Problem set "Electromagnetics waves"

Due April 17, 2025

Problem I

Consider a plane wave with vector potential $A_\mu(x) = a_\mu \mathrm{e}^{-\mathrm{i}k_\alpha x^\alpha}$ where a_μ is a constant 4-vector. Further suppose that $k_\alpha = (\omega/c, 0, 0, -k)$ and choose a (non-orthogonal) set of basis vectors for a_μ:

$$\varepsilon_\mu^{(1)} = (0,1,0,0)\ , \qquad \varepsilon_\mu^{(2)} = (0,0,1,0)$$
$$\varepsilon_\mu^{(L)} = \big(\omega/(ck), 0, 0, -1\big) = \frac{1}{k}\, k_\mu\ , \quad \varepsilon_\mu^{(B)} = \big(1, 0, 0, \omega/(ck)\big) \qquad (k > 0)\,.$$

Write

$$a_\mu = C_1\, \varepsilon_\mu^{(1)} + C_2\, \varepsilon_\mu^{(2)} + C_L\, \varepsilon_\mu^{(L)} + C_B\, \varepsilon_\mu^{(B)}\,.$$

(a) What constraints, if any, does one get on C_1, C_2, C_L, C_B from the Maxwell equation:

(1) $\vec{\nabla} \cdot \vec{B} = 0$

(2) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(3) $\vec{\nabla} \cdot \vec{E} = 0$

(4) $\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$?

(b) Which of the parameters C_1, C_2, C_L, C_B are gauge independent? What constraint does the Lorentz condition impose? The same question for the Coulomb gauge.

(c) Give the average energy density in terms of C_1, C_2, C_L, C_B after imposing $(1-4)$.

Problem II

Consider a monochromatic EM plane wave that is linearly polarized along the x-axis and propagates in the z-direction with respect to an inertial frame K. What is the polarization measured by an observer K' that moves in the x-direction with a velocity v?

Problem III

Two plane monochromatic linearly polarized EM waves of the same frequency propagate along the z-axis. The first wave is polarized along the x-direction and has amplitude $a > 0$, and the second is polarized along the y-axis and has amplitude $b > 0$. The phase of the second wave leads the phase of the first wave by $-\pi < \chi \leq \pi$.[1]

(a) Determine the direction and magnitude of the axes of the polarization ellipse of the resultant wave in terms of a, b and χ in the case $\chi \neq 0, \pi$.

(b) Find the condition for which the resultant wave is right or left circularly polarized.

(c) Discuss the case when the phase difference $\chi = 0, \pi$.

Problem IV

(a) Write down the spatial and temporal dependence of the $\vec{E}$, $\vec{D}$, $\vec{B}$, and $\vec{H}$ fields in an electromagnetic plane wave of angular frequency ω. Specify the magnitude of all vectors, including that of $\vec{k}$, in terms of ω, $\mu > 0$, $\epsilon > 0$, and the electric field amplitude E_0. Specify the direction of all vectors, including the $\vec{k}$-vector. Show explicitly that these fields satisfy Maxwell's equations. Compare the direction of the $\vec{k}$-vector with that of the Poynting vector. (For simplicity, you may choose the propagation direction and polarization direction to be along two of the Cartesian axes.)

(b) In what ways are the fields of part (a) modified in a medium in which the dielectric permittivity ϵ is real and negative, while the magnetic permeability μ is real and positive? (This describes e.g., a free electron metal below the plasma frequency.) Determine the instantaneous and time-averaged Poynting vectors. Can electromagnetic waves propagate in such a medium? Is there energy dissipation? Also consider the case $\epsilon > 0$ and $\mu < 0$.

(c) Repeat part (b) for a material in which both ϵ and μ are real and negative. Can electromagnetic waves propagate in such a medium? Is there energy dissipation? Compared to case (a), what is strangely different here about the $\vec{k}$-vector and the Poynting vector?

(d) At the interface between the vacuum and a material of the type described in part (c) (called a "left-handed material"), energy conservation dictates that the normal component of the Poynting vector be continuous across the interface, while phase matching requires that the parallel component of the $\vec{k}$-vector be continuous across the interface (Snell's law). Consider a plane electromagnetic wave incident from the vacuum onto such an interface at an angle θ with respect to the surface normal. Sketch the $\vec{k}$-vector (characterizing the motion of the wave fronts) and the Poynting vector (characterizing the energy flow) of the incident, reflected, and transmitted waves, with respect to the interface. Assume $\epsilon = -\epsilon_0$ and $\mu = -\mu_0$.

[1] In other words, the electric field of the first and second waves read as $\vec{E}_1 = a\,\vec{e}_x \cos(kz - \omega t)$ and $\vec{E}_2 = b\,\vec{e}_y \cos(kz - \omega t + \chi)$, respectively.