WHAT IS...

a Brane?

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The term "brane" has come to mean many things to many people. Broadly speaking, it refers to a physical object appearing in field theories of gravity and strings. It can refer to widely diverse notions, from solitonic solutions of (super)gravity and (super)string theories, to local boundary conditions in two-dimensional conformal field theory, to objects in certain categories associated with sheaves on algebraic varieties.

The essential physical intuition underlying the notion of a "brane" may be captured by a few simple examples. A p-brane is simply any object of p-dimensional spatial extent. Thus a 0-brane is a point particle, whereas a string is a 1-brane. The etymological root of "brane" is "membrane", the case $p = 2$. The surface of the earth's ocean may be viewed as a 2-brane wrapping the earth and propagating in the $(3+1)$-dimensional spacetime of our solar system. The history of a p-brane may be described mathematically by a map $\phi : \mathcal{W} \to \mathcal{M}$, where $\mathcal{W}$ is some reference $(p+1)$-dimensional manifold, while $\mathcal{M}$ represents a "spacetime" through which the brane propagates. $\mathcal{M}$ is also referred to as the "target space", while $\phi\mathcal{W}$ is referred to as the "worldvolume". Sometimes a brane can have thickness, provided this is small on the scale of the spatial extent in p-dimensions. Thus, a rope is effectively a 1-brane and the earth's ocean is effectively a 2-brane.

Branes play an important role in theories of gravity, so a key physical attribute is the tension T, the energy per unit volume of the brane. The tension of a 0-brane is its mass. Branes can have other attributes, such as "charge". The supergravity and superstring theories in which branes play prominent roles are generalizations of Einstein-Maxwell gauge theories. In addition to the gravitational field, string theories typically include a collection of gauge potentials generalizing the connection 1-form of electromagnetism. Heuristically, these may be thought of as differential form-valued fields on spacetime, although a proper description turns out to require notions of K-theory and differential cohomology theories. Charged branes are sources for these generalized gauge potentials.

Let us translate some of these physical notions into mathematics. The action principle governing a point particle of mass m and electric charge e, moving through a spacetime $\mathcal{M}$, with metric g and Maxwell connection A, is $S_{\text{particle}} = \int_{\mathcal{W}} m ds + \int_{\mathcal{W}} e\phi^*(A)$, where ds is the induced line element on the worldline. When added to the standard action for g and A, namely, $S_{\text{bulk}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} \text{vol}(g)\mathcal{R} - \int_{\mathcal{M}} \frac{1}{2e^2} F \wedge *F$ (where G_N is Newton's constant, $\mathcal{R}$ is the scalar curvature of g, and $F = dA$ is the Maxwell fieldstrength), the action S_{particle} represents a source term in the Einstein-Maxwell equations of motion. Thus the brane may be studied as a solution in field theories of gravity with *localized* energy and charge density. The generalization of the brane action to p-branes is of the form

$$S_{\text{brane}} = \int_{\mathcal{W}} T\text{vol}(\phi^*(g)) + \int_{\mathcal{W}} \varepsilon\phi^*(C) \tag{1}$$

where C is a differential form gauge potential, and ε is the "charge" (which may itself be represented by a differential form). The generalization of S_{bulk} is the action principle of a (super)gravity or (super)string theory on $\mathcal{M}$. The typical supergravity brane solution is a soliton—its stability is guaranteed by topological considerations, which are often intimately connected with supersymmetry.

A central point is that a brane has dynamics: it can wiggle and bend. The oscillations are sections of the normal bundle to $\phi(\mathcal{W}) \subset \mathcal{M}$ and hence are described by a $(p+1)$-dimensional scalar field theory on the brane. For the earth's ocean, the scalar field would represent the height of the waves. Mathematically, these degrees of freedom arise because the soliton solutions come in families.

Physicists consider $\mathcal{W}$ and $\mathcal{M}$ of different signatures. They add various structures to both the

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target space and worldvolume, endowing them with gauge bundles and tensor fields, and generalizing them to supermanifolds. In some cases they propose to discuss the quantum behavior of branes by integrating $e^{-1/\hbar S_{\text{brane}}}$ over the space of all maps ϕ; they even boldly contemplate summing over topologies of $\mathcal{W}$. One instance in which these dreams can be realized, with some degree of rigor, is the case in which the 1-branes are the fundamental strings in a supersymmetric string theory.

One distinguished class of branes are the "D-branes" of string theory. For these one can introduce a fundamentally different viewpoint on the question: "What is a brane?" String theory describes a profound relation between a quantum conformal field theory (CFT) on a two-dimensional worldsheet $\mathcal{W}$ and a corresponding quantum field theory on the target space $\mathcal{M}$. The spacetime field theory includes gravity. In this context, D-branes correspond to CFTs on Riemann surfaces $\mathcal{W}$ with boundary. For example, suppose $\mathcal{W} = [0, \pi] \times \mathbf{R}$ so that ϕ describes the propagation of an open string through spacetime. We now select a submanifold $S \subset \mathcal{M}$ and impose the boundary condition that $\phi : \partial\mathcal{W} \to S$. For certain submanifolds, the associated two-dimensional field theory will be conformal. (Typical examples of such submanifolds include holomorphic subvarieties of complex manifolds and special Lagrangian subvarieties of symplectic manifolds.) The "D" in D-brane refers to the fact that some of the coordinate directions in ϕ thus carry Dirichlet boundary conditions. One may recover the notion of branes as solitons in supergravity via a semiclassical approximation to string field theory.

Now, purely in the context of CFT, a D-brane may be defined to be a local boundary condition preserving conformal invariance. Conformal field theories on Riemann surfaces with boundary can be described axiomatically as a functor from a geometric category to an algebraic category. Simple considerations of gluing show that the boundary conditions should be regarded as objects in an additive category. It is via this route that D-branes are identified with objects in certain categories. Moreover, some CFTs carry a special type of supersymmetry, known as $\mathcal{N} = 2$, which allows a "twisting" or association with a related topological field theory. If the target space is a Calabi-Yau manifold, then some of the branes in the CFT can be interpreted as objects in the derived category of coherent sheaves on the target. This in turn has beautiful applications in the theory of mirror symmetry.

One more crucial point is that the dynamics on the D-brane worldvolume is a *gauge theory*. In addition to the scalar field describing fluctuations of the brane in the normal directions, there is a line bundle with connection on $\mathcal{W}$. When N "elementary" branes are placed on top of each other, new nonabelian degrees of freedom are needed to describe the brane's dynamics. The normal bundle scalars become $N \times N$ hermitian matrices. The connection on a line bundle becomes a nonabelian gauge field, i.e, a connection on a rank N vector bundle over $\mathcal{W}$. This fundamental phenomenon has ultimately led to many startling new insights into gauge theory. Just one example of such an insight is the AdS/CFT correspondence, a vast generalization of the famous relation between three-dimensional Chern-Simons gauge theory and two-dimensional (rational) CFT. The replacement of the normal bundle scalars by $N \times N$ matrices leads to connections between D-branes and noncommutative geometry. Using these insights in the framework of branes within branes leads to new perspectives on hyperkähler quotient constructions and the ADHM construction of instantons.

We began by describing a brane as an object propagating through a spacetime. This puts the spacetime on a primary, and the brane on a secondary footing. However, a common theme in the study of D-branes has been the idea that in fact, the (string) field theory on the brane is the primary concept, whereas the spacetime itself is a secondary, derived, concept. This notion has been given some degree of precision in the so-called Matrix theory formulation of M-theory. A rough analogy of what physicists expect may be described in the context of purely topological branes, where the field theory on a brane is described in terms of a noncommutative Frobenius algebra, and the "spacetime" in which it propagates is derived from the Hochschild cohomology of that algebra. These ideas might ultimately lead to a profound revision of the way we regard spacetime.

The recognition of the importance of branes in string theory has been a central development, one that is still undergoing vigorous evolution. We have focused above on D-branes, but there are other important, but less well-understood, branes. For example, a deeper understanding of the "solitonic 5-branes" will lead to constructions of quantum CFTs and string theories in six-dimensional spacetimes. Further development of the theory is likely to have a wide variety of important mathematical applications.

References

[1] Virtually all relevant research and reviews on this material can be found on the e-print archive, `http://www.arxiv.org`. An example of a recent review is C. V. Johnson, D-brane primer, `http://www.arxiv.org/abs/hep-th/0007170`.

[2] *Mirror Symmetry*, K. Hori et al., AMS/Clay Mathematics Institute, 2003.

[3] *Topology, Geometry and Quantum Field Theory*, U. Tillmann, ed. Cambridge Univ. Press, 2004.