Problem set "Euclidean tensors"

Due February 10, 2025

Problem I

(i) Calculate the integrals

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega \, n^i n^j \ , \qquad \int_{\mathbb{S}^2} \mathrm{d}\Omega \, n^i n^j n^k \ , \qquad \int_{\mathbb{S}^2} \mathrm{d}\Omega \, n^i n^j n^k n^m$$

over the sphere $\mathbb{S}^2$ of unit radius. Here (n^1, n^2, n^3) are the components of a unit vector $\vec{n} \in \mathbb{S}^2$.

(ii) Show that $\epsilon_{ijk}\epsilon_{kmn}$ is a rank 4 invariant tensor and find the numerical coefficients in the formula

$$\epsilon_{ijk}\epsilon_{kmn} = A\,\delta_{ij}\delta_{mn} + B\ \delta_{im}\delta_{jn} + C\ \delta_{in}\delta_{jm}\ .$$

(iii) Show that $\epsilon_{ijk}\epsilon_{lmn}$ is a rank 6 invariant tensor and prove the relation

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\ . \end{aligned}$$

Problem II

(a) Let $T_{j_1\dots j_\ell}$ be a *totally symmetric* and *traceless* tensor of rank ℓ,

$$T_{j_1\dots j_\ell}\ :\quad T_{\dots j_p\dots j_q\dots}=T_{\dots j_q,\dots j_p,\dots}\qquad \&\qquad \delta^{j_p j_q}T_{\dots j_p\dots j_q\dots}=0\qquad \forall p,q \tag{$\star$}$$

and $\vec{n}$ be a unit vector, $|\vec{n}|=1$. Assuming that $T_{j_1\dots j_\ell}$ is a constant tensor (i.e., it is the same at any point in space) consider the non-singular function on the sphere

$$\Psi^{(T)}_\ell(\vec{n})=T_{j_1\dots j_\ell}n^{j_1}\cdots n^{j_\ell}\ .$$

Show that $\Psi^{(T)}_\ell(\vec{n})$ satisfies the equation

$$-\nabla^2_{\vec{n}}\Psi^{(T)}_\ell(\vec{n})=\ell(\ell+1)\ \Psi^{(T)}(\vec{n})\,,$$

where $\nabla^2_{\vec{n}}$ is the Laplacian on the round sphere (i.e. $\nabla^2_{\vec{n}}$ is the spherical part of the Laplacian in $\mathbb{E}^3$.[1]

(b) Show that the number of linearly independent components of a totally symmetric and traceless tensor of rank ℓ is equal to $2\ell+1$. Thus, for given $\ell=0,1,2,\dots$ the functions $\Psi^{(T)}_\ell(\vec{n})$ form a linear space of dimensions $2\ell+1$.

(c) You should know that for given ℓ, the spherical harmonics

$$Y_{\ell,\mathfrak{m}}(\vec{n})\qquad\text{with}\qquad \mathfrak{m}=-\ell,-\ell+1,\dots,\ell-1,\ell$$

form a linear basis in the space of regular solutions of $\nabla^2_{\vec{n}}\Psi=-\ell(\ell+1)\ \Psi$. It follows from (a) and (b) that any spherical harmonic can be written in the form

$$Y_{\ell,\mathfrak{m}}(\vec{n})=T^{(\mathfrak{m})}_{j_1\dots j_\ell}n^{j_1}\cdots n^{j_\ell}\ ,$$

where $T^{(\mathfrak{m})}_{j_1\dots j_\ell}$ are a certain set of (complex) numbers satisfying the conditions ($\star$). Find explicit expressions for $T^{(\mathfrak{m})}_{j}$ ($\mathfrak{m}=-1,0,1$) and $T^{(\mathfrak{m})}_{jk}$ ($\mathfrak{m}=-2,-1,0,1,2$).

[1] Recall that if $\vec{n}$ is parameterized by polar and azimuthal angles θ and φ as

$$\vec{n}=\begin{pmatrix}n^1\\ n^2\\ n^3\end{pmatrix}=\begin{pmatrix}\sin(\theta)\cos(\varphi)\\ \sin(\theta)\sin(\varphi)\\ \cos(\theta)\end{pmatrix}\in\mathbb{S}^2\ ,$$

then

$$\nabla^2_{\vec{n}}=\frac{1}{\sin(\theta)}\ \frac{\partial}{\partial\theta}\left(\sin(\theta)\ \frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\ \frac{\partial^2}{\partial\varphi^2}\ .$$

Problem III

Consider the 2×2 unitary matrices with unit determinant:

$$\boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}_{2\times 2} \qquad \& \qquad \det \boldsymbol{U} = 1 \ .$$

(a) Show that the set of all such matrices $\boldsymbol{U}$ form a group. The latter is denoted by $SU(2)$.

(b) Show that any $SU(2)$ matrix can be written in the form

$$\boldsymbol{U} = \begin{pmatrix} \rho_0 + \mathrm{i}\rho_3 & \rho_2 + \mathrm{i}\rho_1 \\ -\rho_2 + \mathrm{i}\rho_1 & \rho_0 - \mathrm{i}\rho_3 \end{pmatrix} ,$$

where the 4 real numbers $(\rho_0, \rho_1, \rho_2, \rho_3)$ satisfy the condition

$$\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_2^2 = 1 \ .$$

In other words there is a one-to-one correspondence between elements of the group $SU(2)$ and points on the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$ of unit radius.[2]

(c) Show that the matrix exponential

$$\exp\left(\tfrac{\mathrm{i}}{2}\,\phi^k \sigma_k\right) \qquad\qquad (\phi^k = \phi n^k) \ ,$$

where σ_k are conventional Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an $SU(2)$ matrix and express the corresponding set $(\rho_0, \rho_1, \rho_2, \rho_3)$ through the rotational angle $\phi \sim \phi + 2\pi$ and three dimensional unit vector $\vec{n} \in \mathbb{R}^3 : \ |\vec{n}| = 1$. Compare the result with the Euler parameters from Problem IV (HW1).

(d) By the similarity transformation any 4×4 matrix $\boldsymbol{U} \otimes \boldsymbol{U}$ can be brought to the block diagonal form:

$$\boldsymbol{U} \otimes \boldsymbol{U} = \boldsymbol{C} \begin{pmatrix} 1 & \mathbf{O} \\ \mathbf{O} & \boxed{\boldsymbol{S}^{-1}} \end{pmatrix} \boldsymbol{C}^{-1} \ .$$

Here $\boldsymbol{S}$ is the 3×3 matrix of finite rotations from Problem IV (HW1) and the 4×4 matrix $\boldsymbol{C}$ is the same for any $SU(2)$ matrix $\boldsymbol{U}$. Explain why.

(e) Find the explicit form of the 4×4 constant matrix $\boldsymbol{C}$.

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[2] The sphere S^3 is one the simplest examples of a mathematical object called a *manifold*. Hence the group $SU(2)$ possesses the structure of a manifold. Such groups are called continuous groups or Lie groups. The group $SO(3)$ is another example of a Lie group. It has the structure of the 3-dimensional real projective space $\mathbb{RP}^3$. The later can be understood as a 3-dimensional sphere $S^3 \in \mathbb{R}^4$ defined by the equation $\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_2^2 = 1$ with each pair of points $(\rho_0, \rho_1, \rho_2, \rho_3)$ and $(-\rho_0, -\rho_1, -\rho_2, -\rho_3)$ identified (glued together).