Problem set “Space in Classical Physics”

Due January 29, 2024

Problem I

Suppose we are given two different orthogonal coordinate systems that have the same origin. Let \( \vec{e}_i \) and \( \vec{e}_i' \) \((i = 1, 2, 3)\) be unit vectors along the coordinate axes \( OXYZ \) and \( OX'Y'Z' \), respectively.\(^1\)

\[
\vec{e}_i \cdot \vec{e}_j = \vec{e}_i' \cdot \vec{e}_j' = \delta_{ij} .
\]

Introduce the transformation \( \hat{S} \) as a linear operator defined by the conditions,

\[
\vec{e}_i' = \hat{S} \vec{e}_i \quad (i = 1, 2, 3) .
\]

Such transformations are known as orthogonal. The vectors \( \vec{e}_i' \) can be linearly expressed in terms of \( \vec{e}_i \) as

\[
\vec{e}_i' = \vec{e}_j S^i_j ,
\]

\(^1\)One can think of the unit vectors \( \vec{e}_i \) as columns

\[
\begin{align*}
\vec{e}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \vec{e}_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \vec{e}_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .
\end{align*}
\]

Then the position vector would be given by

\[
\vec{r} \equiv \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \sum_{i=1}^{3} x^i \vec{e}_i \equiv x^i \vec{e}_i .
\]

The same vector can be re-written in the other basis \( \vec{r} = x'^j \vec{e}_i' \) with \( x^j = S^j_i x'^i \) or, equivalently, \( x'^j = \Lambda^j_i x^i \) with \( \Lambda \equiv S^{-1} = S^T \).
where the $3 \times 3$ matrix for the operator $\hat{S}$,

$$
S = \begin{pmatrix}
S_1^1 & S_1^2 & S_1^3 \\
S_2^1 & S_2^2 & S_2^3 \\
S_3^1 & S_3^2 & S_3^3
\end{pmatrix},
$$
satisfies the condition

$$
S^T S = 1
$$

(the superscript “$T$” denotes the matrix transposition). For an arbitrary orthogonal transformation one has

$$(\det S)^2 = 1, \quad \text{i.e.,} \quad \det S = \pm 1.$$  

An orthogonal transformation with determinant $+1$ is said to be \textit{proper}.

(a) Show that the linear operator $\hat{S}$ such that

$$
\vec{e}_1' = \hat{S} \vec{e}_1 = \frac{1}{4} \vec{e}_1 + \frac{1+2\sqrt{2}}{4} \vec{e}_2 - \frac{2-\sqrt{2}}{4} \vec{e}_3
$$

$$
\vec{e}_2' = \hat{S} \vec{e}_2 = \frac{1-2\sqrt{2}}{4} \vec{e}_1 + \frac{1}{4} \vec{e}_2 + \frac{2+\sqrt{2}}{4} \vec{e}_3
$$

$$
\vec{e}_3' = \hat{S} \vec{e}_3 = \frac{2+\sqrt{2}}{4} \vec{e}_1 - \frac{2-\sqrt{2}}{4} \vec{e}_2 + \frac{1}{2} \vec{e}_3
$$

is a proper orthogonal transformation.

(b) According to Euler’s theorem an \textbf{arbitrary proper orthogonal transformation is a rotation}.

Illustrate this statement using the transformation from (a), i.e., determine the unit vector $\vec{n}$ along the corresponding axis of rotation and the rotation angle $\phi$.

\textbf{Problem II}

Show that for an arbitrary $3 \times 3$ matrix $A$ with entrees $A^j_i$, the following relation holds true

$$
\det A \ \epsilon_{ijk} = A^j_i A^m_j A^n_k \ \epsilon_{lmn}.
$$
Problem III

The cross product of two vectors $\vec{a} = a^i \vec{e}_i$ and $\vec{b} = b^i \vec{e}_i$ is defined as

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a^j b^k .$$

(a) Show that the 3-component object $c_i \equiv (\vec{a} \times \vec{b})_i$ transforms under the change of coordinate system $OXYZ \mapsto OX'Y'Z'$ according to the rule

$$c_j = (+1) \ S^i_j \ c'_i$$

for a proper orthogonal transformation and

$$c_j = (-1) \ S^i_j \ c'_i$$

for an improper one (here $c_i$ and $c'_i$ are components of the cross product relative to the coordinate systems $OXYZ$ and $OX'Y'Z'$, respectively. For the definition of the transition matrix $S$ see Problem I).

(b) The cross product can also be introduced geometrically as a vector of magnitude $|\vec{a}| |\vec{b}| \sin(\phi)$ with $\phi$ being the smallest angle between $\vec{a}$ and $\vec{b}$ (i.e. $0 \leq \phi \leq \pi$). It is perpendicular to both $\vec{a}$ and $\vec{b}$ and the direction is found in the following way: If one rotates a right handed screw from $\vec{a}$ into $\vec{b}$ through the smallest possible angle then the screw would travel in the direction of $\vec{a} \times \vec{b}$ (in other words the ordered set of vectors $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ is a right-handed triple).

Show that if $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a right-handed triple of basis vectors then the “algebraic” definition is equivalent to the “geometric” one.

Problem IV

For an arbitrary proper orthogonal transformation express the matrix of finite rotations $S$ in terms of the unit vector along the axis of rotation $\vec{n} = (n_1, n_2, n_3)$ and the rotation angle $\phi$. To simplify the final expression, please use the so-called Euler parameters:

$$\rho_0 = \cos(\phi/2) , \quad \rho_i = \sin(\phi/2) \ n_i \quad (i = 1, 2, 3)$$

satisfying the relation

$$\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 1 .$$