derivative of $H(t)$ is zero away from $t = 0$ since H is flat in those regions. Since it takes a finite jump from zero to one over an infinitesimal interval at $t = 0$, dH/dt is somehow infinite at zero and the properties we've ascribed to this object do not seem so crazy. That is the basic idea, but some work (which we just sketched) is required to make that vague feeling hold water.

Now that the delta function is in hand, we may go further and define its derivative by precisely the same trick:

$$\int \frac{d\delta(t)}{dt} f(t)\, dt = -\int \delta(t) \frac{df(t)}{dt}\, dt = -\frac{df}{dt}\Big|_{t=0}.$$

Question Do you see the pattern? Try to continue it to get $d^2\delta/dt^2$, etc.

The delta function and its derivatives are examples of **distributions** or **generalized functions** as they are also known (the latter term being that traditionally favored in the Russian literature). We'll discuss these things briefly before getting back to our Green function.

Given a function $f(x)$, we have been discussing the procedure which takes as input another function $g(x)$ and gives as output the number

$$I^f[g] = \int_{-\infty}^{\infty} f(x)g(x)\, dx. \tag{7.39}$$

We've bumped into things like this before: $I^f[\cdot]$ is a functional. But it has a special property — it is linear. To say that I is a linear functional means that

$$\begin{aligned} I[g+h] &= I[g] + I[h], \\ I[\alpha g] &= \alpha I[g], \end{aligned}$$

where α is a number (real or complex). A functional with this linearity property is called a **distribution.** Clearly,

$$g \mapsto \int \delta(t)g(t)\, dt = g(0).$$

is another linear functional. It is a theorem that any generalized function can be written as a sum of ordinary functions, along with delta functions and their derivatives. The practical usefulness of these things stems largely from the fact that they can be manipulated much as ordinary functions and allow us to differentiate with wild abandon when it would be forbidden under a strict function interpretation.

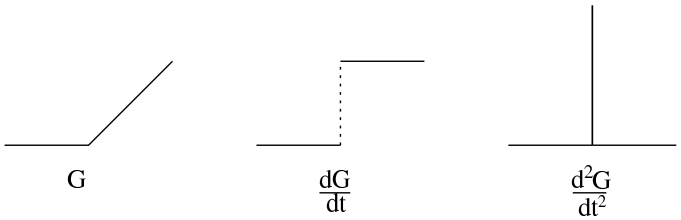

Figure 3.9: The behavior of the Green function $G(t)$ and its derivatives very near $t = 0$. G and dG/dt are accurately portrayed, but the picture of d^2/dt^2 must be taken with a grain of salt. It's not really a function, but inside integrals behaves very like functions of the depicted shape.

3.7.2 Back to Green Functions

At long last, we return to our Green function and see why this is of use. First, you should notice that the Green function was designed from the very beginning to sit inside an integral with a forcing function. So, the derivatives we took in Eq. (7.33) were in the distribution sense whether we realized it or not.

Equation (7.33) tells us the following things:

- Away from $t = 0$, $G(t)$ satisfies the homogeneous oscillator equation.
- At zero, $G(t)$ has a kink, so that its derivative $\dot{G}$ can have a step-function discontinuity, so that its derivative $\ddot{G}$ contains a delta function of unit strength.

The shape of the function $G(t)$ is shown in figure 3.9. Life is not so difficult. $G(t)$ is zero for $t < 0$. All we need to do is find a solution to the oscillator equation for $t \geq 0$ which is zero at $t = 0$ but has slope one there. Since the general solution is

$$G(t) = (a_+ e^{i\omega' t} + a_- e^{-i\omega' t}) e^{-\omega_0 t/2Q},$$

with derivative

$$\dot{G}(t) = \left[\left(i\omega' - \frac{\omega_0}{2Q} \right) a_+ e^{i\omega' t} + \left(-i\omega' - \frac{\omega_0}{2Q} \right) a_- e^{-i\omega' t} \right] e^{-\omega_0 t/2Q},$$

a very short calculation allows us to solve for a_+ and a_- with the result

$$a_- = -a_+ = \frac{1}{2i\omega'}. \tag{7.40}$$

Whence finally, the formula in Eq. (7.32).

Exercise Finish that calculation. Find the coefficients a_+ and a_- so that $G(t)$ satisfies the homogeneous equation away from $t = 0$, has $G(0+) = G(0-) = 0$ and $dG(0+)/dt =$

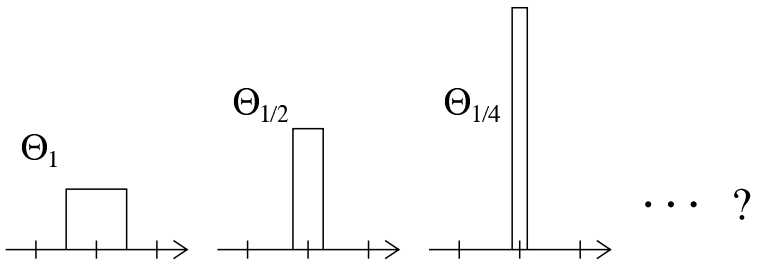

Figure 3.10: A delta sequence

$1 + dG(0-)/dt = 1$. (The argument $0\pm$ means the limit as $t \to 0$ from above $(+)$ or below $(-)$ side.

Precisely the same procedure can be applied to solve inhomogeneous linear differential equations of any order, if we have solutions of the homogeneous equation. For instance, consider

$$c_n(x)\frac{d^n y}{dx^n} + c_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} \ldots + c_0(x)y = f(x). \tag{7.41}$$

This is solved by

$$y(x) = \int_{-\infty}^{\infty} G(x-z)f(z)\,dz,$$

where the Green function $G(\cdot)$ satisfies the equation

$$c_n(x)\frac{d^n G}{dx^n} + c_{n-1}(x)\frac{d^{n-1}G}{dx^{n-1}} \ldots + c_0(x)G = \delta(x).$$

With the solutions to the homogeneous equation, G can be found very easily. This can be quite useful if you have many (or an unknown) f's for which you wish to solve the equation (7.41). That Green function is constructed by pasting together a solution to the homogeneous equation in the region $t \leq 0$ and one in the region $t \geq 0$. The derivatives for the left and right hand pieces agree up to $d^{n-2}G/dx^{n-2}$, but there is a discontinuity of the $(n-1)$-st derivative of size $1/c_n(0)$.

We have basically defined the delta function as the derivative of a step. There are other ways to get at it. In particular, it can be approximated by genuine functions. A sequence of better and better approximations to a delta function is called a delta-sequence. An example is shown in Figure 3.10 The functions Θ_Δ are our friends from the previous subsection. If $f(x)$ is a smooth function

$$\int \Theta_\Delta(x)f(x)\,dx \approx f(0),$$

and the smaller Δ, the closer this will be to an equality. That is the sense in which they approximate a delta function. There are other delta-sequences. Clearly the precise shape is not important. The only thing which matters is that the region where they're nonzero becomes more and more tightly concentrated around zero and the total area under each is one. Sometimes, when dealing with a very sharp spiky signal, it is useful to pretend that it is a delta function so as to avoid worrying about the precise shape. That is an application of this approximation idea.

Exercise Sketch some more delta-sequences.

3.7.3 Green function for the Laplacian

To illustrate another use of distributions, and get a useful result in the bargain, we will establish the formula

$$\nabla^2 \left(\frac{1}{r}\right) = 4\pi\delta^3(\mathbf{r}). \tag{7.42}$$

This provides another proof of the Gauss flux theorem and the Poisson equation. First, it would help to explain what $\delta^3(\mathbf{r})$ means. This is a three-dimensional version of the delta function. If it is put inside an integral over volume with a smooth function g, the integral evaluates to the value of g at the origin:

$$\int \delta^3(\mathbf{r})\, g(\mathbf{r}) d^3r = g(0).$$

We could also write $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Here is how it proves the Poisson equation. The gravitational potential due to a distribution of mass $\rho(\mathbf{r})$ is

$$\Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'.$$

Then

$$\nabla^2\Phi(\mathbf{r}) = -G \int \left(\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \rho(\mathbf{r}')\, d^3r' = -G \int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\, d^3r'.$$

Voila!

Away from $\mathbf{r} = 0$, it is straightforward to calculate $\nabla^2(r^{-1})$, and the result is zero.

Exercise Perform that computation.

At $\mathbf{r} = 0$ it cannot be done because the derivatives don't exist there. However, knowing that $\nabla(r^{-1})$ is zero everywhere else, it is easy to find its value at the origin. We just compute

$$\int \nabla^2 \left(\frac{1}{r}\right) g(\mathbf{r})\, d^3r,$$

where now the factor $\nabla^2(r^{-1})$ must be interpreted as a distribution because it doesn't exist as an old-fashioned function at the origin. Since that factor is at any rate zero away from

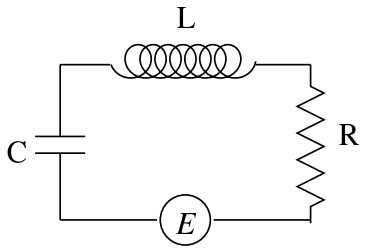

Figure 3.11: A series LRC circuit containing capacitance C, resistance R and inductance L, driven by an emf $\mathcal{E}$

$\mathbf{r}$ we might just as well take g to be one for $r < 1$ and zero for $r > 1$, like a Heaviside step function. Then the integral becomes

$$\cdots = \int_{r<1} \nabla^2 \left(\frac{1}{r}\right) d^3r,$$

and an application of Green's theorem yields

$$\cdots = \int_{r=1} \nabla \left(\frac{1}{r}\right) \cdot \hat{\mathbf{n}}\, dA.$$

Since $\nabla(1/r) = -\hat{\mathbf{r}}/r^2$, and the surface area of the unit sphere is 4π, the value of this integral is -4π. ■

Admission: The use of Green's theorem is justifiable, but it's slightly sleazy.

3.8 LRC Circuits

A nice example of a one degree of freedom linear oscillator is provided by series LRC circuits, such as the one depicted in Figure 3.11.

As you no doubt recall, an inductor is essentially a coil of wire. Current going through it produces a magnetic field. If the current is changing, so is the magnetic flux through the center of the coil and this results in an emf according to Faraday's Law, which is proportional to dI/dt, the proportionality constant L being a property of the device and called inductance. The voltage drop across the resistor is proportional to the current through it and the resistance, according to Ohm's law, $V = IR$. Finally, the voltage drop across the capacitor is proportional to the charge q on one of the capacitor plates, the proportionality constant being defined as the capacitance C of the device. In summary, the voltage drops

across these three circuit elements is

$$V_C = \frac{q}{C}, \quad V_L = L\frac{dI}{dt}, \quad V_R = RI. \tag{8.43}$$

Also, since charge cannot go across the capacitor, nor can it dissapear into nothing, the current through the inductor and resistor must come from a changing charge on the capacitor plate, so that the current through the circuit is related to that charge by

$$I = \frac{dq}{dt}. \tag{8.44}$$

Finally, if the circuit is driven by a voltage source $\mathcal{E}(t)$, the total voltage drop across all the rest of the circuit must equal $\mathcal{E}(t)$. This gives us the following equation of motion for the charge on the capacitor:

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = \mathcal{E}(t). \tag{8.45}$$

This is precisely the same as that of a driven damped oscillator. By some fortuitous accident, the letter q plays the same rôle. All the results we worked out for such an oscillator therefore carry over immediately, though the interpretations can be slightly different. For example the natural frequency of this oscillatory circuit is

$$\omega_0 = \sqrt{\frac{1}{LC}}, \tag{8.46}$$

and the Q factor (not to be confused with the charge!) is

$$Q = \left(\frac{L}{C}\right)^{1/2} \frac{1}{R}. \tag{8.47}$$

More complicated circuits, with various capacitances in parallel are oscillators with many degrees of freedom, each capacitive charge providing one degree of freedom.

3.9 Notes

In large measure, this chapter runs a parallel course to that of chapter 3 in MT. In contrast to MT, the discussion of Green functions given here requires the digestion of some new mathematical ideas, but I think (hope) it is more transparent in the long run.