Problem set “Kinematics of Special Relativity”
Due February 19, 2024

Problem I

(I) If two events are separated by a spacelike interval, show that

(a) there exists a Lorentz frame in which they are simultaneous, and
(b) in no Lorentz frame do they occur at the same point.

(II) If two events are separated by a timelike interval, show that

(a) there exists a frame in which they happen at the same point, and
(b) in no Lorentz frame are they simultaneous.
Problem II

Consider an inertial frame $K$ with coordinates

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

and a frame $K'$ with coordinates $x'^\mu$ related to $K$ by a boost with velocity $\vec{V} = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix}$ along the $y$-axis (we assume $V > 0$). Imagine there is a wall at rest in $K'$, lying along the line $x' = y$.

(a) Suppose there is a ball traveling in the $xy$-plane that elastically hits the wall. In the coordinate frame $K$, what is the relationship between the incident velocity $\vec{v}_\text{in} = \begin{pmatrix} v_{\text{in}}^x \\ v_{\text{in}}^y \\ 0 \end{pmatrix}$ and the outgoing velocity $\vec{v}_\text{out} = \begin{pmatrix} v_{\text{out}}^x \\ v_{\text{out}}^y \\ 0 \end{pmatrix}$ of the ball.

(b) Find the incident angle $\phi_\text{in}$ of the ball and the reflected angle $\phi_\text{out}$. Calculate numerical values of $\phi_\text{in}$ and $\phi_\text{out}$ for

$$v_{\text{in}}^x = -0.5 c, \quad v_{\text{in}}^y = +0.3 c, \quad V = 0.2 c$$
Problem III

For $M^{1,3}$ define the completely antisymmetric Levi-Civita tensor as a set of numbers given in some Minkowski coordinate system by

$$\epsilon^{\mu_1\mu_2\mu_3\mu_4} = \begin{cases} +1 & \text{if } (\mu_1, \mu_2, \mu_3, \mu_4) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (\mu_1, \mu_2, \mu_3, \mu_4) \text{ is an odd permutation of } (0, 1, 2, 3) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4} = -\begin{cases} +1 & \text{if } (\mu_1, \mu_2, \mu_3, \mu_4) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (\mu_1, \mu_2, \mu_3, \mu_4) \text{ is an odd permutation of } (0, 1, 2, 3) \\ 0 & \text{otherwise} \end{cases}$$

Show that

(a) Under the Lorentz transformations

$$\epsilon^{\mu_1\mu_2\mu_3\mu_4} = \pm \epsilon^{\mu_1\mu_2\mu_3\mu_4},$$

where the sign factors $(+)$ or $(-)$ depend on whether the transformation is proper or improper, respectively. In other words $\epsilon^{\mu_1\mu_2\mu_3\mu_4}$ is an invariant pseudotensor of valence $(4, 0)$. Similarly $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ is an invariant pseudotensor of valence $(0, 4)$ and

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4} = \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} \eta_{\mu_3\nu_3} \eta_{\mu_4\nu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4}. $$

(b) It follows immediately from (a) that

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4}$$

is an invariant tensor of type $(4, 4)$. Express this tensor in terms of the invariant tensor $\delta^\mu_\nu$.

Problem IV

Let $F_{\mu\nu}$ be an antisymmetric second rank tensor. From this tensor construct another second rank, antisymmetric tensor, $(*F)_{\mu\nu}$, called the dual of $F_{\mu\nu}$, as follows

$$(*F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}. $$

Show that

$$(*(*F))_{\mu\nu} = -F_{\mu\nu}. $$