## Problem set VI

## Due March 7, 2024

## Problem I

Prolate spheroidal coordinates $\zeta, \eta, \phi$ are related to Cartesian coordinates $x, y, z$ by

$$
\begin{aligned}
x & =g \sqrt{\left(\xi^{2}-1\right)\left(1-\eta^{2}\right)} \cos (\phi) \\
y & =g \sqrt{\left(\xi^{2}-1\right)\left(1-\eta^{2}\right)} \sin (\phi) \\
z & =g \xi \eta,
\end{aligned}
$$

where $g$ is a positive constant. The Euclidean space $\mathbb{E}^{3}$ without the $z$-axis corresponds to

$$
1<\xi<\infty, \quad-1<\eta<1, \quad 0 \leq \phi<2 \pi
$$

(a) Show that $(\zeta, \eta, \phi)$ are orthogonal coordinates and calculate the corresponding Lamé coefficients.
(b) Show that the coordinate surfaces $\xi=$ const are prolate ellipsoids of revolution with foci at the points $(x, y, z)=(0,0, g)$ and $(x, y, z)=(0,0,-g)$. Moreover show that the coordinate surfaces $\eta=$ const are two-sheeted hyperboloids of revolution with the same foci.

(c) Calculate the capacitance of a metallic prolate ellipsoid of revolution with the principal semiaxes $a=b<c$ :

$$
\frac{x^{2}+y^{2}}{a^{2}}+\frac{z^{2}}{c^{2}}=1 \quad(a<c) .
$$

(d) Assuming that $\varepsilon \equiv 1-a / c \ll 1$ calculate the lowest oder correction in $\varepsilon$ to the capacitance of the metallic sphere of radius $c$. Also consider the opposite limiting case when $a / c \ll 1$ (a rod).

## Problem II

A particle of charge $q$ and mass $m$ is travelling with velocity $v \vec{e}_{x}$ when it encounters a constant electric field $\vec{E}$ in the $y$-direction. Find the trajectory $y(x)$, i.e., the shape of the particle's subsequent motion.

## Problem III

A particle of charge $q$ and mass $m$ moves in a circular orbit of radius $R$ in a uniform magnetic field $\vec{B}=B \vec{e}_{z}$.
(a) Find $B$ in terms of $R, q, m$ and the angular frequency $\omega$.
(b) The speed of the particle is constant since the magnetic field can do no work on the particle. An observer moving at velocity $\vec{V}=V \vec{e}_{x}$, however, does not see the speed being as constant. What is $U^{\prime 0}$ measured by this observer?
(c) Calculate

$$
\frac{d U^{\prime 0}}{d s} \quad \text { and thus } \frac{d p^{\prime 0}}{d s}
$$

Explain how the energy of the particle can change since the magnetic field does no work on it.

## Problem IV

A rocket having initially a total mass $M_{0}$ ejects its fuel with constant velocity $-u(u>0)$ relative to its instantaneous rest frame. In Newtonian mechanics, its velocity $V$ relative to the inertial frame in which it was originally at rest, is related to its mass $M(V)$ by the Meshchersky equation

$$
\frac{M}{M_{0}}=\mathrm{e}^{-V / u}
$$

(a) Derive this result.
(b) Assuming the constraint $0<u \leq c$ on the velocity of the ejected fuel, derive the relativistic analogue of the Meshchersky equation. Show that it reduces to the Newtonian result in the appropriate limit.

