# Problem set "Euclidean tensors" 

## Due February 5, 2024

## Problem I

(i) Calculate the integrals

$$
\int_{\mathbb{S}^{2}} \mathrm{~d} \Omega n^{i} n^{j}, \quad \int_{\mathbb{S}^{2}} \mathrm{~d} \Omega n^{i} n^{j} n^{k}, \quad \int_{\mathbb{S}^{2}} \mathrm{~d} \Omega n^{i} n^{j} n^{k} n^{m}
$$

over the sphere $\mathbb{S}^{2}$ of unit radius. Here $\left(n^{1}, n^{2}, n^{3}\right)$ are the components of a unit vector $\vec{n} \in \mathbb{S}^{2}$.
(ii) Show that $\epsilon_{i j k} \epsilon_{k m n}$ is a rank 4 invariant tensor and find the numerical coefficients in the formula

$$
\epsilon_{i j k} \epsilon_{k m n}=A \delta_{i j} \delta_{m n}+B \delta_{i m} \delta_{j n}+C \delta_{i n} \delta_{j m}
$$

(iii) Show that $\epsilon_{i j k} \epsilon_{l m n}$ is a rank 6 invariant tensor and prove the relation

$$
\begin{aligned}
\epsilon_{i j k} \epsilon_{l m n} & =\operatorname{det}\left(\begin{array}{ccc}
\delta_{i l} & \delta_{i m} & \delta_{i n} \\
\delta_{j l} & \delta_{j m} & \delta_{j n} \\
\delta_{k l} & \delta_{k m} & \delta_{k n}
\end{array}\right) \\
& =\delta_{i l}\left(\delta_{j m} \delta_{k n}-\delta_{j n} \delta_{k m}\right)-\delta_{i m}\left(\delta_{j l} \delta_{k n}-\delta_{j n} \delta_{k l}\right)+\delta_{i n}\left(\delta_{j l} \delta_{k m}-\delta_{j m} \delta_{k l}\right)
\end{aligned}
$$

## Problem II

(a) Let $T_{j_{1} \ldots j_{\ell}}$ be a totally symmetric and traceless tensor of rank $\ell$,

$$
T_{j_{1} \ldots j_{\ell}}: \quad T_{\ldots j_{p} \ldots j_{q} \ldots}=T_{\ldots j_{q}, \ldots j_{p}, \ldots} \quad \& \quad \delta^{j_{p} j_{q}} T_{\ldots j_{p} \ldots j_{q} \ldots}=0 \quad \forall p, q
$$

and $\vec{n}$ be a unit vector, $|\vec{n}|=1$. Assuming that $T_{j_{1} \ldots j_{\ell}}$ is a constant tensor (i.e., it is the same at any point in space) consider the non-singular function on the sphere

$$
\Psi_{\ell}^{(T)}(\vec{n})=T_{j_{1} \ldots j_{\ell}} n^{j_{1}} \cdots n^{j_{\ell}}
$$

Show that $\Psi_{\ell}^{(T)}(\vec{n})$ satisfies the equation

$$
-\nabla_{\vec{n}}^{2} \Psi_{\ell}^{(T)}(\vec{n})=\ell(\ell+1) \Psi^{(T)}(\vec{n}),
$$

where $\nabla_{\vec{n}}^{2}$ is the Laplacian on the round sphere (i.e. $\nabla_{\vec{n}}^{2}$ is the spherical part of the Laplacian in $\mathbb{E}^{3}$. ${ }^{1}$
(b) Show that the number of linearly independent components of a totally symmetric and traceless tensor of rank $\ell$ is equal to $2 \ell+1$. Thus, for given $\ell=0,1,2, \ldots$ the functions $\Psi_{\ell}^{(T)}(\vec{n})$ form a linear space of dimensions $2 \ell+1$.
(c) You should know that for given $\ell$, the spherical harmonics

$$
Y_{\ell, \mathfrak{m}}(\vec{n}) \quad \text { with } \quad \mathfrak{m}=-\ell,-\ell+1, \ldots, \ell-1, \ell
$$

form a linear basis in the space of regular solutions of $\nabla_{\vec{n}}^{2} \Psi=-\ell(\ell+1) \Psi$. It follows from (a) and (b) that any spherical harmonic can be written in the form

$$
Y_{\ell, \mathfrak{m}}(\vec{n})=T_{j_{1} \ldots j_{\ell}}^{(\mathfrak{m})} n^{j_{1}} \cdots n^{j_{\ell}}
$$

where $T_{j_{1} \ldots j_{\ell}}^{(\mathfrak{m})}$ are a certain set of (complex) numbers satisfying the conditions ( $\star$ ). Find explicit expressions for $T_{j}^{(\mathfrak{m})}(\mathfrak{m}=-1,0,1)$ and $T_{j k}^{(\mathfrak{m})}(\mathfrak{m}=-2,-1,0,1,2)$.

[^0]then
$$
\nabla_{\vec{n}}^{2}=\frac{1}{\sin (\theta)} \frac{\partial}{\partial \theta}\left(\sin (\theta) \frac{\partial}{\partial \theta}\right)+\frac{1}{\sin ^{2}(\theta)} \frac{\partial^{2}}{\partial \varphi^{2}}
$$

## Problem III

Consider the $2 \times 2$ unitary matrices with unit determinant:

$$
\boldsymbol{U} \boldsymbol{U}^{\dagger}=\boldsymbol{I}_{2 \times 2} \quad \& \quad \operatorname{det} \boldsymbol{U}=1
$$

(a) Show that the set of all such matrices $\boldsymbol{U}$ form a group. The latter is denoted by $S U(2)$.
(b) Show that any $S U(2)$ matrix can be written in the form

$$
\boldsymbol{U}=\left(\begin{array}{cc}
\rho_{0}+\mathrm{i} \rho_{3} & \rho_{2}+\mathrm{i} \rho_{1} \\
-\rho_{2}+\mathrm{i} \rho_{1} & \rho_{0}-\mathrm{i} \rho_{3}
\end{array}\right)
$$

where the 4 real numbers ( $\rho_{0}, \rho_{1}, \rho_{2}, \rho_{3}$ ) satisfy the condition

$$
\rho_{0}^{2}+\rho_{1}^{2}+\rho_{2}^{2}+\rho_{2}^{2}=1 .
$$

In other words there is a one-to-one correspondence between elements of the group $S U(2)$ and points on the 3 -dimensional sphere $S^{3} \subset \mathbb{R}^{4}$ of unit radius. ${ }^{2}$
(c) Show that the matrix exponential

$$
\exp \left(\frac{\mathrm{i}}{2} \phi^{k} \sigma_{k}\right) \quad\left(\phi^{k}=\phi n^{k}\right)
$$

where $\sigma_{k}$ are conventional Pauli matrices:

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

is an $S U(2)$ matrix and express the corresponding set $\left(\rho_{0}, \rho_{1}, \rho_{2}, \rho_{3}\right)$ through the rotational angle $\phi \sim \phi+2 \pi$ and three dimensional unit vector $\vec{n} \in \mathbb{R}^{3}:|\vec{n}|=1$. Compare the result with the Euler parameters from Problem IV (HW1).
(d) By the similarity transformation any $4 \times 4$ matrix $\boldsymbol{U} \otimes \boldsymbol{U}$ can be brought to the block diagonal form:

$$
\boldsymbol{U} \otimes \boldsymbol{U}=\boldsymbol{C}\left(\begin{array}{cc}
1 & \mathrm{O} \\
\mathrm{O} & \boldsymbol{S}^{-1}
\end{array}\right) \boldsymbol{C}^{-1}
$$

Here $\boldsymbol{S}$ is the $3 \times 3$ matrix of finite rotations from Problem IV (HW1) and the $4 \times 4$ matrix $\boldsymbol{C}$ is the same for any $S U(2)$ matrix $\boldsymbol{U}$. Explain why.
(e) Find the explicit form of the $4 \times 4$ constant matrix $\boldsymbol{C}$.

[^1]
[^0]:    ${ }^{1}$ Recall that if $\vec{n}$ is parameterized by polar and azimuthal angles $\theta$ and $\varphi$ as

    $$
    \vec{n}=\left(\begin{array}{c}
    n^{1} \\
    n^{2} \\
    n^{3}
    \end{array}\right)=\left(\begin{array}{c}
    \sin (\theta) \cos (\varphi) \\
    \sin (\theta) \sin (\varphi) \\
    \cos (\theta)
    \end{array}\right) \in \mathbb{S}^{2},
    $$

[^1]:    ${ }^{2}$ The sphere $S^{3}$ is one the simplest examples of a mathematical object called a manifold. Hence the group $S U(2)$ possesses the structure of a manifold. Such groups are called continuous groups or Lie groups. The group $S O(3)$ is another example of a Lie group. It has the structure of the 3-dimensional real projective space $\mathbb{R} \mathbb{P}^{3}$. The later can be understood as a 3 -dimensional sphere $S^{3} \in \mathbb{R}^{4}$ defined by the equation $\rho_{0}^{2}+\rho_{1}^{2}+\rho_{2}^{2}+\rho_{2}^{2}=1$ with each pair of points $\left(\rho_{0}, \rho_{1}, \rho_{2}, \rho_{3}\right)$ and ( $-\rho_{0},-\rho_{1},-\rho_{2},-\rho_{3}$ ) identified (glued together).

