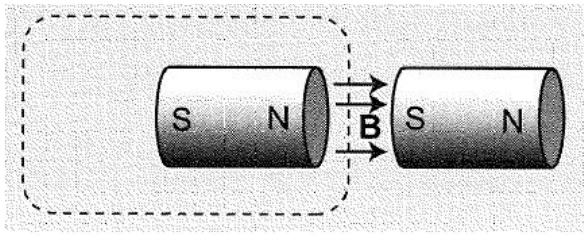
Problem set "Energy-momentum tensor"

Problem I

Consider the attraction between two bar magnets:



Since the components of the symmetric energy-momentum tensor $\Theta^{\mu\nu}$ are quadratic in the fields, we may focus on the region between the poles, where the field is strongest, and ignore the region outside. In the region between the poles, we assume that the magnetic induction has the uniform value B in the x-direction.

(a) Find $\Theta^{\mu\nu}$ in the region between the poles.

(b) Using the Maxwell stress tensor calculate the total force on the left bar magnet (see figure).

(c) As the magnets are moved farther apart, find the force of attraction between them using arguments based on the conservation of energy.

Problem II

(a) In an inertial frame K, calculate the nonzero components of the energy-momentum tensor for a group of particles all moving with the same velocity $\vec{v} = v\,\vec{e}_x$ as seen in K. Let the rest-mass density of these particles be ρ_0 as measured in their comoving frame.[1] Assume a high density of particles and treat them in the continuum approximation.

(b) Consider a gas with a proper number density (i.e. number density as measured in the local rest frame of the gas) n of non interacting particles of mass m. If the particles all have the same speed v but move in random directions (such that the gas is isotropic), what would be the energy-momentum tensor of the gas? (Do not assume $v \ll c$.)

[1] A proper frame, or comoving frame, is the frame of reference that is attached to the object.

Problem III

The Pauli-Lubanski 4-vector (spin 4-vector) is defined by

$$S_\mu \equiv -\frac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} U^\sigma \ .$$

Here U^μ is the "center-of-energy 4-velocity"

$$U^\mu \equiv \frac{P^\mu}{\sqrt{P^\alpha P_\alpha}} \quad : \qquad U^\mu U_\mu = 1$$

and P^μ is the total momentum

$$P^\mu = \frac{1}{c}\int d\Sigma_\lambda \Theta^{\mu\lambda} \ .$$

Show that

(a) The 4-tensor of angular momentum of an isolated system $M^{\mu\nu}$ is not invariant under the coordinate translations $x^\mu \mapsto x^\mu + a^\mu$.

(b) The Pauli-Lubanski 4-vector is both conserved and invariant under translations.

Problem IV

(a) Calculate the angular momentum contained in the fields of a point electric charge q_e in the presence of a point magnetic monopole. Assume that both particles are at rest.

(b) Consider the non-relativistic charge q_e moving in the field of the very heavy monopole. Find the total angular momentum of the system w.r.t. the position of the monopole. How is it related to the Fierz vector $\vec{D}$ (see Problem III from HW5).

Recall that the divergency of the magnetic induction created by the monopole is such that

$$\vec{\nabla}\cdot\vec{B} = q_m\,\delta(\vec{x}-\vec{r}_m) \qquad (q_m - \text{magnetic charge}) \ ,$$

which is similar to the electric field of the point-like charge

$$\vec{\nabla}\cdot\vec{E} = q_e/\epsilon_0\,\delta(\vec{x}-\vec{r}_e) \ .$$