Problem set "Kinematics of Special Relativity"

Due February 24, 2025

Problem I

(I) If two events are separated by a spacelike interval, show that

  (a) there exists a Lorentz frame in which they are simultaneous, and

  (b) in no Lorentz frame do they occur at the same point.

(II) If two events are separated by a timelike interval, show that

  (a) there exists a frame in which they happen at the same point, and

  (b) in no Lorentz frame are they simultaneous.

Problem II

Consider an inertial frame K with coordinates

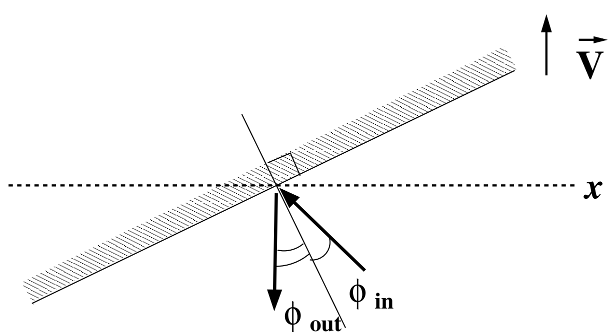
$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

and a frame K' with coordinates x'^μ related to K by a boost with velocity $\vec{V} = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix}$ along the y-axis (we assume $V > 0$). Imagine there is a wall at rest in K', lying along the line $x' = y'$.

(a) Suppose there is a ball traveling in the xy-plane that elastically hits the wall. In the coordinate frame K, what is the relationship between the incident velocity $\vec{v}_{\text{in}} = \begin{pmatrix} v^x_{\text{in}} \\ v^y_{\text{in}} \\ 0 \end{pmatrix}$ and the outgoing velocity $\vec{v}_{\text{out}} = \begin{pmatrix} v^x_{\text{out}} \\ v^y_{\text{out}} \\ 0 \end{pmatrix}$ of the ball.

(b) Find the incident angle ϕ_{in} of the ball and the reflected angle ϕ_{out}. Calculate numerical values of ϕ_{in} and ϕ_{out} for

$$v^x_{\text{in}} = -0.5\,c\,, \qquad v^y_{\text{in}} = +0.3\,c\,, \qquad V = 0.2\,c$$



Problem III

For $\mathbb{M}^{1,3}$ define the completely antisymmetric Levi-Civita tensor as a set of numbers given in some Minkowski coordinate system by

$$\epsilon^{\mu_1\mu_2\mu_3\mu_4} = \begin{cases} +1 & \text{if} \quad (\mu_1,\mu_2,\mu_3,\mu_4) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if} \quad (\mu_1,\mu_2,\mu_3,\mu_4) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4} = -\begin{cases} +1 & \text{if} \quad (\mu_1,\mu_2,\mu_3,\mu_4) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if} \quad (\mu_1,\mu_2,\mu_3,\mu_4) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Show that

(a) Under the Lorentz transformations

$$\epsilon'^{\mu_1\mu_2\mu_3\mu_4} = \pm\,\epsilon^{\mu_1\mu_2\mu_3\mu_4}\,,$$

where the sign factors $(+)$ or $(-)$ depend on whether the transformation is proper or improper, respectively. In other words $\epsilon^{\mu_1\mu_2\mu_3\mu_4}$ is an invariant **pseudotensor** of valence $(4,0)$. Similarly $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ is an invariant pseudotensor of valence $(0,4)$ and

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\eta_{\mu_4\nu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4}\ .$$

(b) It follows immediately from (a) that

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4}\,\epsilon^{\nu_1\nu_2\nu_3\nu_4}$$

is an invariant tensor of type $(4,4)$. Express this tensor in terms of the invariant tensor δ^μ_ν.

Problem IV

Let $F_{\mu\nu}$ be an antisymmetric second rank tensor. From this tensor construct another second rank, antisymmetric tensor, $({}^*F)_{\mu\nu}$, called the dual of $F_{\mu\nu}$, as follows

$$({}^*F)_{\mu\nu} = \tfrac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}\ .$$

Show that

$$({}^*({}^*F))_{\mu\nu} = -F_{\mu\nu}\ .$$