## Problem set "Maxwell's equations"

Due March 6, 2025

## Problem I

Electric and magnetic fields transform under Lorentz transformations as the components of an antisymmetric tensor

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B_3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$$

i.e.,

$$F^{0i} = -E^i/c$$
,  $F^{ij} = -\epsilon_{ijk} B_k$   $(i, j, k \in \{1, 2, 3\})$ .

- (a) Show that  $\vec{B}^2 \vec{E}^2/c^2$  and  $\vec{E} \cdot \vec{B}$  are invariant under the proper Lorentz transformations (which are all possible compositions of space rotations and Lorentz boosts). Are there any invariants which are not merely algebraic combinations of these two?
- (b) Given constant  $\vec{E}$  and  $\vec{B}$  in one frame and assuming that  $\vec{E} \cdot \vec{B} \neq 0$ , find another frame in which  $\vec{E'}$  and  $\vec{B'}$  are parallel to each other (i.e. determine the direction and magnitude of the relative velocity  $\vec{V}$  of the two frames).
- (c) Let the constant fields  $\vec{E}$  and  $\vec{B}$  be such that  $\vec{E} \perp \vec{B}$ . Assuming that  $|\vec{E}|/c > |\vec{B}|$  find the frame in which  $\vec{B}' = 0$ . Similarly for  $|\vec{E}|/c < |\vec{B}|$  find the frame where  $\vec{E}' = 0$ .
- (d) Show that if  $\vec{E} \perp \vec{B}$  and  $|\vec{E}|/c = |\vec{B}|$ , then the same relations hold true in any inertial frame.

## Problem II

A long straight wire carries a current  $I_0$ . At distances a and b from it there are two other wires, parallel to the former one, which are connected together by a resistance R (see figure). A connector slides without friction along the wires with a constant velocity  $\vec{v}$ . Assuming the resistances of the wires, the connector, the sliding contacts, and the self-inductance of the frame to be negligible, find:

- (a) the magnitude and the direction of the current induced in the connector;
- (b) the force required to keep the connector's velocity constant.



## Problem III

A magnetic monopole is defined (if one exists) by a magnetic field singularity of the form

$$\vec{B} = \frac{b\,\vec{r}}{r^3}\,,$$

where b is a constant (a measure of the magnetic charge, as it were). Suppose a non-relativistic particle of mass m and electric charge q moves in the field of a magnetic monopole.

(a) Show that the kinetic energy of the particle  $T = \frac{m\dot{\vec{r}}^2}{2}$  is conserved. Also, by looking at the product  $\vec{r} \times \dot{\vec{r}}$  show that while the mechanical angular momentum is not conserved (the field of force is non-central) there is a conserved vector (Fierz vector)

$$\vec{D} = m \ \vec{r} \times \dot{\vec{r}} - qb \ \frac{\vec{r}}{r} \, .$$

(b) Integrate the equations of motion.