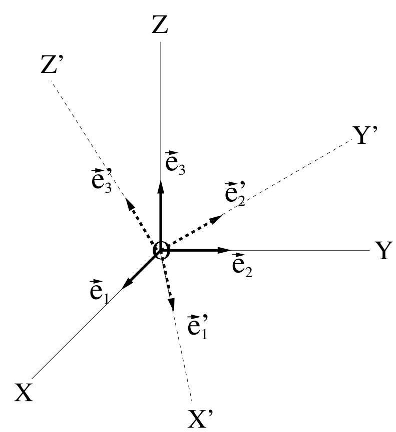
Problem set "Space in Classical Physics"

Due February 3, 2025

Problem I

Suppose we are given two different orthogonal coordinate systems that have the same origin. Let $\vec{e}_i$ and $\vec{e}_i{}'$ $(i = 1, 2, 3)$ be unit vectors along the coordinate axes $OXYZ$ and $OX'Y'Z'$, respectively,[1]

$$\vec{e}_i \cdot \vec{e}_j = \vec{e}_i{}' \cdot \vec{e}_j{}' = \delta_{ij} \ .$$



Introduce the transformation $\hat{S}$ as a linear operator defined by the conditions,

$$\vec{e}_i{}' = \hat{S}\, \vec{e}_i \qquad (i = 1, 2, 3) \ .$$

Such transformations are known as orthogonal. The vectors $\vec{e}_i{}'$ can be linearly expressed in terms of $\vec{e}_i$ as

$$\vec{e}_i{}' = \vec{e}_j\, S^j_i \ ,$$

[1] One can think of the unit vectors $\vec{e}_i$ as columns

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ .$$

Then the position vector would be given by

$$\vec{r} \equiv \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \sum_{i=1}^{3} x^i \vec{e}_i \equiv x^i \vec{e}_i \ .$$

The same vector can be re-written in the other basis $\vec{r} = x'^i \vec{e}_i{}'$ with $x^j = S^j_i x'^i$ or, equivalently, $x'^j = \Lambda^j_i x^i$ with $\boldsymbol{\Lambda} \equiv \boldsymbol{S}^{-1} = \boldsymbol{S}^T$.

where the 3×3 matrix for the operator $\hat{S}$,

$$\boldsymbol{S} = \begin{pmatrix} S^1_1 & S^1_2 & S^1_3 \\ S^2_1 & S^2_2 & S^2_3 \\ S^3_1 & S^3_2 & S^3_3 \end{pmatrix} ,$$

satisfies the condition

$$\boldsymbol{S}^T \boldsymbol{S} = \mathbf{1}$$

(the superscript "T" denotes the matrix transposition). For an arbitrary orthogonal transformation one has

$$(\det \boldsymbol{S})^2 = 1 \ , \qquad \text{i.e.,} \qquad \det \boldsymbol{S} = \pm 1 \ .$$

An orthogonal transformation with determinant $+1$ is said to be *proper*.

(a) Show that the linear operator $\hat{S}$ such that

$$\begin{aligned}
\vec{e_1}' &= \hat{S}\,\vec{e}_1 = \frac{1}{4}\,\vec{e}_1 + \frac{1+2\sqrt{2}}{4}\,\vec{e}_2 - \frac{2-\sqrt{2}}{4}\,\vec{e}_3 \\
\vec{e_2}' &= \hat{S}\,\vec{e}_2 = \frac{1-2\sqrt{2}}{4}\,\vec{e}_1 + \frac{1}{4}\,\vec{e}_2 + \frac{2+\sqrt{2}}{4}\,\vec{e}_3 \\
\vec{e_3}' &= \hat{S}\,\vec{e}_3 = \frac{2+\sqrt{2}}{4}\,\vec{e}_1 - \frac{2-\sqrt{2}}{4}\,\vec{e}_2 + \frac{1}{2}\,\vec{e}_3
\end{aligned}$$

is a proper orthogonal transformation.

(b) According to Euler's theorem **an arbitrary proper orthogonal transformation is a rotation.**

Illustrate this statement using the transformation from (a), i.e., determine the unit vector $\vec{n}$ along the corresponding axis of rotation and the rotation angle ϕ.

Problem II

Show that for an arbitrary 3×3 matrix $\boldsymbol{A}$ with entrees A^j_i, the following relation holds true

$$\det \boldsymbol{A}\ \epsilon_{ijk} = A^l_i A^m_j A^n_k\ \epsilon_{lmn} \ .$$

Problem III

The cross product of two vectors $\vec{a} = a^i\,\vec{e}_i$ and $\vec{b} = b^i\,\vec{e}_i$ is defined as

$$(\vec{a} \times \vec{b}\,)_i = \epsilon_{ijk}\, a^j\, b^k\ .$$

(a) Show that the 3-component object $c_i \equiv (\vec{a} \times \vec{b}\,)_i$ transforms under the change of coordinate system $OXYZ \mapsto OX'Y'Z'$ according to the rule

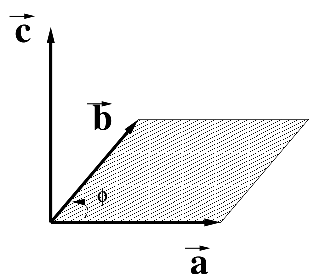
$$c_j = (+1)\ S^j_i\ c'_i$$

for a proper orthogonal transformation and

$$c_j = (-1)\ S^j_i\ c'_i$$

for an improper one (here c_i and c'_i are components of the cross product relative to the coordinate systems $OXYZ$ and $OX'Y'Z'$, respectively. For the definition of the transition matrix $\boldsymbol{S}$ see Problem I).

(b) The cross product can also be introduced geometrically as a vector of magnitude $|\vec{a}||\vec{b}|\ \sin(\phi)$ with ϕ being the smallest angle between $\vec{a}$ and $\vec{b}$ (i.e. $0 \le \phi \le \pi$). It is perpendicular to both $\vec{a}$ and $\vec{b}$ and the direction is found in the following way: If one rotates a right handed screw from $\vec{a}$ into $\vec{b}$ through the smallest possible angle then the screw would travel in the direction of $\vec{a} \times \vec{b}$ (in other words the ordered set of vectors $(\vec{a},\ \vec{b},\ \vec{a} \times \vec{b})$ is a right-handed triple).



Show that if $(\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3)$ is a right-handed triple of basis vectors then the "algebraic" definition is equivalent to the "geometric" one.

Problem IV

For an arbitrary proper orthogonal transformation express the matrix of finite rotations $\boldsymbol{S}$ in terms of the unit vector along the axis of rotation $\vec{n} = (n_1, n_2, n_3)$ and the rotation angle ϕ. To simplify the final expression, please use the so-called *Euler parameters*:

$$\rho_0 = \cos(\phi/2)\,, \qquad \rho_i = \sin(\phi/2)\ n_i \qquad (i = 1, 2, 3)$$

satisfying the relation

$$\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 1\ .$$