Problem set VI

Due March 13, 2025

Problem I

Prolate spheroidal coordinates ζ, η, ϕ are related to Cartesian coordinates x, y, z by

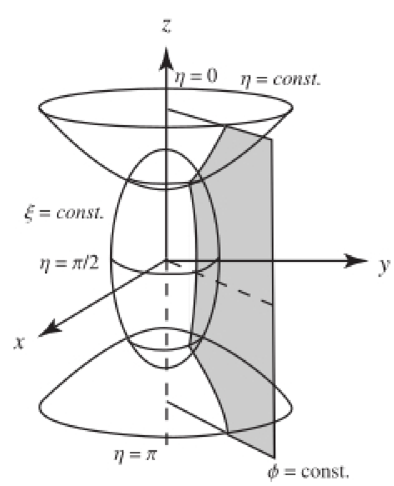
$$\begin{aligned} x &= g\ \sqrt{(\xi^2-1)(1-\eta^2)}\ \cos(\phi) \\ y &= g\ \sqrt{(\xi^2-1)(1-\eta^2)}\ \sin(\phi) \\ z &= g\ \xi\eta\ , \end{aligned}$$

where g is a positive constant. The Euclidean space $\mathbb{E}^3$ without the z-axis corresponds to

$$1 < \xi < \infty\ , \quad -1 < \eta < 1\ , \qquad 0 \le \phi < 2\pi$$

(a) Show that (ζ, η, ϕ) are orthogonal coordinates and calculate the corresponding Lamé coefficients.

(b) Show that the coordinate surfaces $\xi = const$ are prolate ellipsoids of revolution with foci at the points $(x, y, z) = (0, 0, g)$ and $(x, y, z) = (0, 0, -g)$. Moreover show that the coordinate surfaces $\eta = const$ are two-sheeted hyperboloids of revolution with the same foci.



(c) Calculate the capacitance of a metallic prolate ellipsoid of revolution with the principal semiaxes $a = b < c$:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad (a < c)\ .$$

(d) Assuming that $\varepsilon \equiv 1 - a/c \ll 1$ calculate the lowest oder correction in ε to the capacitance of the metallic sphere of radius c. Also consider the opposite limiting case when $a/c \ll 1$ (a rod).

Problem II

A particle of charge q and mass m is travelling with velocity $v\,\vec{e}_x$ when it encounters a constant electric field $\vec{E}$ in the y-direction. Find the trajectory $y(x)$, i.e., the shape of the particle's subsequent motion.

Problem III

A particle of charge q and mass m moves in a circular orbit of radius R in a uniform magnetic field $\vec{B} = B\,\vec{e}_z$.

(a) Find B in terms of R, q, m and the angular frequency ω.

(b) The speed of the particle is constant since the magnetic field can do no work on the particle. An observer moving at velocity $\vec{V} = V\,\vec{e}_x$, however, does not see the speed being as constant. What is U'^0 measured by this observer?

(c) Calculate

$$\frac{dU'^0}{ds} \quad \text{and thus} \quad \frac{dp'^0}{ds} \; .$$

Explain how the energy of the particle can change since the magnetic field does no work on it.

Problem IV

A rocket having initially a total mass M_0 ejects its fuel with constant velocity $-u$ ($u > 0$) relative to its instantaneous rest frame. In Newtonian mechanics, its velocity V relative to the inertial frame in which it was originally at rest, is related to its mass $M(V)$ by the *Meshchersky* equation

$$\frac{M}{M_0} = \mathrm{e}^{-V/u} \; .$$

(a) Derive this result.

(b) Assuming the constraint $0 < u \leq c$ on the velocity of the ejected fuel, derive the relativistic analogue of the Meshchersky equation. Show that it reduces to the Newtonian result in the appropriate limit.