

Superconductivity from energy fluctuations in dilute quantum critical polar metals

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(Dated: June 22, 2021)

Superconductivity in low carrier density metals challenges the conventional electron-phonon theory due to the absence of retardation required to overcome Coulomb repulsion. In quantum critical polar metals, the Coulomb repulsion is heavily screened, while the critical transverse optic phonons decouple from the electron charge. In the resulting vacuum, the residual interactions between quasiparticles are carried by energy fluctuations of the polar medium, resembling the gravitational interactions of a dark matter universe. Here we demonstrate that pairing inevitably emerges from "gravitational" interactions with the energy fluctuations, leading to a dome-like dependence of the superconducting T_c on carrier density. Our estimates show that this mechanism may explain the critical temperatures observed in doped SrTiO₃. We provide predictions for the enhancement of superconductivity near polar quantum criticality in two and three dimensional materials that can be used to test our theory.

Superconductivity exemplifies the dramatic effects of interactions in many-body quantum systems [1]. Conventional superconductors electrons exploit the electron-phonon attraction to overcome the Coulomb repulsion [2, 3] by producing a highly retarded attraction that pairs electrons, a process that requires a large ratio between the Fermi and Debye energies $E_F/\omega_D \gg 1$ [4]. A challenge to this mechanism is posed by superconductivity in low carrier polar metals. These lightly doped insulators, exemplified by doped SrTiO₃ (STO) [5] lie close to a ferroelectric quantum critical point (QCP) and exhibit bulk superconductivity down to carrier densities of order 10^{19}cm^{-3} , where the relevant phonon frequency exceeds the Fermi energy [5] by orders of magnitude. Yet despite this inversion of energy scales, experiments [6, 7] indicate a conventional s-wave condensate, with a ratio of gap to transition temperature $2\Delta/T_c \approx 3.5$ in agreement with BCS theory [7].

Several theories have been advanced to explain superconductivity in polar metals using conventional electron-phonon interaction [8] and its extension to include plasmon effects [9–14]. Alternative phonon coupling mechanisms requiring spin-orbit coupling or multiband effects [15–18] have also been examined. Recently, it has been proposed that the underlying ferroelectric quantum criticality of the polar metal is a key driver in the pairing [19–22]. However, this appealing idea encounters a difficulty, for the critical modes of a polar QCP are transverse optic (TO) phonons, which are neutral and decouple from the electrons [23–25].

These considerations motivate us to reconsider superconductivity in quantum critical polar metals, guided by two key observations: first, the strong ionic screening associated with the enhanced dielectric constant severely weakens the electronic Coulomb interaction (Fig 1 (a)); second, since the critical transverse optic phonon modes decouple from the electron charge, the resulting quan-

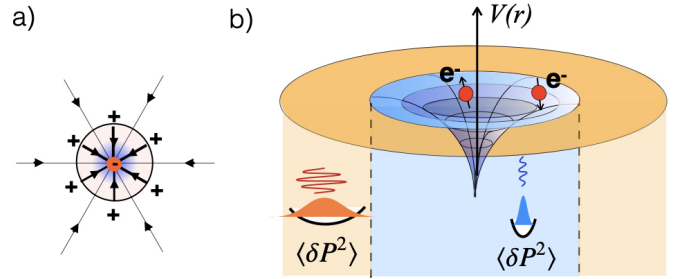


FIG. 1. Interactions between electrons in a quantum critical polar metal: (a) the electric lines of force around an electron are ionically screened, (b) the fluctuations of the phonon energy density around electrons (see (1)) create an attractive potential well.

tum fluid can be likened to dark matter, for like baryons in the cosmos, electron charge does not directly interact with the the intense background of zero-point fluctuations. Moreover, like dark matter, the quantum critical TO modes are only revealed to the electrons via their "gravitational interaction", mediated by the stress-energy tensor. The resulting interaction becomes increasingly intense at a quantum critical point; we model it by the Hamiltonian [26, 27]:

$$H_{\text{En}} = g \int d^3x \rho_e(\mathbf{x}) \vec{P}(\mathbf{x})^2 \quad (1)$$

where $\rho_e(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(x)$ is the electron density, ($\vec{P}(\mathbf{x})$)² is proportional to the energy density of the local polarization \vec{P} and g is a coupling constant with the dimensions of volume. This coupling locally suppresses the zero-point fluctuations of the polarization in the vicinity of electrons, that in turn lowers the chemical potential of electrons passing in the region (Fig. 1 (b)), creating an attractive potential well. To lowest order, the resulting attractive potential is described by the virtual

exchange of pairs of TO phonons [28], allowing us to link these ideas to two recent observations: first, that two-phonon exchange appears to drive the anomalous “high-temperature” T^2 resistivity of polar metals [27, 29], and second that two-phonon processes may drive superconductivity [30] reviving an old idea [26].

Here we study the consequences of the coupling to energy fluctuations of the order parameter (1) in quantum critical polar metals. We find that quantum criticality causes the zero-point fluctuations to intensify, increasing the attractive interaction between electrons. Moreover, while the Fermi liquid behavior is robust against these couplings, the attractive interactions mediated by the energy fluctuations overcome the Coulomb repulsion in the low density regime. Using model parameters for SrTiO₃ we find agreement with experiments both in the magnitude of T_c and in its doping dependence; finally we predict the fingerprints of this novel mechanism to be more pronounced in two-dimensional quantum critical polar systems.

Our theory is built on an isotropic model for the polar metal, with an action $S = S_e + S_C + S_{\text{En}}$, where $S_{\text{En}} = \int d\tau H_{\text{En}}$ is the energy fluctuation term (1), $S_e = \sum_k \psi_k^\dagger (\epsilon_{\mathbf{k}} - i\omega_n) \psi_k$, is the electronic action, in terms of the Fourier transformed electron field ψ_k , where $k \equiv (i\omega_n, \mathbf{k})$ is a four-vector containing the Matsubara frequency and wavevector \vec{k} and

$$S_C = \sum_q \left[\frac{|e\rho_q - (\nabla \cdot \vec{P})_q|^2}{2\varepsilon_0\varepsilon_1\vec{q}^2} + \frac{\nu_n^2 + \omega_T^2(\vec{q})}{2\varepsilon_0\Omega_0^2} |\vec{P}(q)|^2 \right], \quad (2)$$

is the Coulomb interaction between the total charge densities $e\rho_e - \nabla \cdot \vec{P}$, where $q \equiv (i\nu_n, \vec{q})$. ε_1 is the bare dielectric constant, Ω_0 is the ionic plasma frequency and $\omega_T(\vec{q})$ is the energy dispersion of the transverse optical mode. At low momenta $\omega_T^2(\vec{q}) \approx \omega_T^2 + c_s^2\vec{q}^2$, where c_s is the speed of the transverse optic mode and ω_T^2 vanishes at the QCP. The Gaussian coefficients of the polarization, $\delta^2 S_C / \delta P_a(-q) \delta P_b(q) = D_{ab}^{-1}(q)$ in (2), separate into transverse and longitudinal components

$$D_{\alpha\beta}^{-1}(q) = D_L^{-1}(q)\hat{q}_\alpha\hat{q}_\beta + D_T^{-1}(q)(1 - \hat{q}_\alpha\hat{q}_\beta) \quad (3)$$

where $D_{L,T}^{-1}(q) = (\nu_n^2 + \omega_{T,L}^2(\vec{q})) / \varepsilon_0\Omega_0^2$ are the inverse longitudinal and transverse phonon propagators. The longitudinal optic mode frequency $\omega_L^2(\vec{q}) = \omega_T^2 + \Omega_0^2/\varepsilon_1$ is shifted upwards by the Coulomb interaction.

We first consider the case where $g = 0$. Integrating over the longitudinal modes, the Coulomb interaction becomes

$$\tilde{S}_C = \sum_q \left[|\rho_q|^2 \frac{e^2}{2\varepsilon_0\varepsilon(q)\vec{q}^2} + \frac{|\vec{P}^T(q)|^2}{2D_T(q)} \right]. \quad (4)$$

Here, for $|\vec{q}| \ll q_D$

$$\varepsilon(\mathbf{q}, i\nu_n) = \varepsilon_1 + \frac{\Omega_0^2}{\nu_n^2 + c_s^2\vec{q}^2 + \omega_T^2} \quad (5)$$

is the renormalized dielectric constant, $P_a^T(q) = (\delta_{ab} - \hat{q}_a\hat{q}_b)P_b(q)$ are the transverse components of the polarization. Most importantly, in action (4) the quantum critical transverse polar modes are entirely decoupled from the electronic degrees of freedom, exemplifying the dark matter analogy.

Normally, low carrier density metals are considered *strongly interacting*, for the ratio of Coulomb to kinetic energy, determined by $r_s = 1/(k_F a_B)$, where $k_F \sim n_e^{1/3}$ is the Fermi momentum and $a_B = \frac{4\pi\varepsilon\hbar^2}{m^*e^2}$ the Bohr radius, is very large at low densities. However, in a quantum critical polar metal, the large upward renormalization of the dielectric constant, Eq. (5), can severely suppresses the interaction between the electrons. Indeed, the dielectric constant at the relevant electronic scales at low densities is $\varepsilon \sim \varepsilon(\vec{q}, \omega)|_{q=2k_F, \omega=\varepsilon_F} \approx \frac{\Omega_0^2}{(2c_s k_F)^2} \gg 1$ at the polar QCP, leading to $r_s \ll 1$. Furthermore, the electronic corrections to the dielectric constant, given in RPA by $\delta\varepsilon_{RPA} = \frac{e^2}{q^2\varepsilon_0} \Pi_e(\mathbf{q}, \nu_n)$, where $\Pi_e(q)$ is the dynamical susceptibility (Lindhardt function) of the electron gas, can be neglected. Indeed, $\frac{\delta\varepsilon_{RPA}}{\varepsilon}|_{q=2k_F, \omega=\varepsilon_F} \sim r_s \ll 1$.

The regime considered here is in stark contrast to the conventional case of the relevant frequency being of the order of $\omega_L \ll E_F$. There, the strong frequency-dependence of ε leads to a Bardeen-Pines attraction near the Fermi surface. Additionally, at low momentum transfers $v_F q \ll \omega$, a new scale appears in the problem - the electronic plasma frequency; its contribution to pairing is however suppressed by the factor ε^{-1} [11]. Thus, in what follows we neglect this possibility, approximating the electron dynamical susceptibility by its long wavelength, low frequency limit as in (5).

We next consider the effect of turning on the coupling to energy fluctuations in (1). The presence of a finite electron density $n_e = \langle \rho_e(x) \rangle$ leads to a shift in the phonon frequency:

$$\omega_{L,T}^2(n_e) = \omega_{L,T}^2 + 2gn_e\varepsilon_0\Omega_0^2, \quad (6)$$

which naturally explains the suppression of the polar state by charge doping, universally observed in polar metals [21, 31–33].

The coupling of the energy fluctuations to the electron density fluctuations $\delta\rho_e(x) = \rho_e(x) - n_e$, cannot be integrated out exactly. Interactions with critical fluctuations near QCPs can be relevant perturbations in scaling sense [34], destabilizing the Fermi liquid ground state already at weak coupling [34, 35]. In our case, however, the interaction Eq. (1) preserves the Fermi liquid. Assuming the dynamical critical exponent $z = 1$ and taking the scaling dimension of momentum $[q] = 1$, one obtains $[g] = 2 - d$, irrelevant in 3D [28], implying that the system remains a Fermi liquid even at the QCP. Thus, we can consider its effects perturbatively for weak coupling. Integrating out the field $\vec{P}(x)$ to lowest order in g , we obtain an effective

interaction between electrons:

$$\Delta S = \frac{1}{2} \int d^4x d^4x' \delta\rho_e(x) V_{En}(x-x') \delta\rho_e(x') + O(g^3) \quad (7)$$

where

$$V_{En}(x-x') = -2g^2 \text{Tr} \left[D(x-x')^2 \right] \quad (8)$$

is recognized to be an attractive density-density interaction resulting from two-phonon exchange, Fig. 1 (b). At criticality, the contribution to Eq. (8) of the transverse modes stems from their propagator

$$D_{ab}^{tr}(\vec{x}, \tau) = \varepsilon_0 \left(\frac{\Omega_0}{2\pi c_s} \right)^2 \frac{1}{\vec{x}^2 + c_s^2 \tau^2} (\delta_{ab} - \hat{x}_a \hat{x}_b), \quad (9)$$

leading to a long-range interaction of the form

$$V(\vec{x}, \tau) = -\frac{\lambda^2}{(\vec{x}^2 + c_s^2 \tau^2)^2}, \quad (10)$$

where $\lambda = \frac{g\varepsilon_0\Omega_0^2}{2\pi^2 c_s}$. Away from criticality, (10) is valid for space-time separations smaller than the correlation length $\xi = c_s/\omega_T$. The interaction at finite momentum and frequency transfer, relevant for pairing, is obtained by a Fourier transform of this expression

$$\begin{aligned} V(i\omega, \mathbf{q}) &= -\left(\frac{\lambda}{c_s}\right)^2 \int_{a_0}^{\xi} \frac{e^{i[(\vec{q}\cdot\vec{x})+\omega\tau]}}{x^4} d^4x \\ &\sim -\left(\frac{2\pi^2\lambda^2}{c_s}\right) \ln \left[\frac{a_0^{-1}}{\max(\xi^{-1}, \omega_{\mathbf{q}}/c_s)} \right]. \end{aligned} \quad (11)$$

where $\omega_{\mathbf{q}} = \sqrt{\omega^2 + c_s^2 \vec{q}^2}$. The characteristic electronic momentum and energy transfer scales are $q \sim k_F \sim n^{1/3}$ and $\omega \sim E_F \sim n^{2/3}$, respectively, resulting in $|q|^{-1} \sim n^{-1/3}$, $c_s/|\omega_{\mathbf{q}}| \sim n^{-2/3}$. Furthermore, following (6), a finite electron density leads to a finite correlation length of the order $\xi \sim n^{-1/2}$.

Consequently, the interaction character changes with density, and can be described by an effective interaction $V_{En}^{Pair}(\vec{x}, \tau)$ obtained by inverse Fourier transforming the final result of Eq. (11) (Fig. 2). For a polar metal critical at zero doping ($\omega_T(n_e = 0) = 0$), one expects that at low densities the interaction is cut off at the momentum scale of k_F , while being essentially independent of the frequency transferred, since $E_F/c_s \ll k_F$. This is identical to an instantaneous repulsion, nonlocal in real space (Fig. 2, middle). In the high density limit, the frequency-dependence of the interaction can be no longer ignored and is suppressed for frequencies beyond $c_s k_F$, qualitatively similar to a usual phonon-mediated attraction (Fig. 2, rightmost region). Finally, if $\omega_T \gg E_F$, $c_s k_F$ (leftmost region of Fig. 2), which can be realized away from QCP or at intermediate densities, the interaction $V_{En}^{Pair}(\vec{x}, \tau)$ reduces to an instantaneous local attraction.

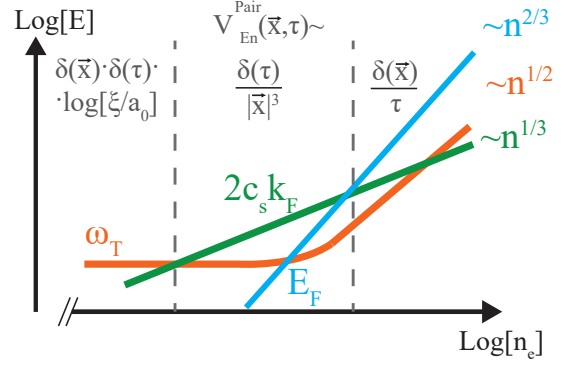


FIG. 2. Density dependence of the effective energy-fluctuation mediated electron-electron attraction $V_{En}^{Pair}(\vec{x}, \tau)$ (see text): colored lines show the relevant energy scales for the interaction; in each region (gray dashed lines) the dominant scale determines the effective form of the interaction. At low densities the interaction can be approximated with a local one. On increasing the density, the momentum dependence of the interaction first becomes prominent, while at the highest densities, strong retardation shows on the scales of order \hbar/E_F .

A more detailed calculation of the interaction potential yields

$$\begin{aligned} V_{2TO}(i\omega, \mathbf{q}) &= -\left(\frac{2\pi^2\lambda^2}{c_s}\right) \left(\log \left[\frac{\Omega_T}{\omega_T} \right] - f\left(\frac{\omega_{\mathbf{q}}}{\omega_T}\right) \right), \\ f(x) &= \frac{\sqrt{4+x^2}}{2x} \log \left[\frac{\sqrt{4+x^2}+x}{\sqrt{4+x^2}-x} \right] - \frac{1}{2}, \end{aligned} \quad (12)$$

where $\Omega_T = \max_{\vec{q}} \omega_T(\vec{q})$. Here it is important to note, that since the integral is logarithmically divergent, large momenta of the order of the Brillouin zone size contribute significantly. At such momenta, the dispersion is expected to deviate from the simple quadratic one. Thus in practical calculations, we need an average value of c_s which we approximate in the spirit of Debye approximation by $\bar{c}_s = \Omega_T / ((6\pi^2)^{1/3}/a_0)$. We remark that the contribution of the longitudinal modes to the induced interaction can be shown to be negligible in the critical regime $\Omega_T \gg \omega_T, c_s k_F$ and assuming $\frac{\Omega_0^2}{c_s^2 \kappa_D^2} \gg 1$ [28]. Additionally, in deriving the above we assumed that the momentum and energy cutoff scales are sufficiently large such that the corrections to the phonon propagator Eq. (9) due to phonon-phonon interactions (logarithmic for a QCP in 3D [36]), can be neglected.

Superconductivity at low densities: We now show that the attractive interaction mediated by the energy fluctuations *always* leads to superconductivity at low densities close to the polar QCP. More specifically, averaging the repulsive Coulomb ((4), (5)) and the attractive (12), interactions over the Fermi surface, i.e. $\langle V(k-k') \rangle = \langle V(k_F, k_F, \theta) \rangle_{\theta}$, we obtain:

$$\lambda_{eff} = N(0) \left(\left(\frac{2\pi^2\lambda^2}{c_s} \right) \log \left[\frac{\Omega_T}{2c_s k_F} \right] - \frac{4\pi e^2 c_s^2}{\Omega_0^2} \right), \quad (13)$$

where $N(0) = \frac{k_F m^*}{2\pi^2 \hbar^2}$ is the density of states. In deriving this we used that $\omega_T \sim (gn)^{1/2}$, $E_F \sim n^{2/3} \ll c_s k_F \sim n^{1/3}$. Most importantly, we find that at low enough dopings the two-phonon attraction inevitably overcomes the Coulomb repulsion due to the logarithmic enhancement of the former. In particular, the total interaction is still attractive for densities below

$$n < n_{cr} = \frac{1}{4a_0^3} e^{-3\alpha}, \quad \alpha = \frac{128\pi^5 c^5 e^2}{\Omega_0^6 g^2} \quad (14)$$

where $\alpha \sim 40 \frac{a_B}{a_0} \frac{E_h \Omega_0^5}{\Omega_0^6 T}$ for $g \sim a_0^3$. For strongly polar materials, $\Omega_0 \gg \Omega_T$ and hence $\alpha \ll 1$, so this restriction is unimportant. Moreover, from (13) the attractive part of the interaction behaves as $\lambda_{eff} \sim n^{1/3} \log(\Omega_T/c_s n^{1/3})$, describing a dome-shaped behavior of the attractive coupling constant peaked at $k_F = \Omega_T/2ec_s$ corresponding to a density:

$$n_{max} = \frac{1}{3\pi^2} \left(\frac{\Omega_T}{2ec_s} \right)^3 \approx \frac{0.01(\bar{c}_s/c_s)^3}{a_0^3}. \quad (15)$$

As the phonon dispersion flattens near the Brillouin zone edges, the average $\bar{c}_s < c_s$ so that $n_{max} \ll 1$ corresponds to a dilute charge concentration well below half filling. Finally, away from the QCP (i.e. $\omega_T(n_e = 0) \neq 0$), the Coulomb screening is reduced, resulting in an additional repulsive term $\sim 2\pi e^2 \omega_T^2 / (\Omega_0^2 k_F^2)$ (c.f. (5)). Due to its singular nature at $k_F \rightarrow 0$ this sets a lower bound on the density $n_{min} \sim \xi^{-3} [\alpha / \log(\Omega_T/\omega_T)]^{3/2} / 3\pi^2$ where the interaction is attractive.

Let us now discuss the critical temperatures of the resulting superconductor. At low densities, the interaction is essentially instantaneous (see Fig. 2). The critical temperature can then be found in the non-adiabatic weak-coupling limit to be $T_c \approx 0.28 E_F e^{-1/\lambda_{eff}}$ [37–39]. Due to the exponential dependence on the coupling constant, one expects T_c to have a dome-like shape with a maximum at n_{max} as in (15). The theory developed here also has important consequences for the dependence of T_c on external tuning parameters (e.g. pressure) in the vicinity of a polar QCP. Neglecting the residual Coulomb term in (13) and thus assuming the dominance of the energy-fluctuation attraction, one obtains for $\omega_T \gg c_s k_F$ that $\frac{d \ln T_c}{d \ln \omega_T} \sim -1 / \log^2 \frac{\Omega_T}{\omega_T}$, leading to a singular dependence of T_c on tuning parameter near the QCP. However, in the high-density limit $2c_s k_F \ll \omega_T$, the coupling constant is almost independent of the TO phonon frequency, so the tuning sensitivity will be much weaker, $\frac{d \log T_c}{d \log \omega_T} \sim \log \left(\frac{c_s k_F}{\omega_T} \right) \omega_T^2 / (c_s k_F)^2$.

2D polar metals: Similar calculations can be performed in two dimensions. While at tree level, Eq. (1) is marginal, the corrections due to quartic interactions between phonons reduce [40] the momentum-space singularities of the energy fluctuations, preserving the Fermi liquid in 2D. The 2D Fourier transformation of expression

(11) yields

$$V_{2TO}^{2D}(i\omega_n, \vec{q}) \sim \frac{g_{2D}^2 \Omega_0^4}{(4\pi)^3 c_s^2} \frac{1}{\max(\omega_T, \omega_{\mathbf{q}})} \quad (16)$$

- a stronger singularity than in 3D. In the limit $c_s k_F \gg \omega_T$ the terms due to LO phonon energy fluctuation also have to be included, being of the same order, but this does not change the qualitative form of the interaction[28]. Finally, the bare Coulomb repulsion in 2D is given by $\frac{2\pi e^2}{q}$. Screening with the polar mode and conduction electrons, however, reduces it to

$$V_C^{2D}(i\omega, \mathbf{q}) = \frac{4\pi}{\frac{\Omega_0^2 l_0 \mathbf{q}^2}{\omega_n^2 + \omega_T^2 + c^2 \mathbf{q}^2} + 4 \frac{m^* e^2}{\hbar^2}}, \quad (17)$$

where l_0 is the 2D layer thickness.

Superconductivity in SrTiO₃: We now apply these results to doped SrTiO₃. Fig. 3 (a) displays the doping dependence of T_c calculated using the parameters from literature [28] and taking $g/a_0^3 = 0.72$, a coupling constant in accord with fits to the low temperature T^2 resistivity[27]. For the energy fluctuation interaction, Eq.(12), we assumed $2c_s k_F \gg E_F$, which holds in SrTiO₃ for densities lower than $2.6 \cdot 10^{19} \text{ cm}^{-3}$. However, even at largest densities considered, $E_F \sim 3(2c_s k_F)$; as the frequency-dependence of the interaction does not alter the leading logarithmic contribution in the low-temperature limit of the pairing problem [28], we expect our approach to be qualitatively correct for all the relevant densities. In this approximation, $2\Delta/T_c = 3.53$ takes the BCS value [37, 38], in accord with STM experiments [7]. Both the magnitude and the doping dependence of the critical temperature are in good agreement with experiment. The dome-like shape of $T_c(n)$ arises from the nonmonotonic dependence of the two-phonon attractive coupling constant on density, initially rising with the density of states, subsequently decreasing as the momentum cutoff $c_s k_F$ approaches Ω_T (with the maximal value expected from (12) reached at n_{max} , Eq. (15)). This is further corroboration of the competition with Coulomb repulsion, which becomes less screened as $q \sim k_F$ grows. We note that experiments which observe a second T_c dome at lower doping also suggest that superconductivity at low doping is highly inhomogeneous and affected by the nature of the dopants [41], effects that lie beyond the current model.

As discussed above, proximity to the QCP should enhance T_c , particularly at low densities. Such a correlation has been observed for the cases of oxygen isotope substitution [20] as well as Ca-Sr substitution [21], pressure [13] or strain [42–44]. In particular, the enhancement is observed to be more pronounced at low dopings [21], in qualitative agreement with the arguments given above. Note also, that in the polar phase away from QCP the interaction (1) would still lead to pairing, however the

mode frequency would grow faster than on the disordered side, according to Landau theory. Finally, [13] in a recent experiment both T_c and ε_0 have been measured as a function of pressure. With the coupling constant being $\lambda \approx 0.25$ from the known E_F , one obtains [28] $\frac{dT_c}{dx} \approx 0.1K/kbar$ consistent with the experimental value $0.06K/kbar$ [13].

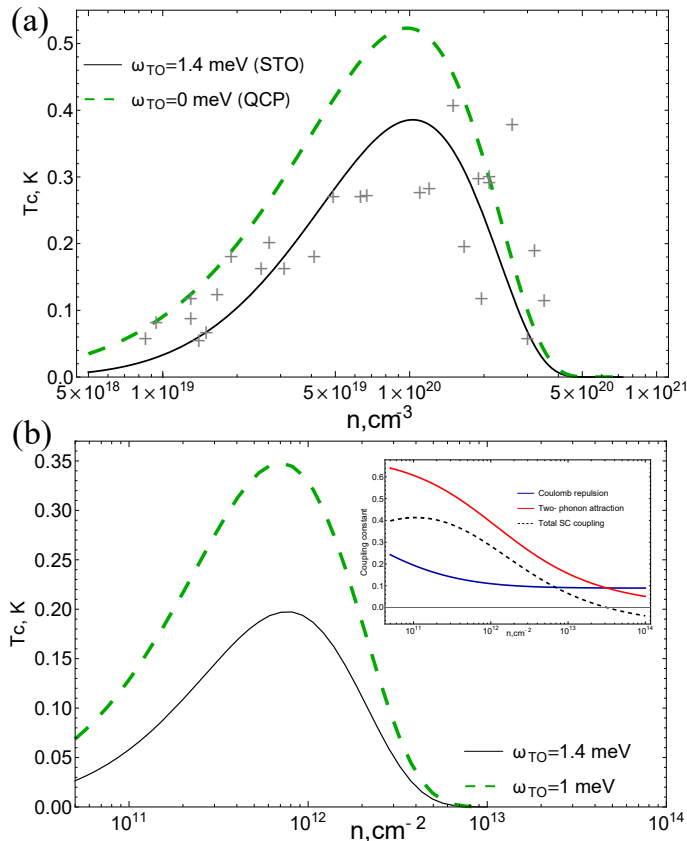


FIG. 3. (a) T_c as a function of carrier density for parameters appropriate for SrTiO₃, where the best fit is obtained for $g/a_0^3 = 0.72$. The experimental T_c is determined from the onset of the Meissner effect [45, 46]. The green line shows the $T_c(n)$ expected at the QCP. (b) same as (a) for a 2D system modeling a film of SrTiO₃ with thickness $2a_0$. $m^* = 2m_e$ is taken due to low densities. The enhancement due to proximity to the QCP (green line) is much stronger, than in 3D case (a).

Thin films of SrTiO₃ will provide a platform to explore the predictions of the energy fluctuation theory. Fig. 3 (b) displays the predicted T_c , using $T_c = 0.15E_F e^{-1/\lambda}$ for 2D instantaneous pairing ($c_s k_F \gg \omega_T$) [37–39]), using coupling constants appropriate for a two-layer thick slab of SrTiO₃. Assuming that the electrons and phonons are in the lowest lateral quantization state, we obtain $g_{2D} = g/(2a_0)$ [28]. The results show that an appreciable T_c is obtained at low densities. Furthermore, T_c is highly sensitive to the approach to criticality: a slight decrease of the TO phonon frequency essentially doubles T_c .

Conclusion: Here we have presented a new mechanism

for superconductivity in quantum critical polar metals that relies neither on retardation nor on momentum-dependence: the electrons interact with quantum critical energy fluctuations of the order parameter similar to the gravitational interactions of baryons with the dark matter. We show that this coupling of the electron density to critical energy fluctuations results naturally in a dome-shaped dependence of T_c on the carrier density n ; our estimates show that this $T_c(n)$ is consistent with that observed in doped superconducting SrTiO₃. We predict that in 2D systems, e.g. epitaxial SrTiO₃ films, the effects of energy fluctuations will be even more marked, with extreme sensitivity of T_c to the vicinity to the QCP. From a broader perspective, large energy fluctuations have been shown to exist near quantum critical points in the strong-coupling regime [47, 48]; our work presents a new mechanism for superconductivity in such strongly correlated systems. The occurrence of superconductivity mediated by energy-density fluctuations can also serve as a tool to probe the "dark matter" aspects of the solid state, involving excitations that do not interact electromagnetically, such as those related to hidden orders.

Acknowledgments: The authors gratefully acknowledge stimulating discussions with D. L. Maslov, and with the late P. W. Anderson during early stages of this work. P. A. V. is supported by a Rutgers Center for Materials Theory Fellowship, P. Chandra is supported by DOE Basic Energy Sciences grant DE-SC0020353 and P. Coleman is supported by NSF grant DMR-1830707.

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