Prove that

$$J_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}\, k!\, (k+3)!} x^{2k+2}$$

on of the Bessel differential equation of order 2:

$$x^2y'' + xy' + (x^2 - 4)y = 0$$

64. Why is it impossible to expand $f(x) = |x|$ as a power series that converges in an interval around $x = 0$? Explain using Theorem 2.

Insights and Challenges

se that the coefficients of $F(x) = \sum_{n=0}^{\infty} a_n x^n$ are *periodic*; for some whole number $M > 0$, we have $a_{M+n} = a_n$. Prove converges absolutely for $|x| < 1$ and that

$$F(x) = \frac{a_0 + a_1x + \cdots + a_{M-1}x^{M-1}}{1 - x^M}$$

Use the hint for Exercise 53.

tinuity of Power Series Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power with radius of convergence $R > 0$.

ve the inequality

$$|x^n - y^n| \le n|x - y|(|x|^{n-1} + |y|^{n-1}) \tag{10}$$

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$.

(b) Choose R_1 with $0 < R_1 < R$. Show that the infinite series $M = \sum_{n=0}^{\infty} 2n|a_n|R_1^n$ converges. *Hint:* Show that $n|a_n|R_1^n < |a_n|x^n$ for all n sufficiently large if $R_1 < x < R$.

(c) Use Eq. (10) to show that if $|x| < R_1$ and $|y| < R_1$, then $|F(x) - F(y)| \le M|x - y|$.

(d) Prove that if $|x| < R$, then $F(x)$ is continuous at x. *Hint:* Choose R_1 such that $|x| < R_1 < R$. Show that if $\epsilon > 0$ is given, then $|F(x) - F(y)| \le \epsilon$ for all y such that $|x - y| < \delta$, where δ is any positive number that is less than ϵ/M and $R_1 - |x|$ (see Figure 6).

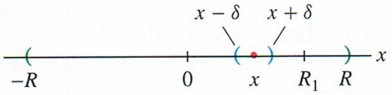

FIGURE 6 If $x > 0$, choose $\delta > 0$ less than ϵ/M and $R_1 - x$.

10.7 Taylor Series

In this section we develop general methods for finding power series representations. Suppose that $f(x)$ is represented by a power series centered at $x = c$ on an interval $(c - R, c + R)$ with $R > 0$:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots$$

According to Theorem 2 in Section 10.6, we can compute the derivatives of $f(x)$ by differentiating the series expansion term by term:

$$\begin{aligned}
f(x) &= a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots \\
f'(x) &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + \cdots \\
f''(x) &= 2a_2 + 2 \cdot 3a_3(x - c) + 3 \cdot 4a_4(x - c)^2 + 4 \cdot 5a_5(x - c)^3 + \cdots \\
f'''(x) &= 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x - 2) + 3 \cdot 4 \cdot 5a_5(x - 2)^2 + \cdots
\end{aligned}$$

In general,

$$f^{(k)}(x) = k!a_k + \Big(2 \cdot 3 \cdots (k + 1)\Big)a_{k+1}(x - c) + \cdots$$

Setting $x = c$ in each of these series, we find that

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad f'''(c) = 2 \cdot 3a_2, \quad \ldots, \quad f^{(k)}(c) = k!a_k, \quad \ldots$$

We see that a_k is the kth coefficient of the Taylor polynomial studied in Section 8.4:

$$\boxed{a_k = \frac{f^{(k)}(c)}{k!}}$$

Therefore $f(x) = T(x)$, where $T(x)$ is the **Taylor series** of $f(x)$ centered at $x = c$

$$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

This proves the next theorem.

THEOREM 1 Taylor Series Expansion If $f(x)$ is represented by a power series centered at c in an interval $|x-c| < R$ with $R > 0$, then that power series is the Tay
series

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n}$$

In the special case $c = 0$, $T(x)$ is also called the **Maclaurin series**:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 +}$$

EXAMPLE 1 Find the Taylor series for $f(x) = x^{-3}$ centered at $c = 1$.

Solution The derivatives of $f(x)$ are $f'(x) = -3x^{-4}$, $f''(x) = (-3)(-4)x^{-5}$, an
general,

$$f^{(n)}(x) = (-1)^n(3)(4)\cdots(n+2)x^{-3-n}$$

Note that $(3)(4)\cdots(n+2) = \frac{1}{2}(n+2)!$. Therefore,

$$f^{(n)}(1) = (-1)^n \frac{1}{2}(n+2)!$$

Noting that $(n+2)! = (n+2)(n+1)n!$, we write the coefficients of the Taylor se

$$a_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^n \frac{1}{2}(n+2)!}{n!} = (-1)^n \frac{(n+2)(n+1)}{2}$$

The Taylor series for $f(x) = x^{-3}$ centered at $c = 1$ is

$$T(x) = 1 - 3(x-1) + 6(x-1)^2 - 10(x-1)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2}(x-1)^n$$

Theorem 1 tells us that if we want to represent a function $f(x)$ by a powe
centered at c, then the only candidate for the job is the Taylor series:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Exercise 92 for an example where a series $T(x)$ converges but does not converge to $f(x)$.

However, *there is no guarantee that $T(x)$ converges to $f(x)$*, even if $T(x)$ converges. To study convergence, we consider the kth partial sum, which is the Taylor polynomial of degree k:

$$T_k(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k$$

In Section 8.4, we defined the remainder

$$R_k(x) = f(x) - T_k(x)$$

Since $T(x)$ is the limit of the partial sums $T_k(x)$, we see that

The Taylor series converges to $f(x)$ if and only if $\lim_{k\to\infty} R_k(x) = 0$.

There is no general method for determining whether $R_k(x)$ tends to zero, but the following theorem can be applied in some important cases.

REMINDER *$f(x)$ is called "infinitely differentiable" if $f^{(n)}(x)$ exists for all n.*

THEOREM 2 Let $I = (c-R, c+R)$, where $R > 0$. Suppose there exists $K > 0$ such that all derivatives of f are bounded by K on I:

$$|f^{(k)}(x)| \le K \qquad \text{for all} \quad k \ge 0 \quad \text{and} \quad x \in I$$

Then $f(x)$ is represented by its Taylor series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \qquad \text{for all} \quad x \in I$$

Proof According to the Error Bound for Taylor polynomials (Theorem 2 in Section 8.4),

$$|R_k(x)| = |f(x) - T_k(x)| \le K\frac{|x-c|^{k+1}}{(k+1)!}$$

If $x \in I$, then $|x-c| < R$ and

$$|R_k(x)| \le K\frac{R^{k+1}}{(k+1)!}$$

We showed in Example 9 of Section 10.1 that $R^k/k!$ tends to zero as $k \to \infty$. Therefore, $\lim_{k\to\infty} R_k(x) = 0$ for all $x \in (c-R, c+R)$, as required. ■

Taylor expansions were studied throughout the seventeenth and eighteenth centuries by Gregory, Leibniz, Newton, Maclaurin, Taylor, Euler, and others. These developments were anticipated by the great Indian mathematician Madhava (c. 1340–1425), who discovered the expansions of sine and cosine and many other results two centuries earlier.

■ **EXAMPLE 2 Expansions of Sine and Cosine** Show that the following Maclaurin expansions are valid for all x.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Solution Recall that the derivatives of $f(x) = \sin x$ and their values at $x = 0$ form a repeating pattern of period 4:

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$\cdots$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cdots$
0	1	0	-1	0	$\cdots$

In other words, the even derivatives are zero and the odd derivatives alternate in sign: $f^{(2n+1)}(0) = (-1)^n$. Therefore, the nonzero Taylor coefficients for $\sin x$ are

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

For $f(x) = \cos x$, the situation is reversed. The odd derivatives are zero and the even derivatives alternate in sign: $f^{(2n)}(0) = (-1)^n \cos 0 = (-1)^n$. Therefore the nonzero Taylor coefficients for $\cos x$ are $a_{2n} = (-1)^n/(2n)!$.

We can apply Theorem 2 with $K = 1$ and any value of R because both sine and cosine satisfy $|f^{(n)}(x)| \le 1$ for all x and n. The conclusion is that the Taylor series converges to $f(x)$ for $|x| < R$. Since R is arbitrary, the Taylor expansions hold for all x.

■ **EXAMPLE 3** **Taylor Expansion of $f(x) = e^x$ at $x = c$** Find the Taylor series $T(x)$ at $f(x) = e^x$ at $x = c$.

Solution We have $f^{(n)}(c) = e^c$ for all x, and thus

$$T(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$$

Because e^x is increasing for all $R > 0$ we have $|f^{(k)}(x)| \le e^{c+R}$ for $x \in (c - R, c + R)$. Applying Theorem 2 with $K = e^{c+R}$, we conclude that $T(x)$ converges to $f(x)$ for all $x \in (c - R, c + R)$. Since R is arbitrary, the Taylor expansion holds for all x. For $c = 0$ we obtain the standard Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Shortcuts to Finding Taylor Series

There are several methods for generating new Taylor series from known ones. First of all, we can differentiate and integrate Taylor series term by term within its interval of convergence, by Theorem 2 of Section 10.6. We can also multiply two Taylor series or substitute one Taylor series into another (we omit the proofs of these facts).

In Example 4, we can also write the Maclaurin series as

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

■ **EXAMPLE 4** Find the Maclaurin series for $f(x) = x^2 e^x$.

Solution Multiply the known Maclaurin series for e^x by x^2.

$$x^2 e^x = x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \frac{x^7}{5!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}$$