June 2005

Amsterdam Talk

Based on Work w/ D. Belov, D. Freed, G. Segal.

1. Introduction
2. Generalized Maxwell + D.C.S.
3. Electric + Magnetic Flux
4. RR Fields
   a.) Lagrangian
   b.) Hamiltonian

1// Introduction.

I'm going to be talking about generalized abelian gauge theories. These theories are free theories so in a sense they are trivial, but I'll be focussing on topological questions which seem nontrivial, at least to me.

My motivation is:

(a.) We should know what we're talking about. Here it is possible to be very precise about the proper mathematical formulation of B-fields and RR-fields, and it is nontrivial.

(b.) Possible applications to flux compactifications. These topological subtleties do constrain flux compact's. So far the constraints have been modest. But there is more to understand.

Results of this talk are:

(a.) I'll point out an interesting subtlety in the formulation of the Hilbert space of GAGT's. The main application at present is this: Consider type II strings on $X_9 \times \mathbb{R}$, with X_q compact. Then it is usually said that the superselection sectors of the theory are given by a K-theory class. So the Hilbert space is graded:

$$\mathcal{H} = \bigoplus_{x \in K_B^{\epsilon}(X_q)} \mathcal{H}_x \qquad \epsilon = 0/1 \quad \text{IIA/IIB}$$

This is actually not true!

(b.) I will comment on the action for self-dual fields, like the RR field, especially the "Chern-Simons term." Recent literature shows that even some of the best people ~~in~~ our field don't understand it.

<u>Framework</u>

The right framework for GAGT's appears to be "differential generalized cohomology."

Singular cohomology H^* can be characterized by axioms - essentially naturality + gluing, or Meyer-Vietoris, plus the dimension axiom

$$H^k(pt; G) = G\,\delta_{k,0} \qquad G = \text{coefficient group.}$$

Notation $H^k(M)$ means $H^k(M; \mathbb{Z})$ unless explicitly said otherwise.

Generalized cohomology retains all axioms but the last and is important in topology. K-theory is an example. Great development of the 60's.

Of course to do physics we need local fields constrained by certain topological considerations - this is what leads to differential generalized cohomology. ~~Examples include:~~

This is a recent mathematical development - foundational paper is Hopkins + Singer.

nice review of Freed

Examples include

3A

- Deligne–Cheeger–Simons $\check{H}^{\ell}(M)$ · B-fields · Self-dual 3-form on M5
- Differential K: $\check{K}^{\epsilon}(M)$ type II RR
- Differential KO: $\check{KO}$ type I, orientifolds.

⋮

Field theory whose gauge inv't space of fields is a G.C.T. = GAGT.

2/1 Generalized Maxwell Theory

Maxwell theory = theory of connection 1-form on a line bundle.

For a fixed line bundle $\mathcal{L} \to M$, the space of gauge equivalence classes is

$$\underbrace{\mathcal{A}(\mathcal{L})}_{\text{conn's}} / \underbrace{\mathcal{G}}_{\text{gauge group}}$$

If we consider all line bundles together

$$\{\text{gauge equiv. classes}\} = \bigcup_{c_1 \in H^2(M)} \mathcal{A}(\mathcal{L}_{c_1})/\mathcal{G}$$

This is known as the D-C-S differential cohomology ~~group~~ $\check{H}^2(M)$.

The gauge equivalence class of A is given by its holonomy function

$$Z_1 \longrightarrow U(1) = \exp(2\pi i\, \mathbb{R}/\mathbb{Z})$$

$$\Sigma \longrightarrow \exp\left(2\pi i \int_{\Sigma} A\right)$$

In general

Def: $\check{H}^{\ell}(M)$ = group of homomorphisms $Z_{\ell-1}(M) \to U(1)$

s.t. $\Sigma = \partial \mathcal{B}$ then $\chi_A(\Sigma) = \exp\left(2\pi i \int_{\mathcal{B}} F\right)$

for some $F \in \Omega^{\ell}_{d}(M)$. = d-closed ℓ-forms, called the fieldstrength.

Def: $\check{H}^{\ell}(M) = \{ \text{Hom's } \chi : Z_{\ell-1}(M) \longrightarrow U(1) \}$

Any such homomorphism has the following property:

$\exists\ F \in \Omega^{\ell}(M)$ s.t. $\Sigma = \partial \mathcal{B}$

$$\chi(\Sigma) = \exp\left(2\pi i \int_{\mathcal{B}} F\right)$$

Now – by "generalized Maxwell" I mean a fieldtheory ~~with fields $[\check{A}]$~~ whose gauge equivalence classes of fields are

$$[\check{A}] \in \check{H}^{\ell}(M)$$

I denote χ by $[\check{A}]$ because I think in terms of some underlying "gauge potential"

Now I'm going to give a small math lesson on differential cohomology $\check{H}^{\ell}$ because we'll need several facts to make the basic physics point.

1.) Note that small changes in $\mathcal{B}$ don't change holonomy so $F \in \Omega^{\ell}_{d} \Longrightarrow [F] \in H^2_{DR} \cong H^2(M;\mathbb{R})$

Moreover, big changes in $\mathcal{B}$ don't matter so $F \in \Omega^{\ell}_{\mathbb{Z}}$ has integral periods: so $[F]$ lies in

~~$[F] \in \bar{H}^2(M) = H^2(M) / H^2_T(M)$~~ ~~← torsion group~~

The image $H^2(M;\mathbb{Z}) \hookrightarrow H^2(M;\mathbb{R})$ full lattice

You might ask – why not just work with fluxes with integral periods? But this misses some information –

What we miss are the topologically trivial flat fields

$$0 \to H^{\ell-1}(M) \otimes \mathbb{R}/\mathbb{Z} \to \check{H}^\ell(M) \to \mathcal{R} \to 0$$

This can be put differently in a useful way

$$0 \to \underbrace{H^{\ell-1}(M, \mathbb{R}/\mathbb{Z})}_{\text{flat}} \to \check{H}^\ell(M) \to \underset{F}{\Omega^\ell_{\mathbb{Z}}} \to 0$$

$$0 \to \underbrace{\Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}}}_{\text{top. trivial}} \to \check{H}^\ell(M) \to \underset{a}{H^\ell(M, \mathbb{Z})} \to 0$$

keep

$$A \rightsquigarrow \left(\Sigma \to \exp\left(2\pi i \int_\Sigma A\right) \right)$$

Notice two things
Flat fields need not be topologically trivial

1.) $H^{\ell-1}(M, \mathbb{R}/\mathbb{Z})$ = abelian group = $\underset{1 \atop H^{\ell-1}(M)\otimes \mathbb{R}/\mathbb{Z}}{(0)} + \underbrace{(0) + (0) \cdots + (0)}_{\text{group of cpts } = H^\ell_T(M)}$

2.) Topologically trivial fields are not just a v.s.

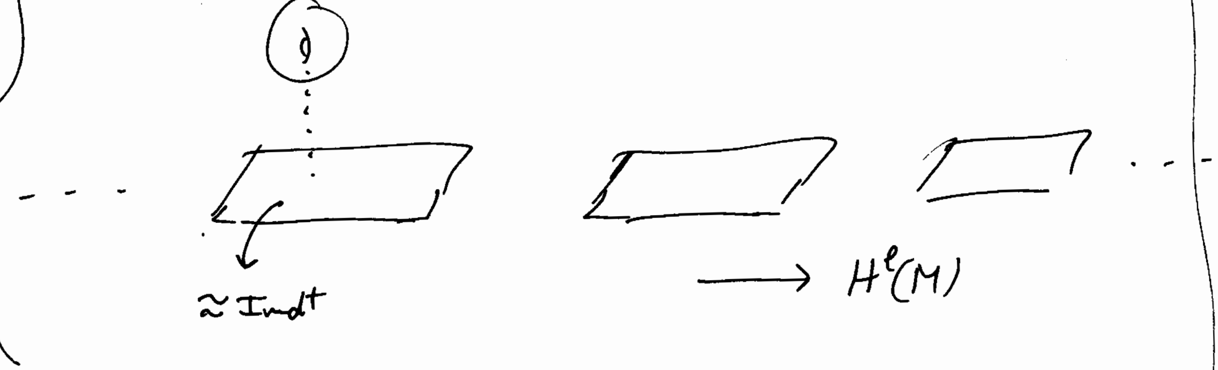
$\dfrac{\mathcal{H}^{\ell-1}}{\mathcal{H}^{\ell-1}_{\mathbb{Z}}}$ = torus

$$\cong \frac{\Omega^{\ell-1}_d}{\Omega^{\ell-1}_{\mathbb{Z}}} \to \Omega^{\ell-1}/\Omega^{\ell-1}_{\mathbb{Z}}$$

$$\downarrow$$

$$\Omega^{\ell-1}/\Omega^{\ell-1}_d \cong \mathrm{Im}\, d^\dagger = \text{vector space}$$

So the picture of $\check{H}^\ell$ is



→ keep this picture

A little math lesson on diff'l cohomology:

From $$\exp\left(2\pi i \int_\gamma A\right) = \exp\left(2\pi i \int_{\mathcal{B}} F\right)$$

We see that F is closed.

Thus it defines a class $[F] \in H^2_{DR} \cong H^2(M;\mathbb{R})$

Moreover it has integral periods: $[F] \in \bar{H}^2(M) := H^2(M)/H^2_T(M)$

do we use this notation

$H^2_T(M)$ ↖ torsion group.

But the Cheeger Simons character χ_A has more information:

Suppose $k\Sigma = \partial \mathcal{B}$ then

$$\left(\exp 2\pi i \int_\Sigma A\right)^k = \exp\left(2\pi i \int_{\mathcal{B}} F\right)$$

But how to take the k^{th} root? That extra information is contained in the "torsion part" of the characteristic class:

$$a(\chi_A) \in H^k(M;\mathbb{Z})$$

So we have maps

$$\begin{tikzcd}
\check{H}^\ell(M) \arrow[r] \arrow[d] & \Omega^\ell_d(M) \ni F \arrow[d] \\
a \in H^\ell(M) \arrow[r] & H^\ell(M;\mathbb{R}) \; ?
\end{tikzcd}$$

But $\check{H}^\ell(M)$ is NOT the setwise fiber product.

Denote that by

$$\mathcal{R} = \{ (a, F) \mid \bar{a} = [F]_{DR} \}$$

Examples

1.) $\check{H}^1(M) = \{ f : M \to U(1) \}$

$$a = f^*[d\theta] \qquad \omega = \frac{1}{2\pi i} f^{-1} df$$

1.) $\check{H}^2(M) = \{\text{line bundles w/ connection}\}/\sim$

Now, there is a multiplication on this cohomology theory:

$$\check{H}^{\ell_1} \times \check{H}^{\ell_2} \longrightarrow \check{H}^{\ell_1+\ell_2}$$

$$a_1 \cup a_2$$

$$F_1 \wedge F_2$$

but the holonomy is subtle.

<u>Exercise</u>: Construct a line bundle w/ connection from two circle-valued functions.

Moreover, if M is <u>oriented</u>, $\dim M = n$, then there is an integration

$$\int_M^{\check{H}} : \check{H}^{n+1}(M) \longrightarrow \mathbb{R}/\mathbb{Z}$$

Again - it is subtle - <u>not</u> the integral of the fieldstrength. Examples

1.) $\int_{pt}^{\check{H}} : \check{H}^1(pt) \longrightarrow \mathbb{R}/\mathbb{Z}$

2.) $\int_{S^1}^{\check{H}} : \check{H}^2(S^1) \to \mathbb{R}/\mathbb{Z}$ is $\frac{1}{2\pi i}\log\left(\mathrm{hol}(A)\right)$ on

3.) $\dim M = 2p+1$ $\quad [\check{A}] \in \check{H}^p(M)$ then

$$\int_M^{\check{H}} \check{A} \cdot \check{A} = \text{Chern-Simons term. Usually written } \int_M^{H} A\,dA$$

In general – in (differential) generalized cohomology theory, $\check{h}$, if

$$\begin{array}{ccl} M \longrightarrow & \mathcal{X} & \\ & \downarrow & \\ & S & \end{array}$$

is a <u>family</u> of n-manifolds that are properly oriented for that cohomology theory

Then there is a notion of integration in that ~~[illegible]~~ theory:

$$\int^{\check{h}} : \check{h}^{k}(\mathcal{X}) \longrightarrow \check{h}^{k-n}(S)$$

So here

$$\int_M^{\check{H}} : \check{H}^{n+1}(M) \longrightarrow \check{H}^{1}(pt) = \mathbb{R}/\mathbb{Z}$$

Finally, we have Poincaré duality

M cpt, oriented

$$\check{H}^{\ell}(M) \times \check{H}^{n+1-\ell}(M) \longrightarrow U(1)$$

<u>$\dim M = n$</u>.

is a perfect-pairing.

$$\boxed{\begin{array}{c} 0 \to H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^{\ell}(M) \longrightarrow \underset{}{\overset{F}{\overset{\wedge}{\Omega^{\ell}_{\mathbb{Z}}}}} \to 0 \\[2ex] 0 \to \underset{\underset{A_D}{\Psi}}{\Omega^{n-\ell}}/\Omega^{n-\ell}_{\mathbb{Z}} \to \check{H}^{n+1-\ell}(M) \longrightarrow H^{n+1-\ell}(M) \to 0 \end{array}}$$

keep.

pairing of A_D, F is $\exp\left(2\pi i \int_M^{H} A_D F\right)$

Note that the pairing on topologically trivial and flat is trivial.

3/1 Hamiltonian Formulation of Generalized Maxwell

Now assume $M = X \times \mathbb{R}$ X = space = compact, oriented $\dim X = n-1$.

Consider ~~Def~~ Gen. Maxwell $[\check{A}] \in \check{H}^\ell(M)$ and

$$S = \int \tfrac{1}{2}\lambda^{-1} F * F \Rightarrow$$

The Hilbert space is $\mathcal{H} = L^2(\check{H}^\ell(X))$

Let's note that it is naturally graded by the topological class of the magnetic flux.

$$\pi_0(\check{H}^\ell(X)) = H^\ell(X) \ni m$$

$$\mathcal{H} = \bigoplus_m \mathcal{H}_m$$

But now we have Poincaré duality

$$\check{H}^\ell(X) \times \check{H}^{n-\ell}(X) \to U(1)$$

So there is a dual formulation w/ $[\check{A}_D] \in \check{H}^{n-\ell}(M)$

So $\mathcal{H} = \bigoplus_e \mathcal{H}_e \qquad e \in H^{n-\ell}(X)$

Can we grade by both?

$$\mathcal{H} \stackrel{?}{=} \bigoplus_{e,m} \mathcal{H}_{e,m}.$$

~~Key Observation: $\check{H}^\ell(X)$ is just an abelian group.~~

Intuition should be - yes

$$\Big[\underbrace{\int_X \omega_1 F}_{\text{measures magnetic flux}}, \underbrace{\int_X \omega_2 * F}_{\text{measures electric flux}}\Big] = \Big[\int_X \omega_1 F, \int_X \omega_2 \Pi\Big] = \int_X \omega_1 d\omega_2 = 0$$

for ω_1, ω_2 closed

Let's have a closer look.

A key remark: $\check{H}^{\ell}(X)$ = abelian group

Now - simple mathematical fact.

Let S be any abelian group. Then $L^2(S)$ is a rep$^{\underline{n}}$ of:

of S: $s_0 \in S$ $\quad (L_{s_0}\psi)(s) = \psi(s+s_0)$

of $\check{S}$: $\chi \in \check{S}$ $\quad (m_\chi \psi)(s) = \chi(s)\psi(s)$

But $L^2(S)$ is not a rep$^{\underline{n}}$ of $S \times \check{S}$ because

$$L_{s_0} m_\chi = \chi(s_0)\, m_\chi L_{s_0}$$

For exactly this reason it is a rep$^{\underline{n}}$ of

~~(cocycle: $c((\chi_1,s_1),(\chi_2,s_2)) = \chi_1(s_2)$)~~

~~Because we have a perfect pairing $L^2(S)$ = unique irrep of~~

$$1 \to U(1) \to \mathrm{Heis}(S\times\check{S}) \to S\times\check{S} \to 1$$

defined by cocycle $c\big((\chi_1,s_1),(\chi_2,s_2)\big) = \chi_1(s_2)$

Moreover, we could define $L^2(S)$ as the unique irrep of $\mathrm{Heis}(S\times\check{S})$. This uniqueness relies on the perfect pairing

So you can decompose $L^2(S)$ wrt characters of S or characters of $\check{S}$, but not both.

Let's apply that to our case

Now $S = \check{H}^{\ell}(X)$

A state which transforms as a character of the group of translations is a state of definite electric flux:

Def (a) A state of definite electric flux is a translation eigenstate

$$\forall \check{\phi} \in \check{H}^{\ell} \qquad \psi(\check{A}+\check{\phi}) = \exp\left(2\pi i \int^{\check{H}} \check{\phi}\check{\mathcal{E}}\right) \psi(\check{A})$$

$\check{\mathcal{E}} \in \check{H}^{n-\ell}(X)$ is the electric flux.

Working with states of definite electric flux is not useful. We are only interested in

Def. (b) A state of definite topological electric flux is

$$\forall \check{\phi}_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \qquad \psi(\check{A}+\check{\phi}) = \exp\left(2\pi i \int^{H} \phi_f e\right) \psi(\check{A})$$

$$e = a(\check{\mathcal{E}}) \in H^{n-\ell}(X, \mathbb{Z}).$$

Now, suppose we are in a state of definite topological class of magnetic flux $m \in H^{\ell}(X)$ — that means

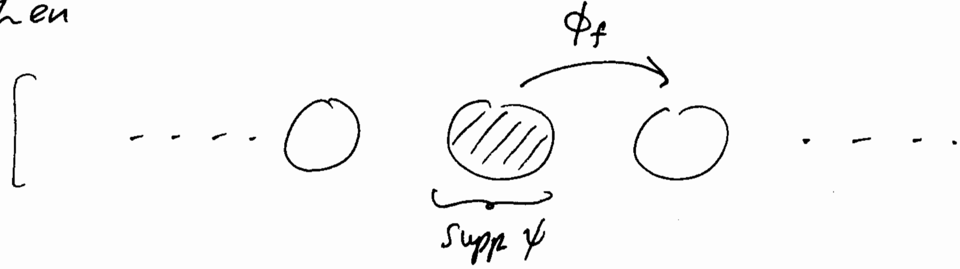
$$\text{Support}(\psi) \in \check{H}^{\ell}(X)_m$$

Now we easily see that it is impossible for such a state to be in an eigenstate of electric flux because

There can be topologically nontrivial flat fields

If $\phi_f \in$ a nontrivial component of $H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$ then

refer to previous figure



$\Longrightarrow$ Electric and magnetic flux cannot be simultaneously measured!

What's going on:

$\bigoplus_m \mathcal{H}_m$ grading by characters of $H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z}) \subset \check{H}^{n-\ell}(X)$

char $= H^{\ell}(X)$

$\bigoplus_e \mathcal{H}_e$ " " " " $H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \subset \check{H}^{\ell}(X)$

char $= H^{n-\ell}(X)$

But the Heisenberg pairing on

$$H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z}) \times H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$$

is <u>nontrivial</u>.

It is trivial on the connected cpt. of 1, but given by the <u>torsion pairing</u>:

$$H^{\ell}_{T}(X) \times H^{n-\ell}_{T}(X) \longrightarrow U(1)$$

which is a perfect pairing. This is what prevents simultaneous measurement.

In fact more is true:

In fact $L^2(\check{H}^{\ell}(X))$ is the unique irrep of $\mathrm{Heis}\left(\check{H}^{\ell}(X) \times \check{H}^{n-\ell}(X)\right)$. Since

$H^{\ell-1}(X) \otimes \mathbb{R}/\mathbb{Z} \times H^{n-\ell-1}(X) \otimes \mathbb{R}/\mathbb{Z}$ commutes we can grade by characters of this group and write

$$\mathcal{H} = \bigoplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}}$$

Then the finite Heisenberg group $\mathrm{Heis}\left(H^{\ell}_{T} \times H^{n-\ell}_{T}\right)$ acts on this, permuting factors

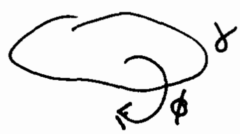
Example: 3+1 Maxwell $[\check{A}] \in \check{H}^2(M)$

Let $M = S^3/\mathbb{Z}_k \times \mathbb{R}$. $H^2(S^3/\mathbb{Z}_k) = \mathbb{Z}_k$.

We can grade $\bigoplus_e \mathcal{H}_e$ or $\bigoplus_m \mathcal{H}_e$, but not both. $\mathrm{Heis}(H^2_T \times H^2_T)$ is the familiar finite Heisenberg group of clock + shift operators.

Close with 2 remarks

1.) A closely related observation of Gukov-Rangamani-Witten: Maxwell on $S^3/\mathbb{Z}_k$. 't Hooft operators?



$g \sim e^{i\phi}$ but impossible to extend.

$\longrightarrow$ they defined an 't Hooft operator so that Wilson + 't Hooft formed a finite Heisenberg group. Would be hard to generalize that discussion.

2.) We've given a manifestly e-m dual description of the Hilbert space. This leads to an obvious definition of the H.S. for the selfdual p-form in 2p dimensions: $\dim X = 2p-1$

$$\mathcal{H}_{SD} = \text{Unique irrep of } \mathrm{Heis}(\check{H}^p(X))$$

4// RR Fields

Now let's apply these ideas to type II RR

First we have

$$[\check{B}] \in \check{H}^3(M_{10})$$

Next - by anomaly cancellation, et. al. the proper mathematical def$^{\underline{n}}$ of the RR field is

$$[\check{C}] \in \check{K}^{\epsilon}_{\check{B}}(M) \qquad \epsilon = 0/1 \quad IIA/IIB$$

What is this group?

Similar picture to what went before

Useful to introduce the ring $R = \mathbb{R}[u, u^{-1}]$ $\deg u = 2$ "Bott-element"

$$\Omega^{0}_{d_H}(M; R) = \{ d_H \text{ closed } \overset{\text{even}}{\cancel{\text{even}}} \text{ forms w/ coeff's in } R \}$$

$$d_H = d - H$$

Similarly for $\Omega^1_{d_H}$

$$= \{ \tilde{F} = \tilde{F}_0 + u^{-1}\tilde{F}_2 + \cdots \qquad d_H \tilde{F} = 0 \}$$

$$K^{\epsilon}_{B}(M) = \text{twisted topological } K$$

Take the setwise fiber product

$$\begin{tikzcd}
R_K \arrow[r] \arrow[d] & \Omega^{\epsilon}_{d_H}(M; R) \arrow[d] & \ni \tilde{F} \arrow[d] \\
K^{\epsilon}_{B}(M) \arrow[r] & H^{\epsilon}_{d_H}(M) & {[\tilde{F}]_{d_H}}
\end{tikzcd}$$

$$x \longmapsto ch_B(x)\sqrt{\hat{A}} \; = [\tilde{F}]_{d_H}$$

As before, we must extend by the topologically trivial flat fields

$$0 \to K^{\epsilon-1}(M) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow \check{K}_B^{\epsilon}(M) \longrightarrow \mathcal{R}_K \to 0$$

Example:

$$\check{K}^0(pt) = \mathbb{Z} \qquad \check{K}^{-1}(pt) = \check{K}^{+1}(pt) = \mathbb{R}/\mathbb{Z}$$

Again we can write two exact sequences.

Let $\Omega^{\epsilon}_{d_H,\mathbb{Z}}(M,R)$ be the forms whose classes $[\tilde{F}]$ are images of $ch_B\sqrt{\hat{A}}$. Then the analog of before is:

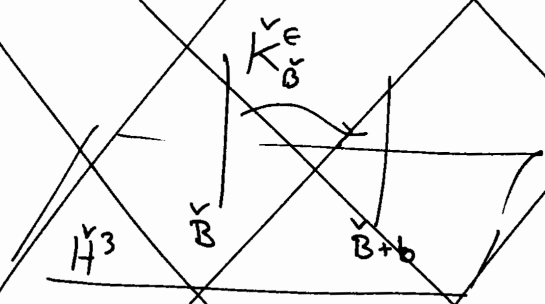
$$0 \to K_B^{-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}_B^0(M) \longrightarrow \Omega^0_{d_H,\mathbb{Z}}(M;R) \to 0$$

keep

$$0 \to \Omega^{odd}(M)/\Omega^1_{d_H,\mathbb{Z}}(M) \longrightarrow \check{K}_B^0(M) \longrightarrow K_B^0(M) \longrightarrow 0$$

$$0 \to K^{\epsilon-1}(M)\otimes \mathbb{R}/\mathbb{Z} \longrightarrow \check{K}^{\epsilon}_{\check{B}}(M) \longrightarrow R_K \to 0$$

The whole set of fields is fibered over the B-fields



omit

needed?

Connection takes $R \to e^b R$.

Example:

$$\check{K}^0(pt) = \mathbb{Z} \qquad \check{K}^{-1}(pt) = \mathbb{R}/\mathbb{Z}$$

Now there is a new ingredient: The RR field is selfdual both in IIA and in IIB, so it must be handled with care, since in general we don't understand selfdual fields very well at the global level.

There are two issues to discuss:

4A// How do we form the action?

4B// What is the Hilbert space? How is it graded?

4A: Advertisement: General viewpoint on selfdual actions.

- nice way to get conformal blocks in SD theories
- gives Chern-Simons terms in type II sugra for general backgnd fluxes.

4A// RR action

Because the field is self-dual there is no natural action – the action depends on a "Lagrangian splitting" of the space of fields.

I'll take the point of view that the action is essentially a period matrix:

To justify this lets consider a self-dual p-form in $2p$.

The best way to look at this theory is that it is holographically dual to $[\check{A}] \in \check{H}^{p+1}(M)$ in $\dim M = 2p+1$

$$S = \int \frac{1}{e^2} F * F + \pi \int^{\check{H}} k\, \check{A} \cdot \check{A}$$

Heuristic:

Focus on long distance $\longrightarrow$ $\int_M k\, A dA$.

$$\delta \int_M A dA = \int_M \delta A\, dA + \int_X \delta A\, A$$

p odd: BVP good for $A = *A$.

Gauge transformations $A \to A + \omega$ $\quad \omega \in \Omega^p_{\mathbb{Z}}$

a.) Gauge modes on the boundary are dynamical

b.) B.C. $\Rightarrow$ $\omega = * \omega$.

In fact a careful analysis really derives the conformal blocks of the S.D. theory

Gauss law: $\Psi(A+\omega) = e_\omega(A)\Psi(A)$

Groundstate $\quad \Psi(A) = e^{g(A,A)} \Phi(A^{1,0}) \quad A \in \mathcal{H}^p(X)$

$e^2, * \Rightarrow$ Kähler structure on $\mathcal{H}^p(X)$

Those two conditions $\Rightarrow$ $\Psi \sim \Theta$-function

$$\sim \sum_{\omega \in \Lambda_1} \underbrace{e^{i\pi\tau(\omega+\theta)}}_{\text{interpret as } e^{-S}} e^{2\pi i(\omega+\theta,\phi)}$$

Examples:

(1.) M5 brane

(2.) (B_2, C_2) on "$AdS_5 \times T_5$"

(3.) $p=1$: Classification of abelian spin C.S.

(4.) Type II RR fields.

Details:

$$g(R_1, R_2) = \int R_1 * R_2$$

$$\omega(\check{x}_1, \check{x}_2) = \int R_1 \wedge \overline{R_2}$$

Choose a splitting $\mathcal{R}_K = \Lambda_1 + \Lambda_2$

a.) Fields valued in Λ_1

b.) $e^{-S} = e^{i\pi\tau(R+\check{\theta}/2)} e^{-\pi \mathrm{Re}\,\tau(\check{\theta}/2)} \Omega(x)$

For IIA there is a canonical splitting at large distance:

$$\Lambda_2 = \left\{ \begin{array}{l} \check{x} \text{ torsion on 5-skeleton} \\ R_0 = R_2 = R_4 = 0 \end{array} \right\}$$

Λ_1 = any complementary subspace $\Rightarrow$ local solutions of Bianchi are determined by C_1, C_3

Compute:

$$\tau(R) = i \int R_0 * R_0 + R_2 * R_2 + R_4 * R_4 + \int R_0 R_{10} - R_2 R_8 + R_4 R_6$$

In fact it is locally only a function of C_1, C_3

$$\delta\tau(R) = 2i\int \delta R_0 * R_0 + \delta R_2 * R_2 + \delta R_4 * R_4$$

$$+ 2 \int \delta R_0 R_{10} - \delta R_2 R_8 + \delta R_4 R_6$$

$$+ \cancel{R_0 \delta R_{10} - R_2 \delta R_8 + R_4 \delta R_6}$$

So we recover self-duality!

Expanding out in local fields: $R = R_\bullet + d_H C$ gives the standard terms of the action plus corrections to C.S. that confused people before.

4B// <u>Hamiltonian Picture</u>. Now $M = X \times \mathbb{R}$

RR fields are in $\check{K}^{\vee\epsilon}_{\check{B}}(X)$ and evolve in time. How do we impose self-duality?

$\check{K}(X)$ is the space of electric <u>and</u> magnetic fields. Now – there is a multiplication and integration, so a pairing

$$\check{K}^{\vee\epsilon}_{\check{B}}(X) \times \check{K}^{\vee\epsilon}_{\check{B}}(X) \longrightarrow \check{K}^{\vee 0}(X) \xrightarrow{\int_X^{\check{K}}} U(1)$$

$$(\check{x}_1, \check{x}_2) \longrightarrow \check{x}_1 \cdot \bar{\check{x}}_2 \longrightarrow \int_X^{\check{K}} \check{x}_1 \bar{\check{x}}_2$$

<u>Note</u>:

$$\int_X^{\check{K}} : \check{K}^{\vee 0}(X) \longrightarrow \check{K}^{\vee - 9}(pt) = \check{K}^{\vee -1}(pt) = \mathbb{R}/\mathbb{Z}.$$

$$\tilde{F}(\check{x}_\bullet) = \tilde{F}_0 - u^{-1}\tilde{F}_2 + u^{-2}\tilde{F}_4 - \cdots$$

– What is the pairing more explicitly?

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{odd}(M) / \text{“}\Omega^1_{d_H \mathbb{Z}}(M)\text{”} \arrow[r] \arrow[drr, dashed, leftrightarrow, "①"] & \check{K}^0(M) \arrow[r] & K^0(M) \arrow[r] \arrow[dll, dashed, leftrightarrow, "②"'] & 0 \\
0 \arrow[r] & K^{-1}(M; \mathbb{R}/\mathbb{Z}) \arrow[r] & \check{K}^0(M) \arrow[r] & \Omega^{ev}_{d_H, \mathbb{Z}}(M) \arrow[r] & 0
\end{tikzcd}$$

$c_1 \in \Omega^{odd}(M)/\text{“}\Omega^1_{d_H\mathbb{Z}}(M)\text{”}$, $\quad R_2 \in \Omega^{ev}_{d_H,\mathbb{Z}}(M)$

$$① = \int^H_X c_1 \wedge \bar{R}_2$$

so, e.g. the pairing on two topologically trivial fields is

$$\int c_1 \wedge d_H c_2 = \int \left(c_1^{(1)} + c_1^{(3)} + \cdots + c_1^{(9)}\right) \wedge (d - H) \wedge \left(c_2^{(1)} - c_2^{(3)} \pm \cdots\right)$$

② $K^0(X) \times K^{-1}(X; \mathbb{R}/\mathbb{Z}) \to K^{-1}(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{\int^K_X} U(1)$

Element of $K^{-1}(X; \mathbb{R}/\mathbb{Z})$: $y \in \{ (E_1, \nabla_1), (E_2, \nabla_2), \psi$

$\psi : E_1^{\oplus n} \to E_2^{\oplus n} \}$

$$\int^K_X y = \eta(\not{D}_{\nabla_1}) - \eta(\not{D}_{\nabla_2}) - \frac{1}{n} \int CS\left(\psi^*(\nabla_2^{\oplus n}), \nabla_1^{\oplus n}\right) \hat{A} \mod \mathbb{Z}$$

<u>Nontrivial Claim</u>: This is a perfect pairing.

~~[illegible]~~

We can use this as a cocycle to define an Heisenberg extension $\mathrm{Heis}(\check{K}^{\epsilon}_{\beta}(X))$ — Because the pairing is perfect, we <u>define</u>

$$\mathcal{H}_{RR}(X) := \text{Unique irrep of that group.}$$

is it graded by components?

$$\bigoplus_{x \in K^{\epsilon}_{\beta}(X)} \mathcal{H}^{\bullet}_{RR,x}$$

$K^{\epsilon}_{\beta}(X)$ — topological classes of electric <u>and</u> magnetic fluxes. The class $x \in K^{0}_{\bullet}(X)$ is Pontryagin dual to $K^{-1}_{\bullet}(X; \mathbb{R}/\mathbb{Z})$ but the Heisenberg commutator is nonzero —

$$\mathrm{Tors}\, K^{0}_{\bullet}(X) \times \mathrm{Tors}_{\bullet}\, K^{0}_{\beta}(X) \longrightarrow U(1)$$

is a perfect pairing

$$\boxed{\text{Q.M.} \Rightarrow \text{Cannot be in a definite state of a K-theory class!}}$$

Currently Belov and I are trying to justify this using more traditional physics methods — Compute momenta, Page charges, commutators

In particular - the situation is a little more dramatic in the presence of a B-field.

For example we get

$$\text{IIA} \qquad \left[\int \tilde{F}_6 \omega_8^{(1)} , \int \tilde{F}_6 \omega_3^{(2)} \right] = \frac{i}{2\pi} \int H \omega_3^{(1)} \omega_3^{(2)}$$

In fact this is the reduction of the Page charge algebra from M-theory on a circle

$\exists$ Similar statements for IIB.

5// Conclusion :

What does it mean?

The classical picture of a brane as a cycle with vector bundle is not valid quantum mechanically, if there is torsion