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Introduction To Chern-Simons Theories

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Abstract: These are lecture notes for a series of talks at the 2019 TASI school. They contain introductory material to the general subject of Chern-Simons theory. They are meant to be elementary and pedagogical. ************* THESE NOTES ARE STILL IN PREPARATION. CONSTRUCTIVE COMMENTS ARE VERY WELCOME. (In particular, I have not been systematic about trying to include references. The literature is huge and I know my own work best. I will certainly be interested if people bring reference omissions to my attention.) *************** June 7, 2019

Contents

"This whole book is but a draught nay, but the draught of a draught." - Herman Melville

1. Introduction: The Grand Overview

Chern-Simons theory is a quantum gauge theory involving a rather subtle action principle. It leads to quantum field theory in which many, many, natural questions can be explicitly answered. The usual difficulties of quantum field theory are exchanged for subtle questions in topology, but the latter turn out to be fairly accessible. It is thus very computable but highly nontrivial example of a quantum field theory. Moreover, it has proven to have an astonishingly wide variety of applications, from condensed matter physics, through string theory, and pure mathematics.

The story goes back, on the mathematical side to the work of S.S. Chern and J. Simons and, on the physics side it goes back (at least) to an observation that three-dimensional Yang-Mills theory admits an interesting gauge invariant deformation of its action [15, 14]. ♣say what problem they were addressing ♣

In 1988 Witten wrote a foundational paper on the subject [67]. This paper answered a major question posed by Michael Atiyah: "What is the physical interpretation of the Jones polynomial?" In the process Witten gave some important new insight into some constructions of conformal field theory, especially rational CFT, that were undergoing vigorous development at the time. (See, for examples, [6, 41, 42, 43, 60, 61, 62] and all the many many competitors of Moore and Seiberg ....)

Witten's paper was a major breakthrough - worthy of a Fields medal - and, together with the physical interpretation of Donaldson invariants (answering another of Michael Atiyah's questions) it gave a strong impetus to the development of the subject of topological field theory. [1] In some sense three-dimensional CS was the first and most important example of a topological quantum field theory. [2]

At some level, the story line is very simple:

Consider a gauge theory for a Lie group G. Locally the gauge field A - is a 1-form valued in the Lie algebra $\mathfrak{g}$ that transforms under gauge transformations like [3]

$$d + A^g := g^{-1}(d + A)g \tag{1.1}$$

so that $F := dA + A^2$ transforms like $F \to g^{-1}Fg$. Then, suppose that $\wp$ is a conjugation invariant polynomial on the Lie algebra of order two so that we can make the 4-form $\wp(F)$. A typical example is to introduce a trace on the a simple Lie algebra and write

$$\wp(F) = \mathrm{tr}\, F^2 \tag{1.2}$$

Then $\wp(F)$ is gauge invariant and you can check that, formally, it is a total derivative:

$$\mathrm{tr}\, F^2 = d\left(\mathrm{tr}\, AdA + \frac{2}{3}A^3\right) = dcs(A) \tag{1.3}$$

and more generally $\wp(F) = dcs_{\wp}(A)$. Now consider a three-dimensional manifold M_3 together with an orientation $\mathfrak{o}(M_3)$. You can then define an action principal for a gauge theory path integral:

$$\int_{\mathcal{A}/\mathcal{G}} e^{\mathrm{i}\int_{M_3} cs_{\wp}(A)} \tag{1.4}$$

Here the integral is over the space of inequivalent gauge fields, commonly denoted $\mathcal{A}/\mathcal{G}$. Using Appendix **** you compute that

$$cs(A^g) = cs(A) + cs(g^{-1}dg) - d(\mathrm{tr}\, Adgg^{-1}) \tag{1.5}$$

Restricting to the case that M_3 has no boundary, for quantized choices of $\wp$ (e.g. with a properly normalized trace on a simple Lie algebra) $\int cs(g^{-1}dg)$ will be 2π times an integer so that the exponentiated action $\exp[\mathrm{i}\int_{M_3} cs(A)]$ is gauge invariant, that is, is a well-defined function on $\mathcal{A}/\mathcal{G}$.

The equations of motion follow from

$$\delta cs(A) = 2\mathrm{tr}\,(\delta AF) - d(\mathrm{tr}\, A\delta A) \tag{1.6}$$

[1] Albert Schwarz had previously studied topological field theories for Abelian gauge theories in [54]. In an announcement at a conference in [1988 ?? Odessa ?? CHECK!!!] he independently suggested that Wilson line observables might be related to the Jones polynomial.

[2] Four-manifold enthusiasts will argue that Donaldson-Witten theory is the first and most important example of a TQFT.

[3] Note that in our conventions we have chosen a right-action of the group of gauge transformations on gauge fields: $(A^{g_1})^{g_2} = A^{g_1 g_2}$.

The equations of motion therefore state that the fieldstrength vanishes: $F = 0$. Such gauge fields are called *flat gauge fields*. They can nevertheless be very interesting and nontrivial.

No metric is used in forming the action principal. Naively, one might expect the path integral to be a topological invariant of M_3. Moreover, you can introduce topological Wilson line defects: Choose an oriented closed curve $\gamma \subset M_3$ and a representation R of the Lie group and construct the gauge invariant function of A:

$$W(R, \gamma) = \mathrm{Tr}_R \mathrm{Pexp} \oint_\gamma A \tag{1.7}$$

We can then formally consider the correlators

$$\langle \prod_\alpha W(R_\alpha, \gamma_\alpha) \rangle = \int_{\mathcal{A}/\mathcal{G}} e^{\mathrm{i} \int_{M_3} cs_\wp(A)} \prod_\alpha W(R_\alpha, \gamma_\alpha) \tag{1.8}$$

and formally this should be a topological invariant of the configuration of labeled loops in M_3.

All of the above is sort of true. It is true in spirit, but overlooks a lot of important sublteties. For example, regularization questions spoil the topological invariance and introduce extra (”framing”) data to define the Wilson loop observables. In these lectures we will try to explain some of the above claims more carefully and in more detail.

The net result is the following: One can define a three-dimensional topological field theory that depends on the data:

1. A compact group G. The structure of the general compact Lie group is described in Appendix J.1.
2. A choice of “level.” Roughly, the level is a measure of $1/\hbar$. More precisely, the “level” lives in a certain finitely-generated Abelian group associated to G. (Technically, it is a “nondegenerate element of $H^4(BG; \mathbb{Z})$.”)
3. The data needed to obtain topological invariants are: An orientation and 2-framing of M_3 together with ribbons in M_3 labeled by representations of G.

This, of course, is only the data needed to make sense of partition functions on closed three-manifolds. The theory extends to give mathematical objects for 2- , 1-, and 0-dimensional manifolds.

Chern-Simons theory has been immensely popular and influential in both physics and mathematics. As one (superficial) measure, a cursory literature search on ”Chern-Simons” turns up thousands of scholarly papers. [4]

The theory has had numerous applications and can be generalized in a large number of ways. A few of them are:

1. It is a paradigmatic example of an(extended) TQFT.

[4]On May 8, 2019: MathSciNet: 5521; e-Print ArXiv: 5395; scholar.google.com: 45700.

2. It therefore allows for the construction of important topological invariants ("quantum invariants") of 3-manifolds and links in 3-manifolds. (These must carry some extra structure to be defined, as we will see.)

3. The connection between 2d rational conformal field theory and 3d topological Chern-Simons theory is a kind of "holographic correspondence," and was, historically, one of the first examples of holography.

4. There are numerous further applications to more modern versions of holography as in the AdS/CFT correspondence. (For example, in the theory of singletons.)

5. Physically, 3d CS turns out to be very relevant to describing certain topological properties of the FQHE. This motivated a much more general application of topological field theory ideas to phases of matter with particles with nontrivial statistics [52].

6. There is a developing story for Chern-Simons theory for noncompact groups. These have numerous applications to supersymmetric gauge theories, knot polynomials, BPS invariants,...

7. One of the most striking examples of noncompact Chern-Simons is in formulating various versions of three-dimensional quantum gravity.

8. Chern-Simons theory has been used to address the problem of putting chiral fermions on the lattice.

9. Chern-Simons theories are relevant to describing the spacetime physics of topological string theory.

10. Some versions of string field theory are of Chern-Simons form.

These lectures are meant to be very pedagogical and very basic. I am definitely not talking to the experts (usually).

1.1 Assumed Prerequisites

1. Quantum mechanics.

2. Basic knowledge of differential forms: Definition. Exterior derivative. Integration on manifolds and Stokes' theorem.

3. Some elementary quantum field theory, but not much.

4. Simple notions from topology: Homotopy groups. Bundles. Cohomology groups. Connections on bundles.

5. Some basic notions from group theory. Exact sequence. Projective representations. The basic concepts of group actions on manifolds.

6. Some knowledge of symplectic geometry would be useful, but is not essential.

Some sources for mathematical background:
Differential forms, bundles, homology, homotopy and cohomology:

1. R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, [12]

2. M. Nakahara, *Geometry, Topology and Physics, Second Edition (Graduate Student Series in Physics)* , [51]

3. Lectures 1-4 by J. Morgan in [50]

Some standard, and not-so-standard things from group theory and linear algebra that might be freely used in these lectures are explained in my online lecture notes for a group theory course I recently gave. See, especially, [48] and [49].

2. Chern-Simons Theories For Abelian Gauge Fields

2.1 Topological Terms Matter

We consider the quantum mechanics of a charged particle confined to a ring that surrounds a solenoid. This is a good pedagogical toy example that illustrates some of the subtle effects associated with the θ-term in QCD and QED and in topological insulators. This is a very standard example. We are giving a slightly condensed version of the discussion of sections 11.3.1-11.3.2, pages 140-158 of [48]

2.1.1 Charged Particle On A Circle Surrounding A Solenoid: Hamiltonian Quantization

Consider a particle of mass m confined to a ring of radius r in the xy plane. The position of the particle is described by an angle ϕ, so we identify $\phi \sim \phi + 2\pi$, and the action is

$$S = \int \frac{1}{2} m r^2 \dot{\phi}^2 = \int \frac{1}{2} I \dot{\phi}^2 \tag{2.1}$$

with $I = mr^2$ the moment of inertia.

Let us also suppose that our particle has electric charge e and that the ring is threaded by a solenoid with magnetic field B, so the particle moves in a zero B field, but there is a nonzero gauge potential [5]

$$A = \frac{B}{2\pi} d\phi \tag{2.2}$$

The action is therefore:

$$\begin{aligned} S &= \int \frac{1}{2} I \dot{\phi}^2 dt + \oint eA \\ &= \int \frac{1}{2} I \dot{\phi}^2 dt + \frac{eB}{2\pi} \dot{\phi} dt \end{aligned} \tag{2.3}$$

[5] For readers not familiar with differential form notation this means, in cylindrical coordinates that $A_z = 0$, $A_r = 0$ and $A_\phi = B/2\pi$.

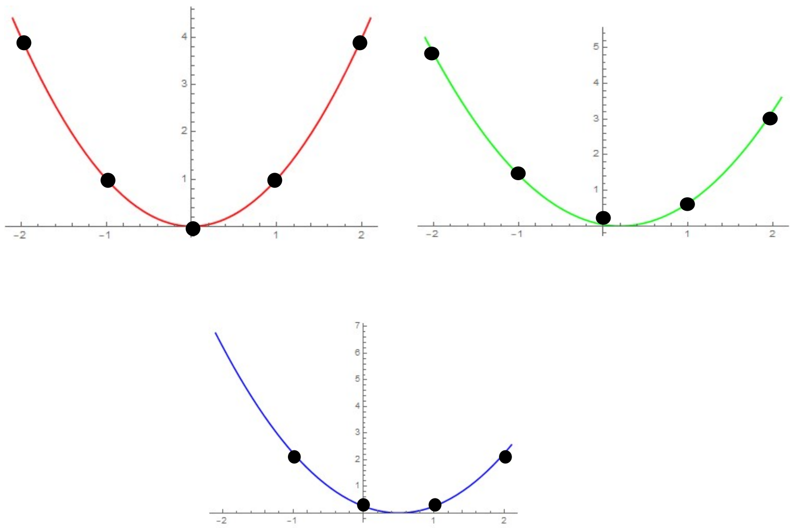

Figure 1: Spectrum of a particle on a circle as a function of $\mathcal{B} = eB/2\pi$. The upper left shows the low-lying spectrum for $\mathcal{B} = 0$. It is symmetric under $m \to -m$. The upper right shows the spectrum for $\mathcal{B} = 0.2$. There is no symmetry in the spectrum. The lower figure shows the spectrum for $\mathcal{B} = 1/2$. There is again a symmetry, but under $m \to 2\mathcal{B} - m = 1 - m$. In general there will be no symmetry unless $2\mathcal{B} \in \mathbb{Z}$. If $2\mathcal{B} \in \mathbb{Z}$ the spectrum is symmetric under $m \to 2\mathcal{B} - m$.

The second term is an example of a "topological term" or a "θ-term." Classically, the second term does not affect physical predictions, since it is a total derivative. However, quantum mechanically, it will have an important effect on physical predictions.

We are going to analyze the symmetries of this system and compare their realization in the classical and quantum theories.

We begin by analyzing the classical symmetries. Because the θ-term does not affect the classical dynamics the classical system has $O(2)$ symmetry. We can rotate: $R(\alpha)$: $e^{\mathrm{i}\phi} \to e^{\mathrm{i}\alpha} e^{\mathrm{i}\phi}$, or, if you prefer, translate $\phi \to \phi + \alpha$ (always bearing in mind that α and ϕ are only defined modulo addition of an integral multiple of 2π). If we think of the circle in the $x - y$ plane centered on the origin, with the solenoid along the z-axis then we could also take

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{2.4}$$

as usual.

Also we can make a "parity" or "charge conjugation" transformation $P : \phi \to -\phi$. The second term in the Lagrangian is not invariant but this "doesn't matter" because it is a total derivative. Put differently: $\phi \to -\phi$ is a symmetry of the equations of motion.

Note that these group elements in $O(2)$ satisfy

$$
\begin{aligned}
R(\alpha)R(\beta) &= R(\alpha+\beta) \\
P^2 &= 1 \\
PR(\alpha)P &= R(-\alpha)
\end{aligned}
\tag{2.5}
$$

and indeed, as we have seen, $O(2)$ is a semidirect product:

$$
O(2) = SO(2) \rtimes \mathbb{Z}_2 \tag{2.6}
$$

with $\omega : \langle P \rangle \cong \mathbb{Z}_2 \to \mathrm{Aut}(SO(2)) \cong \mathbb{Z}_2$ acting by taking the nontrivial element of $\mathbb{Z}_2$ to the outer automorphism that sends $R(\alpha) \to R(-\alpha)$.

Viewing the system as a field theory.

We have introduced this system as describing the quantum mechanics of a particle. However, it is important to note that it can also be viewed as a special case of a quantum field theory. In general, in a field theory we have a spacetime M and the fields ϕ are functions on M valued in some *target space* $\mathcal{X}$. (So the term "target space" means nothing more or less than the codomain of the fields.) In our case we have $M = \mathbb{R}$, interpreted as the real line of time while $\mathcal{X}$ is the circle. So our fields are maps

$$
e^{\mathrm{i}\phi} : M \to S^1 \tag{2.7}
$$

where M is the real line of time, or perhaps a time interval. Later we will also take M to be the circle. The space of fields is $Map[M \to \mathcal{X}]$. Since M is the one-dimensional manifold of time we refer to this as a "0+1 dimensional field theory. We have been referring to $\phi \to -\phi$ as "parity" because that is the appropriate term in the context of the quantum mechanics of a particle constrained to a circle in the plane. However, if we take the point of view that we are discussing a $0+1$ dimensional "field theory" then it would be better to refer to the operation as "charge conjugation" because it complex conjugates the $U(1)$-valued field $e^{\mathrm{i}\phi}$.

In addition there are (in the field theory interpretation) "worldvolume symmetries" of time translation invariance and time reversal. These form the group $\mathbb{R} \rtimes \mathbb{Z}_2$. We will put those aside. (Note that time reversal is not a symmetry of the second term in the Lagrangian but is a symmetry of the space of solutions of the equations of motion.)

Symmetries In The Quantum Theory.

Now let us consider the quantum mechanics with the "θ-term" added to the Lagrangian. Our goal is to see how that term affects the quantum theory.

We will first analyze the quantum mechanics in the Hamiltonian approach. See the remark below for some remarks on the path integral approach. The conjugate momentum is

$$
L = I\dot{\phi} + \frac{eB}{2\pi} \tag{2.8}
$$

We denote it by L because it can be thought of as angular momentum.

Note that the coupling to the flat gauge field has altered the usual relation of angular momentum and velocity. Now we obtain the Hamiltonian from the Legendre transform:

$$
\int L\dot{\phi}dt - S = \int \frac{1}{2I}(L - \frac{eB}{2\pi})^2 dt \tag{2.9}
$$

Upon quantization [6] $L \to -\mathrm{i}\hbar\frac{\partial}{\partial\phi}$, so the Hamiltonian is

$$H_{\mathcal{B}} := \frac{\hbar^2}{2I}\left(-\mathrm{i}\frac{\partial}{\partial\phi} - \mathcal{B}\right)^2 \tag{2.10}$$

where $\mathcal{B} := \frac{eB}{2\pi\hbar}$.

The eigenfunctions of the Hamiltonian $H_{\mathcal{B}}$ are just

$$\Psi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}m\phi} \qquad m \in \mathbb{Z} \tag{2.11}$$

They give energy eigenstates states with energy

$$E_m = \frac{\hbar^2}{2I}(m - \mathcal{B})^2 \tag{2.12}$$

There is just one energy eigenstate for each $m \in \mathbb{Z}$.

Remarks:

1. The action (2.31) makes good sense for ϕ valued in the real line or for $\phi \sim \phi + 2\pi$, valued in the circle. Making this choice is important in the choice of what theory we are describing. Where - in the above analysis - did we make the choice that the target space is a circle? [7]

2. Taking $\phi \sim \phi + 2\pi$, even though the θ-term is a total derivative it has a nontrivial effect on the quantum physics as we can see since B has shifted the spectrum of the quantum Hamiltonian in a physically observable fashion: *This is how we see that topological terms matter.*

3. Note that when $2\mathcal{B}$ is even the energy eigenspaces are two-fold degenerate, except for the ground state at $m = \mathcal{B}$. On the other hand, when $2\mathcal{B}$ is odd all the energy eigenspaces are two-fold degenerate, including the ground state. If $2\mathcal{B}$ is not an integer all the energy eigenspaces are one-dimensional. See Figure 1.

4. The total spectrum is *periodic* in $\mathcal{B}$, and shifting $\mathcal{B} \to \mathcal{B}+1$ is equivalent to $m \to m+1$. To be more precise, we can define a unitary operator on the Hilbert space by its action on a complete basis:

$$U\Psi_m = \Psi_{m+1} \tag{2.13}$$

   and

$$UH_{\mathcal{B}}U^{-1} = H_{\mathcal{B}+1} \tag{2.14}$$

[6] One could add a constant here. It would introduce a second parameter into the quantization story, but it is redundant with $\mathcal{B}$.

[7] *Answer*: If we took the case where ϕ is valued in $\mathbb{R}$ and not the circle then there would be no quantization on m and the spectrum of the Hamiltonian would be continuous. In this case the Chern-Simons term would not affect the physics in the quantum mechanical version as well.

5. The quantum mechanics problem (2.31) and the spectrum (2.12) arise in the discussion of the "Coulomb blockade" in physics of quantum dots. See Yoshimasa Murayama, *Mesoscopic Systems*, Section 10.10.

6. *Relations to higher dimensional field theories and string theory.* The θ-term we have added as a very interesting analog in $1+1$ dimensional field theory, where it is known as a coupling to the B-field. It can also be obtained from a Kaluza-Klein reduction of $1+1$ dimensional Maxwell theory:

$$S = \frac{1}{e^2}\int F * F + \int \frac{\theta}{2\pi} F \tag{2.15}$$

In $1+1$ dimensional theory we can choose $A_0 = 0$ gauge and gauge away the x^1 dependence so that on $S^1 \times \mathbb{R}$ the only gauge invariant quantity is

$$e^{\mathrm{i}\phi(t)} = e^{\mathrm{i}\oint_{S^1} A} = e^{\mathrm{i}\oint_{S^1} A_1 dx^1} \tag{2.16}$$

With this in mind we can say

$$\theta = 2\pi\mathcal{B} \tag{2.17}$$

Remark: More generally, in $1+1$ dimensional Yang-Mills theory on $S^1 \times \mathbb{R}$ we can always go to $A_0 = 0$ gauge and then the only gauge invariant observable is the conjugacy class of the holonomy around the circle.

The theta term also has a close analog in $3+1$-dimensional gauge theory. In the case of $3+1$ dimensional Maxwell theory we can write

$$S = \int d^4x \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \int \frac{\theta}{(2\pi)^2} F \wedge F \tag{2.18}$$

In fact, in the effective theory of electromagnetism in the presence of an insulator a very similar action arises with a θ term. If a parity- and/or time-reversal symmetry is present then θ is zero or π, corresponding to our case $2\mathcal{B} \in \mathbb{Z}$. The difference between a normal and a topological insulator is then, literally, the difference between $2\mathcal{B}$ being even (normal) and odd (topological), respectively. Finally, in the 3+1-dimensional Yang-Mills theories that describe the standard model of weak and strong interactions one can add an analogous θ-term. One of the great unsolved mysteries about nature is why the (effective) theta angle for the strong interactions in the standard model is unmeasurably small.

Now, if we consider how the $O(2)$ symmetry is realized in the quantum theory we find a surprise:

1. The classical theory has an $O(2)$ symmetry.

2. In the quantum theory when $2\mathcal{B} = \theta/\pi$ is not an integer the symmetry is broken to $SO(2)$.

3. In the quantum theory, when $2\mathcal{B} = \theta/\pi$ is an even integer the theory still has $O(2)$ symmetry.

4. In the quantum theory, when $2\mathcal{B} = \theta/\pi$ is an odd integer, the classical $O(2)$ relations no longer hold, but the covering group $\mathrm{Pin}^+(2)$ is a symmetry.

We stress that the particle on the ring is NOT a spin one-half!! Having said that, if we define an angular momentum $\mathcal{L}$ so that $H = \frac{\mathcal{L}^2}{2I}$ then indeed when $\mathcal{B}$ is half-integral the angular momentum has half-integral eigenvalues, as one expects for a spin representation. So, what we are finding is that the half flux quantum is inducing a half-integral spin of the system so that the classical $O(2)$ symmetry of the classical system is implemented as a $\mathrm{Pin}^+(2)$ symmetry in the quantum theory. This is an intriguing phenomenon appearing in quantum symmetries with nontrivial gauge fields and Chern-Simons terms: The statistics and spins of particles can be shifted from their classical values, often in ways that involve curious fractions.

2.1.2 Charged Particle On A Circle Surrounding A Solenoid: Path Integrals

It turns out to be very instructive to compute the Euclidean time propagator in this theory in two ways. Thus, we consider the quantity $\langle\phi_2|e^{-\beta H}|\phi_1\rangle$. On the one hand, we have completely diagonalized the Hamiltonian, so inserting a complete set of states we immediately find:

$$\langle\phi_2|e^{-\beta H_{\mathcal{B}}}|\phi_1\rangle = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2I}(m-\mathcal{B})^2 + \mathrm{i}m(\phi_1-\phi_2)} \tag{2.19}$$

On the other hand, using the Wick rotation of the Feynman path integral representation of the propagator we also have an expression in terms of a path integral:

$$\langle\phi_2|e^{-\beta H}|\phi_1\rangle = Z(\phi_2,\phi_1|\beta) := \int [d\phi(t)]_{\phi(0)=\phi_1}^{\phi(\beta)=\phi_2} e^{-\frac{1}{\hbar}\int_0^\beta \frac{1}{2}I\dot{\phi}^2 dt - \mathrm{i}\int \mathcal{B}\dot{\phi}dt} \tag{2.20}$$

Note that in the rotation to Euclidean signature the topological term is still imaginary. One must be careful to get the sign of the imaginary term, and it matters.

Viewed as a field theory, this is a free field theory and the path integral can be done exactly by semiclassical techniques:

The equation of motion is simply $\ddot{\phi} = 0$. Again, the θ-term has not changed it. Thus, the classical solutions to the equations of motion with boundary conditions $\phi(0) = \phi_1, \phi(\beta) = \phi_2$ are:

$$\phi_{cl}(t) = \phi_1 + \left(\frac{\phi_2 - \phi_1 + 2\pi w}{\beta}\right)t \qquad\qquad w \in \mathbb{Z} \tag{2.21}$$

or more to the point:

$$e^{\mathrm{i}\phi_{cl}(t)} = e^{\mathrm{i}\left((1-\frac{t}{\beta})\phi_1 + \frac{t}{\beta}\phi_2\right) + \frac{2\pi \mathrm{i} t w}{\beta}} \tag{2.22}$$

These are solutions of the Euclidean equations of motion, and are known as "instantons" for historical reasons. Notice that because of the compact nature of the spacetime on which we define our $0+1$ dimensional field theory there are infinitely many solutions labeled by $w \in \mathbb{Z}$. There are two circles in the game: The spacetime of this $0+1$-dimensional field theory is the Euclidean time circle. Then the target space of the field theory is also a circle. The quantum field $e^{\mathrm{i}\phi(t)}$ is a map $M \to \mathcal{X}$. M, which is spacetime is S^1 and $\mathcal{X}$, which is the target space is also $\mathcal{X}$. Recall that $\pi_1(S^1) \cong \mathbb{Z}$. There can be topologically inequivalent field configurations. That is the space of maps $Map(M \to \mathcal{X})$ has different connected components. The different topological sectors are uniquely labelled by the winding number of the map (2.22). In the path integral we sum over all field configurations so we should sum over all these instanton configurations.

A straightforward computation (details in the online lecture notes [48]) shows that the path integral expression is exactly given by

$$Z(\phi_2, \phi_1|\beta) = \sqrt{\frac{I}{2\pi\beta}} \sum_{w\in\mathbb{Z}} e^{-\frac{2\pi^2 I}{\beta}(w+\frac{\phi_2-\phi_1}{2\pi})^2 - 2\pi\mathrm{i}\mathcal{B}(w+\frac{\phi_2-\phi_1}{2\pi})} \tag{2.23}$$

Now compare (2.19) with (2.23). These expressions look very different! One involves a sum of exponentials with β in the numerator and the other with β in the demoninator. One is well-suited to discussing the asymptotic behavior for $\beta \to \infty$ (low temperature) and the other for $\beta \to 0$ (high temperature), respectively. Nevertheless, we have computed the same physical quantity, just using two different methods. So they must be the same. But the mathematical identity that says they are the same appears somewhat miraculous. We now explain how to verify the two expressions are indeed identical using a direct mathematical argument.

We introduce the Riemann theta function

$$\vartheta[{\theta \atop \phi}](z|\tau) := \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi\tau(n+\theta)^2 + 2\pi\mathrm{i}(n+\theta)(z+\phi)} \tag{2.24}$$

The Riemann theta function is an absolutely convergent analytic function of τ in the upper half-plane. It is also an entire function of z. Using the Poisson summation formula one verifies the crucial modular transformation law:

$$\vartheta[{\theta \atop \phi}](\frac{-z}{\tau}|\frac{-1}{\tau}) = (-i\tau)^{1/2} e^{2\pi i\theta\phi} e^{i\pi z^2/\tau} \vartheta[{-\phi \atop \theta}](z|\tau) \tag{2.25}$$

One can easily check that the above formulae for the Euclidean propagator are related by the modular transformation for the Riemann theta function.

We will find theta functions and their modular properties will play an important role in what follows.

Exercise

Suppose $2\mathcal{B}$ is an odd integer.

a.) Show that as $\beta \to \infty$ we have

$$\langle \phi_2 | e^{-\frac{\beta H}{\hbar}} | \phi_1 \rangle \sim 2 e^{\frac{\mathrm{i}}{2}(\phi_1 - \phi_2)} \cos\left(\frac{\phi_1 - \phi_2}{2}\right) e^{-\beta E_{\text{ground}}} + \cdots \tag{2.26}$$

b.) Note that this expression is not invariant under $\phi \to -\phi$. But in $\mathrm{Pin}^{+}(2)$ there is an element P which corresponds to $\phi \to -\phi$. How is this compatible with our argument that $\mathrm{Pin}^{+}(2)$ is a valid symmetry of the quantum theory? [8]

2.1.3 Gauging The Global $SO(2)$ Symmetry And Chern-Simons Terms

When a theory has a symmetry one can implement a procedure called "gauging the symmetry." This is a two-step process:

1. Make the symmetry local and couple to a gauge field.

2. Integrate over the gauge fields.

It is not necessary to proceed to step (2) after completing step (1). In this case, we say that we are coupling to nondynamical external gauge fields. It makes perfectly good sense to introduce nondynamical, external gauge fields for a symmetry. We do this all the time in quantum mechanics courses where we couple our quantum system to an electromagnetic field, but do not try to quantize the electromagnetic field.

For the more mathematically sophisticated reader the two-step process can be summarized as saying that given a field theory with a symmetry group G we can make the theory G-equivariant by changing the bordism category of the domain of definition:

1. Identify the symmetry group with the structure group of a principal bundle and we change the geometric domain of the field theory to bundles with connection.

2. Sum over isomorphism classes of principal bundles and integrate over the connections.

We will explain all this a bit more in section ****.

In the present simple case we can "gauge" the global $SO(2)$ symmetry $\phi \to \phi + \alpha$ that is present for all values of $\mathcal{B}$. [9]

So in our simple example we make the shift symmetry local, that is, we attempt to make

$$\phi(t) \to \phi(t) + \alpha(t) \tag{2.27}$$

into a symmetry where $\alpha(t)$ is not a constant but an "arbitrary" function of time. Now, of course the action $\sim \int \dot{\phi}^2$ is not invariant under such transformations. We introduce a gauge field $A_t^{(e)} dt$ where the superscript e - for "external" - reminds us that this is <u>NOT</u>

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[8] *Answer*: See [48].

[9] It is then interesting to see how the external gauge field tells us about the charge conjugation symmetry. For a discussion of this see Appendix D of D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, "Theta, Time Reversal, and Temperature," https://arxiv.org/pdf/1703.00501.pdf

the gauge field of electromagnetism. (That field has already produced our theta term.) Rather, it is a new field in our system: A gauge field for the rotation symmetry.

The gauged action is

$$S = \int \frac{1}{2} I(\dot{\phi} + A_t^{(e)})^2 dt + \oint \mathcal{B}(\dot{\phi} + A_t^{(e)}) dt \tag{2.28}$$

This is gauge invariant with gauge transformations:

$$\begin{aligned} \phi &\to \phi + \alpha(t) \\ A_t^{(e)} &\to A_t^{(e)} - \partial_t \alpha(t) \end{aligned} \tag{2.29}$$

or, better

$$\begin{aligned} e^{\mathrm{i}\phi(t)} &\to e^{\mathrm{i}\phi(t)} e^{\mathrm{i}\alpha(t)} \\ A_t^{(e)} &\to A_t^{(e)} + \mathrm{i} e^{-\mathrm{i}\alpha(t)} \partial_t e^{\mathrm{i}\alpha(t)} \end{aligned} \tag{2.30}$$

This is better because when working on topologically nontrivial spacetimes, such as the Euclidean time circle, it is $e^{\mathrm{i}\alpha(t)}$ which should be single-valued.

In the absence of the external gauge field $A_t^{(e)}$ we found that $\mathcal{B}$ was periodic modulo an integer. We can restore a kind of periodicity in $\mathcal{B}$ by adding a Chern-Simons term to the action. In Euclidean space the new action is:

$$e^{-S} = e^{-\int \frac{1}{2} I(\dot{\phi} + A_t^{(e)})^2 dt - \mathrm{i} \oint \mathcal{B}(\dot{\phi} + A_t^{(e)}) dt} e^{\mathrm{i}k \int A_t^{(e)} dt} \tag{2.31}$$

We have restored the invariance by hand so that the new theory with $(\mathcal{B}, k)$ is equivalent to the theory with $(\mathcal{B}+1, k+1)$. But now we must worry about the gauge invariance of the new term. Here is a key point:

It is not necessary for the action to be invariant. All that is necessary for a well-defined path integral is that the exponentiated action must be invariant.

We will invoke this observation several more times, when we quantize the coefficients of other Chern-Simons terms in actions, as well as when we discuss the WZ term in 2D CFT. We will call it the multi-valued action principle: The action S can be multivalued, so long as $e^{\mathrm{i}S}$ is single-valued.

Therefore, we study gauge invariance of

$$e^{\mathrm{i}k \oint A_t^{(e)} dt} \tag{2.32}$$

Now we must decide which 1-manifold we are working on.

On an interval $[t_1, t_2]$ the expression

$$\exp[\mathrm{i} \int_{t_1}^{t_2} A_t^{(e)}(t') dt'] \tag{2.33}$$

is not gauge invariant. Rather:

$$\exp[\mathrm{i}\int_{t_1}^{t_2} A_t^{(e)}(t')dt'] \to e^{-\mathrm{i}\alpha(t_2)}\exp[\mathrm{i}\int_{t_1}^{t_2} A_t^{(e)}(t')dt']e^{\mathrm{i}\alpha(t_1)} \tag{2.34}$$

Therefore, on the real line $\exp[\mathrm{i}\int_{\mathbb{R}} A_t^{(e)}(t')dt']$ is gauge invariant if we choose boundary conditions so that $\alpha(t)$ vanishes as $t \to \pm\infty$.

Similarly, on the circle, if $\alpha(t)$ is single valued then

$$\exp[\mathrm{i}\oint_{S^1} A_t^{(e)}(t')dt'] \tag{2.35}$$

is gauge invariant. However, for the circle, the true gauge parameter is $g(t) = e^{\mathrm{i}\alpha(t)}$ for a $U(1)$ gauge group. Therefore, $\alpha(t) = 2\pi wt/\beta$ with $w \in \mathbb{Z}$ is a valid gauge transformation but under such gauge transformations:

$$\exp[\mathrm{i}k\oint_{S^1} A_t^{(e)}(t')dt'] \to e^{2\pi\mathrm{i}wk}\exp[\mathrm{i}k\oint_{S^1} A_t^{(e)}(t')dt'] \tag{2.36}$$

and therefore, if we are going to allow our theory to make sense on a circle then k should be quantized to be an integer. In general: $t \mapsto e^{\mathrm{i}\alpha(t)}$ defines a map $S^1_{time} \to U(1)$. But $\pi_1(U(1),*) \cong \mathbb{Z}$ is characterized by winding number. We say that $e^{\mathrm{i}\alpha(t)}$ defines a *large gauge transformation* when this winding number is nonzero.

Gauge invariance of the "Chern-Simons term" under large gauge transformations implies that the coupling is quantized: $k \in \mathbb{Z}$.

For reasons related to the implementation of charge conjugation symmetry it turns out to be desirable to extend the definition to k which is half-integral. There is a way to make sense of the half-integer quantized Chern-Simons term by viewing the $0+1$ dimensional theory as the boundary of a well-defined $1+1$ dimensional theory. By Stokes' theorem we have:

$$\exp[ik\oint_{S^1} A_t^{(e)}dt] = \exp[\mathrm{i}k\int_{\Sigma} F^{(e)}] \tag{2.37}$$

where $F^{(e)} = dA^{(e)}$. The RHS makes sense even if k is not an integer, but now the expression depends on details of the gauge field in the "bulk" of the $1+1$ dimensional spacetime Σ.

A very analogous phenomenon is observed in real condensed matter systems where the boundary theory of a 3+1 dimensional topological insulator is described by a Chern-Simons theory with half-integral level. (That is, half the level allowed by naive gauge invariance.)

Exercise *Puzzle*

Resolve the following paradox:

We first argued that, if $k \notin \mathbb{Z}$ then the LHS of (2.37) is not invariant under large gauge transformations. Then we proceeded to define the LHS by the expression on the RHS which is manifestly gauge invariant.

How can these two statements be compatible? [10]

2.2 $U(1)$ Chern-Simons Theory In 3 Dimensions

2.2.1 Some $U(1)$ Gauge Theory Preliminaries

We are going to need to be very careful about normalizations and periodicities throughout these notes, so we spell out our normalizations for $U(1)$ gauge theory.

We begin with a gauge theory on some spacetime M based on an Abelian Lie group $U(1)$ or $\mathbb{R}$. Locally, the gauge field is a real 1-form denoted $A = A_\mu dx^\mu$. We stress that, while $F = dA$ is a globally well-defined 2-form, in general A is not a globally well-defined one-form so F might well be a non-exact closed 2-form.

Put more formally, A is the local one-form representation of a connection on a principal $U(1)$ bundle or complex line bundle over M. In our normalization the covariant derivative is locally $D = d + \mathrm{i}A$ and changes on patch overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ as $D_\alpha = g_{\alpha\beta}^{-1} D_\beta g_{\alpha\beta}$.

Remark: An important convention. Although it is not the best normalization from the mathematician's viewpoint, here we will take (the locally defined one form) A to be a real 1-form when the gauge group is $G = U(1)$. When $G = U(1)^d$ we continue to take it to be a d-tuple A^I of locally-defined 1-forms. However, when working with the general nonabelian Lie group we take A to be a one-form valued in the Lie algebra $\mathfrak{g}$. Now, since we are dealing with compact Lie groups, if G is a matrix group then $\mathfrak{g}$ will be a subalgebra of a Lie algebra of anti-hermitian matrices. With that convention the covariant derivative is $D_A = d + A$.

As mentioned above, because A is only locally defined, the 2-form F, while globally defined, need not be exact. Our gauge field is normalized so that $F/2\pi$ has integral periods, that is, the integrals around closed 2-cycles are always integers:

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z} \tag{2.38}$$

for any closed oriented 2-cycle $\Sigma_2 \subset M$. Moreover, if the gauge group is $G = U(1)$ there will always exist manifolds M, line bundles $L \to M$, with 2-cycles $\Sigma_2 \subset M$ so that the integer $\int_{\Sigma_2} \frac{F}{2\pi}$ is exactly one. Thus this is the optimal normalization. If the gauge group is, instead $G = \mathbb{R}$, the periods of F will always be zero. That is, F will always be exact, and A can be taken to be a globally well-defined 1-form.

With the above conventions the gauge transformations are such that the covariant derivative $D = d + \mathrm{i}A$ transforms by $D \to g^{-1} D g$ where

$$g : M \to U(1) \tag{2.39}$$

[10] *Answer*: The gauge transformation $e^{\mathrm{i}\alpha(t)}$ must extend to a continuous map $\Sigma \to U(1)$. If Σ is a smooth manifold whose only boundary is S^1, as we have tacitly assumed in writing equation (2.37), then such maps always restrict to small gauge transformations on the bounding S^1.

is a gauge transformation. If we can take a logarithm and take $g = e^{i\epsilon}$ for a globally well-defined function

$$\epsilon : M \to \mathbb{R} \tag{2.40}$$

then

$$A \to A - d\epsilon \tag{2.41}$$

we will call these "small gauge transformations." If the gauge group is $\mathbb{R}$ all gauge transformations are of this form. On the other hand, if the gauge group is $U(1)$ then there will be maps $g : M \to U(1)$ which do not admit a global logarithm. A simple test is whether the one-form $\omega = -ig^{-1}dg$ has nonzero periods or not. If $g(x)$ cannot be written as $g(x) = e^{i\epsilon(x)}$ for a globally well-defined function $\epsilon(x)$ we say that $g(x)$ is a "large gauge transformation." In particular, if $\pi_1(M, x_0)$ is nontrivial there will be large gauge transformations. In this case the gauge transformations are better thought of as shifts

$$A \to A + \omega \qquad \omega \in \Omega^1_{2\pi\mathbb{Z}}(M) \tag{2.42}$$

where $\Omega^1_{2\pi\mathbb{Z}}(M)$ is the space of all (suitably differentiable) closed one-forms whose periods are all in $2\pi\mathbb{Z}$.

In addition, when M is noncompact or has a nonempty boundary then we need to discuss boundary conditions on gauge fields and the gauge group. We will be considering different kinds of boundary conditions so we return to this later.

2.2.2 From θ-term To Chern-Simons

Now, since $F = dA$ we observe that the topological density in four-dimensions can be written as:

$$F \wedge F = d\,(AdA) \tag{2.43}$$

If we consider a 4d path integral with $\partial M_4 = M_3$ (where M_3 might or might not be connected) then we are asked to consider

$$\int_{M_3} AdA \tag{2.44}$$

as a term in the action.

At first (2.44) looks unpromising as a term in the action for a gauge theory: The term involves an explicit factor of A and hence is not obviously gauge invariant. On the other hand, $F \wedge F$ is gauge invariant so the change of AdA must be d-closed. We can hope it is d-exact and that a change by total derivatives "doesn't matter."

So, let us compute:

1. Under small gauge transformations (2.41) we have

$$A \wedge F \to A \wedge F + d\,(\epsilon F) \tag{2.45}$$

   This is indeed a total derivative, just as we had hoped. If we neglect boundary contributions (which, in general, we <u>cannot do</u>) then gauge invariance is assured. In any case, this is a good start.

2. Under large gauge transformations (2.42) we have

$$A \wedge F \to A \wedge F + \omega \wedge F \tag{2.46}$$

Now, any 1-form with quantized periods must necessarily be a closed 1-form, but in general it need not be globally exact, so $\omega \wedge F$ need not be globally exact. Along with the issues of boundary terms we will need to deal with this problem.

Quantization Of κ

We are considering a possible Chern-Simons term in the action of a $U(1)$ gauge theory on an oriented 3 manifold M_3 and we normalize the coupling constant as:

$$\exp[\mathrm{i}\frac{\kappa}{2\pi}\int_{M_3} A \wedge F] \tag{2.47}$$

For the moment we take M_3 to have no boundary. Even in that case, as we have just discussed the action is not gauge invariant. In the path integral we do not integrate over $\mathcal{A}$ - the space of all gauge fields (i.e. all connections on a fixed principal bundle) - but over $\mathcal{A}/CG$ - the space of all gauge-equivalence classes of gauge fields. Here $\mathcal{G}$ is the group of gauge transformations. Via a Fadeev-Popov or BRST type procedure we can push down a natural measure $[dA]$ on $\mathcal{A}$ to one on $\mathcal{A}/\mathcal{G}$, but then when we weight this measure with $e^{\mathrm{i}S}$ that factor must be a well-defined and single-valued function on the space $\mathcal{A}/\mathcal{G}$.

When we see an explicit gauge field in an action all the alarm bells should go off in your head: The theory probably does not make sense. Chern-Simons dodges the problem because of the multi-valued action principle mentioned above. This principle applies because the path integral of the theory is an integral over $\mathcal{A}/\mathcal{G}$, the space of gauge equivalence classes of gauge fields, and all we need is a (formally) well-defined measure $[dA]e^{\mathrm{i}S}$ on this space.

Let us return to (2.46). Then, as we just noted, the Chern-Simons action $\int_{M_3} AdA$ is not gauge invariant. The change in the exponentiated action under (2.46) is a multiplicative factor:

$$e^{2\pi \mathrm{i}\frac{\kappa}{4\pi^2}\int_{M_3} A\wedge dA} \to e^{2\pi \mathrm{i}\frac{\kappa}{4\pi^2}\int_{M_3} A\wedge dA} \cdot e^{2\pi \mathrm{i}\kappa \int_{M_3} \frac{\omega}{2\pi}\wedge\frac{F}{2\pi}} \tag{2.48}$$

Thus if the factor

$$e^{2\pi \mathrm{i}\kappa \int_{M_3} \frac{\omega}{2\pi}\wedge\frac{F}{2\pi}} \tag{2.49}$$

is always one for all gauge transformations $\omega \in \Omega^1_{2\pi\mathbb{Z}}(M_3)$ and all gauge fields then we are assured gauge invariance of the path integral measure. But, once again, this is all we really need for a well-defined path integral, and hence a well-defined theory. (At least, when M_3 has no boundary.)

By assumption we take $\omega \in \Omega^1_{2\pi\mathbb{Z}}(M_3)$ to be an arbitrary closed 1-form with $2\pi\mathbb{Z}$ periods. Now, a theorem in topology [12, 51] assures us that, if the gauge group is $U(1)$, then the form $F/2\pi$ has arbitrary integral periods.

Therefore, the path integral measure will be well-defined iff $\kappa \in \mathbb{Z}$.

Remarks

1. A crucial point is that the dimensionless coupling constant k in (2.51) and (2.52) is quantized. It is sometimes claimed that this is not the case in Chern-Simons with Abelian gauge group. That is the case if one considers $G = \mathbb{R}$, or rather relatively trivial three-manifolds (like $M_3 = \mathbb{R}^3$) that are simply connected. But, if the gauge group is $U(1)$ and we consider topologically interesting situations it must, in fact, be quantized.

2. We should mention here that there is an important refinement of this quantization story. For some purposes in physics (e.g. in the fractional quantum Hall effect) we need to take κ to be a half-integer. If we just blindly set $\kappa = 1/2$ then

$$\exp[\mathrm{i}\frac{1}{4\pi}\int_{M_3} AF] \tag{2.50}$$

is not gauge invariant: For perhaps the simplest example take $M_3 = T^3$ with $F/2\pi = d\sigma^1 d\sigma^2$ and make a gauge transformation with $\omega = 2\pi d\sigma^3$. Here $\sigma^i \sim \sigma^i + 1$ are coordinate on the torus. Nevertheless, one can define expressions like (2.50) in a gauge invariant way provided one adds data in defining the theory. Namely, if one chooses a "spin structure" on M_3 then it is possible to define a gauge invariant measure in the path integral. The price one pays is that the action and the theory then depend on the choice of spin structure, albeit in a computable way. We will return to this important point in detail in section 2.5 below.

2.2.3 3D Maxwell-Chern-Simons For U(1)

Fields And Action

Let us begin with the "Maxwell-Chern-Simons" $U(1)$ gauge theory in three dimensions. Thus, let M_3 be an oriented Riemannian three-manifold. Then the contribution of the action to the path integral is via a factor $e^{\mathrm{i}S}$ where:

$$S = \int_{M_3} \frac{-1}{8\pi^2 e^2} dA \wedge *dA + \frac{\kappa}{2\pi} A \wedge dA \tag{2.51}$$

Here we are working on a Lorentzian signature $(-,+,+)$ 3-manifold and the Lorentzian metric induces the Hodge star $*$. We choose our local coordinates so that the orientation, thought of as a globally non-vanishing 3-form is vol $= dx^0 \wedge dx^1 \wedge dx^2$.

In the Euclidean theory the contribution of the action to the path integral is a factor e^{-S} with

$$S_E = \int_{M_3} \frac{-1}{8\pi^2 e^2} dA \wedge *dA - \mathrm{i}\frac{\kappa}{2\pi} A \wedge dA \tag{2.52}$$

Remark: Note that, in contrast with the standard Maxwell term, the Chern-Simons term requires a choice of orientation. Orientation-reversing diffeomorphisms, such as parity-reversal, will change the sign of the Chern-Simons action. Moreover, note that the contribution of the Chern-Simons term to the measure of the path integral is a phase both in Euclidean and Minkowskian signatures.

Plane Waves And Local Degrees Of Freedom

The equation of motion in Lorentzian signature is

$$d * F - mF = 0 \tag{2.53}$$

where we define $m := 4\pi k e^2$, with units of mass. Consider the theory in Minkowski space $M_3 = \mathbb{M}^{1,2}$. To analyze local degrees of freedom we consider plane wave solutions:

$$A = \epsilon_i(\vec{p}) e^{ip\cdot x} + c.c. \tag{2.54}$$

where $p \cdot x = p_0 t + p_1 x^1 + p_2 x^2$. Then (2.53) becomes:

$$\begin{aligned}(p_1 p_2 + imp_0)\epsilon_1 + (p_0^2 - p_1^2)\epsilon_2 = 0 \\ (p_0^2 - p_2^2)\epsilon_1 + (p_1 p_2 - imp_0)\epsilon_2 = 0 \\ (p_1 p_0 + imp_2)\epsilon_1 + (p_0 p_2 - imp_1)\epsilon_2 = 0\end{aligned} \tag{2.55}$$

There are nonzero solutions for ϵ_i iff

$$p_0^2(p_0^2 - p_1^2 - p_2^2 - m^2) = 0 \tag{2.56}$$

There are two branches of solutions.

1. If $p_0 \neq 0$ then define $\omega(\vec{p}) := +\sqrt{\vec{p}^2 + m^2}$. For $p_0 = \pm\omega(\vec{p})$ the gauge is completely fixed, and ϵ_i is determined up to an overall multiplicative constant:

$$\begin{aligned}\epsilon_1^{\pm}(\vec{p}) = mp_1 + ip_0 p_2 \\ \epsilon_2^{\pm}(\vec{p}) = mp_2 - ip_0 p_1\end{aligned} \tag{2.57}$$

   Note that $\epsilon_i = \epsilon_i^+ = (\epsilon_i^-)^*$. Since the two polarizations are related we have the degree of freedom of a single massive scalar of mass $|m|$. This is the main result of [15, 14]. It generalizes the well-known equivalence of a $U(1)$ gauge field in 3d to a compact scalar field.

♣SHOULD I EXPLAIN THE DUALITY BETWEEN A $U(1)$ GAUGE FIELD AND A COMPACT SCALAR FIELD ? ♣

2. Another branch of solutions to the equations of motion has $p_0 = 0$. In this case, $p_1\epsilon_2 - p_2\epsilon_1 = 0$, that is, $F = 0$ and we have a flat gauge field. This gives the topological sector of the theory. We can detect that sector on $\mathbb{M}^{1,2}$ with nonlocal operators line Wilson lines in topologically nontrivial situations, or in other physical situations where the topology is nontrivial.

Long-Distance/Strong-Coupling Limit: The Topological Sector

The coupling e^2 has dimensions of mass. Under a conformal rescaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ of the 3-dimensional metric the first term in the action scales as Ω^{-1}, while the second is invariant. Note two things:

1. The long distance limit is obtained by taking a fixed metric $g_{\mu\nu}$ on M_3 and scaling $\Omega \to \infty$.

2. In that limit, at least formally, the first term vanishes and only the second term remains: We have a Chern-Simons theory.
3. Scaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ is equivalent to scaling $e^2 \to \Omega e^2$. So the long-distance limit is the same as the strong coupling limit.

When $e^2 \to \infty$ the mass $m \to \infty$ and the propagator goes to zero killing all Feynman diagrams and all local correlations. So the correlation functions of local gauge invariant observables like

$$\langle F_{\mu_1\nu_1}(p_1) F_{\mu_2\nu_2}(p_2) \cdots F_{\mu_n\nu_n}(p_n) \rangle \tag{2.58}$$

all will vanish. Another way of thinking of this is that, when the insertion points $p_1, \dots, p_n$ as very distant, by inserting a complete set of states and using the fact that these are states of a massive scalar field we see that the correlator must vanish like

$$\mathrm{Max}_{i \neq j} e^{-const.|m| \cdot \| p_i - p_j \|} \tag{2.59}$$

But this does not make the theory trivial in this limit. If R is a finite-dimensional representation of the gauge group $U(1)$ and $\gamma \subset M_3$ is a closed oriented loop then we can define the classical observable

$$W(R, \gamma) := \mathrm{Tr}_R \exp \left(\mathrm{i} \oint_\gamma A \right) \tag{2.60}$$

Of course, for $U(1)$ every finite dimensional representation is fully reducible to a sum of one-dimensional irreducibles and the irreducible representations are $\rho_n(z) = z^n$ for $n \in \mathbb{Z}$ so WLOG we can consider the classical Wilson lines

$$W(n, \gamma) := \exp \left(\mathrm{i}n \oint_\gamma A \right) \tag{2.61}$$

♣r is a poor choice of notation since it looks like a radius. ♣

Before we take $e^2 \to \infty$ the correlation functions of these Wilson lines in the path integral will depend strongly on the curve γ in spacetime, as well as the proximity of all its points to local operators such as $F_{\mu\nu}(p)$. However, the $e^2 \to \infty$ limit of the correlators of (2.61) will only depend on γ up to "isotopy." That means they will only depend on γ up to continuous deformation of γ insider M_3 minus the location of other operators. In particular, one cannot deform one curve γ through another curve γ'. To deduce these things we now proceed to the quantization of the "pure" Chern-Simons term where we formally take the limit $e^2 \to \infty$ in the action (2.52).

2.2.4 The Formal Path Integral Of The $U(1)$ Chern-Simons Theory

We now consider the Chern-Simons theory with path integral measure defined by the Chern-Simons term on its own. That is, we aim to make sense of the path integrals:

$$\int_{\mathcal{A}/\mathcal{G}} [dA] e^{2\pi \mathrm{i} \frac{\kappa}{4\pi^2} \int_{M_3} A \wedge dA} \prod_\alpha W(n_\alpha, \gamma_\alpha) \tag{2.62}$$

where the integral is, formally, over the space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of connections on a fixed line bundle over M_3.

A few immediate comments:

1. This is a free field theory with a quadratic action. The path integral can be rigorously defined and everything should be completely solvable. There is no excuse.
2. The coupling constant κ plays the role of $1/\hbar$.
3. It is useful to note the classical equation of motion when we couple to a source J. If we regard J as a 2-form then
$$\frac{\kappa}{2\pi}\int AdA + \int A \wedge J \tag{2.63}$$
gives equation of motion
$$F = -\frac{\pi}{\kappa} J \tag{2.64}$$
4. The action $\int_{M_3} A \wedge dA$ does not make reference to any spacetime metric. This suggests that the theory is a "topological field theory" where all transition functions, amplitudes, correlation functions etc. are independent of distances. (We will give a more formal and mathematical definition later.)
5. One should be cautious about jumping to this conclusion because after all, one needs to define a measure, and when defining the correlation functions of operators some regularization might be required. Indeed, quantum mechanically, there will be some very mild metric dependence, but it is easily described and isolated, and in essence this will turn out to be a true topological field theory. See sections 2.2.17 and 3.2.1 below
6. As an example of the kind of quantum subtleties we will find, the definition of the Wilson line observables $W(n, \gamma)$, will depend on extra data known as a "framing" and in fact they are observables associated to ribbons, not curves. See section 2.2.16.

2.2.5 First Steps To The Hilbert Space Of States

The next several sections, up to, but not including section 2.2.14 below will be concerned with the following problem: Suppose $M_3 = \Sigma_2 \times \mathbb{R}$, where Σ_2 is a closed topological surface. There should be a (Hilbert) space of quantum states associated with Σ_2. Since the theory depends on orientation of M_3 and such an orientation is determined by an orientation $\mathfrak{o}(\Sigma_2)$ we expect our space to depend on $\Sigma_2, \mathfrak{o}(\Sigma_2)$ and, of course, κ. We will denote it by
$$\mathcal{H}(\Sigma_2, \mathfrak{o}(\Sigma_2), \kappa) \tag{2.65}$$

Remark: Later when we introduce a complex structure J on Σ_2 we can write $\mathcal{H}(\Sigma_2, J, \kappa)$ and this will be the fiber of a bundle of Hilbert spaces over the moduli space of complex structures on Σ_2.

The first observation we should make about the action
$$\int_{M_3} A \wedge dA = \int_{M_3} d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{2.66}$$

is that it is first order in "time." If we choose local coordinates (x^0, x^1, x^2) with x^0 serving as a time then we can write $A = A_0 dx^0 + A_s$ where A_s involves only "spatial components" and

$$A \wedge dA = A_s dx^0 \partial_0 A_s - A_0 F_s \tag{2.67}$$

where F_s is the spatial component.

With this in mind we will focus on Hamiltonian quantization - breaking Lorentz symmetry by choosing a time. So we consider our 3-manifold to be of the form

$$M_3 = \Sigma_2 \times \mathbb{R} \tag{2.68}$$

where Σ_2 is a two-dimensional oriented surface. We can write

♣Notation question: Σ_2 oriented surface. Later use (Σ_2, J) for surface with a complex structure, i.e. Riemann surface?? ♣

$$\int_{M_3} A \wedge dA = -\int dx^0 \left(\int_{\Sigma_2} A_s \partial_0 A_s \right) - \int dx^0 \left(\int_{\Sigma_2} A_0 F_s \right) \tag{2.69}$$

Now, the action is <u>first order</u> in time derivatives. This is quite significant.

In general an action first order in time derivatives looks like

$$S = \int dt \lambda_i(\phi) \frac{d\phi^i}{dt} \tag{2.70}$$

We should view the action as an action for trajectories in a <u>phase space</u> $\mathcal{P}$. A phase space is the same thing as a symplectic manifold. See Appendix A for a summary of some definitions and facts from symplectic geometry. Note that the variation of the action is

$$\delta S = \int dt (\partial_i \lambda_j - \partial_j \lambda_i) \delta\phi^i \frac{d\phi^j}{dt} \tag{2.71}$$

and if the action is nondegenerate we can read off the symplectic form

$$\omega = (\partial_i \lambda_j - \partial_j \lambda_i) \delta\phi^i \wedge \delta\phi^j \tag{2.72}$$

Put differently, if we write (locally) $\omega = d\lambda$ for some one-form λ then the action for a trajectory along a path γ in phase space is

$$S = \int_\gamma \lambda - \int dt H(\gamma(t)) \tag{2.73}$$

Locally, in Darboux coordinates we can write $\lambda = p_i dq^i$ and hence our action is

$$S = \int p_i \frac{dq^i}{dt} dt - \int H dt = \int p\dot{q} dt - \int H(p(t), q(t)) dt \tag{2.74}$$

What we conclude from this is that

We should view the space $\mathcal{A}(P \to \Sigma)$ of gauge fields on Σ (connections on P) as a phase space and the action is the action of a particle moving on a phase space subject to a constraint.

2.2.6 General Remarks On Quantization Of Phase Space And Hamiltonian Reduction

What Does It Mean To "Quantize A Phase Space?"

Suppose we are given a symplectic manifold $\mathcal{P}$. So, it is a manifold with a closed 2-form $\omega = \frac{1}{2}\omega_{ij}(x)dx^i \wedge dx^j$ such that the antisymmetric matrix $\omega_{ij}(x)$ is invertible at all points x on the manifold. We want to"quantize $\mathcal{P}$." Let us say precisely what we mean by this:

First, we consider a family of symplectic forms $\omega^{(k)} = k\omega$ so that we can discuss a semiclassical limit. In our examples k will be an integer.

Want to assign a Hilbert space $\mathcal{H}_k$ and to suitable functions $f : \mathcal{P} \to \mathbb{C}$ corresponding linear operators $Q^{(k)}(f)$ on $\mathcal{H}_k$. We require that:

$$\lim_{k\to\infty} \parallel Q^{(k)}(f)Q^{(k)}(g) - Q^{(k)}(fg) \parallel < \infty \tag{2.75}$$

and

$$[Q^{(k)}(f), Q^{(k)}(g)] = -\frac{\mathrm{i}}{k}Q^{(k)}(\{f,g\}) + \mathcal{O}(1/k^2) \tag{2.76}$$

where

$$\{f,g\} = \omega^{ij}\partial_i f \partial_j g \tag{2.77}$$

are the Poisson brackets. Furthermore we want

$$(Q^{(k)}(f))^\dagger = Q^{(k)}(f^*) \tag{2.78}$$

so that real functions correspond to self-adjoint operators.

How To Quantize Of Phase Space

There are many approaches to quantizing phase spaces. As far as we know, there is no known general procedure. [11] However, if we are given more structure then there are some general methods:

1. *Schrödinger Quantization*: Sometimes, there is a global separation of p_i, q^i. As a simple case $\mathcal{P} = T^*M$ where M is a manifold. Then $\mathcal{H} = L^2(M)$. States are normalizable wavefunctions $\Psi(q^i)$ and $p_i = -\mathrm{i}\hbar\nabla_{q^i}$ are first order differential operators. Up to operator ordering we can translate classical observables - i.e. functions on $\mathcal{P}$ – into operators on $\mathcal{H}$. [12]

2. Sometimes the phase space has extra structures on it that allow quantization. One good example is when $\mathcal{P}$ is also a Kähler manifold and there is a holomorphic line bundle on $L \to \mathcal{P}$ with a suitable connection. Then we can apply the method of *geometric quantization* or *Kähler quantization* or *Berezin-Toeplitz quantization*. In Appendix B we recall some essential points from the theory of geometric quantization and coherent states.

[11] If we are content to work with formal series in $1/k$ the results of Fedosov, and Kontsevich, (see, especially the physical interpretation of Kontsevich's theorem by Cattaneo and Felder), do provide answers.

[12] In our application of the Schrödinger quantization there will be an important added subtlety that the momenta themselves are periodic, so we will not be quantizing a phase space of the form T^*M.

These approaches are not universally applicable. Thanks to advances in the theory of four-manifolds there are now plentiful examples of compact symplectic four-dimensional manifolds which do not admit the structure of a Kähler manifold. Nevertheless, in the examples of symplectic spaces that we will encounter in these notes, the above two procedures will suffice. In fact, we will mostly use the Kähler quantization approach.

Hamiltonian Reduction

See Appendix A for background on Hamiltonian reduction of symplectic spaces with symplectic group action.

When we are quantizing a phase space with first order constraints generating a symplectic Lie group action there are two ways to quantize:

1. First perform Hamiltonian reduction classically: Work out the space of fields with $\mu = 0$ then work out the quotient $\mu^{-1}(0)/G = \bar{\mathcal{P}}$ then quantize the quotient. Somehow.

2. First quantize on big phase space "upstairs" and impose then impose the constraints via operator constraints on the allowed states. In our case we quantize the space of gauge fields using (2.87). Then we impose the Gauss law on the wavefunctionals $\Psi[A]$, by $U(g)\Psi = \Psi$ for all g in the gauge group.

3. There are also mixed cases, where we impose constraints classically on some degrees of freedom before quantization, but then we quantize the remaining degrees of freedom, and then quantize. [13]

It is not *a priori* obvious that one will obtain the same results from the different procedures. One would certainly hope for this to be the case, and, thankfully, in Chern-Simons theories things turn out well. ♣Give example where the procedures do not commute? ♣

Application To Chern-Simons Theory

The symplectic form on that phase space is

$$\omega = \frac{\kappa}{2\pi} \int_{\Sigma} \delta A_s \wedge \delta A_s \tag{2.79}$$

We record here the generalization to the nonabelian theory. Using a normalization of an Ad-invariant trace on the Lie algebra $\mathfrak{g}$ discussed in section 3.1 we have

$$\omega = \frac{k}{4\pi} \int_{\Sigma} \mathrm{tr}\, (\delta A_s \wedge \delta A_s) \tag{2.80}$$

So we go from nonabelian to $U(1)$ by dropping tr and taking $k = -2\kappa$. To understand the minus sign, recall that for $G = U(1)$ we take A to be real but for general G we take A to be Lie-algebra valued. This accounts for the relative sign between (2.79) and (3.37).

For many of you the notation δA_s will be self-evident: It is a one-form on space and simultaneously a one-form on the space of all gauge fields corresponding to a small variation of connection.

[13] This applies to certain "Abelianization procedures" of the nonabelian theory. NEED TO CITE SOME REFERENCES FOR THIS. Includes last section of [17].

Remark: *For mathematically-minded fussbudgets.* Most physicists, when presented with expressions like (2.79) just know what is the right thing to do, but for those who start wondering about what a symbol like δA actually means, and "where it lives" the following remarks might be useful. Given a principal bundle $P \to M$ the tangent space to the space $\mathcal{A} = Conn(P \to M)$ of all connections on P is canonically

$$T_A \mathcal{A} \cong \Omega^1(M; \mathrm{ad}P) \tag{2.81}$$

at every point $A \in \mathcal{A}$. The space of connections is an affine space so it makes sense that the tangent space "doesn't depend" on the point A. To justify this note that if we have a continuous family of connections $d + A(t)$ then the difference at time $t + \epsilon$ and t is

$$(d + A(t+\epsilon)) - (d + A(t)) = A(t+\epsilon) - A(t) = \epsilon\alpha + \mathcal{O}(\epsilon^2) \tag{2.82}$$

for $\epsilon \to 0$ where α is some globally-defined one-form in the adjoint representation. Now suppose that $\gamma(t) = d + A(t)$ is a path of connections so that the tangent vector is $\dot{\gamma}(0) = \alpha$. Now, δA is the one-form such that

$$\delta A(\dot{\gamma}(0)) = \alpha \tag{2.83}$$

When you see a differential form on $\mathcal{A}$ written in terms of δA, if you want the mathematically precise version, apply it to polyvector fields made from $\alpha_1, \alpha_2, \dots$ and use (2.83) (and some common sense) and you will get the kinds of formulae found in the math papers. Another way of thinking about this is the following: If we imagine trivializing the bundle, choosing local coordinates and choosing a basis t_a for the Lie algebra $\mathfrak{g}$ then we have local "coordinates" $A^a_\mu(x)$ on $\mathcal{A}$ where a, μ, x are "indices". If you are uncomfortable with continuous indices like x then use lattice gauge theory to combine (x, μ) into a link variable. Then we have locally defined differentials $\delta A^a_\mu(x)$ where δ means an exterior derivative in the space of connections. So, (3.37) is shorthand for

$$\omega|_A = \frac{\kappa}{2\pi} \int_\Sigma \delta A^a_i(x) \wedge \delta A^a_j(x) dx^i \wedge dx^j \tag{2.84}$$

where we choose t_a to be an orthonormal basis for $\mathfrak{g}$ in the nondegenerate form tr.

Written in local coordinates, the Poisson brackets implied by (2.79) are

$$\{A_i(x), A_j(y)\} = \frac{\pi}{\kappa} \epsilon_{ij} \delta^{(2)}(x, y) \tag{2.85}$$

Here $\delta^{(2)}(x, y)$ is a Dirac delta function of weight one and we need to choose an orientation on Σ_2 to define ϵ_{ij}. We choose local coordinates so that $\epsilon_{12} = 1$. To get the normalization right it helps to recall that if $\omega = dp \wedge dq$ then $\{p, q\} = 1$.

In the nonabelian case we have:

$$\{A^a_i(x), A^b_j(y)\} = \frac{2\pi}{k} \epsilon_{ij} \delta^{ab} \delta^{(2)}(x, y) \tag{2.86}$$

where $A = A_i^a t_a dx^i$ and t^a are an orthonormal basis with respect to the trace tr. where $x, y \in \Sigma_2$. Thus, in the most straightforward approach to quantization we would say

♣Make sure this is geometrically right - section of correct bundle on RHS. Also ϵ is local gauge parameter and also tensor ϵ_{ij}. ♣

$$[\hat{A}_i(x), \hat{A}_j(y)] = -\frac{\mathrm{i}\pi}{\kappa}\epsilon_{ij}\delta^{(2)}(x, y) \tag{2.87}$$

Moreover, there is a group action on the phase space: The group of spatial gauge transformations. An infinitesimal gauge transformation by $\epsilon : M_3 \to \mathbb{R}$ generates $\delta A = D_A\epsilon = d\epsilon + [A, \epsilon]$ Geometrically, there is a vector field $V(\epsilon)$ whose value on the one-form δ on $\mathcal{A}$ is

$$V(\epsilon)(\delta A) = D_A\epsilon \tag{2.88}$$

With the symplectic form (2.79) we can work out the corresponding moment map:

$$\begin{aligned} \iota(V(\epsilon))\omega &= \frac{k}{2\pi}\int_\Sigma \mathrm{Tr} D_A\epsilon\delta A \\ &= \frac{k}{2\pi}\int_\Sigma [d\,(\mathrm{tr}\,\epsilon\delta A) - \mathrm{tr}\,\epsilon D_A\delta A] \\ &= -\frac{k}{2\pi}\int_\Sigma \mathrm{tr}\,(\epsilon D_A\delta A) + \frac{k}{2\pi}\int_\Sigma d\,(\mathrm{tr}\,\epsilon\delta A) \end{aligned} \tag{2.89}$$

In the case that $\partial\Sigma_2 = \emptyset$ we have that

$$\iota(V(\epsilon))\omega = \delta\left(-\frac{k}{2\pi}\int_\Sigma \mathrm{Tr}\epsilon F\right) \tag{2.90}$$

So:

♣NEED TO CHECK SIGNS ♣

$$\langle\mu, \epsilon\rangle = -\frac{k}{2\pi}\int_{\Sigma_2} \mathrm{tr}\,(\epsilon F) \tag{2.91}$$

That is, for all points $p \in \Sigma_2$,

$$\{\mu(\epsilon), A_i(p)\} = (-D_A\epsilon)_i(p) \tag{2.92}$$

where we have written this in a way that will generalize to the nonabelian case.

Now recall that in Hamiltonian reduction we consider the quotient of the subspace where $\mu = 0$ by the symmetry group. In our case $\mu = 0$ means we have a flat connection, $F = 0$ and therefore the Hamiltonian reduction gives us the phase space

$$\mathcal{M}_{flat} = \{A \in \mathcal{A} | F(A) = 0\}/\mathcal{G} \tag{2.93}$$

Thus we arrive at a key result: The Hamiltonian reduction of the phase space is the moduli space of <u>flat</u> gauge fields on Σ_2.

Remarks:

1. In the simple $U(1)$ case all this talk of Hamiltonian reduction is a bit of overkill: One could just note that A_0 functions as a Lagrange multiplier in the action and therefore integration over A_0 in the path integral gives a delta function $\delta(F_{12})$ localizing the action on flat gauge fields.

2. If we work with the Maxwell-Chern-Simons theory the story is quite different because the action is second-order in time derivatives. The canonical conjugate of the gauge field is the electric field, but there is a nontrivial line bundle with connection on phase space. Some of this is explained in section 2.2.10.

♣NEED TO EXPLAIN MORE HERE! ♣

3. We will write down the explicit symplectic form on the finite-dimensional moduli space of flat connections below. In the nonAbelian case, in contrast to the relatively simple Poisson brackets (3.38), it is nontrivial to write down explicitly the symplectic structure on the finite dimensional moduli space of flat connections.

♣This remark probably goes in the nonabelian section and one should discuss Goldman brackets etc. ♣

2.2.7 The Space Of Flat Gauge Fields On A Surface

One more preliminary before we make a few remarks about moduli spaces of flat gauge fields. Our conventions for path-ordered exponentials are in Appendix H. Note in particular that A is a locally defined form in $\mathfrak{g}$ and is thus valued in anti-hermitian matrices.

♣They might need to be changed: Better to have successive paths lead to successive right-multiplications of Pexp. ♣

We begin with a G gauge theory on any manifold X.

Quite generally, the holonomy of a flat gauge field for group G defines a map from closed curves on X to G. If we choose a basepoint x_0 for the curve so that $\gamma(t)$ goes from x_0 to x_0 then we define the holonomy:

$$\mathrm{Hol}_{x_0}(A;\gamma) := P\exp \oint_\gamma A \tag{2.94}$$

where A is the flat gauge field. In general

$$\mathrm{Hol}_{x_0}(A^g;\gamma) := g^{-1}(x_0) Hol_{x_0}(A;\gamma) g(x_0) \tag{2.95}$$

Therefore, if we make a gauge transformation, or change the trivialization of our bundle near x_0, then the holonomy function gets conjugated by an element $g(x_0) \in G$. Therefore only the conjugacy class is gauge invariant. Similarly, if x_0, y_0 are two points in the same connected component of X and we have chosen a path then the holonomy function for curves based at x_0 is just a conjugate of the holonomy function for curves based at y_0.

A key and useful general fact:

Theorem 1: The conjugacy class of the holonomy function is a complete gauge invariant description of the gauge field in the following sense: A' is gauge equivalent to A, i.e. $A' = A^g = g^{-1}Ag + g^{-1}dg$ for some $g \in \mathcal{G}$ iff there exists a $g_0 \in G$ such that, for all based loops $\gamma \in \Omega_{x_0}(X)$

$$\mathrm{Hol}_{x_0}(A';\gamma) = g_0^{-1} \mathrm{Hol}_{x_0}(A;\gamma) g_0 \tag{2.96}$$

Proof: A connection is, by definition, a parallel transport law. We will demonstrate that all parallel transports are gauge equivalent. Choose a point $x_0 \in X$. Let $\mathbb{U}(A^{(i)}, \gamma_{x_0,x_0})$, with $i = 1, 2$, be the parallel transport along a based closed loop γ_{x_0,x_0} at x_0. Then

$$\mathbb{U}(A^{(1)}, \gamma_{x_0,x_0}) = h(x_0)^{-1} \mathbb{U}(A^{(2)}, \gamma_{x_0,x_0}) h(x_0) \tag{2.97}$$

for some $h(x_0) \in G$ that does not depend on the choice of closed loop γ_{x_0,x_0}. By making a suitable gauge transformation on $A^{(2)}$ we can put $h(x_0) = 1$. This fixes the gauge (framing) at x_0. Now, for each $x \in X$ choose a "fiducial path" $\bar{\gamma}_{x_0,x}$ from x_0 to x and choose a local framing at x. Then, trivially, there is a $g(x)$ so that

$$\mathbb{U}(A^{(1)}, \bar{\gamma}_{x_0,x}) = \mathbb{U}(A^{(2)}, \bar{\gamma}_{x_0,x})g(x) \tag{2.98}$$

Now, if $\tilde{\gamma}(x_0, x)$ is any other path from x_0 to x then $\bar{\gamma}_{x_0,x} \star \tilde{\gamma}^{-1}_{x_0,x}$ is a closed path from x_0 to x_0, and so from (2.97) and (2.98) we learn that

$$\mathbb{U}(A^{(1)}, \tilde{\gamma}_{x_0,x}) = \mathbb{U}(A^{(2)}, \tilde{\gamma}_{x_0,x})g(x) \tag{2.99}$$

for any other path $\tilde{\gamma}_{x_0,x}$. So $A^{(2)} \to (A^{(2)})^g$ is the gauge transformation that lines up all parallel transports. It remains to check that $g(x)$ is actually well-defined. Suppose we have a family of fiducial paths $\bar{\gamma}_{x_0,x(s)}$ with $x(0) = x(1) = x$. We note that

$$\bar{\gamma}_{x_0,x(s=1)} = \bar{\gamma}_{x_0,x(s=0)} \star \bar{\gamma}_{x_0,x(s)}(t = 1) \tag{2.100}$$

where in the second factor on the RHS we have a closed path, parametrized by s, of the endpoints of the fiducial paths (each of which is parametrized by t). But we know that

$$\mathbb{U}(A^{(1)}, \bar{\gamma}_{x_0,x(s)}(t = 1)) = g(x)^{-1}\mathbb{U}(A^{(2)}, \bar{\gamma}_{x_0,x(s)}(t = 1))g(x) \tag{2.101}$$

From this we learn that if we define a closed loop of fiducial paths we nevertheless have the same gauge transformation $g(x)$. ♠

Remarks

****

For a fixed connection $D_A = d + A$, denote holonomy function for closed curves based at any point,

$$[Hol_{x_0}(A, \cdot)] : \Omega_{x_0}(X) \to \mathcal{C} \tag{2.102}$$

where $\mathcal{C}$ is the set of conjugacy classes of G.

DISCUSS WHEN THIS IS AND IS NOT A COMPLETE GAUGE INVARIANT.

ALSO: Consider the functions defined for R a finite-dimensional representation of G:

$$W(R) : \mathcal{A}/\mathcal{G} \times \Omega_{x_0}(X) \to \mathbb{C} \tag{2.103}$$

defined by

$$W(R)(A, \gamma) := \mathrm{Tr}_R Pexp \oint_\gamma A \tag{2.104}$$

Discuss when this separates points. That is, if $W(R)(A', \gamma) = W(R)(A, \gamma)$ for all R, γ then A' is a gauge transformation of A. Reference: [56]

****

Because the parallel transport composes nicely under composition of curves (again see Appendix H for precise statements) and because the parallel transport around a small closed curve is a series in curvatures and their covariant derivatives it follows that we have

Theorem 2: If $F(A)$ is a flat gauge field then the function

$$\mathrm{Hol}_{x_0}(A;\cdot) : \Omega_{x_0}X \to G \tag{2.105}$$

only depends on γ up to homotopy and descends to a group homomorphism $\pi_1(X, x_0) \to G$.

Proof: Let $\gamma(t;s)$ be a homotopy of $\gamma_0(t)$ to $\gamma_1(t)$ where $0 \le s \le 1$. Then consider the function of s given by $\mathrm{Hol}_{x_0}(A, \gamma(\cdot, s))$. From properties of the path ordered exponential we can derive:

$$\frac{d}{ds}\mathrm{Hol}_{x_0}(A,\gamma(\cdot,s)) = \int_0^1 \left(P\exp \int_{x_0}^{\gamma(t,s)} A_\mu \frac{dx^\mu}{dt} dt \right) \frac{dx^\mu}{ds}\frac{dx^\nu}{dt} F_{\mu\nu}(\gamma(t,s)) \left(P\exp\int_{\gamma(t,s)}^{x_0} A_\mu \frac{dx^\mu}{dt} dt \right) \tag{2.106}$$

If $F_{\mu\nu} = 0$ then this variation is zero so $\mathrm{Hol}_{x_0}(A, \gamma)$ only depends on the homotopy class of γ. It follows from the composition properties of the holonomy that, as a function on $\pi_1(X, x_0)$ for fixed A this is a group homomorphism ♠.

We conclude that, if we restrict to flat gauge fields then the holonomy map is a group homomorphism $\pi_1(X, x_0) \to G$ that captures all the gauge invariant information in the flat gauge connection. Therefore, the moduli space of flat G connections on X is, always,

$$\mathcal{M}_{\mathsf{flat}} \cong \mathrm{Hom}(\pi_1(X, x_0), G)/G \tag{2.107}$$

where on the RHS G acts on a homomorphism $\phi : \pi_1 \to G$ by conjugation:

$$(g \cdot \phi)([\gamma]) := g^{-1}\phi([\gamma])g \tag{2.108}$$

Remarks:

1. Equation (2.107) also holds when G is disconnected. If G is finite we are talking about finite covering spaces. Each G bundle has a unique connection, and it is necessarily flat. (One way to see this is that the Lie algebra of G is the zero vector space and the curvature of a connection is always valued in the Lie algebra.)

2. Note that we always have the trivial homomorphism $\phi([\gamma]) = 1_G$ for all $[\gamma]$. Therefore, the moduli space is always nonempty.

3. If π_1 is a finitely generated group then one can present $\mathcal{M}_{\mathsf{flat}}$ more explicitly as follows: Choose generators γ_i, $i = 1, \dots, N$, and relations R_a, $a = 1, \dots, M$, for $\pi_1(X, x_0)$. A homomorphism ϕ is characterized by the images $\phi(\gamma_i) = g_i \in G$. If G is finitely generated then $\mathrm{Hom}(\pi_1(X, x_0), G)$ can be viewed as a subspace of G^N. For we can fix N generators $(\gamma_1, \dots, \gamma_N)$ and our homomorphism is completely determined by the images $(g_1, \dots, g_N) \in G^N$ under ϕ. However, the equation $\phi(\gamma_i) = g_i$ will only define a consistent homomorphism if the images g_i are consistent with the relations R_a. Since ϕ is a homomorphism we must have $\phi(R_a) = 1_G$. On the other hand,

since the R_a are words in the γ_i and ϕ is a homomorphism the image of R_a under ϕ must be a corresponding word in the g_i. This defines constraint equations on the possible choices of g_i. For matrix groups, they are polynomial equations on the matrix elements of the g_i. Thus, $\mathrm{Hom}(\pi_1(X, x_0), G)$ is an algebraic subspace of G^N defined by the relations. This subspace is then invariant under overall conjugation by G. Note that, since it is an algebraic variety we can expect there will be singularities in general.

4. When G is a positive dimensional Lie group we can, typically, deform the g_i (consistent with the relations), and so $\mathcal{M}_{\text{flat}}$ will be some positive dimensional space. However, it will clearly be singular because of the quotient by the conjugation action of G. These are singularities above and beyond the algebraic singularities of the previous remark. For example, the trivial homomorphism $\phi(\gamma) = 1_G$ for all $\gamma \in \pi_1$ is a fixed point under the conjugation action of all of G. So, the space is always singular at least at one point. And there will typically be other singularities from the quotient by G at other homomorphisms where the centralizer jumps. The "moduli space" is more properly understood as a "moduli stack" for exactly this reason.

5. One criterion for determining if we are working at a nonsingular point is to choose a faithful representation $\rho : G \to GL(V)$ of G. Then $\rho \circ \phi$ makes V the carrier space of a representation of $\pi_1(X, x_0)$. If this representation is irreducible then ϕ will be a *stable* point of $\mathrm{Hom}(\pi_1(X, x_0), G)$ and it should be a smooth point.

♣Clear for surface groups. Be careful about making too general a statement. ♣

We can now specialize in two independent ways:

First, we take $X = \Sigma_2$ to be a two-dimensional surface, but G is not necessarily Abelian. In this case the holonomy is completely determined by the holonomy on the generators of $\pi_1(\Sigma_2, x_0)$, so, choosing α_I, β^I cycles as in figure 2 with corresponding holonomies A_I, B_I this space is quite explicitly:

$$\{(A_I, B_I) \in G^{2g} | \prod_{I=1}^{g} [A_I, B_I] = 1_G\}/G \tag{2.109}$$

If there are punctures $x_1, \ldots, x_n$ then we have holonomies C_i around the punctures and we have

$$\{(A_I, B_I) \in G^{2g} | \prod_{I=1}^{g} [A_I, B_I] \prod_{i=1}^{n} C_i = 1_G\}/G \tag{2.110}$$

If G has positive dimension this space has dimension

$$\dim \mathcal{M}_{\text{flat}} = (2g+n) \dim G - \dim G - \dim G = (2g-2+n) \dim G = -\chi(\Sigma_2) \dim G \tag{2.111}$$

Second, in the case that G is Abelian, but X is general, the homomorphism $\pi_1(X, x_0) \to G$ factors through a homomorphism from the homology: $H_1(X; \mathbb{Z}) \to G$. This follows from

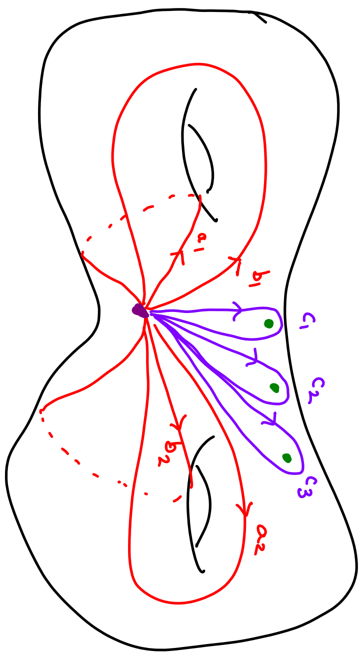

Figure 2: A choice of generators of $\pi_1(\Sigma_2, x_0)$ for the case of a genus two surface with three punctures.

Stokes' theorem: If $\gamma - \gamma' = \partial\Sigma$ then [14]

$$\exp[\mathrm{i}\oint_\gamma A]/\exp[\mathrm{i}\oint_{\gamma'} A] = \exp[\mathrm{i}\int_\Sigma F] = 1 \tag{2.112}$$

In fact, this argument shows that in the Abelian case, the holonomy only depends on the homology class $[\gamma] \in H_1(\Sigma_2;\mathbb{Z})$.

Now, taking $G = U(1)$ and $X = \Sigma_2$ we therefore have

$$\mathcal{M}_{\text{flat}} = \text{Hom}(H_1(\Sigma_2;\mathbb{Z}), U(1)) \tag{2.113}$$

Now Σ_2 has no torsion in its homology so we can safely say:

$$\text{Hom}(H_1(\Sigma_2;\mathbb{Z}), U(1)) \cong H^1(\Sigma_2;\mathbb{R})/H^1(\Sigma_2;\mathbb{Z}) \tag{2.114}$$

This can be understood more directly as follows: We can directly solve $F = dA = 0$. First of all, this implies that the first Chern class, which is measured by $\int_{\Sigma_2} F/2\pi = 0$.

[14] Recall our convention that for the special case when G is a torus group we consider A to be real.

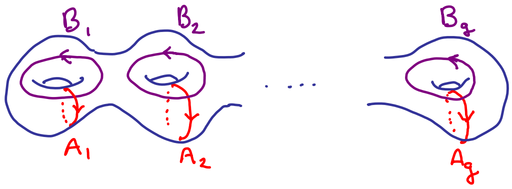

Figure 3: A standard choice of basis for the homology of a genus g surface (given its presentation as the surface of a handlebody.

Therefore, we can think of A as a globally well-defined 1-form. Since $dA = 0$ it is a closed one-form. But we must mod out by the shifts $A \to A + \omega$ where $\omega \in \Omega^1_{2\pi\mathbb{Z}}(\Sigma_2)$. The resulting space is just

$$\mathcal{M}_{flat} = H^1(\Sigma_2; \mathbb{R})/H^1(\Sigma_2; \mathbb{Z}) \tag{2.115}$$

This is just a torus of dimension $2g$. If we choose explicit generators A_I, B_I for the homology (as, for example, in 3) then we have coordinates

$$e^{\mathrm{i}\theta_I} = e^{\mathrm{i}\oint_{A_I} A} \qquad\qquad e^{\mathrm{i}\phi_I} = e^{\mathrm{i}\oint_{B_I} A} \tag{2.116}$$

In these coordinates we can write

$$A = \theta_I \alpha_I + \phi_I \beta_I \tag{2.117}$$

where α_I, β_I is a dual basis of degree one cohomology classes and

$$\omega = \frac{\kappa}{\pi} \delta\theta_I \wedge \delta\phi_I \tag{2.118}$$

Remarks:

1. Dimension Formula. Alert readers will note that (2.111) can sometimes be negative. Indeed, this is a virtual dimension formula. If the conjugation acts freely at a smooth point of $\mathrm{Hom}(\pi_1, G)$ then it truly gives the dimension of a manifold. However, if the group action is ineffective then the true dimension is not given by the formula (if we even have a manifold at all). For example, consider the case where Σ_2 is S^2 with no punctures. Then π_1 is the group with one element and there is only one homomorphism. The moduli stack is that stack $pt//G$ and formally has negative dimension. In the next sections we will study the torus with $G = U(1)$. In this case, the commutator equation in fact poses no constraint and conjugation acts trivially. So, the true dimension is
$$2g \dim G = 2 \times 1 \times 1 = 2 \tag{2.119}$$
and the two formal subtractions $-\dim G - \dim G$ contribute -2 to the virtual dimension, which is zero. In a sense, every point is "singular" since the division by the conjugation action is trivial everywhere.

2. Intrinsic definition of the symplectic structure. We have seen from (2.93) that the space of flat gauge fields on Σ_2 is a Hamiltonian reduction of a symplectic space and therefore the symplectic form is inherited from (2.79). It is rather straightforward to write it down for the Abelian case, as we did above. It is more challenging to write an explicit formula in the nonabelian case. This was done in [24, 25] and we give a brief description here. Denote $\pi := \pi_1(\Sigma_2, x_0)$. Suppose we have a homomorphism $\phi : \pi \to G$. By constructing a tangent vector v to the space of homomorphisms at ϕ one easily sees that v should be considered as a group cocycle $\phi \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)})$. Here we are using group cohomology with coefficients in a representation of π obtained by taking $\rho(\gamma) := \mathrm{Ad}(\phi(\gamma))$ acting on $\mathfrak{g}$. In order to understand this, suppose we have a family of homomorphisms $\phi_t : \pi \to G$ defining a path through $\phi = \phi_0$ at time $t = 0$. Then, for every $\gamma \in \pi$ we have a path of group elements in G: $\phi_t(\gamma) = g_\gamma(t)$. Now, define
$$v : \pi \to \mathfrak{g} \tag{2.120}$$
by
$$v(\gamma) := \phi(\gamma)^{-1} \frac{d}{dt}|_0 \phi_t(\gamma) = g_\gamma(0)^{-1} \frac{d}{dt}|_0 g_\gamma(t) \tag{2.121}$$
Now we must impose the group homomorphism property:
$$\phi_t(\gamma_1)\phi_t(\gamma_2) = \phi_t(\gamma_1\gamma_2) \tag{2.122}$$
for all $\gamma_1, \gamma_2 \in \pi$. Taking a derivative at $t = 0$ and multiplying the equation on the left by $\phi_0(\gamma_1\gamma_2)^{-1}$ gives:
$$\mathrm{Ad}(\phi_0(\gamma_2))v(\gamma_1) - v(\gamma_1\gamma_2) + v(\gamma_2) = 0 \tag{2.123}$$

This is precisely the group cocycle condition for coefficients in $\mathfrak{g}$ where $\mathfrak{g}$ is thought of as a π-representation via $\gamma \to \mathrm{Ad}(\phi(\gamma))$. The space of such cocycles is denotes $Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)})$. On the other hand, if we make a family of homomorphisms

$$\phi_t(\gamma) = g(t)^{-1}\phi(\gamma)g(t) \tag{2.124}$$

with some family $g(t) = e^{-tX}$ where $X \in \mathfrak{g}$ we clearly should mod out by such tangent vectors since our space is obtained with a quotient by the conjugation action of G. Again taking a derivative at $t = 0$ shows that such tangent vectors are

$$v(\gamma) = \mathrm{Ad}(\phi(\gamma))X - X \tag{2.125}$$

This is precisely the definition of coboundaries $B^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)})$. Thus, we conclude that there is a natural isomorphism

$$T_{[\phi]}\mathcal{M}_{\mathrm{flat}} \cong H^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)}) \tag{2.126}$$

Now, because Σ_2 has a simply connected cover [15] we can identify the group cohomology with the singular cohomology $H^k(\pi; M) \cong H^k(\Sigma_2; M)$. Indeed, if we have a flat G bundle $P \to \Sigma_2$ then there is an adjoint bundle and the tangent space to the space of flat G bundles is $H^1(\Sigma_2; \mathrm{ad}P)$. Using the trace form on $\mathfrak{g}$ and multiply and integrate we get a map

$$H^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}(\phi)}) \to \mathbb{R} \tag{2.127}$$

and this is Goldman's description of the symplectic form. One useful way to describe it is in terms of the Poisson brackets of the classical Wilson lines. In general, if $f : G \to \mathbb{R}$ is an Ad-invariant function let $df : G \to \mathfrak{g}$ be the function defined by

$$\frac{d}{dt}|_0 f(ge^{tX}) = \mathrm{tr}\,(df(g)X) \tag{2.128}$$

Then in [25] it is shown that if $W_R(\gamma, A) = \mathrm{Tr}_R P \exp \oint_\gamma A$ then one can write a beautiful explicit formula for the Poisson brackets of these functions:

$$\{W_R([\gamma_1], \cdot), W_R([\gamma_2], \cdot)\} = \sum_{p \in \gamma_1 \cap \gamma_2} \iota_p(\gamma_1, \gamma_2)\mathrm{tr}\,\left(dW_R(\gamma_1(p), \cdot)dW_R(\gamma_2(p), \cdot)\right) \tag{2.129}$$

In this formula we have chosen generic representatives γ_1, γ_2 for the homotopy class that intersect transversally in a finite number of points. For $p \in \gamma_1 \cap \gamma_2$ we denote by $\gamma_i(p)$ the representative of γ_i based at p. It is not obvious, but it is true that the choice of representative used on the RHS does not matter.

The bracket (2.129) is very reasonable given the Poisson brackets on the gauge fields in (3.38): The delta-function support means there can only be contributions from the intersections. One can then do a simple local analysis to derive the formula.

♣SHOULD DO THIS IN DETAIL!! ESSENTIALLY THIS IS WHAT IS DONE IN APPENDIX B OF GMN2 ♣

[15] This is the uniformization theorem: For $g = 0$ it is simply connected, for $g = 1$ the cover is $\mathbb{C}$, for > 1 it is the upper half-plane, or equivalently the disk. Technically, the condition we are using is that, so long as Σ_2 is not the sphere with no punctures, the surface is a $K(\pi, 1)$ space.

For the case where $f(\gamma, A) = 2\mathrm{Re}\chi_{fund}(\gamma, A)$ for $G = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ this simplifies to

$$\{f(\gamma_1, \cdot), f(\gamma_2, \cdot)\} = \sum_{p \in \gamma_1 \cap \gamma_2} \iota_p(\gamma_1, \gamma_2) f(\gamma_1(p) \cdot \gamma_2(p), \cdot) \tag{2.130}$$

2.2.8 Quantization Of Flat Connections On The Torus: The Real Story

We now specialize to $\Sigma_2 = T^2$, the two-dimensional torus.

Here we have one of the most basic and most easily accessible examples of a quantization of a Chern-Simons theory. Therefore, we will cover it in excruciating detail.

We describe it as $T^2 = \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ that is, we choose coordinates (σ^1, σ^2) with

$$\sigma^i \sim \sigma^i + 1 \qquad\qquad i = 1, 2 \tag{2.131}$$

There are two generators of π_1 and we can take as representatives:

$$\begin{aligned} \alpha : \gamma(t)(\sigma^1(t), \sigma^2(t)) &= (t, 0) \qquad\qquad 0 \le t \le 1 \\ \beta : \gamma(t)(\sigma^1(t), \sigma^2(t)) &= (0, t) \qquad\qquad 0 \le t \le 1 \end{aligned} \tag{2.132}$$

The gauge equivalence class of the flat $U(1)$ gauge field is described by the holonomies around the α, β cycles:

$$U_1 = \mathrm{Hol}(\alpha) = \exp \mathrm{i} \oint_\alpha A \qquad\qquad U_2 = \mathrm{Hol}(\beta) = \exp \mathrm{i} \oint_\beta A \tag{2.133}$$

We can define logarithms:

$$U_i = e^{\mathrm{i} a_i} \qquad a_i \in \mathbb{R}/2\pi\mathbb{Z} \tag{2.134}$$

Put differentialy, the most general flat $U(1)$ gauge field on T^2 is

$$A = a_1 d\sigma^1 + a_2 d\sigma^2 \tag{2.135}$$

where we must identify $a_i \sim a_i + 2\pi\mathbb{Z}$ because of large gauge transformations. To get the correctly normalized symplectic form we substitute into the action:

$$\frac{\kappa}{2\pi} \int_{T^2 \times \mathbb{R}} A dA = \frac{\kappa}{2\pi} \int dt (a_1 \dot{a}_2 - a_2 \dot{a}_1) \tag{2.136}$$

so we recover a specialization of the general result **** above:

$$\omega = \frac{\kappa}{\pi} da_1 \wedge da_2 \tag{2.137}$$

Remarks

1. In our normalization the semiclassical number of states in the Hilbert space is given by the symplectic volume of $\omega/(2k)$. In the present case that is:

$$\int_{\mathcal{M}_{flat}} \frac{\omega}{2\pi} = 2\kappa \tag{2.138}$$

2. To quantize we will need to choose which coordinate on phase space is "position" and which is a "momentum." Clearly there is an ambiguity here. Classically we can make symplectic transformations on the a_i valued in $Sp(2,\mathbb{R})$. Because of the periodicity conditions on the a_i we are only allowed to make transformations in $Sp(2,\mathbb{Z})$. So we take

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{2.139}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) = Sp(2,\mathbb{Z}) \tag{2.140}$$

♣This remark now redundant with material in 2.2.12. ♣

3. These canonical transformations can also be identified with the action of large diffeomorphisms of the torus. The transformation (2.139)(2.140) is equivalent to

$$\begin{pmatrix} \tilde{\sigma}^1 \\ \tilde{\sigma}^2 \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \tag{2.141}$$

   Clearly, this is a diffeomorphism of the torus. In fact, it is a nontrivial diffeomorphism in the following sense:

   In general, if X is a smooth manifold one can put a topology on the group of all diffeomorphisms of X. This is an infinite-dimensional (Banach) Lie group. The connected component of the identity is a normal subgroup and the quotient $Diff(X)/Diff_0(X)$ is often called the group of *large diffeomorphisms.* [16] In the case of the torus, using the simple relation to the universal cover, one can show that $Diff^+(T^2)/Diff_0^+(T^2) \cong SL(2,\mathbb{Z})$.

4. We will discuss later how these canonical transformations/diffeomorphisms act on the Hilbert space of the theory.

For the moment, let us choose a_2 to be the coordinate, so the measure becomes

$$\exp\left(\mathrm{i}\frac{\kappa}{\pi}\int dt a_1 \dot{a}_2\right) \tag{2.142}$$

Easy enough: Upon quantization we must have

$$[\hat{a}_1, \hat{a}_2] = -\mathrm{i}\frac{\pi}{\kappa} \tag{2.143}$$

We describe the quantum states by normalizable wavefunctions $\psi(a_2)$ and momentum

$$p_{a_2} = \frac{\kappa}{\pi} a_1 \tag{2.144}$$

But now we need to impose periodicity: First of all, $\psi(a_2)$ must be periodic under $a_2 \sim a_2 + 2\pi n$, $n \in \mathbb{Z}$.

[16] More properly, a large diffeomorphism is an element of $Diff(X)$ that passes to a nontrivial element of the quotient.

However, and this is very different from the quantization of a particle moving on a circle, we also must have periodicity in a_2.

Normally, periodicity in position space implies that momenta are quantized, so that the De Broglie waves can fit in the periodic space. Therefore, we expect that periodicity in momenta should quantize the positions.

Indeed, if we quantized with a_1 as coordinate then

$$\hat{a}_2 = \mathrm{i}\frac{\pi}{\kappa}\frac{\partial}{\partial a_1} \tag{2.145}$$

so $\exp(-\mathrm{i}\frac{\kappa}{\pi}\alpha\hat{a}_2)$ translates $a_1 \to a_1 + \alpha$ and therefore, if we wish to have periodicity $a_1 \sim a_1 + 2\pi$ then, returning to the Schrödinger picture with a_2 as coordinate and states as wavefunctions $\psi(a_2)$ we must have

$$e^{-\mathrm{i}2\kappa\hat{a}_2}\psi = \psi \tag{2.146}$$

But, since we are in the Schrödinger representation with a_2 as coordinate this implies

$$e^{-\mathrm{i}2\kappa a_2}\psi(a_2) = \psi(a_2) \tag{2.147}$$

This implies that ψ can only have support on the values of a_2 given by $a_2 = \frac{n\pi}{\kappa}\mathrm{mod}2\pi\mathbb{Z}$ so the different possibilities are $n = 0, \dots, 2\kappa - 1\mathrm{mod}2\kappa$.

Conclude: Since both momentum and coordinate are periodic, the general wavefunction is of the form

$$\psi = \sum_{n=0}^{2\kappa-1}\psi_n\delta\left(a_2 - \frac{n\pi}{\kappa}\right) \qquad \psi_r \in \mathbb{C} \tag{2.148}$$

♣DOESN'T LOOK VERY NORMALIZABLE! JUSTIFY ♣

We conclude that the Hilbert space of states on the torus $\mathcal{H}(T^2)$ is a complex Hilbert space of dimension 2κ.

The problem is isomorphic to the quantization of a particle on a discrete approximation to a ring: We think of the ring as $U(1)$ and the "discrete approximation" as the embedding of the cyclic group $\mathbb{Z}_{2\kappa}$.

2.2.9 Quantization Of Flat Connections On The Torus: The Coherent Story

In this section we endow the topological surface Σ_2 with a complex structure J. Together (Σ_2, J) defines a *Riemann surface*.

We first consider the case where Σ_2 is a torus, as in the previous section. The general complex structure produces on the torus gives a Riemann surface which can be written as an elliptic curve:

$$(\Sigma_2, J) = E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \tag{2.149}$$

where τ is a complex number and it must have a nonzero imaginary part so the quotient is a nice manifold. The complex structure is determined by saying that

$$d\zeta = d\sigma^1 + \tau d\sigma^2 \tag{2.150}$$

is a $(1,0)$ form. We need to choose an orientation on Σ_2 and we choose $d\sigma^1 \wedge d\sigma^2$. Given

♣Find a bettter font: Need a fancy z here. ♣

a complex structure there is a canonical orientation $id\zeta \wedge d\bar{\zeta}$. Working this out, we see that we should choose $\mathrm{Im}\tau > 0$.

Math fact: τ, τ' define equivalent complex structures if

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.151}$$

That is E_τ is isomorphic to $E_{\tau'}$ as a complex manifold iff(2.151) holds. Thus we have a moduli space ("stack") of complex structures

$$\mathcal{M}_{cplx}(\Sigma_2) \cong \mathcal{H}/SL(2,\mathbb{Z}) \tag{2.152}$$

****

Keyhole. Explain word "stack" in terms of the two order 2,3 orbifold points.
****

The complex structure on Σ_2 induces a complex structure on the space of flat gauge fields:

$$\begin{aligned} A &= a_1 d\sigma^1 + a_2 d\sigma^2 \\ &= a_\zeta d\zeta + a_{\bar{\zeta}} d\bar{\zeta} \\ a_\zeta &= \frac{a_2 - \bar{\tau} a_1}{\tau - \bar{\tau}} \\ a_{\bar{\zeta}} &= -\frac{a_2 - \tau a_1}{\tau - \bar{\tau}} \end{aligned} \tag{2.153}$$

The a_ζ and $a_{\bar{\zeta}}$ serve as (anti-)holomorphic coordinates on the moduli space of flat connections.

It will actually be useful to define slightly modified coordinates:

$$\begin{aligned} \tilde{a}_{\bar{\zeta}} &= (a_2 - \tau a_1)/2\pi \\ \tilde{a}_\zeta &= (a_2 - \bar{\tau} a_1)/2\pi \end{aligned} \tag{2.154}$$

Because we want our wavefunctions to vary holomorphically with complex structure, and not anti-holomorphically [17] we will take $z = \tilde{a}_{\bar{\zeta}}$. We normalize it so that it has periodicity

$$z \sim z + n + m\tau \qquad n, m \in \mathbb{Z} \tag{2.156}$$

Now the Poisson brackets are

$$\{z, \bar{z}\} = \frac{\mathrm{Im}\tau}{2\pi i} \frac{1}{\kappa} \tag{2.157}$$

[17] The more conceptual reason for this is that the Cauchy-Riemann operator $\bar{\partial}_A$ changes under variation of complex structure by

$$\bar{\partial}_A \to \bar{\partial}_A + \mu \partial_A \tag{2.155}$$

where μ is a Beltrami differential: A form of type $(-1,1)$. The space of closed Beltrami differentials can be canonically identified with the holomorphic tangent space to the space of complex structures on Σ_2.

and we have, using (2.79)

$$\begin{aligned}\frac{\omega}{2\pi} &= \frac{\kappa}{(2\pi)^2}\int_{T^2}\delta A_s\wedge\delta A_s \\ &= \frac{\kappa}{2\pi^2}\delta a_1\wedge\delta a_2 \\ &= \mathrm{i}\kappa\frac{dz\wedge d\bar{z}}{\mathrm{Im}\tau} \\ &= (2\kappa)\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}K\end{aligned} \tag{2.158}$$

where we define:

$$K = 2\pi\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau} \tag{2.159}$$

so that

$$\partial\bar{\partial}K = \frac{\pi}{\mathrm{Im}\tau}dz\wedge d\bar{z} \tag{2.160}$$

One way (out of many) to derive the wavefunctions is the following: Berezin-Toeplitz quantization tells us to find a holomorphic line bundle $\mathcal{L}_\kappa \to E_\tau$ with an invariant Hermitian metric

$$\parallel s(z) \parallel^2 = e^{-K_\kappa}|s(z)|^2 \tag{2.161}$$

where, up to a Kähler transformation,

$$K_\kappa = (2\kappa)K = 4\pi\kappa\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau} \tag{2.162}$$

The inner product on the Hilbert space is

$$\langle\psi_1,\psi_2\rangle = \int_{E_\tau} e^{-K_\kappa}\psi_1^*(z)\psi_2(z)\frac{\omega}{2\pi} \tag{2.163}$$

where we use a local trivialization of the line bundle so that we think of $\psi_1(z)$, and $\psi_2(z)$ as locally defined holomorphic functions. Equivalently: We pull back to the universal cover and integrate over a fundamental domain in $\mathbb{C}$ for the action of $\mathbb{Z}+\tau\mathbb{Z}$ by translation. ♣OVERALL NORMALIZATION OF THE INNER PRODUCT ???? ♣

It is useful to consider a basic holomorphic line bundle $\mathcal{L}_\theta \to E_\tau$ where we use the metric with

Then we can take $\mathcal{L}_\kappa = \mathcal{L}_\theta^{\otimes 2\kappa}$.

How shall we describe holomorphic sections of $\mathcal{L}_\theta$? Well,

$$e^{-2\pi\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}}|\psi(z)|^2 \tag{2.164}$$

must be well-defined. So we can lift $\psi(z)$ to an entire function on the universal cover, namely, the complex z-plane and then we require invariance under $z\to z+a+b\tau$ where $a,b\in\mathbb{Z}$. Now under $z\to z+a$ we must have $\psi(z)$ transform at most by a phase, and since it is holomorphic that phase must be 1. On the other hand, under $z\to z+b\tau$ a short computation shows that

$$\begin{aligned}e^{-2\pi\frac{(\mathrm{Im}(z+b\tau))^2}{\mathrm{Im}\tau}} &= e^{-2\pi\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}}e^{2\pi\mathrm{i}b(z-\bar{z})+\mathrm{i}\pi b^2(\tau-\bar{\tau})} \\ &:= e^{-2\pi\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}}e_b(z,\tau)^{-1}\overline{e_b(z,\tau)}^{-1}\end{aligned} \tag{2.165}$$

where

$$e_b(z,\tau) = e^{-\mathrm{i}\pi b^2\tau - 2\pi \mathrm{i}bz} \tag{2.166}$$

Again, using holomorphy to eliminate a possible extra phase dependence we learn that the lift of the section to the universal cover must be an entire function that is quasiperiodic and satisfies the functional equation:

$$\psi(z+a+b\tau) = e_b(z,\tau)\psi(z) \qquad a,b\in\mathbb{Z} \tag{2.167}$$

We will set $e_1(z,\tau) = e(z,\tau)$. We can easily solve this functional equation: Invariance under $z\to z+1$ means that $\psi(z)$ must have a Fourier expansion:

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n e^{2\pi \mathrm{i}nz} \tag{2.168}$$

and then the transformation under $z\to z+\tau$ implies

$$\psi_{n+1} = e^{\mathrm{i}\pi(2n+1)\tau}\psi_n \qquad n\in\mathbb{Z} \tag{2.169}$$

and hence $\psi_n = \psi_0 e^{\mathrm{i}\pi n^2\tau}$.

Thus, in this basic case there is a one-dimensional space of sections, and it is spanned by the Riemann theta function:

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi n^2\tau + 2\pi \mathrm{i}nz} \tag{2.170}$$

Although we have written $\theta(z,\tau)$ as an entire function on the z-plane it can also be viewed as a holomorphic section of a holomorphic line bundle. ♣Redundant? ♣

Then we define a complex line bundle associated to a principal $U(1)$ bundle of Chern class 1 by

$$L = (\mathbb{C}\times V)/(\mathbb{Z}\oplus\mathbb{Z}) \tag{2.171}$$

where we think of V as the one-dimensional representation of $U(1)$. The a pair (g_1,g_2) of generators of $\mathbb{Z}\oplus\mathbb{Z}$ act by

$$\begin{aligned} g_1 &: (z,\psi)\to(z+1,\psi) \\ g_2 &: (z,\psi)\to(z+\tau, e(z,\tau)\psi) \end{aligned} \tag{2.172}$$

If we take V_ℓ to be $V^{\otimes\ell}$, the charge ℓ representation of $U(1)$ then we have

$$L^{\otimes\ell} = (\mathbb{C}\times V_\ell)/(\mathbb{Z}\oplus\mathbb{Z}) \tag{2.173}$$

where now the generators act by

$$\begin{aligned} g_1 &: (z,\psi)\to(z+1,\psi) \\ g_2 &: (z,\psi)\to(z+\tau, e(z,\tau)^\ell\psi) \end{aligned} \tag{2.174}$$

Consequently, holomorphic sections of $L^{\otimes\ell}$ obey the identities:

$$\begin{aligned} \psi(z+1) &= \psi(z) \\ \psi(z+\tau) &= e^{-\ell\pi\mathrm{i}\tau - 2\ell\pi\mathrm{i}z}\psi(z) \end{aligned} \tag{2.175}$$

When $\ell \geq 0$ there will be global holomorphic sections. When $\ell < 0$ there are no global holomorphic sections.

Now, as we said, in quantizing level κ $U(1)$ Chern-Simons we have $\mathcal{L}_\theta^{2\kappa}$ as the relevant holomorphic line bundle so if $\psi(z)$ is a wavefunction then

$$\langle \psi_1, \psi_2 \rangle = \int_{E_\tau} e^{-4\pi i k \frac{(\mathrm{Im} z)^2}{\mathrm{Im}\tau}} \psi_1^*(z) \psi_2(z) \frac{\omega}{2\pi} \tag{2.176}$$

The line bundle has $c_1(\mathcal{L}_\theta^{2\kappa}) = 2\kappa c_1(\mathcal{L}_\theta) = 2\kappa$ and there will be a 2κ-dimensional space of holomorphic sections. A natural basis of sections is given by the level κ theta functions, $\Theta_{\mu,\kappa}$. See Appendix D for more details about level κ theta functions.

Remark: Here we are assuming that $\kappa > 0$. If we flip the orientation of Σ_2 that changes the orientation of $\mathcal{M}_{flat}$ and then we would need to take $\kappa < 0$ to have nonzero sections.

In conclusion, when we choose the extra data of a complex structure J on Σ_2 we can construct a space of physical states $\mathcal{H}(\Sigma_2, J)$ of the Chern-Simons theory. A basis for the physical states on the torus, in holomorphic quantization, is

$$\chi_{\mu,\kappa}(z,\tau) = \frac{\Theta_{\mu,\kappa}(z,\tau)}{\eta(\tau)} \qquad \mu \sim \mu + 2\kappa \tag{2.177}$$

The factor of $1/\eta(\tau)$ is not at all obvious at this point. We will discuss it below. It can be obtained by:

1. Parallel transport over the moduli space of complex structures.
2. Normalizing the wavefunctional $\Psi[A]$ in the quantization of the space of all gauge fields (subject to the Gauss law).

Important Remark: In this way of quantizing we get a space of states $\mathcal{H}(\Sigma_2, J)$ that depends on the complex structure J on Σ_2. But J is not part of the data of the problem! In a topological field theory we need to "get rid of" the J dependence. We should view $\mathcal{H}(\Sigma_2, J)$ as a fiber of a vector bundle over $\mathcal{M}_{cplx}(\Sigma_2)$. The statement that the space of states is "independent of J" is implemented by the existence of a projectively flat connection on this bundle. See below for more about that connection.

****

SHOULD DISCUSS THE COVARIANT DERIVATIVE: CHERN-CONNECTION FOR THE ABOVE HERMITIAN METRIC

****

Generalization To Compact Riemann Surface Of Genus g

One advantage of the coherent state approach is that it is readily generalized to a compact oriented surface Σ_2 of genus g.

Recall that the phase space can be identified with the $2g$ dimensional torus $\mathcal{P} = H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$.

If we give Σ_2 a complex structure, so that it becomes a *Riemann surface* then, as in the torus case we described explicitly, $\mathcal{M}_{\rm flat}$ inherits a complex structure with

$$T^{1,0}\mathcal{M}_{\rm flat} \cong H^{0,1}(\Sigma_2) \tag{2.178}$$

Thus, our torus, $\mathcal{P} \sim (S^1)^{2g}$ is now a complex torus. The prequantum line bundle $L_k \to \mathcal{M}_{flat}$ becomes $L_k = L_\theta^{\otimes 2k}$ where L_θ is again a standard holomorphic line bundle with a one-dimensional space of holomorphic sections. This fits into a general story: (See Appendix E for more details.)

1. Given a symplectic lattice with an integral symplectic form one can canonically form a Heisenberg group as a $U(1)$ extension of $\Lambda \otimes \mathbb{R}/\Lambda$.

2. That Heisenberg group is a principal $U(1)$ bundle over the torus $\Lambda \otimes \mathbb{R}/\Lambda$.

3. To represent that Heisenberg group we need to choose a Lagrangian decomposition of the lattice.

4. Moreover, if we in addition have a compatible complex structure then the complex line bundle associated to the Heisenberg group by the defining representation of $U(1)$ is a holomorphic line bundle with canonical connection, whose curvature represents $c_1(L)$.

5. The holomorphic section can be lifted to the universal cover $\Lambda \otimes \mathbb{R} \cong \mathbb{C}^n$ and in terms of the complex structure is an entire function. It is a quasi-periodic function with factor of automorphy determined by the cocycle of the Heisenberg group. This defines the basic theta function.

6. A basis of the holomorphic sections of $L_\theta^{\otimes 2\kappa}$ are the level κ theta functions defined relative to a Lagrangian splitting of Λ.

In our case, the symplectic lattice is just $H^1(\Sigma;\mathbb{Z})$ with the symplectic form provided by the intersection form. (Remember: Σ is oriented.) Then a complex structure on Σ induces a complex structure on $H^1(\Sigma;\mathbb{R})$ as mentioned above. The rest is just running the machine.

In very concrete terms: Choose a basis of A,B cycles on Σ and dual 1-forms α_I, β^I, $I = 1, \dots, g$. Then there is a basis of harmonic $(0,1)$ forms:

$$f_I := \alpha_I + \tau_{IJ}\beta^J \qquad I = 1, \dots, g \tag{2.179}$$

while the harmonic $(1,0)$ forms are spanned by

$$\bar{f}_I := \alpha_I + \bar{\tau}_{IJ}\beta^J \qquad I = 1, \dots, g \tag{2.180}$$

Here τ_{IJ} is an element of the Siegel upper half space. [18]

Once one has the period matrix there is a straightforward definition of the level κ theta functions given in (E.65). The Chern-Simons wavefunctions, as holomorphic functions on the space of flat gauge fields on Σ, are linear combinations of these. It is a bit trickier to include properly the dependence on complex structure in the Chern-Simons wavefunctions. This gets us into sections of determinant line bundles over the moduli space of complex structures. A basis for the space of physical wavefunctions is

$$\chi_{\beta,\kappa}(z,\tau) = \frac{\Theta_{\beta,\kappa}(z,\tau)}{\mathrm{Det}\bar{\partial}} \tag{2.181}$$

where $\mathrm{Det}\bar{\partial}$ is the holomorphic section of the determinant line bundle $\mathrm{DET}(\bar{\partial}) \to \mathcal{M}_{cplx}$. ♣THIS NEEDS MUCH MORE EXPLANATION. ♣

In particular:

1. The complex dimension of the Hilbert space $\mathcal{H}(\Sigma)$ is $(2\kappa)^g$. ♣Forward ref. to section 2.2.12 here. ♣

2. It is straightforward to compute the action of the modular group on these.

3. The action of the modular group factors through $Sp(2g;\mathbb{Z})$ and indeed there is an action of the entire group $Sp(2g;\mathbb{Z})$ because this is just the group of symplectic automorphisms of the symplectic torus.

4. There is an analogue of the heat equation (**??**) for the higher genus theta functions.

2.2.10 Quantization Of Flat Connections On The Torus: Landau's Story

We can also obtain the results in an interestingly different way using the nontopological massive Chern-Simons theory described in section 2.2.3.

Here we are following [26]. This method is very useful for describing singleton degrees of freedom in the AdS/CFT correspondence [26, 8], as well as describing certain topological aspects of M5-branes and the M-theory C-field [45]. We have somewhat simplified the general story here. For a completely general quantization of Maxwell-Chern-Simons theories including their p-form generalization see section 6 of [20].

In the present theory, thanks to linearity the space of (not necessarily gauge inequivalent) solutions of the equations of motion is a product

$$S = S_f \times S_{nf} \tag{2.182}$$

where S_f is the space of flat solutions $F = 0$. These are the solutions of the topological sector. We can decompose $A = A_f + A_{nf}$ where A_{nf} is orthogonal to the flat subspace in, say, the Hodge metric.

Let us work out the Hamiltonian formulation on a spacetime of the form $X \times \mathbb{R}$, with metric $-dt^2 + g_{ij}dx^i dx^j$ and orientation $dt dx^1 dx^2$. The canonically conjugate momentum as a vector-density is ($\epsilon^{12} = +1$): ♣CHANGE NOTATION FROM X TO Σ_2. Also $k \to \kappa$ in this subsubsection!!! ♣

[18] In the most concrete terms, the Siegel upper half-space is the space of symmetric $g \times g$ complex matrices whose imaginary part is positive definite.

$$\Pi^i = \frac{1}{e^2}\sqrt{g}g^{ij}(\dot{A}_j - \partial_j A_0) + 2\pi k\epsilon^{ij}A_j \tag{2.183}$$

We find a Hamiltonian density

$$\mathcal{H} = \frac{e^2}{2\sqrt{g}}g_{ij}E^iE^j + \frac{1}{2e^2}F \wedge *_2 F \tag{2.184}$$

where $*_2$ is the Hodge star on X and

$$E^i := \Pi^i - 2\pi k\epsilon^{ij}A_j \tag{2.185}$$

(We will also denote $E^i = \tilde{\Pi}^i$.) The Gauss law is:

$$\partial_i \Pi^i + 2\pi k\epsilon^{ij}\partial_i A_j = 0 \tag{2.186}$$

If we formulate the theory "upstairs" in $A_0 = 0$ gauge then phase space has coordinates (Π^i, A_i) and symplectic form:

$$\omega = \int_X \delta\Pi^i \wedge \delta A_i \tag{2.187}$$

where δ is exterior derivative on the infinite dimensional phase space. Notice that when (2.187) is restricted to the subspace of flat gauge fields we get second class constraints and the phase space is the Chern-Simons symplectic form

$$\omega_f = \int_X 2\pi k\delta A \wedge \delta A \tag{2.188}$$

This is gauge invariant on the subspace $F = 0$ and one may then perform Hamiltonian reduction.

****

THIS IS NOT QUITE RIGHT! CORRECT VERSION IS DESCRIBED IN [20]

****

It is instructive to reconsider the $e^2 \to \infty$ limit. Using (2.184), we see that if we restrict to finite energy field configurations then we must set $E^i = 0$. Then, by the Gauss law we must put $F = 0$. As we have said, restriction to this subspace imposes second class constraints and we are restricting to the flat factor in phase space.

If we quantize on phase space and then impose the Gauss law we have wavefunctionals $\Psi[A_i]$, and we quantize using the symplectic form (2.187). Thus

$$\Pi^i = -i\frac{\delta}{\delta A_i} \tag{2.189}$$

Since we can split $A = A_f + A_{nf}$ and the Hamiltonian does not mix these, the Hilbert space of the theory is naturally thought of as a product

$$H = H_f \otimes H_{nf} \tag{2.190}$$

where H_f is the space of wavefunctions of flat potentials.

The Gauss law is:

$$\Psi(A+\omega) = e^{-2\pi i k \int \omega\wedge A}\Psi(A) \tag{2.191}$$

This is valid also for large gauge transformations. [19] Here ω is a closed 1-form with integral periods. Note that this does not affect the A_{nf} variable.

We will determine the Hamiltonian for the singletons by considering the Euclidean path integral of the theory on the solid torus, and then interpreting that path integral in terms of Hamiltonian evolution in the radial direction.

Since our action is second order in derivatives, when formulating the path integral on the solid torus we should specify all of A_X on the boundary X. This is to be contrasted with the Chern-Simons path integral which is a phase space path integral, and in which we specify just one component of A_X on the boundary X.

Let us consider the Euclidean partition function of the theory on a solid torus with radius ρ: Y_ρ. We assume the torus has a metric that behaves asymptotically like $d\rho^2 + \Omega^2(\rho) g_X$. The path integral defines a state $\Psi_{Y_\rho}(A)$ defined by

$$\Psi_{Y_\rho}(A) = \int \frac{dA_Y}{\mathrm{vol}\,(\mathcal{G}(Y))} e^{-\int \frac{1}{2e^2} dA * dA + 2\pi i k \int A dA} \tag{2.192}$$

where $\mathcal{G}(Y)$ is the gauge group on Y_ρ. We can understand the behavior for $\rho \to \infty$ just from the above understanding of the spectrum.

We can view the evolution to large ρ as evolution in a Euclidean time direction. The large ρ behavior projects onto the lowest energy states.

$$\lim_{\rho\to\infty} \Psi_{Y_\rho}(A) = e^{-\rho 2\pi |k| e^2}\Psi_0 \tag{2.193}$$

with Ψ_0 in the space of ground states on the torus. The insertion of local operators such as Wilson lines or other disturbances induces transitions between vectors within this space of ground states.

Hilbert Space For Quantization Of The Flat Gauge Fields

We now consider quantization on $T^2 \times \mathbb{R}$. Our wavefunction is $\Psi(A_f) \otimes \Psi(A_{nf})$. The spectrum of the nonflat sector is clear, and we take the unique groundstate wavefunction for this factor: It is the product of harmonic oscillator groundstates for the oscillators of the massive scalar described by [15, 14]. In this section we drop this factor so we can focus on the dependence on A_f.

To simplify matters, we work on a torus $X = T^2$ with $z = \sigma^1 + \tau\sigma^2$ and metric $\Omega^2 |dz|^2$, $\sigma^i \sim \sigma^i + 1$. We fix the small gauge transformations by assuming A_f is constant.

[19] This requires explanation. The proper mathematical formulation involves regarding Ψ as a section of a line bundle over the space of gauge potentials $\mathcal{A}(X)$ on X. We then lift the group action, and find that a lift only exists when $c_1(P) = 0$. There is a canonical trivialization of the line in this case, as well as a canonical connection, and the wavefunction becomes a function. A similar discussion holds for the more subtle case of the M-theory C-field [16].

In complex coordinates $A = A_z dz + A_{\bar{z}} d\bar{z}$ we have

$$\begin{aligned} \tilde{\Pi}^z &= -i \left(\frac{\partial}{\partial A_z} - 4\pi k \mathrm{Im}\tau A_{\bar{z}} \right) \\ \tilde{\Pi}^{\bar{z}} &= -i \left(\frac{\partial}{\partial A_{\bar{z}}} + 4\pi k \mathrm{Im}\tau A_z \right) \end{aligned} \tag{2.194}$$

so that the Hamiltonian density is:

$$\mathcal{H} = \frac{e^2}{4\mathrm{Im}\tau} (\tilde{\Pi}^z \tilde{\Pi}^{\bar{z}} + \tilde{\Pi}^{\bar{z}} \tilde{\Pi}^z) \tag{2.195}$$

Note that these do not commute: $[\tilde{\Pi}^z, \tilde{\Pi}^{\bar{z}}] = -8\pi k \mathrm{Im}\tau$. The ground state energy density is $2\pi|k|e^2$ and is infinitely degenerate, as in the standard Landau-level problem.

If $k > 0$ we have

$$\tilde{\Pi}^{\bar{z}} \Psi = 0 \Rightarrow \Psi = e^{-4\pi k \mathrm{Im}\tau A_z A_{\bar{z}}} \psi(A_z) \tag{2.196}$$

If $k < 0$ we have

$$\tilde{\Pi}^z \Psi = 0 \Rightarrow \Psi = e^{4\pi k \mathrm{Im}\tau A_z A_{\bar{z}}} \psi(A_{\bar{z}}) \tag{2.197}$$

Where ψ are holomorphic. Indeed, if we take $\psi = \psi_\lambda$, where

$$\psi_\lambda(x) := e^{\lambda x} \tag{2.198}$$

then the set of wavefunctions $\{\Psi_\lambda | \lambda \in \mathbb{C}\}$ is an overcomplete set spanning the lowest Landau level.

The set of states spanned by (2.198) is infinite dimensional, but when we consider gauge invariant wavefunctions on the torus the LLL becomes finite dimensional. We have already enforced the invariance under small gauge transformations by choosing our flat connections to be constants on the torus. We can impose the invariance under large gauge transformations by averaging over large gauge transformations. Given *any* wavefunction $\Psi(A)$ the average:

$$\bar{\Psi}(A) := \sum_{\omega \in \mathcal{H}^1_Z} \Psi(A + \omega) e^{2\pi i k \int \omega A} \tag{2.199}$$

where $\mathcal{H}^1_Z$ are the harmonic 1-forms with integral periods, transforms according to the Gauss law (2.191).

If we consider the space of functions of A_z spanned by the expressions of the form (2.199) where we take functions in the LLL as our test function then, because the LLL is an infinite-dimensional vector space, we might at first think that the vector space of wavefunctions spanned by $\bar{\Psi}(A)$ is also infinite-dimensional. After all, λ in (2.198) can be any complex number. However, applying the Poisson summation formula and a little bit of algebra we find that (we assume $k > 0$ here):

$$\overline{\Psi_\lambda} = e^{-4\pi k \mathrm{Im}\tau (A_z A_{\bar{z}} + A_{\bar{z}}^2) - \frac{\lambda^2}{16\pi k \mathrm{Im}\tau}} \sqrt{\frac{\tau_2}{k}} \sum_{0 \le \mu < 2k} \Theta_{-\mu,k}(-2i\tau_2 A_z, -\bar{\tau}) \Theta_{\mu,k}(\frac{-\lambda}{4\pi i k}, \tau) \tag{2.200}$$

This formula demonstrates that, as a function of A_z, we only produce a finite-dimensional space. Indeed, it can be shown (see [17, 26]) that a basis of normalized wavefunctions for the sector of flat gauge fields is:

$$\psi_\mu = \frac{k^{3/4}}{\bar{\eta}} e^{-4\pi k \mathrm{Im}\tau (A_z A_{\bar{z}} + A_z^2)} \Theta_{-\mu,k}(-2i\tau_2 A_z, -\bar{\tau}) \tag{2.201}$$

2.2.11 Quantization Of Flat Connections On The Torus: Heisenberg's Story

In case you are uncertain of the result we just derived, let us consider the viewpoint from that of Heisenberg groups.

Classically, the holonomies are described by $U_i = \exp[\mathrm{i}a_i]$. Quantum mechanically, we get operators:

$$\widehat{U}_i = \exp[\mathrm{i}\hat{a}_i] \tag{2.202}$$

It follows from the commutator (2.143) that

$$\widehat{U}_1 \widehat{U}_2 = e^{\frac{2\pi \mathrm{i}}{2\kappa}} \widehat{U}_2 \widehat{U}_1 \tag{2.203}$$

In particular,

$$\widehat{U}_1 \widehat{U}_2 \widehat{U}_1^{-1} = e^{\frac{2\pi \mathrm{i}}{2\kappa}} \widehat{U}_2 \tag{2.204}$$

implying that U_1 translates a_2 by $2\pi/2\kappa$, and similarly for U_2 translating a_1. Therefore we also have the relations:

$$\widehat{U}_1^{2\kappa} = \widehat{U}_2^{2\kappa} = 1 \tag{2.205}$$

The equations (2.204) and (2.205) define a finite nonabelian group called the finite Heisenberg group. In general, for any integer N there is a Heisenberg group of order N^3. It fits in a (nonsplit) exact sequence

$$1 \to \mathbb{Z}_N \to \mathrm{Heis}_N \to \mathbb{Z}_N \oplus \mathbb{Z}_N \to 0 \tag{2.206}$$

and is a finite-group analog of the usual Heisenberg relations of quantum mechanics.

There is a standard theorem, the Stone-von Neumann theorem that says that, up to isomorphism, there is a unique unitary irreducible representation of a Heisenberg group where the scalars in $U(1)$ act simply as multiplication by that scalar. The same is true for all Heisenberg groups, and so Heis_N has a unique irrep.

Clock And Shift Matrices

One standard way of presenting the Stone-von Neumann representation of the Heisenberg group Heis_N is via the $N \times N$ "clock" and "shift" matrices P and Q. To define these introduce an N^{th} root of unity, say $\omega = \exp[2\pi i/N]$. Then

$$P_{i,j} = \delta_{i=j+1 \mathrm{mod} N} \tag{2.207}$$

$$Q_{i,j} = \delta_{i,j}\omega^j \tag{2.208}$$

Note that $P^N = Q^N = 1$ and no smaller power is equal to 1. Further note that [20]

$$QP = \omega PQ \tag{2.209}$$

For $N = 4$ the matrices look like

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.210}$$

with $\omega = e^{2\pi i/4}$. The group of matrices generated by P, Q is a finite subgroup of $GL(N, \mathbb{C})$ isomorphic to a *finite Heisenberg group*. One naturally encounters these quantizing a quantum particle moving on the group $U(1)$ but confined to the cyclic subgroup of order N.

♣More details about that? ♣

Remark: The operator algebra generated by P, Q is the full matrix algebra $Mat_{N\times N}(\mathbb{C})$. That is, any linear transformation on the space of states can be expressed in terms of a linear combination of products of $P^m Q^n$.

♣Perhaps say more about how this is done with the finite Fourier transform. ♣

Hilbert Space For Case Of General Compact Oriented Σ_2

♣Might want to put hats on these operators $\widehat{W(n,\gamma)}$ for consistency with above, and to distinguish between line defects inserted in the path integral and operators on Hilbert space ♣

We can readily generalize the above discussion to a general compact oriented surface Σ_2: If $\gamma \subset \Sigma_2$ is an oriented closed loop and V_n is a charge n representation of $U(1)$, with $n \in \mathbb{Z}$ then we can form the Wilson line operator $W(n,\gamma) = \exp[in \oint_\gamma A]$. (See (2.61).) Because the gauge field is flat this only depends on the homology class of γ (in the absence of other operators). In particular, rather trivially

$$\exp[in \oint_\gamma A] = \exp[i \oint_{n\gamma} A] \tag{2.211}$$

There are corresponding quantum operators. We can write them as:

$$\widehat{W}(n,\gamma) := \exp[in \oint_\gamma \widehat{A}] \tag{2.212}$$

Again, this operator only depends on the product, so we can simplify by just working with an operator-valued function $\widehat{W(\gamma)}$ of a single variable $\gamma \in H_1(\Sigma_2; \mathbb{Z})$.

These operators depend on the orientation of γ and we have

$$\widehat{W}(-\gamma) = \widehat{W}(\gamma)^\dagger = \widehat{W}(\gamma)^{-1} \tag{2.213}$$

[20] The fastest way to check that - and thereby to check that you have your conventions under control - is to compute QPQ^{-1} because $(Q^{-1}PQ)_{ij} = Q_{ii}P_{ij}(Q_{jj})^{-1} = \omega P_{ij}$.

Then the generalization of (2.204) uses the Baker-Campbell-Hausdorff formula and the brackets of (2.87). The result is

$$\widehat{W}(\gamma_1)\widehat{W}(\gamma_2) = e^{\frac{2\pi \mathrm{i}}{2\kappa} I(\gamma_1,\gamma_2)}\widehat{W}(\gamma_2)\widehat{W}(\gamma_1) \tag{2.214}$$

where

$$I(\gamma_1,\gamma_2) = \sum_{p\in\gamma_1\cap\gamma_2} \iota_p(\gamma_1,\gamma_2) \tag{2.215}$$

is the oriented intersection number of γ_1 and γ_2.

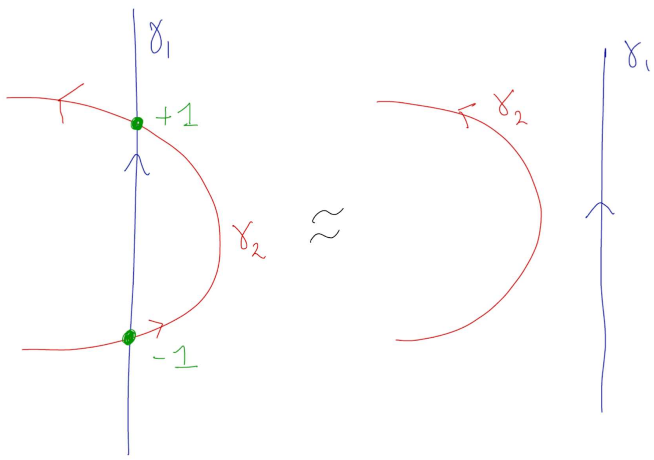

Figure 4: Because intersection numbers are signed sums, with the sign determined by the orientation to total intersection number is invariant under deformation, as shown in a simple case here. Summing over the oriented intersections of transversally intersecting curves gives and invariant that only depends on the homology classes of those curves.

To define the oriented intersection number we assume that γ_1 and γ_2 are in general position so that there are a finite number of intersection points and they are all transversal. Then, at each intersection point $p \in \gamma_1 \cap \gamma_2$ the tangent vectors $\{\dot{\gamma}_1(p), \dot{\gamma}_2(p)\}$ form an ordered basis for $T_p\Sigma_2$. Therefore, they define an orientation $o_{12}(p)$ for the vector space $T_p\Sigma_2$. On the other hand, we assume Σ_2 has an orientation o_Σ so we can compare $o_{12}(p)$ with $o_\Sigma(p)$. We can define

$$\iota_p(\gamma_1,\gamma_2) := \begin{cases} +1 & o_{12}(p) = o_{\Sigma_2}(p) \\ -1 & o_{12}(p) \neq o_{\Sigma_2}(p) \end{cases} \tag{2.216}$$

One can then show that (2.215) in fact only depends on the homology class of γ_1 and γ_2. The basic idea is shown in Figure 4. The oriented intersection number makes the

♣Explain or cite reference here. ♣

Abelian group $H_1(\Sigma_2;\mathbb{Z})$ into a symplectic lattice so we will often denote the intersection by $\langle\gamma_1,\gamma_2\rangle := I(\gamma_1,\gamma_2)$.

Since the operators $\widehat{W(\gamma)}$ are unitary, the operator algebra they generate is an algebra known as the noncommutative torus algebra. See below for some general remarks. One can show that the operators satisfy the relations:

♣should we try to justify that it is a C* or W* algebra? ♣

$$\widehat{W}(\gamma_1)\widehat{W}(\gamma_2) = e^{\frac{\pi \mathrm{i}}{2\kappa}I(\gamma_1,\gamma_2)}\widehat{W}(\gamma_1+\gamma_2) \tag{2.217}$$

(This does not follow immediately from the BCH formula. However, one can choose a basis for $H_1(\Sigma_2;\mathbb{Z})$ and define $\widehat{W}(\gamma)$ for general γ in a suitable way to that (2.217) holds.)

Now, the center of the algebra of unitary operators satisfying (2.214) is generated by $W(n,\gamma)^{2\kappa}$ so we can put these to one. Indeed, we have seen in the case of the torus that we must put these to one. Therefore $W(n,\gamma)$ only depends on $n\sim n+2\kappa$ and we have the product rule:

$$\widehat{W}(n_1,\gamma)\widehat{W}(n_2,\gamma) = \widehat{W}(n_1+n_2,\gamma) \tag{2.218}$$

where addition is modulo 2κ.

Once again the general theory of Heisenberg groups becomes very helpful. We have a nonsplit central extension

$$1 \to \mathbb{Z}/2\kappa\mathbb{Z} \to \mathrm{Heis}(H_1(\Sigma_2;\mathbb{Z}/2\kappa\mathbb{Z})) \to H_1(\Sigma_2;\mathbb{Z}/2\kappa\mathbb{Z}) \to 1 \tag{2.219}$$

where we consider the central group $\mathbb{Z}/2\kappa\mathbb{Z}\subset U(1)$ to be the group of $(2\kappa)^{th}$ roots of unity. There is a unique (up to isomorphism) Stone-von Neumann representation and it can be given explicitly by choosing a maximal Lagrangian decomposition

$$H_1(\Sigma_2;\mathbb{Z}/2\kappa\mathbb{Z}) \cong L_1\oplus L_2 \tag{2.220}$$

Such a Lagrangian subspace amounts to a distinction between a-cycles and b-cycles as in Figure 3. So we will denote: $L_1 = L_a$ and $L_2 = L_b$:

$$H_1(\Sigma_2;\mathbb{Z}/2\kappa\mathbb{Z}) \cong L_a\oplus L_b \tag{2.221}$$

To give an explicit SvN representation we consider L^2 functions on, say, the Lagrangian subgroup L_a. As an Abelian group L_a is (noncanonically):

$$L_a \cong \mathbb{Z}_{2\kappa}\oplus\cdots\oplus\mathbb{Z}_{2\kappa} \tag{2.222}$$

with g summands. The space of L^2 functions on this finite group is therefore $(2\kappa)^g$ dimensional. The representation is determined by using the rule (2.217). and then specifying the action of $W(\gamma)$ for γ restricted to the two Lagrangian subspaces:

$$\begin{aligned} \widehat{W}(\gamma)\Psi(\gamma_0) &= \Psi(\gamma_0+\gamma) & \gamma\in L_a \\ \widehat{W}(\gamma)\Psi(\gamma_0) &= e^{2\pi\mathrm{i}\frac{\langle\gamma,\gamma_0\rangle}{2\kappa}}\Psi(\gamma_0) & \gamma\in L_b \end{aligned} \tag{2.223}$$

♣Again, should have an appendix on general SvN rep for Heisenberg groups ♣

Remark: *The noncommutative torus algebra.* Suppose Λ is an integral symplectic lattice. The noncommutative torus algebra, also known as the *irrational rotation algebra* is the C^*-algebra generated by unitary operators X_γ satisfying

$$X_{\gamma_1} X_{\gamma_2} = q^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2} \tag{2.224}$$

for some complex number q. This is a fascinating and nontrivial infinite-dimensional algebra with an interesting representation theory. When q is not a root of unity it is a simple algebra, but when q becomes a root of unity there is an infinite-dimensional center. If $q^N = 1$ then X_γ for $2\gamma \in N\Lambda$ is in the center. The quotient by this center is a finite-dimensional Heisenberg algebra. In our example we should use the character on the center which is just given by unity.

♣PUT THIS INTO BROADER CONTEXT: NON-COMMUTATIVE TORUS ALGEBRA IS UBIQUITOUS. ♣

2.2.12 Symplectic Transformations, Large Diffeomorphisms, And Modular Transformations

The spaces of physical states at genus g form projective representations of the symplectic group $Sp(2g, \mathbb{Z})$. Note that for $g = 1$ $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$.)

There are three distinct reasons for this:

1. Origin In Topological Field Theory: Since the Chern-Simons theory is (formally) "topological" the (oriented!) diffeomorphism group of Σ_2, denoted $\mathrm{Diff}^+(\Sigma_2)$ should be a symmetry. So there should be a projective representation of this group on the Hilbert space of states. Moreover, the diffeomorphisms "isotopic to the identity" should act trivially, because they can be continuously deformed to the trivial diffeomorphism. [21] As usual, the connected component of the identity $\mathrm{Diff}_0^+(\Sigma_2)$ is a normal subgroup and the quotient

$$\Gamma_g := \mathrm{Diff}^+(\Sigma_2)/\mathrm{Diff}_0^+(\Sigma_2) \tag{2.225}$$

   is known as the (oriented) *mapping class group* of the (oriented) surface. The infinite-dimensional group of diffeomorphisms can be given a topology so that this is also the group of connected components:

$$\Gamma_g \cong \pi_0(\mathrm{Diff}^+(\Sigma_2)). \tag{2.226}$$

2. Now, the mapping class group acts by pullback on the homology $H^1(\Sigma_2; \mathbb{Z})$ preserving the intersection form. There is a kernel of this action that defines an important normal subgroup $N_g \triangleleft \Gamma_g$. [22] The quotient group

$$\Gamma_g/N_g \tag{2.227}$$

acts symplectically on the lattice $H^1(\Sigma_2;\mathbb{Z})$ and in fact, using Dehn twists and a set of generators for $Sp(2g;\mathbb{Z})$ one can verify that

♣REFER TO APPENDIX FOR GENERATORS ♣

$$\Gamma_g/N_g \cong Sp(2g;\mathbb{Z}) \tag{2.228}$$

When $g=1$ there are no separating curves so $N_g=1$ and in this case $\Gamma_1 \cong Sp(2;\mathbb{Z}) = SL(2,\mathbb{Z})$. In the Abelian case the moduli space of flat connections is a symplectic torus built from $H^1(\Sigma_2;\mathbb{Z})$ and hence we expect the wavefunctions to transform in projective representations of this group.

3. Symplectic Geometry Viewpoint. In Abelian Chern-Simons theory the phase space $\mathcal{M}_{flat}$ is a symplectic torus. The group of symplectic automorphisms of the phase space should act (perhaps with central extension) on the space of states obtained by quantizing that torus. This recovers the expectation from the more general topological field theory viewpoint.

4. Operator Algebra Viewpoint. From the operator algebra viewpoint, the Heisenberg group is an extension of a symplectic lattice. The automorphisms of the symplectic lattice will (under some circumstances) lift to be automorphisms of the Heisenberg group. The automorphisms of the operator algebra that we are representing should act (again, in general, projectively) on the Hilbert space of states.

♣THERE IS A GREAT DEAL MORE TO SAY ABOUT THIS, AND IT IS NOT IN THE LITERATURE. ♣

Example: We study the example when Σ_2 is genus one in detail. In this case, the large diffeomorphisms can be shown to act on the coordinates (σ^1,σ^2) with fixed identifications $\sigma^{1,2}\sim\sigma^{1,2}+1$ on the σ^i as:

$$f(\sigma^1,\sigma^2) = (\delta\sigma^1+\beta\sigma^2, \gamma\sigma^1+\alpha\sigma^2)\mathrm{mod}\mathbb{Z}\oplus\mathbb{Z} \tag{2.229}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.230}$$

In this case the relation to $SL(2,\mathbb{Z})$ can be seen directly since the transformation should preserve the oriented lattice.

[21] One nice way of saying what such diffeomorphisms look like is the following: Two diffeomorphisms f,f' are isotopic if, for every closed curve $\wp\subset\Sigma_2$ the images $f(\wp)$ and $f'(\wp)$ can be smoothly deformed into each other.

[22] The group N_g is known as the *Torelli group*. To describe it recall the notion of a *Dehn twist* around closed curve $\gamma\subset\Sigma_2$. We cut out a small annulus around γ and then rotate one side of the annulus by 2π holding the other side fixed. Then we glue the new annulus back in. [DRAW WHAT HAPPENS TO A CURVE TRAVERSING THE ANNULUS.] A famous theorem of Max Dehn states that the mapping class group is generated by the Dehn twists around closed curves. A simple example of an element in the Torelli group is a Dehn twist around *separating curves*, that is, curves such that, if we cut along them then the surface falls apart into more than one connected component. In fact, the Dehn twist around separating curves and around "bounding pair maps" generate the Torelli group. This is known as the Birman-Powell thoerem. See [27] for an exposition and a proof.

Now, to learn how this acts on τ define the holomorphic coordinate:

$$\zeta = \sigma^1 + \tau\sigma^2 \tag{2.231}$$

For surfaces, a conformal class of a metric is equivalent to a holomorphic structure. So consider the action on the metric $ds^2 = |d\zeta|^2$:

$$f^*(ds^2) = |\gamma\tau + \delta|^2 \left| d\sigma^1 + \tilde{\tau} d\sigma^2 \right|^2 \tag{2.232}$$

so we have a Weyl scaling and a modular transformation on τ:

$$\tau \to \tilde{\tau} := \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \tag{2.233}$$

(Note: We could have dispensed with the conformal classes of metrics and just observed that:

$$f^*(d\zeta) = (\gamma\tau + \delta)(d\sigma^1 + \tilde{\tau} d\sigma^2) \tag{2.234}$$

and some readers will prefer to think of it this way.)

Similarly, we can derive the action of the diffeomorphism on the flat gauge fields:

$$A \to \tilde{A} := f^*(A) \tag{2.235}$$

Since $A = a_1 d\sigma^1 + a_2 d\sigma^2$ we get $f^*(A) = \tilde{a}_1 d\sigma^1 + \tilde{a}_2 d\sigma^2$ so that

$$\tilde{z} := \frac{\tilde{a}_2 - \tilde{\tau}\tilde{a}_1}{2\pi} = \frac{z}{\gamma\tau + \delta} \tag{2.236}$$

This derives the action of $SL(2,\mathbb{Z})$ on (z,τ) found in Appendices C and D. (See, especially (D.6).) Now, appendix D gives the detailed expressions for the transformation of the wavefunctions $\chi_\mu(z,\tau)$. We note a few things:

1. Very importantly, the S transform, which corresponds to the symplectic transformation exchanging a and b cycles, that is, that exchanges coordinates and momenta in the real polarization, is a finite Fourier transform in the canonical basis of level κ theta functions.

2. The action of $-1 \in SL(2,\mathbb{Z})$ is nontrivial: $(z,\tau) \to (-z,\tau)$. It acts by a kind of "charge conjugation" $\chi_\mu(z,\tau) \to \chi_{-\mu}(z,\tau)$. Indeed, when we discuss the relation of these quantum states to Wilson lines we will see this is indeed exactly charge conjugation. The relation $S^2 = -1$ then follows because the Fourier transform is order four.

3. It is also instructive to check the other relation

$$(ST)^3 = S^2 \tag{2.237}$$

which can be written as

$$STS = T^{-1}S^{-1}T^{-1} \tag{2.238}$$

viewed this way, the relation boils down to a *Gauss sum*. The $\mu\lambda$ matrix element of the LHS of (2.238) is

$$\frac{1}{2\kappa}e^{-2\pi\mathrm{i}\frac{1}{24}}\sum_{\nu}e^{2\pi\mathrm{i}\frac{\nu^2}{4\kappa}}e^{2\pi\mathrm{i}\frac{\nu(\lambda+\mu)}{2\kappa}} \tag{2.239}$$

while the $\mu\lambda$ matrix element of the RHS of (2.238) is

$$\frac{1}{\sqrt{2\kappa}}e^{-2\pi\mathrm{i}\frac{2}{24}}e^{-2\pi\mathrm{i}\frac{(\mu+\lambda)^2}{4\kappa}} \tag{2.240}$$

Are (2.239) and (2.240) the same? This is not obvious. We can of course simplify by shifting the summation variable $\nu \to \nu - (\lambda+\mu)$ and completing the square. But a nontrivial finite Gaussian sum remains.

4. The finite Gaussian sum in (2.239) is a special case of more general sums that are familiar in elementary number theory. See [58] for a nice summary of facts about finite Gaussian sums. The proper context is that of finite Abelian groups with bilinear forms valued in $\mathbb{Q}/\mathbb{Z}$. These forms admit quadratic refinements and there is a general result for

$$\frac{1}{\sqrt{|\mathcal{D}|}}\sum_{x\in\mathcal{D}}e^{2\pi\mathrm{i}q(x)} \tag{2.241}$$

where q is a quadratic refinement of a bilinear form on a finite Abelian group $\mathcal{D}$. In general this expression will be a subtle eighth root of unity. In the present case, the relevant version is

$$\frac{1}{\sqrt{2\kappa}}\sum_{\nu}e^{2\pi\mathrm{i}\frac{\nu^2}{4\kappa}} = e^{2\pi\mathrm{i}/8} \tag{2.242}$$

for any integer κ.

♣WE NEED TO HAVE AN APPENDIX ON THE GAUSS-MILGRAM FORMULA. GIVE TWO DEFINITIONS INCLUDING THE ONE FROM THE MODULAR TRANSFORMATIONS OF LATTICE THETA FUNCTIONS. ♣

5. We thus find the matrices $\rho(S)$ and $\rho(T)$ on the wavefunctions of the Chern-Simons theory actually satisfy:

$$(\rho(S)\rho(T))^3 = e^{-2\pi\mathrm{i}/6}\rho(S)^2 \tag{2.243}$$

We thus have a <u>projective</u> representation of the mapping class group.

****

OTHER THINGS TO COVER HERE:

****

1. Lifting canonical transformations to the Heisenberg group. Obstructions. And all that.

2. Generalization to higher genus $Sp(2g;\mathbb{Z})$. Siegel upper half plane. (That should have been discussed when we discussed the higher genus conformal blocks.)

2.2.13 The Bundle Of Physical States Over The Moduli Space Of Curves

When we carried out the Kähler quantization of $\mathcal{P}(\Sigma_2)$ we introduced some extra data: The complex structure of the curve. The quantization appears to depend on the complex structure in the sense that the theta functions definitely do.

However:

1. Different period matrices τ related by $Sp(2g;\mathbb{Z})$ transformations correspond to the same complex structure on Σ_2. It is thus very important that the space of theta functions forms a representation of $Sp(2g;\mathbb{Z})$. This means that there is a well-defined space of conformal blocks associated to a given complex structure.

♣SOMEWHERE NEED TO EXPLAIN ABOUT THE EXTRA DATA OF A MARKING AND A CHOICE OF BASIS FOR H_1 ♣

2. Moreover, the theta functions are holomorphic in the period matrix. For example, the level κ theta functions at $g=1$ satisfy

$$\mathcal{D}\Theta_{\mu,\kappa}(z,\tau)=0 \tag{2.244}$$

where $\mathcal{D}$ is the heat equation operator:

$$\mathcal{D}=\frac{\partial}{\partial\tau}+\frac{\mathrm{i}}{4\pi}\frac{1}{2\kappa}\left(\frac{\partial}{\partial z}\right)^2 \tag{2.245}$$

Note that $\mathcal{D}$ does not depend on μ, so that it applies uniformly to all states in the space $\mathcal{H}_{\kappa,\tau}$.

3. So, the spaces $\mathcal{H}_{\kappa,\tau}$ are the fibers of a holomorphic bundle over the space of complex structures with a flat connection $\mathcal{D}$. This connection is not quite modular invariant, and we want a connection that descends to a bundle over the moduli space of curves. When considering the τ-dependence it is better to take a basis of physical states as

$$\chi_\mu(z,\tau)=\frac{\Theta_{\mu,\kappa}(z,\tau)}{\eta(\tau)} \tag{2.246}$$

Since $\partial_\tau\log\eta(\tau)=\frac{2\pi\mathrm{i}}{24}E_2(\tau)$ the $\chi_\mu(z,\tau)$ are parallel-transported with respect to

$$\widetilde{\mathcal{D}}=D_\tau+\frac{\mathrm{i}}{4\pi}\frac{1}{2\kappa}\left(\frac{\partial}{\partial z}\right)^2 \tag{2.247}$$

$$D_\tau=\partial_\tau+\frac{2\pi\mathrm{i}}{24}E_2(\tau) \tag{2.248}$$

****

NEED TO DISCUSS DESCENT TO THE MODULI STACK OF CURVES AND THE ORBI BUNDLE.

WHY IT IS A PROJECTIVELY FLAT CONNECTION.

****

4. The extension of this structure to higher genus and to the nonabelian case is nontrivial.

♣NEED TO PUT IN MORE INFO ABOUT THESE. NEED CONNECTION ON SECTION OF DET LINE BUNDLE. ♣

2.2.14 Correlation Functions Of Line Defects And Framing Anomalies

Now we consider the path integral on a general 3-manifold M_3 with the insertion of Wilson line defects $W(r,\gamma)$. These provide sources in the action. The path integral ***** is a Gaussian integral, so the dependence on the Wilson lines can be done exactly by stationary phase.

To begin, let $M_3 = S^3$ or, more generally, a homology sphere where $H_1(M_3,\mathbb{R}) = 0$ so there are no nontrivial flat connections.

Now we consider the formal path integral

$$\langle \prod_\alpha W(r_\alpha,\gamma_\alpha)\rangle := \int [dA] e^{\mathrm{i}\frac{\kappa}{2\pi}\int_{M_3} A\wedge dA + \mathrm{i}\sum_\alpha r_\alpha \oint_{\gamma_\alpha} A} \tag{2.249}$$

If we are only interested in the dependence on Wilson lines then we can do this Gaussian integral by stationary phase. Using a suitable Green's function we can find the classical gauge field such that

$$\frac{\kappa}{\pi} dA_{class} = -J = -\sum_\alpha r_\alpha \eta(\gamma_\alpha \hookrightarrow M_3) \tag{2.250}$$

Here $\eta(\gamma \hookrightarrow M_3)$ is a differential form representative of the Poincaré dual of γ. It can be written in terms of delta functions:

$$\eta(\gamma \hookrightarrow \mathbb{R}^3) = \frac{1}{2} dx^m dx^n \epsilon_{mnj} \oint_\gamma \frac{dx^j(t)}{dt} \delta^{(3)}(\vec{x} - \vec{x}(t)) \tag{2.251}$$

Plugging back in to the action we get

$$\langle \prod_\alpha W(r_\alpha,\gamma_\alpha)\rangle = \mathcal{N} e^{-\frac{\mathrm{i}\kappa}{2\pi}\int_{M_3} A_{class} dA_{class}} \tag{2.252}$$

so it is expressed as the classical Chern-Simons invariant of the classical solution.

Here $\mathcal{N}$ is the value of the partition function with no Wilson lines. It is formally just some numerical constant, independent of the metric, but we will need to revisit this naive expectation below.

Let us continue to understand this value of this path integral. We write:

$$\begin{aligned} \frac{\mathrm{i}\kappa}{2\pi} \int_{M_3} A_{class} dA_{class} &= -\frac{\mathrm{i}}{2} \int_{M_3} A_{class} J \\ &= -\frac{\mathrm{i}}{2} \sum_\alpha r_\alpha \oint_{\gamma_\alpha} A_{class} \end{aligned} \tag{2.253}$$

Now, if we are in $\mathbb{R}^3$ we can choose an unambiguous disk D_α so that $\partial D_\alpha = \gamma_\alpha$. (This will not be true in a general 3-manifold.) Then we can continue our evaluation:

$$\begin{aligned} \frac{\mathrm{i}\kappa}{2\pi} \int_{M_3} A_{class} dA_{class} &= -\frac{\mathrm{i}}{2} \sum_\alpha r_\alpha \int_{D_\alpha} dA_{class} \\ &= \frac{\mathrm{i}\pi}{2\kappa} \sum_\alpha r_\alpha \int_{D_\alpha} J \\ &= \frac{\mathrm{i}\pi}{2\kappa} \sum_{\alpha,\beta} r_\alpha r_\beta \int_{D_\alpha} \eta(\gamma_\beta \hookrightarrow \mathbb{R}^3) \end{aligned} \tag{2.254}$$

and so

$$\langle \prod_\alpha W(r_\alpha, \gamma_\alpha) \rangle = \mathcal{N} \cdot \exp \left[\sum_{\alpha,\beta} \frac{2\pi \mathrm{i}}{4\kappa} r_\alpha r_\beta L(\gamma_\alpha, \gamma_\beta) \right] \tag{2.255}$$

where

$$L(\gamma_\alpha, \gamma_\beta) := \int_{D_\alpha} \eta(\gamma_\beta \hookrightarrow \mathbb{R}^3) \tag{2.256}$$

is called the *Gauss linking number*. It is discussed in detail in section 2.2.15. It is in fact symmetric, and an elegant formula for it is

$$L(\gamma_1, \gamma_2) = \int_{\gamma_1 \times \gamma_2} \omega(x_1 - x_2) \tag{2.257}$$

where $\omega(x)$ is the 2-form

$$\omega(x) := \frac{1}{8\pi} \frac{\epsilon_{ijk} x^i dx^j dx^k}{|x|^3} \tag{2.258}$$

Exercise *Holonomy Around The Source*

Show that the classical gauge field obtained by solving the equations of motion with source $W(r)$ have a holonomy

$$e^{-\frac{2\pi i r}{2\kappa}} \tag{2.259}$$

around the line supporting $W(r)$.

2.2.15 Digression On The Gauss Linking Number

♣SHORTEN THIS ♣

The Gauss linking number was already considered by J.C. Maxwell himself in his great book. [23]

The problem is this: *Consider a closed oriented loop $C \subset \mathbb{R}^3$ carrying a current I. Next, consider a second oriented loop $C' \subset \mathbb{R}^3$ as in Figure 5. What is the work done by the magnetic field when transporting a magnetic pole of unit charge around the curve C'?*

The result is given by a *topological invariant*, the *Gauss linking number*, which measures the amount by which the loops C and C' are linked. Using the Biot-Savart law you can work out that the result is that the work is

$$\oint \vec{B} \cdot d\vec{\ell} = I L(C, C') \tag{2.260}$$

where

$$L(C_1, C_2) := -\frac{1}{4\pi} \int_{C_1} \int_{C_2} \frac{(\vec{x}_1 - \vec{x}_2) \cdot (\frac{d\vec{x}_1}{ds_1} \times \frac{d\vec{x}_2}{ds_2})}{|\vec{x}_1 - \vec{x}_2|^3} ds_1 ds_2 \tag{2.261}$$

where $\vec{x}_1 := \vec{x}_1(s_1)$ describes the loop C_1 and $\vec{x}_2 := \vec{x}_2(s_2)$ describes the loop C_2. Note that $L(C_1, C_2) = L(C_2, C_1)$.

[23] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Section 419. Dover 1954, vol. 2, p.43

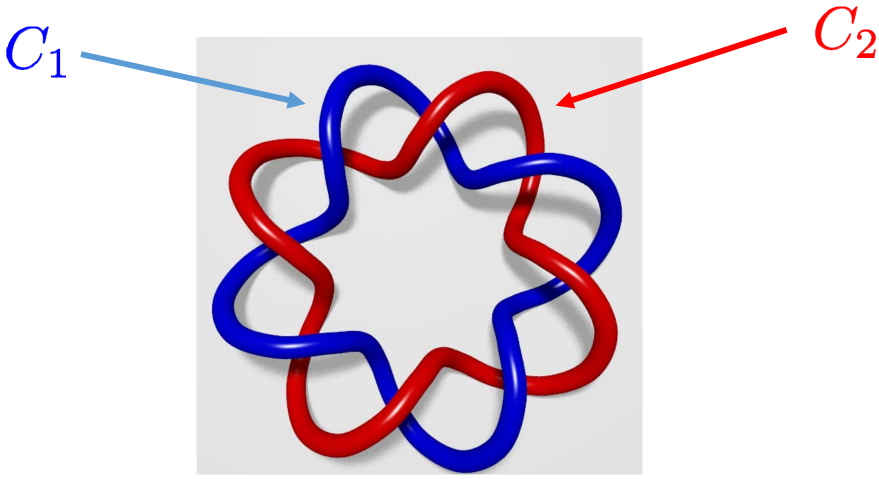

Figure 5: Two linked curves C_1 and C_2 in $\mathbb{R}^3$

Remark: The integral formula for $L(C_1, C_2)$ was discovered by Gauss in 1833. It has some similarities with, but is really quite different from, the Neumann formula for the mutual inductance of two current loops.

We claim that $L(C, C')$ is in fact an *integer* which measures the linking of C and C'. In magnetostatics, if we do not worry about orientation-reversing spacetime transformations, we can think of $\vec{B}$ as defining a one-form on $\mathbb{R}^3$, using $B = B^i dx^i$. Similarly, we can think of the current $\vec{J}$ as defining a two-form $J = \frac{1}{2}\epsilon_{ijk} J^i dx^j dx^k$. We are using here the Hodge duality between one-forms and two-forms in $\mathbb{R}^3$ and the equivalence of vectors and one-forms from a Euclidean metric.

In any case, identifying B with a one-form and J with a two-form, the Biot-Savart law is equivalent to:

$$dB = J \tag{2.262}$$

and the work done is just $\int_{C'} B$.

Now put $I = 1$ and let D' be an oriented disk spanning C' as in Figure 6. Then we evaluate the total current flowing through D':

$$\begin{aligned}\oint_{C'} B &= \int_{D'} dB = \int_{D'} J \\ &= \int_{D'} d\xi^\alpha \wedge d\xi^\beta \frac{\partial x^m}{\partial \xi^\alpha} \frac{\partial x^n}{\partial \xi^\beta} \frac{1}{2}\epsilon_{mnj} \oint_C \frac{dx^j(t)}{dt} \delta^{(3)}(\vec{x}(\xi) - \vec{x}(t))\end{aligned} \tag{2.263}$$

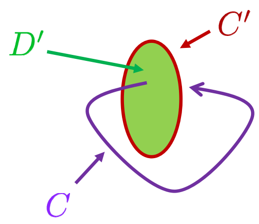

Figure 6: If we fill in C' with a disk D' then we can show that $L(C, C')$ counts signed intersections of C' with D', and is therefore an integer.

where ξ^α are some coordinates on D'. It is easy to see that in the last expression each transverse *intersection* of D' with C contributes ± 1 according to orientation: The orientation of C' induces one on D', and C is oriented. This oriented intersection number is one of the definitions of the linking number. From this interpretation $L(C, C')$ is clearly invariant under continuous deformation of D' or C or C', so long as C and C' do not cross.

Now that $L(C, C')$ is a *continuous* function of the locations of C and C'. On the other hand it is an *integer*. Therefore, it is a *topological invariant*. Note that this topological invariant can change, if we allow C and C' to cross. When C and C' cross the formula (2.261) becomes ill-defined, and then the integer can jump.

Exercise *Explicit verification of topological invariance*

Show that $I(C_1, C_2)$ is invariant under small deformations of $\vec{x}_1(t)$ by an explicit variation of the formula (2.261).

Exercise

Define the 2-form on $\mathbb{R}^3 - \{0\}$

$$\omega(x) := \frac{1}{8\pi} \frac{\epsilon_{ijk} x^i dx^j dx^k}{|x|^3} \tag{2.264}$$

a.) Show that, when restricted to the unit sphere $S^2 \subset \mathbb{R}^3 - \{0\}$ the form ω restricts to the standard volume form with unit volume.

b.) Show that the Gauss linking number can be expressed elegantly as:

$$L(C_1, C_2) = \int_{C_1} \int_{C_2} \omega(x_1 - x_2) \tag{2.265}$$

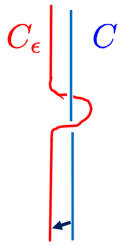

Figure 7: Displacing C infinitesimally to C_ϵ in the normal direction might lead to nontrivial self-linking because the normal vector might twist around. For this reason $L(C,C)$ is ill-defined.

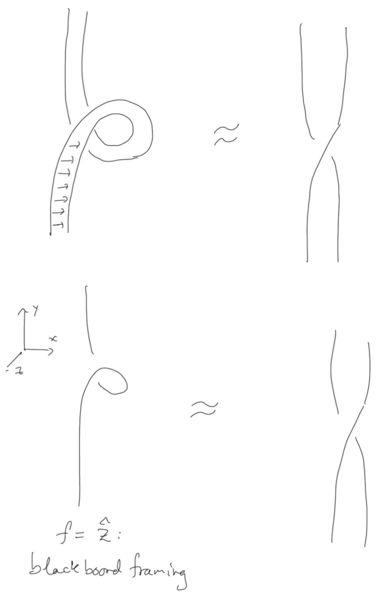

Figure 8: Different, but equivalent ways of illustrating the effect of the framing. Note in the lower left, without the framing, one could pull the string tight to a straight line. The fact that quantum line defects depend on a framing leads to a quantum correction to the first Reidemeister move as descrbed below.

2.2.16 The Framing Anomaly And Ribbons

All of the above works nicely for the terms with $\alpha \neq \beta$. However, the linking number for $\alpha = \beta$ requires further discussion. Just as self-inductance is rather more subtle than mutual inducance, self-linking numbers are a good deal more subtle than mutual linking numbers. The formula for the self-linking number clearly has singularities at coincident points for the Green's function. That makes perfectly good sense, as explained in Figure 7. To define $L(C, C)$ one must displace C infinitesimally in a normal direction and evaluate the mutual linking number of C and its displaced version. but this is clearly ill-defined because C could link around itself several times. In Chern-Simons theory this is known as the *framing anomaly*.

Definition: The *framing of the Wilson line operator* is a choice of nowhere zero section of the normal bundle of $\gamma \subset M_3$ up to homotopy (continuous deformation). If $s : \gamma \to N(\gamma)$ is a nowhere zero section of the normal bundle we denote the framing by $f = [s]$.

More colloquially: We can say that in the quantum theory the Wilson line operators are associated with *ribbons*. This will have an important implication when we try to write link invariants. See Figure 8 for a preview.

Remarks

1. Given a nonvanishing section $s : \gamma \to N(\gamma)$ we can twist it around n times for any integer n. Here we use orientation: The Wilson line is defined for an oriented curve γ and there is therefore an orientation in the disk normal to the line. Then a twist by n is a twist by a $2\pi n$ counterclockwise rotation in the disk. Note that the framings do not form an Abelian group: You cannot add $f_1 + f_2$. For, if you tried to add two sections of the normal bundle there is no guarantee the sum will be nonzero. Nevertheless, it does make sense to add an integer to a framing $f \to f + n$, $n \in \mathbb{Z}$, and moreover, any two framings can be related by shifting by an integer. Mathematically, we say that the framings form a $\mathbb{Z}$-*torsor*. [24]

2. What we have discovered is that in the quantum definition of $W(r, \gamma)$ we need to add an extra piece of data to define the line defect, namely, the framing of the curve γ. We accordingly change our notation to

$$W(n, \gamma) \to W(n, \gamma, f) \tag{2.266}$$

   Note that given γ and f there is a well-defined isotopy class $\gamma + f$ and we can define the self-linking by

$$L_f(\gamma) := L(\gamma, \gamma + f) \tag{2.267}$$

   Therefore, a more accurate version of (2.255) is therefore

$$\langle \prod_\alpha W(r_\alpha, \gamma_\alpha, f_\alpha) \rangle = \mathcal{N} \cdot \exp \left[\sum_{\alpha<\beta} \frac{2\pi \mathrm{i}}{2\kappa} r_\alpha r_\beta L(\gamma_\alpha, \gamma_\beta) + \sum_\alpha \frac{2\pi \mathrm{i}}{4\kappa} r_\alpha^2 L_{f_\alpha}(\gamma_\alpha) \right] \tag{2.268}$$

3. Note that

$$L_{f+n}(\gamma) = L_f(\gamma) + n \tag{2.269}$$

♣Could it be $-n$? Need to check. ♣

[24] In general for any group G, a set S is said to be a G-*torsor* if there is a G action on S that is both free and transitive. One can think of S as a copy of G without a choice of identity element. The fibers of a principal G-bundle are good examples of G-torsors.

and therefore

$$W(n,\gamma,f+N) = \left(e^{2\pi \mathrm{i}\frac{n^2}{4\kappa}}\right)^N W(n,\gamma,f) \tag{2.270}$$

We say that $e^{2\pi \mathrm{i}\frac{n^2}{4\kappa}}$ is the *spin of the line defect.* (Some authors will say that $\frac{n^2}{4\kappa}\mathrm{mod}\mathbb{Z}$ is the spin of the line defect. This is the sense in which it is used, for example, in [28]. This will be related to the spin of a corresponding conformal field and quantum dimension of a representation of a chiral algebra when we discuss the relation to RCFT.)

4. If $\gamma \subset \Sigma_2$ in a 3-manifold of the form $\Sigma_2 \times \mathbb{R}$ then we can regard the line defect $W(n,\gamma,f)$ as an operator on the Hilbert space. If we choose the natural framing that points forward in time then this operator can be identified with the operator $\widehat{W}(n\gamma)$ we discussed in Heisenberg's story.

5. Identification Of Line Defect Labels: Note that $n \to n+2\kappa$ leaves correlators unchanged. This is in accord with the quantum identification:

$$W(n+2\kappa,\gamma,f) \cong W(n,\gamma,f) \tag{2.271}$$

that we found when considering the operators on the Hilbert space $\mathcal{H}(\Sigma_2)$. While similar, (2.271) is conceptually distinct since it concerns line defects that are inserted in a path integral, rather than operators on the physical Hilbert space of a surface.

From the formula (2.259) we see that we could give a definition of Wilson line observables by imposing boundary conditions on the gauge fields, rather than integrating over smooth gauge fields and adding source terms (as we have been doing thus far). From this point of view, a line defect supported on γ is defined by cutting out a small tubular neighborhood around γ of radius ϵ and, to define the defect of charge n, we require that the gauge field have boundary condition that if $C(\epsilon)$ is a small linking curve on the boundary of this tubular neighborhood then the path integral with $W(n,\gamma,f)$ inserted has boundary conditions

$$\lim_{\epsilon\to 0} e^{\mathrm{i}\oint_{C(\epsilon)} A} = e^{-2\pi \mathrm{i} n/(2\kappa)} \tag{2.272}$$

Note that from this viewpoint we automatically have the identification $n \sim n+2\kappa$. [25]

[25] There is a sloppy version of this argument that goes as follows: We make a singular gauge transformation by a transformation that looks like $g \sim e^{\mathrm{i}\phi}$ near the Wilson line, where ϕ is an azimuthal angle. The shift of the Chern-Simons action is written as

$$\frac{\kappa}{2\pi}\int d(Ad\phi) + 2\cdot\frac{\kappa}{2\pi}\int Ad^2\phi \tag{2.273}$$

when we say that $d(d\phi) = 2\pi\delta^{(2)}(x)$ where $\delta^{(2)}(x)$ is the Dirac measure on supported on the Wilson line. You get the constant in this slightly dubious equation by integrating over a disk and imposing Stokes' theorem. In this approach, there is no boundary, so we can drop the first term. The net effect is to shift $n \to n+2\kappa$.

6. "Operator Product" of line defects. In general, when we have "parallel" defects we would like to define a notion of "operator product" of the defects. In the present case we can use (2.268) to define an operator product of the Wilson line defects. To define what we mean by "parallel curves" we assume that γ_2 can be obtained from γ_1 by an infinitesimal framing displacement so that γ_2 is isotopic to $\gamma_1 + f_{12}$. We can equally well say that $\gamma_2 = \gamma_1 + f_{21}$. Note that

$$L_{f_{12}}(\gamma_1) = L_{f_{21}}(\gamma_2) = L(\gamma_1, \gamma_2) \tag{2.274}$$

Now, consider the product of line defects:

$$W(n_1, \gamma_1, f_1) W(n_2, \gamma_2, f_2) \tag{2.275}$$

Suppose that all other line defects link γ_1 and γ_2 in the same way, so that:

$$L(\gamma_1, \gamma_\beta) = L(\gamma_2, \gamma_\beta) \qquad \beta > 2 \tag{2.276}$$

If γ_2 is gotten from γ_1 by an infinitesimal displacement this is automatically understood. Under the condition (2.276) we can replace the product (2.275) of line defects by a single line defect. To do this we write

$$\begin{aligned} f_1 &= f_{12} + N_{12} \\ f_2 &= f_{21} + N_{21} \end{aligned} \tag{2.277}$$

with $N_{12}, N_{21} \in \mathbb{Z}$. Then we have

$$\begin{aligned} W(n_1, \gamma_1, f_1) W(n_2, \gamma_2, f_2) &\cong e^{2\pi \mathrm{i}(\frac{n_1^2}{4\kappa} N_{12} + \frac{n_2^2}{4\kappa} N_{21})} W(n_1 + n_2, \gamma_1, f_{12}) \\ &\cong e^{2\pi \mathrm{i}(\frac{n_1^2}{4\kappa} N_{12} + \frac{n_2^2}{4\kappa} N_{21})} W(n_1 + n_2, \gamma_2, f_{21}) \end{aligned} \tag{2.278}$$

7. Monopole Operators. An important class of local operators in three-dimensional gauge theories are the monopole operators. For a general gauge theory with gauge group G they are obtained by choosing an embedding $\mu : U(1) \to G$. This allows to embed a Dirac monopole of magnetic charge μ, where $F^{\mathrm{Dirac}} = 2\pi \frac{\sin\theta d\theta d\phi}{4\pi}$ into the G gauge theory. We can then define a singular G connection at a point p by surrounding p with a small sphere of radius ε are requiring that, as $\varepsilon \to 0$

$$F \to \mu(F^{\mathrm{Dirac}}) \tag{2.279}$$

This singular boundary condition defines a point defect "monopole operator" of charge μ: $M_\mu(p)$.

However, if we recall the gauge variation of the Chern-Simons action (2.45) then we see that the operator $M_\mu(p)$ is not gauge invariant, but rather transforms by

$$M_\mu(p) \to e^{\mathrm{i}\kappa\mu(\epsilon(p))} M_\mu(p) \tag{2.280}$$

The operator is thus not gauge invariant. In a non-topological theory this problem can be cured by adjoining a Wilson line or multiplying by some other charged operator of opposite charge, but in the topological Chern-Simons theory one cannot introduce these operators.

♣NEED TO EXPLAIN A LOT MORE HERE.... ♣

8. One-form Symmetries. In [23] it was suggested that many familiar facts about global symmetries of quantum field theories generalize nicely to transformations of mathematical structures associated with higher dimensional defects. In particular, "one-form symmetries" are associated with codimension two defects. When these link dimension one defects along an infinitesimal link the result is a new codimension two defect. These are the "one-form symmetries" of [23]. In the present case the objects that implement the one-form symmetries are themselves the one-dimensional defects, hence the Wilson lines. They form an Abelian group $\cong \mathbb{Z}/2\kappa\mathbb{Z}$ which is just the group under the "operator product" described above. Because of the simple linking rule above, $W(n,\gamma)$ has "charge n" under the one-form symmetry group. That is, it is in the n^{th} power of the defining representation.

9. COMMENT ON WHAT HAPPENS FOR WILSON LINES IN A GENERAL 3-MANIFOLD

2.2.17 A Topological Anomaly: The Normalization Factor: $\mathcal{N}$

As we have said, the normalization factor is formally metric independent since, thus far, we have not needed to use the metric. However, to define the path integral we must gauge-fix the gauge symmetry and any choice of gauge fixing will break the formal topological symmetry of the path integral. Fortunately, the topological invariance is spoiled in a computable way, and a way that makes good physical sense.

The most standard way to gauge fix is to introduce a Riemannian metric on M_3 and choose (in the Abelian case) the gauge slice:

$$d * A = \mathrm{vol}\,(g) g^{\mu\nu} D_\mu A_\nu = 0 \tag{2.281}$$

This will fix all gauge freedom up to harmonic forms $A \to A + \omega$. So there will be a finite-dimensional torus $\mathcal{H}^0(M_3; U(1))$ of unfixed gauge transformations.

The standard way to gauge fix is to introduce Fadeev-Popov ghosts in the BRST procedure: We introduce anticommuting zero-forms (in the adjoint representation of the gauge group) c, b of ghost numbers $+1, -1$ respectively with a differential ♣should say where they live, etc. ♣

$$\begin{aligned} Q(A) &= Dc \\ Q(c) &= 0 \\ Q(b) &= H \\ Q(H) &= 0 \end{aligned} \tag{2.282}$$

and we add to the action

$$Q(\mathrm{i}\int b d * A) = \mathrm{i}\int b d * dc + \mathrm{i}\int H d * A \tag{2.283}$$

We should think of H as a Lagrange multiplier. [26] The integral over b, c gives

$$\det{}' * d * d \tag{2.284}$$

where $*d*d$ is the standard metric Laplacian on 0-forms.

♣Should discuss zeromode ♣

Meanwhile we can rescale the Lagrange multiplier field $H = const. * \phi$ where $\phi \in \Omega^3(M_3)$ to get the action

$$\exp[\frac{\mathrm{i}k}{2\pi}\int_{M_3}(AdA + \phi * d * A) = \exp[\frac{\mathrm{i}k}{2\pi}\int_{M_3}\Phi * D\Phi] \tag{2.285}$$

where

$$\Phi = \begin{pmatrix} A \\ \phi \end{pmatrix} \qquad\qquad D_+ = \begin{pmatrix} *d & 0 & 0 \\ 0 & 0 & d* \\ 0 & d* & 0 \end{pmatrix} \tag{2.286}$$

Here

$$D_+ : \Omega^{\mathrm{odd}}(M_3) \to \Omega^{\mathrm{even}}(M_3) \tag{2.287}$$

is $D_+ = *d + d*$. This can be interpreted as a twisted Dirac operator. We will return to that when we revisit this computation in the nonabelian theory.

Later, in section **** we will view D_+ as a twisted Dirac operator and use some general facts about the determinant of Dirac operators on odd-dimensional manifolds. For the moment, we use some Hodge theory instead. See section F for a summary of some relevant facts. The operator $*d : \Omega^1 \to \Omega^1$ is self-adjoint. [27] and can be diagonalized with spectrum $\mu_n^{(1)}$ on $\mathrm{Im}d^\dagger \subset \Omega^1$. This is a Dirac-like operator and has spectrum unbounded from above and below. Meanwhile, $dd^\dagger$ on $\mathrm{Im}d \subset \Omega^1$ has spectrum $(\lambda_n^{(0)})^2$.

♣Check a sign here. Anti-self-adjoint?? ♣

Integration over H gives a delta function so we get a Jacobian

$$\int_{\Omega^1}[dA]\delta(d*A)[....] = \int_{\mathrm{Im}d^\dagger}[dA]\frac{1}{\prod_n|\lambda_n^{(0)}|}[....] \tag{2.288}$$

then doing the remaining integrals gives determinant:

$$\frac{\det' *d*d}{\det^{1/2}(\frac{1}{2e^2}d^\dagger d + 2\pi\mathrm{i}k*d)|_{\mathrm{Im}d^\dagger}}\frac{1}{\prod_n|\lambda_n^{(0)}|} \tag{2.289}$$

The first factor can give a phase. We regularize using the ζ-function so

$$\zeta(s) = \sum_n(\frac{1}{2e^2}(\mu_n^{(1)})^2 + 2\pi\mathrm{i}k\mu_n^{(1)})^{-s} \tag{2.290}$$

Note that there is spectral asymmetry. In the limit $e^2 \to \infty$ we get

$$\mathrm{Im}(-\zeta'(0)) = \frac{\pi}{2}\eta(*d + d*) \tag{2.291}$$

This gives a nice derivation of the framing anomaly of the CS theory.

[26]Usually, in the BRST procedure one adds a Gaussian term in H. In this case that would spoil the first-derivative nature of the action and complicate the beautiful story.

[27]$(*d)^\dagger = d^\dagger *^\dagger = \pm * d * * = \pm * d$. SIGN????

The η invariant does have metric dependence, but the metric dependence is local. We will discuss it when we revisit this in the nonabelian case in section 3.2.1. The absolute value is

$$\prod_n \frac{|\lambda_n^{(0)}|}{\sqrt{|\mu_n^{(1)}|}} = \exp[-\frac{1}{4}\tau(M_3)] \tag{2.292}$$

where $\tau(M_3)$ is the Ray-Singer torsion, or analytic torsion. It is a topological invariant, as shown in Appendix F.

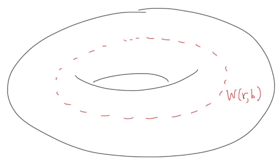

Figure 9: The path integral on a handlebody (in this figure, a solid torus) defines a wavefunctional of the boundary values of the fields. In the case of Chern-Simons theory, it defines a specific state in the space of conformal blocks of the bounding surface. We can change the state by inserting nonlocal operators such as Wilson line defects.

2.2.18 Uniting The Path Integral And Hamiltonian Viewpoints: States Created By Wilson Lines

Quite generally, suppose we have a field theory in d space-time dimensions with fields ϕ valued in some target space $\mathcal{T}$. If this theory is defined via path integrals, then, if X is a d-dimensional spacetime with boundary $\partial X = M$, the path integral on X defines a specific state in the Hilbert space $\mathcal{H}(M)$ associated to the $(d-1)$-dimensional manifold M:

$$\partial X = M \qquad \Rightarrow \qquad \psi_X \in \mathcal{H}(M). \tag{2.293}$$

Our notation here is highly schematic. For example, if we insert local operators or other defects in the interior of the manifold X then that changes what we mean by X and in general such insertions will certainly change the state.

Equation (2.293) is easily understood: In order to define the path integral we need to specify some boundary values ϕ_∂ so that there is a suitable stationary phase approximation and a suitable classical variational problem. (The ϕ_∂ might be more subtle than the simple restriction of the field ϕ to ∂X.) We can identify $\mathcal{H}(M)$ as the space of (suitably normalizable) wavefunctions on the space of boundary conditions, and the value of the path integral with specific boundary conditions ϕ_∂ is declared to be the value of the wavefunction of the state ψ_X on ϕ_∂:

$$\psi_X(\phi_\partial) := PATH\ INTEGRAL\ WITH\ b.c. = \phi_\partial \qquad \Rightarrow \qquad \psi_X \in \mathcal{H}((M) \tag{2.294}$$

In the axiomatic approach to field theory based on functors from a geometric category to a tensor category the above remark becomes axiomatic. (More precisely, it is an immediate consequence of some more basic axioms.

The implementation of this idea in the present case of $U(1)$ Chern-Simons theory is the following. Suppose our 3-manifold has a bounding surface Σ_2. Suppose, initially, that it is connected. Then the 3-manifold M_3 is what is known as a *handlebody*. Note that there is a group homomorphism

$$\iota : \pi_1(\Sigma_2, x_0) \hookrightarrow \pi_1(M_3, x_0) \tag{2.295}$$

which induces a homomorphism

$$\iota : H_1(\Sigma_2) \hookrightarrow H_1(M_3) \tag{2.296}$$

For both homomorphisms there will be a large kernel: In Figure 3 if we imagine filling in the surface in the natural way, then all the a-cycles will map to zero, while the b-cycles will remain nontrivial. Since all the a-cycles map to zero we expect that, in the operator approach $W(\gamma)\psi_{M_3} = \psi_{M_3}$ for all $\gamma \in L_1$. This indeed selects a one-dimensional line in the Stone-von Neumann representation, hence, a pure state. Represent this state by some nonzero vector Ψ_{M_3} in that line.

It is interesting to alter X in the way suggested above by inserting a Wilson line of charge $n_0 \in \mathbb{Z}/2\kappa\mathbb{Z}$ as in Figure 9. In general, we can consider $W(n_0\gamma)$ where $\gamma \in H_1(M_3;\mathbb{Z})$. This will produce a new state which can be represented by a nonzero vector $\Psi_{M_3,n_0\gamma}$. Let $\hat{\gamma} \in H_1(\Sigma_2)$ be in the pre-image of ι. It is clear that

♣Need to fix figure so insertion charge is n_0 ♣

$$\Psi_{M_3,n_0\gamma} = \widehat{W}(n_0\gamma)\Psi_{M_3} \tag{2.297}$$

because if the Wilson line is applied to the surface then we can let it "sink in" to get the picture in Figure 9. This is, of course, the pure state (one-dimensional line in $\mathcal{H}(\Sigma_2)$) determined by the condition that it is the eigenline of $W(\gamma')$ of eigenvalue $e^{2\pi i \frac{\langle \gamma', n_0\gamma\rangle}{2\kappa}}$ for all $\gamma' \in L_b$.

$$\widehat{W}(\gamma')\Psi_{M_3,n_0\gamma} = e^{2\pi i \frac{\langle \gamma', n_0\gamma\rangle}{2\kappa}}\Psi_{M_3,n_0\gamma} \tag{2.298}$$

In fact, (2.298) can also be derived rather directly by letting γ' also "sink in to the handlebody" until it links γ.

****

1. DISCUSS THE BOUNDARY CONDITIONS ϕ_∂ IN THE PARTICULAR EXAMPLE OF THE TORUS.

2. NOW GENERALIZE TO SEVERAL CONNECTED COMPONENTS. THEY CAN HAVE ORIENTATIONS THAT AGREE OR DIFFER WITH THAT OF THE 3-MANIFOLD M_3 SO WE GET SPACES AND DUAL SPACES. IN AND OUTGOING SPACES. PATH INTEGRAL AS A LINEAR TRANSFORMATION FROM IN TO OUT.

♣Need to fix figure: Want ribbons. Representations are labeled by n_1, n_2 not μ, ν ♣

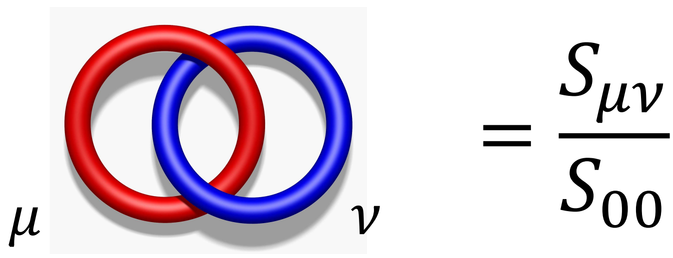

Figure 10: Two linked circles carrying Wilson line defects of charges μ, ν, and linking nothing else, are equivalent to a scalar, and this scalar is just the matrix element $S_{\mu\nu}$ of the modular transformation by S in the canonical basis for $\mathcal{H}^{(}T^2)$.

3. Relation To The Modular S-Matrix. Of particular interest is an identity relating the correlators to the modular S-matrix.

ALREADY HERE YOU CAN NOTE THE RELATION OF THE LINKED CIRCLES TO THE MODULAR S-MATRIX.

****

2.3 Relation To Rational Conformal Field Theory

2.3.1 Quantization When $\partial\Sigma$ Is Nonempty: Emergence Of Edge States

We continue to consider the quantization of $U(1)$ level k Chern-Simons theory.

In this section we consider $M_3 = D_2 \times \mathbb{R}$ where D_2 is a disk. We now have a boundary, and therefore must choose boundary conditions.

One very general principle: *Boundary conditions should be chosen so that the stationarity of the action is a well-posed problem*

In other words, the boundary variations of the action should vanish.

♣Perhaps explain in terms of very general symplectic path integrals that this always entails a choice of Lagrangian submanifold. ♣

In our case we have

$$\delta \int_{M_3} AdA = \int_{M_3} 2\delta AF + \int_{M_3} d(\delta AA) \tag{2.299}$$

and therefore our boundary conditions must be such that

$$\int_{\partial M_3} \delta A A = 0 \tag{2.300}$$

The most straightforward way is to enforce $\delta A A|_{\partial M_3} = 0$ locally.

On the disk choose polar coordinates (r, ϕ) and let x^0 be the time direction so

$$\delta A A = (\delta A_\phi A_t - \delta A_t A_\phi)\, d\phi \wedge dx^0 \tag{2.301}$$

Next, we must carefully declare what the group of gauge transformations is. We regard A_0 as a Lagrange multiplier, enforcing $F = 0$ on M_3. Then the remaining gauge group involves time-independent gauge transformations.

One interesting choice is to consider

$$\mathcal{G} = Map(g : D_2 \to U(1) \quad | \quad g|_{\partial D_2} = 1\} \tag{2.302}$$

♣This remark possibly out of place here. ♣

Remark: In general if X is a manifold with boundary and we consider the group

$$Map(X \to G) \tag{2.303}$$

then there is a subgroup of maps $g : X \to G$ such that $g|_{\partial X} = 1$. In gauge theories this is often taken as the group of local gauge transformations that describe redundancies in our parametrization of local physics. This is a normal subgroup of $Map(X \to G)$ and the quotient group is often identified with the group of global gauge transformations. We expect states and operators to transform in (possibly projective) representations of this group.

In this case, it is straightforward to show that $A_s = ig^{-1}dg = dX$ where X is a periodic scalar field and the value of the action depends only on the restriction of X to the boundary:

$$\int_{D_2\times\mathbb{R}} A_s \partial_0 A_s dx^0 = \int_{D_2\times\mathbb{R}} d_s\,(A_s\partial_0 X) = \int_{\partial D_2\times\mathbb{R}} \partial_\phi X \partial_0 X \tag{2.304}$$

Now let us return to the boundary conditions: but express them in terms of X. We have

$$\partial_\phi \delta X \partial_t X - \partial_t \delta X \partial_\phi X = 0 \tag{2.305}$$

and we can solve this if X is a chiral (or anti-chiral) scalar field

$$\partial_\phi X \pm \partial_t X = 0 \tag{2.306}$$

where we choose one, or the other, but not both.

****

MENTION THE ISSUE OF ACTIONS FOR SELF-DUAL FIELDS.

****

1. As we have seen, on a closed three-manifold M_3 the value of the Chern-Simons invariant of a gauge field $\frac{1}{4\pi^2}\int_{M_3} AdA$ only makes sense as an element of $\mathbb{R}/\mathbb{Z}$. When M_3 has a nonempty boundary the story is different and more subtle: The value of the action defines a section of a line bundle.

   EXPLAIN THAT HERE OR LATER???

2. Anomaly cancellation. We could couple any system to the edge with a suitable gauge anomaly and have a nonanomalous system.

2.3.2 Quantization With Wilson Lines Piercing Spatial Surface

Disk x R with Wilson line: Nontrivial holonomy.

Sources in Gauss law.

2.4 The Gaussian Model For R^2 Rational

The *Gaussian model* is a term that is often used to refer to the conformal field theory of a single massless scalar field with a periodic identification.

This section is, in part, a rapid tour of some simple aspects of conformal field theory. For more extensive accounts see:

REFS: Polchinski, vol. 1; DiFrancesco et. al. ; Fuchs ; ... ?

The action for this model is

$$S = K\int dX * dX = K\int d\tau \int_0^{2\pi} d\sigma \left[(\partial_\tau X)^2 - (\partial_\sigma X)^2\right] \tag{2.307}$$

with X dimensionless and periodic (the period can be anything, so long as it is fixed) and K is a positive constant.

The Green's function in two dimensions is $\frac{1}{4\pi}\log|z_1 - z_2|^2$ so

$$\begin{aligned} \langle X(z_1)X(z_2)\rangle &= \frac{1}{2\pi K}\log|z_1 - z_2|^2 \\ \langle \partial X(z_1)\partial X(z_2)\rangle &= -\frac{1}{2\pi K}\frac{1}{(z_1 - z_2)^2}dz_1\wedge dz_2 \\ \langle \bar\partial X(z_1)\bar\partial X(z_2)\rangle &= -\frac{1}{2\pi K}\frac{1}{(\bar z_1 - \bar z_2)^2}d\bar z_1\wedge d\bar z_2 \end{aligned} \tag{2.308}$$

One standard normalization in physics is to take $X \sim X + 2\pi$ and $K = \frac{r^2}{4\pi\ell_s^2}$. Then X is a map of the worldsheet to a single circle of radius r, and ℓ_s is the string length.

The momentum and winding zero modes of the Gaussian field defined by the general solution of the equation of motion:

$$X = x_0 + \frac{p_L}{\sqrt{2}}(\tau + \sigma) + \frac{p_R}{\sqrt{2}}(\tau - \sigma) + X^{\rm osc} \tag{2.309}$$

where we have set $\ell_s = 1$ and $X^{\rm osc}$ is the sum of solutions with nonzero Fourier modes. The zero modes have the property that the vectors (p_L, p_R) are valued in an even unimodular lattice embedded in $\mathbb{R}^{1;1}$. The lattice of zero modes can be written as

$$\Gamma(r) := \{ne_r + wf_r | n, w \in \mathbb{Z}\} \subset \mathbb{R}^{1;1} \tag{2.310}$$

where

$$e_r = \frac{1}{\sqrt{2}}(1/r; 1/r), \qquad f_r = \frac{1}{\sqrt{2}}(r; -r) \tag{2.311}$$

Note that $e_r^2 = f_r^2 = 0$, $e_r \cdot f_r = 1$ so that $\Gamma(r)$ is indeed an embedding of the even unimodular (a.k.a. self-dual Lorentzian) lattice $II^{1,1}$ of rank 2 and signature $(1,1)$.

This model has a $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ current algebra generated by currents

♣SOME ANNOYING FACTORS OF 2 MISSING HERE. ♣

$$J_L = \sqrt{\kappa} \sum_{n\in\mathbb{Z}} \alpha_n z^{-n} \frac{dz}{z} \qquad J_R = \sqrt{\kappa} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n \bar{z}^{-n} \frac{d\bar{z}}{\bar{z}} \tag{2.312}$$

We have

$$J_L \sim -\mathrm{i}\partial X \qquad J_R \sim -\mathrm{i}\bar{\partial} X \tag{2.313}$$

♣Annoyingly, in our conventions, there is a nonzero constant here. ♣

****

NEED TO GIVE MORE CAREFUL AND FORMAL DISCUSSION OF WHAT IS MEANT BY A "CHIRAL ALGEBRA"

****

The "chiral algebra" they generate is the smallest operator product algebra generated by these fields. Ignoring possible contact terms from contraction of left- and right-moving fields we have a direct sum of the "$\mathfrak{u}(1)$ chiral algebra". The fields are all obtained from polynomials in (anti-) holomorphic derivatives of X. We call it $\mathcal{A}(\mathfrak{u}(1))$.

♣GIVE FORMAL DEFINITION OF CHIRAL ALGEBRA HERE OR DEFER AND JUST SAY THIS IS A SPECIAL CASE OF A "CHIRAL ALGEBRA" TO BE DEFINED LATER ? ♣

We compute the two-point function in the $SL(2) \times SL(2)$ - invariant vacuum in radial quantization: [28]

$$\begin{aligned} \langle 0|J_L(z_1)J_L(z_2)|0\rangle &= \kappa \sum_{n=1}^{\infty} n \left(\frac{z_2}{z_1}\right)^n \frac{dz_1}{z_1} \otimes \frac{dz_2}{z_2} \\ &= \frac{\kappa}{(z_1-z_2)^2} dz_1 \otimes dz_2 \end{aligned} \tag{2.315}$$

Note that in radial quantization we first work in the domain with $|z_1| > |z_2|$ so that the series converges. Then we analytically continue from there. For the currents there is a single-valued analytic continuation to the entire space $\mathbb{C} \times \mathbb{C} - Diag$ with no monodromy.

That's how you define all correlators in radial quantization: The operators in the theory are such that the correlation functions are single-valued.

Any chiral algebra has $T(z)$ and all its derivatives and products of derivatives. Note that in this case

$$T(z) = -\frac{1}{2}(\partial x)^2 = \frac{1}{2\kappa} : J_L^2 : \tag{2.316}$$

This is called the *Sugawara form of the stress tensor* and will hold, in a slightly modified form, in the nonabelian case.

[28] This corresponds to standard normalization

$$\langle \partial x(z_1) \partial x(z_2)\rangle = -\frac{1}{(z_1-z_2)^2} \tag{2.314}$$

****

Unitary Representations Of The $U(1)$ Chiral Algebra:

Representations: V_p with $p \in \mathbb{C}$. Unitary representations: $p \in \mathbb{R}$. Character

$$\begin{aligned} \chi_p &:= \mathrm{Tr}_{V_p} q^{L_0 - \frac{c}{24}} e^{2\pi \mathrm{i} \xi J_0} \\ &= \frac{q^{\frac{1}{2}p^2} e^{2\pi \mathrm{i}\xi p}}{\eta} \end{aligned} \tag{2.317}$$

In addition, the model has vertex operators

$$V_p(z,\bar{z}) = c(p) : e^{\mathrm{i}p_L X(z)} \otimes e^{\mathrm{i}p_R \tilde{X}(\bar{z})} : \tag{2.318}$$

where the momentum $p = (p_L; p_R)$ is confined to be in the even unimodular lattice $\Gamma(r)$: So

$$\begin{aligned} p_L &= \frac{1}{\sqrt{2}}(\frac{n}{r} + wr) \\ p_R &= \frac{1}{\sqrt{2}}(\frac{n}{r} - wr) \end{aligned} \tag{2.319}$$

where $n, w \in \mathbb{Z}$ are momentum, and winding quantum numbers, respectively. (Here $c(p)$ is a "cocycle operator" - IGNORE IT OR EXPLAIN?) We indicated standard normal ordering symbols. Henceforth our exponentiated vertex operators are all assumed to have this normal ordering and we drop it from the notation.

The operator product of these fields is, exactly:

$$V_{p_1}(1)V_{p_2}(2) = \varepsilon(p_1,p_2)(z_1 - z_2)^{p_L^1 \cdot p_L^2}(\bar{z}_1 - \bar{z}_2)^{p_R^1 \cdot p_R^2} c(p_1,p_2) : e^{\mathrm{i}(p_L^1 X_L(1) + p_L^2 X_L(2))} \otimes e^{\mathrm{i}(p_R^1 X_R(1) + p_R^2 X_R(2))} \tag{2.320}$$

Here the cocycle operators satisfy $c(p_1)c(p_2) = \varepsilon(p_1,p_2)c(p_1 + p_2)$. They define a $\mathbb{Z}_2$ central extension of the lattice $\Gamma(r)$, that is, the $\varepsilon(p_1,p_2)$ are valued in $\{\pm 1\}$ and must satisfy

$$\frac{\varepsilon(p_1,p_2)}{\varepsilon(p_2,p_1)} = (-1)^{p_1 \cdot p_2} \tag{2.321}$$

EXPLAIN:

1. This is single-valued because $p_1 \cdot p_2 \in \mathbb{Z}$, because $\Gamma(r)$ is an integral lattice.

2. Consequently, using radial quantization and the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant vacuum:

$$\langle \prod_i V_{p_i}(z_i,\bar{z}_i)\rangle = \mathcal{F}_L(z_1,\dots,z_N)\tilde{\mathcal{F}}_R(\bar{z}_1,\dots,\bar{z}_N) \tag{2.322}$$

   where

$$\mathcal{F}_L(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^{p_L^i p_L^j} \tag{2.323}$$

   are known as *conformal blocks for the correlation function.*

3. Stress that the correlation function is single-valued, but the conformal blocks have monodromy.

For special radii some of the V_{p_L,p_R} become purely holomorphic operators. This happens if there is a solution to the Diophantine equation $p_R = 0$. Clearly this requires

$$r^2 = \frac{m}{w} \tag{2.324}$$

for some integers m, w, and therefore $r^2 \in \mathbb{Q}$. These are called the *rational Gaussian models*. They are among the simplest of the *rational conformal field theories.*

♣AGAIN, GIVE A FORMAL DEFINITION OF RCFT? OR DEFER? ♣

So, suppose $r^2 = p/q$ is rational in lowest terms. Then the holomorphic operators are obtained from $m = \ell p$ and $w = \ell q$. These are all powers of the basic holomorphic vertex operators:

$$V_\pm := c_\pm e^{\mathrm{i}\sqrt{2pq}X} \tag{2.325}$$

Set $\kappa = pq$. Then these operators have integer conformal dimension κ and no monodromy around the operators generated by $J_L(z)$, so they can be added to the chiral algebra $\mathcal{A}(\mathfrak{u}(1))$ to produce a larger chiral algebra known as the $\mathfrak{u}(1)$ *level* κ *chiral algebra* and denoted as $\mathcal{A}(\mathfrak{u}(1))_\kappa$.

The representations of $\mathcal{A}(\mathfrak{u}(1))_\kappa$ can be found among the holomorphic parts of the vertex operators of the Gaussian model at radius $r^2 = p/q$. They are generated by the "chiral vertex operators":

$$V_n(z) = c e^{\mathrm{i}\frac{n}{\sqrt{2\kappa}}X(z)} \tag{2.326}$$

Warning: The logical status of (2.326) is different from (2.325) in an important way. The fields (2.326) by themselves are not part of the spectrum of the Gaussian model. The left-movers and right-movers must be combined into single valued non-holomorphic fields. This is quite different from (2.325). These fields are in the spectrum of the theory.

Note that $n \sim n + 2\kappa$ and the character is

$$\begin{aligned}
\chi_n(z,\tau) &:= \mathrm{Tr}_{\mathcal{H}_n} q^{L_0 - \frac{c}{24}} e^{2\pi \mathrm{i}\xi J_0} \\
&= \frac{\sum_{\ell\in\mathbb{Z}} q^{\frac{1}{2}p_L^2} e^{2\pi\mathrm{i}\xi p_L}}{\eta} \\
&= \frac{\sum_{\ell\in\mathbb{Z}} e^{2\pi\mathrm{i}\kappa\tau\left(\ell + \frac{r}{2\kappa}\right)^2 + 2\pi\mathrm{i}z(2\ell\kappa+n)}}{\eta} \\
&= \frac{\Theta_{n,\kappa}(z,\tau)}{\eta}
\end{aligned} \tag{2.327}$$

where we have defined $z = \xi/\sqrt{2\kappa}$ as a more convenient normalization for working with modular transformations and periodicities.

Now let us consider the partition function of the RCFT where we combine left- and right-movers. At a generic radius we have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}} e^{2\pi\mathrm{i}\tau(L_0 - \frac{c}{24}) + \mathrm{i}\xi J_0 - 2\pi\mathrm{i}\bar{\tau}(\tilde{L}_0 - \frac{\tilde{c}}{24}) - 2\pi\mathrm{i}\tilde{\xi}\tilde{J}_0} &= \frac{\sum_{(p_L,p_R)\in\Gamma(r)} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} e^{2\pi\mathrm{i}\xi p_L - 2\pi\mathrm{i}\tilde{\xi}p_R}}{\eta\bar{\eta}} \\
&= \sum_{(p_L,p_R)\in\Gamma(r)} \chi_{p_L}(z,\tau)\overline{\chi_{p_R}(z,\tau)}
\end{aligned} \tag{2.328}$$

Now, this expression also has an interpretation as a path integral on a torus:

$$Z(\tau,\xi;\bar\tau,\bar\xi) = e^{-2\frac{(\mathrm{Im}\xi)^2}{\mathrm{Im}\tau}}\mathrm{Tr}_{\mathcal{H}}e^{2\pi i\tau(L_0-\frac{c}{24})+i\xi J_0-2\pi i\bar\tau(\tilde L_0-\frac{\tilde c}{24})-2\pi i\tilde\xi\tilde J_0} \tag{2.329}$$

where

$$Z(\tau,\xi;\bar\tau,\bar\xi) = \int [dX]e^{-K\int_\Sigma(\partial X+A^{1,0})\wedge(\bar\partial X+A^{0,1})} \tag{2.330}$$

and we have coupled to a flat external gauge field $A = a_1 d\sigma^1 + a_2 d\sigma^2$ and

$$\xi = constant\frac{1}{2\pi}(a_2 - \bar\tau a_1) \tag{2.331}$$

♣NEED TO CHECK THIS EQUATION! ♣

The path integral is clearly diffeomorphism invariant and gauge invariant under the gauge transformations for the external, nondynamical, $U(1)$ gauge field A.

When a path integral coupled to a metric and/or gauge field fails to be properly gauge invariant we say there is an *anomaly*. Because we have both left- and right-moving bosons coupled in a symmetric way there is no possibility of an anomaly in this case.

****

****

HERE PROVIDE DETAILS OF COMPUTATION OF THE PATH INTEGRAL

1. The relevant field space is not connected and has topologically distinct sectors: $X = X_q + X_{sol}$ with $X_{sol} = 2\pi n^1\sigma^1 + 2\pi n^2\sigma^2$, $n^1, n^2 \in \mathbb{Z}$.

2. So $Z = Z_q\sum_{n,m} e^{-S_{class}}$.

   GIVE FORMULA FOR CLASSICAL ACTION

3. Poisson resum on m.

4. Evaluate Z_q using ζ-function regularization to get

$$Z_q = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2} \tag{2.332}$$

5. Comment: For noncompact boson we have

$$Z_q = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2} \tag{2.333}$$

   We can recover this from the $r\to\infty$ limit: It comes from the Gaussian integral

$$\int_{-\infty}^{+\infty} dp e^{-\pi\mathrm{Im}\tau p^2} \tag{2.334}$$

****

****

Now when $r^2 = p/q$ the infinite sum over the Narain lattice $\Gamma(r)$ simplifies in an interesting way

$$Z(z,\tau;\bar{z},\bar{\tau}) = e^{-4\pi k \frac{(\mathrm{Im} z)^2}{\mathrm{Im}\tau}} \sum_{r,\tilde{r}\in\mathbb{Z}/2k\mathbb{Z}} D_{r,\tilde{r}} \chi_r(z,\tau) \overline{\chi_{\tilde{r}}(z,\tau)} \tag{2.335}$$

The matrix $D_{r,\tilde{r}}$ is discussed further in section 2.4.1 below.

We can draw some important conclusions from the above computation:

1. In both (2.328) with (2.335) we see the typical feature of conformal field theories that the partition function is a sum of holomorphic times anti-holomorphic objects, up to a simple local factor. Notice they are holomorphic in both z and τ. These holomorphic factors are called *conformal blocks* and a (defining) property of rational conformal field theory is that all correlators on all Riemann surfaces are such sums. The local prefactor represents an anomaly in requiring both gauge invariance and holomorphic factorization. This observation goes back, at least, to [5].

♣REALLY NEED TO DISCUSS MODULAR TRANSFORMATIONS TO SEE THAT ONLY THE VECTOR SPACE OF CONFORMAL BLOCKS IS WHAT IS INVARIANT. ♣

2. Moreover, as a function of z the vector space of conformal blocks is identical to the Hilbert space of the $U(1)_k$ Chern-Simons theory. This will turn out to be a general principle:

   There is an isomorphism between the vector space of physical states in a Chern-Simons gauge theory quantized on $\Sigma \times \mathbb{R}$ and the space of conformal blocks of a corresponding rational conformal field theory. Moreover, if the Chern-Simons theory is quantized using Kähler quantization induced by a choice of complex structure on Σ then this is an identification of holomorphic vector bundles over the moduli space of Riemann surfaces.

   One of our goals is to understand this statement better in the context of more general Chern-Simons theories.

3. In this sense there is a kind of equivalence between the topological Chern-Simons theory on three-manifolds with boundary, and the (chiral half of) a rational conformal field theory on the boundary. It is a kind of "holography."

2.4.1 Gluing The Same Chiral Algebra To Get Different Radii

Enquiring minds will notice that, the integer k will have different factorizations as $k = pq$ (even when k is prime). What picks out the different radii?

<u>Answer Within The Framework Of RCFT</u>

In (2.335) the matrix $D_{r,r'}$ tells us how to glue left- and right-movers. As shown in [43] this must be done via an automorphism of the $\mathbb{Z}/(2k\mathbb{Z})$ fusion rule algebra. (This is a general result on gluing together representations in RCFT.) Thinking of that Abelian group additively the automorphisms are just $x \mapsto yx$ where y is relatively prime to $2k$. It is claimed in [43] that if we think of a fundamental domain $0 \le r \le 2k-1$ then $D_{r,r'} = 1$ if $r = pr_2 + qr_1$ and $r' = qr_1 - pr_2$ with (r_1, r_2) valued in the diamond region in $\mathbb{Z} \oplus \mathbb{Z}$ determined by $(\pm p, 0)$ and $(0, \pm q)$.

♣Better recheck. There is a related paper by Dijkgraaf-Verlinde - need to cite this. ♣

Answer Within The Framework Of Maxwell-Chern-Simons Theory

As discussed at length in [26], deriving the lowest Landau Level wavefunctions in the framework of Chern-Simons theory actually selects out a preferred radius. Let us return to (2.199) and substitute the formula for the wavefunctions in the LLL. We get:

$$\bar{\Psi}(A) := \mathcal{N} e^{-4\pi k \mathrm{Im}\tau A_z A_{\bar z}} \sum_{\omega \in \mathcal{H}^1_Z} e^{-4\pi k \mathrm{Im}\tau \omega_z \omega_{\bar z}} e^{-8\pi k \mathrm{Im}\tau \omega_{\bar z} A_z} \psi(A_z + \omega_z) \tag{2.336}$$

We can recognize this as the instanton sum for a periodic scalar field (with both left- and right-moving degrees of freedom) at a specific radius, namely,

$$S = \frac{k}{4\pi} \int d\phi * d\phi \tag{2.337}$$

where $\phi \sim \phi + 2\pi$ with a chiral coupling to $A^{1,0}$:

$$2\mathrm{i}k \int \bar\partial \phi \wedge A^{1,0} \tag{2.338}$$

We can unpack this information by matching the partition function of the boson on the torus with this action with (2.336). The radius is $r^2 = k\ell_s^2$. For further discussion, including the nontrivial generalization to a torus gauge group see [26].

Answer Within The Framework Of Topological Field Theory: Surface Defects

Nontrivial automorphisms of the fusion rules can be interpreted as defining surface defects within the three-dimensional theory. See [35].

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EXPLAIN MORE ABOUT THE KAPUSTIN-SAULINA PICTURE

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2.4.2 The Free Fermion Radius: Level $1/2$ Theta Functions And Spin Structures On The Torus

It is clear from (2.310) that there are radii of special interest:

1. There is the renowned $r \to 1/r$ T-duality symmetry, so we need only look at $r \ge 1$.

2. At the fixed point of T-duality, $r = 1$ there is an enhanced $SO(4) = (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ symmetry.

3. At the radius $r^2 = 2$ we have $\sqrt{2pq} = 2$ so that we have holomorphic operators $e^{\pm \frac{I}{2} X_L(z)}$ of conformal dimension $1/8$ and their squares

$$\psi(z) = e^{\mathrm{i}X_L(z)} \qquad \bar{\psi}(z) = e^{-\mathrm{i}X_L(z)} \tag{2.339}$$

of dimension $h = 1/2$.

4. At the radius $k = 6$ the theory actually has $N = 2$ supersymmetry. The $r = \pm 6$ representations give

$$G^{\pm}(z) \sim e^{\pm \mathrm{i}\sqrt{3}X} \tag{2.340}$$

which, remarkably, satisfy the $N = 2$ superconformal algebra.

The last two examples motivate the idea of introducing a super-chiral algebra. In the free fermion case we have

$$\begin{aligned}\psi(z_1)\psi(z_2) &\sim (z_1 - z_2) : \psi(z_1)\psi(z_2) : \\ \psi(z_1)\bar{\psi}(z_2) &\sim (z_1 - z_2)^{-1} + : \partial\psi\psi : + \mathcal{O}(z_1 - z_2)\end{aligned} \tag{2.341}$$

Thus, the ψ and $\bar{\psi}$ "almost" form a chiral algebra: Their correlators are single valued but there is nontrivial braiding: The ordering of the ψ and $\bar{\psi}$ in the correlators matters.

3. In any case, there is a "double cover" of the Gaussian model at this special "free fermion radius" . However, rather obviously to define correlators we must define spin structures.

4. Perhaps the most obvious case of this is in the computation of the partition function. First of all we must choose fermion boundary conditions for

$$\psi(z) = \sum d_r z^{-r-1/2} dz^{1/2} \tag{2.342}$$

when $z = e^{\mathrm{i}\phi + t}$ as a function of ϕ.

5. As explained in Appendix G there are two spin structures on the circle, the *Neveu-Schwarz*(NS) and *Ramond*(R) spin structures corresponding to anti-periodic and periodic boundary conditions on ψ respectively. In terms of the mode expansion NS corresponds to $r \in \mathbb{Z} + \frac{1}{2}$ and R corresponds to $r \in \mathbb{Z}$.

6. $\mathrm{Tr}e^{-\beta H}$ naturally chooses AP bc's. [See any QFT textbook, or demonstrate it].

7. Altogether we have <u>four</u> natural torus partition functions, corresponding to the four spin structures on the torus (Convention: $Z(g_t, g_s; \tau)$ for twisted boundary conditions on the torus.)

♣Get the box macro and use it for torus with boundary conditions ♣

$$\begin{aligned}Z(+,+) &= \frac{\vartheta_1(z,\tau)}{\eta(\tau)} \\ Z(-,+) &= \frac{\vartheta_2(z,\tau)}{\eta(\tau)} \\ Z(-,-) &= \frac{\vartheta_3(z,\tau)}{\eta(\tau)} \\ Z(+,-) &= \frac{\vartheta_4(z,\tau)}{\eta(\tau)}\end{aligned} \tag{2.343}$$

ETC.

Here we have introduced the half-integer level theta functions.

Define:

♣Depending on how much we said in the section on the charged particle round the ring we just refer back. ♣

$$\vartheta[{\theta \atop \phi}](z|\tau) := \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\theta)^2+2\pi i(n+\theta)(z+\phi)} \tag{2.344}$$

$$\begin{aligned}
\vartheta_1(z|\tau) &:= \vartheta[{1/2 \atop 1/2}](z|\tau) = 2q^{1/8}\cos[\pi(z+\frac{1}{2})] + 2q^{9/8}\cos[3\pi(z+\frac{1}{2})] + \cdots \\
\vartheta_2(z|\tau) &:= \vartheta[{1/2 \atop 0}](z|\tau) = 2q^{1/8}\cos(\pi z) + 2q^{9/8}\cos(3\pi z) + \cdots \\
\vartheta_3(z|\tau) &:= \vartheta[{0 \atop 0}](z|\tau) = 1 + 2\cos[2\pi z]q^{1/2} + 2\cos[4\pi z]q^2 + \cdots \\
\vartheta_4(z|\tau) &:= \vartheta[{0 \atop 1/2}](z|\tau) = 1 - 2q^{1/2}\cos[2\pi z] + 2q^2\cos[4\pi z] + \cdots
\end{aligned} \tag{2.345}$$

8. Discuss the three index 3 subgroups of $PSL(2,\mathbb{Z})$ that preserve the three even spin structures. Use this to explain the S and T-transformations of the above theta functions.

♣Also give infinite product representation. All these formulae should go into an appendix. ♣

9. Comment on spin operators and their correlators.

2.5 Spin Theories

In important variation on Chern-Simons theories are the *spin-Chern-Simons theories*.

In this case, M_3 has, in addition to an orientation, also a *spin structure* and the Chern-Simons action really depends on the spin structure.

See Appendix G for mathematical background on spin structures.

We can define the Chern-Simons invariant on a closed 3-manifold by a bordism of the gauge bundle to zero. This means that we find a four-manifold W_4 such that

$$\partial W_4 = M_3 \tag{2.346}$$

and we extend the bundle over W_4. Once the bundle extends there is no problem extending the connection, by a partition of unity argument. Now given this extra data we can lift the Chern-Simons invariant

$$CS(A) = \frac{1}{(2\pi)}\int_{M_3} AdA \in \mathbb{R}/2\pi\mathbb{Z} \tag{2.347}$$

to an actual real number

$$\widehat{CS}(A; W_4) = 2\pi \int_{W_4} \frac{F}{2\pi}\wedge\frac{F}{2\pi} \in \mathbb{R} \tag{2.348}$$

There can be different choices of W_4 and the differences

♣COMMENT ON DEPENDENCE ON EXTENSION OF A TO W_4 ♣

$$(\widehat{CS}(A; W_4) - \widehat{CS}(A; W_4')) = 2\pi \int_{Z_4} \frac{F}{2\pi}\wedge\frac{F}{2\pi} \in 2\pi\mathbb{Z} \tag{2.349}$$

are the integral over a closed four-manifold Z_4 (gotten by gluging W_4 to W_r') of the wedge product of representatives of the first Chern class of the bundle. This is the intersection of an integral form and hence is an integer. For example, if $Z_4 = \mathbb{CP}^2$ and $[F/2\pi] = c_1(H)$ where H is the basic tautological hyperplane bundle then the integer is $+1$.

We would like to divide the CS action by 2 to define half-integer level Chern-Simons theory. But this will lead to an ill-defined action. The above example of $Z_4 = \mathbb{CP}^2$ shows that this can be the case.

It is "only" a sign, since we are "only" trying to take a square-root of $\exp[\mathrm{i}CS(A)]$ but sloppiness here will lead to serious errors.

$H^2(Z_4;\mathbb{Z})$ is an integral lattice. In general given an integral lattice Λ, a vector c such that

$$v^2 = c \cdot v \bmod 2 \tag{2.350}$$

is called a *characteristic vector*. There are always infinitely many characteristic vectors and any two, say, c, c' differ by a vector divisible by 2. That is, $c - c' \in 2\Lambda$. When you have a characteristic vector $\frac{1}{2}v \cdot (v + c)$ is an integer.

It turns out that any integral lift $\widehat{w_2}$ of $w_2 \in H^2(Z_4;\mathbb{Z}_2)$ is a characteristic vector. This motivates the definition of the half-integer Chern-Simons term:

We choose a spin structure $\mathfrak{s}$ on W_4 and set

$$\frac{1}{2}\widehat{CS}(A;W_4;\mathfrak{s}) := 2\pi \int_{W_4} \frac{1}{2}\frac{F}{2\pi} \wedge (\frac{F}{2\pi} + \widehat{w_2}) \in \mathbb{R} \tag{2.351}$$

♣You are using $\widehat{w_2}$. Need to explain that different lifts for the same spin structure don't matter. ♣

Now, if we have two extensions so that all the data (including the spin structure) agree on the common boundary then

$$\frac{1}{2}\widehat{CS}(A;W_4;\mathfrak{s}) - \frac{1}{2}\widehat{CS}(A;W_4';\mathfrak{s}') = 2\pi \int_{Z_4} \frac{1}{2}\frac{F}{2\pi} \wedge \left(\frac{F}{2\pi} + \widehat{w_2}\right) \in 2\pi\mathbb{Z} \tag{2.352}$$

as desired. We can therefore define the spin Chern-Simons action to be

$$\exp[2\pi\mathrm{i}\frac{1}{2}CS(A;\mathfrak{s})] := \exp\left[2\pi \int_{W_4} \frac{1}{2}\frac{F}{2\pi} \wedge (\frac{F}{2\pi} + \widehat{w_2})\right] \tag{2.353}$$

The price we have paid is:

1. Restricting attention to W_4 with a choice of spin structure. This is not a serious restriction since the spin bordism group in three-dimensions is trivial, as is the bordism group of a spin manifold with principal $U(1)$ bundle. (In higher dimensions that kind of restriction can become quite significant.)

2. We need to include the extra data of an explicit lift $\widehat{w_2}$ of w_2 to a closed 2-form on W_4.

3. In a neighborhood of the boundary $M_3 = \partial W_4$, since M_3 admits a spin structure, it must be that $w_2(M_3)$ is trivializable. Therefore, in the neighborhood we can trivialize $\widehat{w_2} = d\eta$. However, η will not vanish on the boundary ∂W_4. The value of η depends on the choice of spin structure $\mathfrak{s}$ on M_3. Two different choices of spin structure $\mathfrak{s}, \mathfrak{s}'$ differ by an element of $\Delta\mathfrak{s} \in H^1(M_3;\mathbb{Z}_2)$. Now, $\Delta\mathfrak{s}$ can be lifted to a closed one-form $\Delta\eta$ with integer periods. Then $\eta(\mathfrak{s}) - \eta(\mathfrak{s}') = \Delta\eta$ can have nontrivial periods, although only the value of these periods modulo two is meaningful.

4. From the definition of the spin Chern-Simons term it is easy to see how it depends on spin structure. If we shift by $\epsilon \in H^1(M_3; \mathbb{Z}/2\mathbb{Z})$ then we can prove

$$\exp[2\pi \mathrm{i}\frac{1}{2}CS(A; \mathfrak{s}+\epsilon)] = \exp[\mathrm{i}\pi \int_{M_3} \epsilon \cup c_1(L)] \cdot \exp[2\pi \mathrm{i}\frac{1}{2}CS(A;\mathfrak{s})] \tag{2.354}$$

The pre-factor on the RHS is "just" a sign. To prove (2.354) note that the shift of $\eta(\mathfrak{s})$ by $\Delta\eta$ descends to ϵ and $\frac{F}{2\pi}$ is a representative of $c_1(L)$ where $L \to M_3$ is the principal $U(1)$ bundle over M_3. So the change of the exponentiated action, which is just a factor of

$$\exp[\mathrm{i}\pi \int_{M_3} \Delta\eta \wedge \frac{F}{2\pi}] \tag{2.355}$$

can just be written as a cup product in cohomology:

$$\exp[\mathrm{i}\pi \int_{M_3} \Delta\eta \wedge \frac{F}{2\pi}] = \exp[\mathrm{i}\pi \int_{M_3} \epsilon \cup c_1(L)] \tag{2.356}$$

****

Comment: Phase of fermion path integral in 3d. APS AND ETA EXPLAIN INDEX THEOREM ARGUMENT

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2.6 Generalization: Torus Gauge Group

Let us generalize the gauge group to be $U(1)^d$. Then locally there are d connection forms, locally written as A^I, $I = 1, \dots, d$, and the general Chern-Simons action is

♣Can't the $U(1)$ factors mix through monodromy? ♣

$$S = \frac{1}{4\pi}\int_{M_3} K_{IJ} A^I dA^J \tag{2.357}$$

where K^{IJ} is a symmetric integral matrix.

1. K^{IJ} determines an integral lattice Λ.

2. For nonspin theories, the diagonal matrix elements K^{II} must be even. So we are speaking of even integral lattices.

3. For spin theories, we allow the diagonal matrix elements K^{II} to be odd.

4. Spin structure dependence: Also need to choose a characteristic vector c_Λ for Λ. Then $c_\Lambda \otimes \widehat{w}_2$ will be a characteristic vector for $\Lambda \otimes H^2(Z_4;\mathbb{Z})$ and hence we write the action:

$$\exp\left[2\pi \mathrm{i}\int_{W_4} \frac{1}{2}K_{IJ}\frac{F^I}{2\pi} \wedge \left(\frac{F^J}{2\pi} + c^J \otimes \widehat{w}_2\right)\right] \tag{2.358}$$

5. Wilson operators: Character group $\mathrm{Hom}(U(1)^d, U(1)) \cong \mathbb{Z}^d$ so define

$$W(\vec{n},\gamma,f) = \exp[2\pi \mathrm{i}\oint_\gamma n_I A^I] \tag{2.359}$$

The singular gauge transformation ♣NEED TO DISCUSS FRAMING DEPENDENCE ♣

$$A^I \to A^I + \ell^I d\phi \tag{2.360}$$

where ϕ is the azimuthal angle around γ induces the identification

$$W(\vec{n}, \gamma, f) \sim W(\vec{n} + K\vec{\ell}, \gamma, f) \tag{2.361}$$

6. Coupling to external gauge field: "Spin-charge relation" of [9], see also [55].

Quantum equivalence statement of Belov-Moore:

1. Discriminant group with quadratic refinement.

2. Gauss-Milgram formula. Need lift to $c$mod24.

3. Some nontrivial equivalences: $II^{1,1}$ (Witten) and the Niemeier lattices. Revisit Chetan Nayak's paper.

Boundary conditions:

1. Nontopological: edge states.

when discriminant group has a square root: equivalence to finite group gauge theory [Maldacena-Moore-Seiberg; Banks-Seiberg].

1-form symmetries.

Time reversal.

Extra outer automorphisms: Automorphisms of the lattice Λ act as outer automorphisms of phase space. How do they act on the Hilbert space?

****

2.6.1 Applications To the FQHE

how much of FQHE to recall?

the "statistical gauge fields"

duality. densities of quasiparticles.

2.6.2 Coupling To External Abelian Gauge Fields

Witten's Sp(2n,Z) action.

3. Chern-Simons Theories For NonAbelian Gauge Fields

3.1 Some Chern-Weil Theory

More details in Appendix I. Here we just summarize some key points.

We will eventually want to talk about an arbitrary compact Lie group.

For this section, let G be a compact, connected, simple Lie group with Lie algebra $\mathfrak{g}$.

We consider a principal G-bundle $P \to M$ over a manifold M of dimension ≥ 4.

We introduce an Ad-invariant bilinear form on $\mathfrak{g}$

$$\mathrm{tr}\,(x, y) = -\frac{1}{2h^\vee}\mathrm{Tr}_{\mathfrak{g}} Ad(x)Ad(y) \tag{3.1}$$

where $h^\vee$ is the dual Coxeter number. This is a good normalization because for compact groups it is positive definite and for simple Lie algebras the norm-square of the longest coroots is $+2$. ♣CHECK! CHECK! ♣

****

As a sanity-check, let us consider the case of $\mathfrak{g} = \mathfrak{su}(2)$:

****

Another good aspect of this is that when G is in addition simply connected then

$$\wp(A) := \frac{1}{8\pi^2} \mathrm{tr}\, F^2 \tag{3.2}$$

will have integral periods on all four-cycles, and moreover there will exist bundles and four-manifolds where the period is 1.

Now, as explained in detail in Appendix I for two connections on P we have a well-defined form $CS(A_1, A_2) \in \Omega^3(M)$ so that

$$dCS(A_1, A_2) = \wp(A_1) - \wp(A_2) \tag{3.3}$$

If $A_2 = A$ and $A_1 = A + \alpha$ where $\alpha \in \Omega^1(M; \mathrm{ad}P)$ is a globally well-defined section of the adjoint bundle $\mathrm{ad}P$ then

$$\mathrm{tr}\, F(A+\alpha)^2 - \mathrm{tr}\, F(A)^2 = d\left\{ \mathrm{tr}\, \left(2\alpha F + \alpha D_A \alpha + \frac{2}{3}\alpha^3 \right) \right\} \tag{3.4}$$

Informally, this is ♣Spell out $D_A\alpha$ ♣

$$\delta\left(\mathrm{tr}\, F^2\right) = 2d\left(\mathrm{tr}\, \delta A F\right) \tag{3.5}$$

So:

$$CS(A+\alpha, A) = \frac{1}{8\pi^2} \mathrm{tr}\, \left(2\alpha F + \alpha D_A \alpha + \frac{2}{3}\alpha^3 \right) \tag{3.6}$$

Now we would like to write a Chern-Simons action which depends on a single gauge field A. Informally, we would like to study the exponentiated action:

$$\exp[2\pi \mathrm{i} \frac{k}{8\pi^2} \int_{M_3} \mathrm{tr}\, \left(AdA + \frac{2}{3}A^3 \right)] \tag{3.7}$$

So, why not just consider $CS(A, 0)$? If the principal G bundle $P \to M_3$ is nontrivial then there is no such thing as the $A = 0$ connection on P. Only when P is trivial can one take the trivial connection. In general the 3-form $\mathrm{tr}\, \left(AdA + \frac{2}{3}A^3\right)$ will not be a well-defined form over M_3. Nevertheless, there is a well-defined meaning to the integrated expression

$$\frac{1}{8\pi^2} \int_{M_3} \mathrm{tr}\, \left(AdA + \frac{2}{3}A^3 \right) \tag{3.8}$$

provided we only consider it as an element of $\mathbb{R}/\mathbb{Z}$. Consequently, the constant k must be an integer. We can define (3.8) as an element of $\mathbb{R}/\mathbb{Z}$ in three different ways (all of which will agree):

1. We write M_3 as the boundary of an oriented 4-fold W_4 and extend the bundle and connection to W_4. (There are no topological obstructions to the existence of such extensions.) Then we define

$$\int_{M_3} \frac{1}{8\pi^2} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) := \int_{W_4} \wp(A) \qquad \in \mathbb{R} \tag{3.9}$$

The problem here is that there are many potential extensions and choices of W_4. The difference between any two choices will be:

$$\int_{W_4} \wp(A) - \int_{W_4'} \wp(A') = \int_{Z_4} \wp(A) \in \mathbb{Z} \tag{3.10}$$

where Z_4 is a closed four-manifold. Since $\wp$ is normalized so that we can get any integer (for G simply connected), these periods will be integers. So the definition is only independent of the choice of extension as an element of $\mathbb{R}/\mathbb{Z}$.

2. While there is no globally-defined $\mathfrak{g}$-valued connection one-form on M_3 there is such a connection on the total space of the principal G-bundle. Therefore, there is a well-defined form $CS(A)$ on the total space of P. [29] Now, if we could choose a section $s : M \to P$ then we could pull back and integrate $\int s^*CS(A)$. There are two problems with trying to do this: First, if $P \to M$ is a nontrivial bundle there is no continuous section. This is not too serious because we can remove a set of measure zero in M where the connection is smooth and trivialize the bundle on the complement. However, the resulting number will depend on how we trivialized. If s_1, s_2 are two different trivializations they will differ by $s_1 = s_2 \cdot g$ where g is an automorphism (gauge transformation) of P. In the simplest (and most common) cases that means $g : M \to G$ is a well-defined map. Now, when A is globally well-defined so that the 3-form $CS(A)$ makes sense one can easily check the gauge transformation property:

$$CS(A^g) - CS(A) = CS(g^{-1}dg) + d\left(\frac{1}{8\pi^2}\mathrm{tr}\, Adgg^{-1}\right) \tag{3.11}$$

and

$$CS(g^{-1}dg) = -\frac{1}{24\pi^2}\mathrm{tr}\,(g^{-1}dg)^3 \tag{3.12}$$

Therefore $\int s^*CS(A)$ will change by $-\frac{1}{24\pi^2}\int \mathrm{tr}\,(g^{-1}dg)^3$ and the period of this form on M will be a nonzero integer. Thus, $\int_{M_3} CS(A)$ makes sense, but only as an element of $\mathbb{R}/\mathbb{Z}$.

♣FACTOR OF TWO CONFUSION. IT SEEMS THAT WITH OUR NORMALIZATIONS THIS IS ALWAYS AN EVEN INTEGER!!! ♣

How does the Chern-Simons action itself vary? Again, using formulae from the Appendix we have

$$\frac{d}{dt}CS(A + \alpha(t), A) = \frac{1}{8\pi^2}\left[2\mathrm{tr}\,\dot\alpha(t)F(A + \alpha(t)) - d\left(\mathrm{Tr}\alpha(t)\dot\alpha(t)\right)\right] \tag{3.13}$$

[29] A more elegant way to construct the globally defined Chern-Simons form on the total space of P is the following: If $\pi : P \to M$ is a principal G bundle then the principal G bundle $\pi^*(P) \to P$ is canonically trivializable since $s(p) = (p, p)$ is a global section. If A is a connection on $P \to M$ it will pull back to a connection on $\pi^*(P) \to P$ and we can define $CS(\pi^*(A), 0)$ as a globally well-defined form $CS(A)$ on the total space of P.

Again, this is usually informally written as:

$$\delta CS(A) = \frac{1}{8\pi^2}\left[2\mathrm{tr}\,\delta A F - d\left(\mathrm{tr}\left(A\delta A\right)\right)\right] \tag{3.14}$$

****

NEED TO COMMENT ON SPIN THEORIES AGAIN IN THE NONABELIAN CASE

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3.1.1 The Chern-Simons Action On M_3 With A Nonempty Boundary

EXPLAIN THAT THE ACTION MUST NOW BE CONSIDERED A SECTION OF A LINE BUNDLE

3.2 The Semiclassical Approximation To The Chern-Simons Path Integral

So, we now consider the formal path integral generalizing (3.15)

$$\int_{\mathcal{A}/\mathcal{G}} [dA] \exp[\mathrm{i}\frac{k}{4\pi}\int_{M_3} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)] \prod_\alpha W(R_\alpha, \gamma_\alpha) \tag{3.15}$$

For the moment we take γ_α to be closed curves and M_3 to have no boundary.

Formally we expect this to be a topological field theory so it should only depend on the topology of M_3 and that of the link $\amalg_\alpha \gamma_\alpha$.

From our experience with the $U(1)$ case we expect, of course, some anomalies in the topological invariance.

In the semiclassical approximation $k \to \infty$ we should expand around the solutions of the equations of motion. These are the flat connections on M_3. In sharp contrast to simple cases like $\Sigma_2 \times S^1$ there will, in fact, tend to be just isolated flat connections.

****

Examples

1. *Rational homology spheres*: Take $M_3 = S^3/\Gamma$ where Γ is a discrete subgroup of $SU(2)$. Then $\pi_1(M_3) \cong \Gamma$. If $G = SU(2)$ then, up to conjugation, there is a unique subgroup, so the flat connections are isolated.

2. *Mapping torus.* Let S be a surface and $f : S \to S$ a diffeomorphism. Then we can form the mapping torus $M_f = (S \times [0,1])/\sim$ where $(x, 0) \sim (f(x), 1)$. This can be viewed as a fibration over a circle by Riemann surfaces. AGOL THEOREM ON HYPERBOLIC STRUCTURES. NEED TO SAY SOMETHING ABOUT FUNDAMENTAL GROUP IN THIS CASE.

3. *Heegaard splittings.*

3.2.1 Semiclassical Approximation: The Framing Anomaly Revisited

We now consider the $k \to \infty$ asymptotics. Since $\hbar \sim 1/k$ this is the semiclassical approximation. We are following the discussion in Witten's paper [67], with a few modifications.

We should expand around the solutions to the equations of motion. These are the flat connections. For simplicity, we assume the flat connections are isolated - up to gauge equivalence.

So let us consider expanding around an isolated flat connection A_*. We write $A = A_* + \alpha$, and, using (3.6) the action is

$$\frac{k}{4\pi}\int_{M_3} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) = \frac{k}{4\pi}\int_{M_3} \mathrm{tr}\left(A_* dA_* + \frac{2}{3}A_*^3\right) + \frac{k}{4\pi}\int \mathrm{tr}\left(\alpha D_{A_*}\alpha + \frac{2}{3}\alpha^3\right) \tag{3.16}$$

The first term on the RHS is the classical action $S(A_*)$. We then have a nondegenerate quadratic term and an interaction. To do the one loop approximation we introduce BRST ghosts exactly as in equation *** above and introduce a metric by using the gauge fixing condition

$$D_A * \alpha = D_A^i \alpha_i \mathrm{vol}\,(g) = 0 \tag{3.17}$$

leading to the quadratic action:

$$\exp\left[\mathrm{i}\frac{k}{4\pi}\int_{M_3} \mathrm{tr}\,\left(\alpha D_{A_*}\alpha + H D_{A_*}\alpha + b D_{A_*} * D_{A_*} c\right)\right] \tag{3.18}$$

here $\alpha \in \Omega^1(M_3; \mathrm{ad}P)$ while $H, b, c \in \Omega^0(M_3; \mathrm{ad}P)$ with b, c anticommuting fields, while H is commuting.

Now, because A_* is a flat connection $D_{A_*} : \Omega^k(M_3; \mathrm{ad}P) \to \Omega^{k+1}(M_3; \mathrm{ad}P)$ squares to zero:

$$(D_{A_*})^2 = 0 \tag{3.19}$$

and therefore D_{A_*} serves as a differential. So the situation is very closely analogous to the Abelian case discussed in section 2.2.17 above. The Hodge theory discussion of Appendix F goes through in the same way and we have the one-loop determinant

$$\frac{\det'(*D_{A_*} * D_{A_*})}{\det^{1/2}\mathcal{D}} \tag{3.20}$$

where

$$\mathcal{D} : \Omega^{\mathrm{odd}}(M_3; \mathrm{ad}P) \to \Omega^{\mathrm{odd}}(M_3; \mathrm{ad}P) \tag{3.21}$$

is defined by $\mathcal{D} = *D_{A_*} + D_{A_*}*$. The Hodge theory arguments of appendix F show that the absolute value of (3.20) is a topological invariant, but the phase turns out to be continuously metric dependent.

♣NEED REF TO ALBERT SCHWARZ SOMEWHERE ♣

To investigate the metric dependence we interpret $\mathcal{D}$ as a twisted Dirac operator. Recall that antisymmetric tensors can be regarded as bispinors:

$$S \otimes S \cong \oplus_k \Lambda^k T^* M_3 \tag{3.22}$$

Under this isomorphism, the Dirac operator coupled to spinors becomes a first order differential operators on differential forms and in fact becomes $d * + * d$. The same is true if we couple to ad$P$.

Now, there is a general formula for the phase of 3d Dirac operator [4, 72]

$$\mathrm{Det}\gamma \cdot D = |\mathrm{Det}\gamma \cdot D| e^{\frac{i\pi}{2}\eta(\gamma\cdot D)} \tag{3.23}$$

****

NEED TO DEFINE ETA INVARIANT

EXPLAIN ARGUMENT FROM [4]

MAYBE EXPLAIN PHASE OF FERMI DET AND ETA EARLIER????

****

The η invariant $\eta(\mathcal{D})$ depends on both the flat connection A_* and the metric. The dependence on A_* can be evaluated by considering a one-parameter family of connections $A(s)$ from A_* to the trivial connection $A = 0$. (Since we have flat connections the bundle P is trivializable.) We then apply the APS index theorem to the four-manifold $M_3 \times [0, 1]$ where s is a coordinate along the interval. Denoting $\eta(\mathcal{D})$ by $\eta(A_*, g_{\mu\nu})$ to emphasize the dependence on these variables we get from the APS theorem:

♣Wait a minute! Only true if G is simply connected. There is a gap here!! ♣

♣NEED TO SUPPLY SOME DETAILS HERE!!! ♣

$$\frac{\pi}{2}\eta(A_*, g_{\mu\nu}) - \frac{\pi}{2}\eta(0, g_{\mu\nu}) = \frac{1}{2}c_2(G)2\pi \int_{M_3} CS(A_*) \tag{3.24}$$

The RHS is a topological invariant, once again. However, we must now confront the metric dependence of $\eta(0, g_{\mu\nu})$. So we consider a one-parameter family of metrics $g_{\mu\nu}(s)dx^\mu \otimes dx^\nu$ and once again apply the APS index theorem to the four-manifold $M_3 \times [0, 1]$ where s is a coordinate along the interval. Now, when we have set the gauge field to $A = 0$ we have decoupled the gauge indices so

$$\eta(0, g_{\mu\nu}) = (\dim G)\eta(g_{\mu\nu}) \tag{3.25}$$

where $\eta(g_{\mu\nu})$ is the eta invariant for the operator $*d + d*$ restricted to odd forms. Now the APS index theorem gives us:

$$\mathrm{Index}\gamma \cdot D = \int_{M_3\times[0,1]} \hat{A} - \frac{1}{2}[\eta(g^{(1)}_{\mu\nu}) - \eta(g^{(1)}_{\mu\nu})] \tag{3.26}$$

but

$$\int_{M_3\times[0,1]} \hat{A} = -\frac{1}{24}\int_{M_3\times[0,1]} p_1(R) = -\frac{1}{24}\int_{M_3\times[0,1]} p_1(R) \tag{3.27}$$

where

$$p_1(R) = -\frac{1}{8\pi^2}\mathrm{Tr}_{vector} R^2 \tag{3.28}$$

is the Chern-Weil representative of Pontryagin class. Then

$$\int_{[0,1]} \mathrm{Tr}_{vector} R^2 = CS(\omega^1, \omega^2) \tag{3.29}$$

where ω is the spin connection for the Levi-Civita connection so $R = d\omega + \omega^2$. Informally,

$$CS(\omega) = \mathrm{Tr}_{vector}(\omega d\omega + \frac{2}{3}\omega^3) \tag{3.30}$$

It follows that, if we trivialize the tangent bundle so we can make global sence of $CS(\omega)$ then

$$\mathcal{F} := \frac{1}{2}\eta(g_{\mu\nu}) + \frac{1}{24}\int_{M_3} \frac{1}{8\pi^2}\mathrm{Tr}_{vector}(\omega d\omega + \frac{2}{3}\omega^3) \tag{3.31}$$

♣NEED TO EXPLAIN TO WHAT EXTENT YOU CAN INDEED TRIVIALIZE THE TANGENT BUNDLE ♣

is metric-independent. But $\mathcal{F}$ does depend on the framing of the tangent bundle. Given one framing and a map $\rho : M_3 \to SO(3)$ we can produce another one. The Chern-Simons invariant of the spin connection changes by a multiple of the winding number $w(\rho)$ of ρ and so

$$\mathcal{F} \to \mathcal{F} + \frac{w(\rho)}{12} \tag{3.32}$$

Remarks

1. Witten argues that we should add a local counterterm proportional to the gravitational Chern-Simons action to eliminate the metric dependence. Then the one-loop approximation becomes:

$$Z \sim_{k\to\infty} e^{\mathrm{i}\pi\mathcal{F}} \sum_{A_*} T(A_*) e^{2\pi\mathrm{i}(k+\frac{1}{2}C_2(G))\int_{M_3} CS(A_*)} \tag{3.33}$$

where $T(A_*)$ is the analytic torsion of D_{A_*}.

2. The phase shifts by

$$e^{\mathrm{i}\pi\mathcal{F}} \to e^{2\pi\mathrm{i}\frac{d}{24}w(\rho)} e^{\mathrm{i}\pi\mathcal{F}} \tag{3.34}$$

Now, from the relation to RCFT described below we actually expect that this shift should be

$$e^{\mathrm{i}\pi\mathcal{F}} \to e^{2\pi\mathrm{i}\frac{c}{24}w(\rho)} e^{\mathrm{i}\pi\mathcal{F}} \tag{3.35}$$

where c is the central charge of the WZW theory:

$$c = \frac{k}{k+h^\vee}\dim G \tag{3.36}$$

There is no immediate contradiction since we are looking at the leading $k \to \infty$ asymptotics, but it does raise the question of the metric dependence of the $1/k$ corrections.

3. Thus, the choices of topological data that the path integral depends on include: A choice of orientation and a choice of framing. When we include Wilson lines, we will need to includings framings of the loops, as we have already seen in the Abelian case.

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Remarks from

1. Atiyah's paper on 2-framings.

2. Segal on riggings
3. p1 structures: Blanchet, Habegger, Masbaum, Vogel

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3.2.2 Higher Loops

Comment: Axelrod-Singer

3.3 Quantization In Some Important Cases

3.3.1 Quantization On $\Sigma_2 \times \mathbb{R}$ With Σ A Compact Surface

We now quantize the theory when $M_3 = \Sigma_2 \times \mathbb{R}$ where Σ_2 is an oriented surface. We therefore assume we have a principal G-bundle $P \to \Sigma_2$ and we will first quantize the space of all gauge fields $\mathcal{A} = Conn(P \to \Sigma_2)$ and then impose the Gauss law.

Recall we have symplectic form on $\mathcal{A}$:

$$\omega = \frac{k}{4\pi} \int_{\Sigma} \mathrm{tr}\ (\delta A_s \wedge \delta A_s) \tag{3.37}$$

giving Poisson brackets:

$$\{A_i^a(x), A_j^b(y)\} = \frac{2\pi}{k} \epsilon_{ij} \delta^{ab} \delta^{(2)}(x, y) \tag{3.38}$$

where $A = A_i^a t_a dx^i$ and t^a are an orthonormal basis with respect to the trace tr. The moment map for gauge transformations $\epsilon : \Sigma_2 \to \mathfrak{g}$ is

$$\mu(\epsilon) = -\frac{k}{2\pi} \int_{\Sigma} \mathrm{tr}\,(\epsilon F) \tag{3.39}$$

Now we need to quantize. A method available for all surfaces Σ_2 is to choose a complex structure, making it a Riemann surface. Locally we choose we can define a holomorphic coordinate $z = x^1 + \mathrm{i}x^2$ where $dx^1 dx^2$ is oriented and, also choosing a local trivialization of the bundle and defining $A = A_z dz + A_{\bar{z}} d\bar{z}$ we find

$$[A_z^a(x), A_{\bar{z}}^b(y)] = \frac{\pi}{k} \delta^{ab} \delta^{(2)}(x - y) \tag{3.40}$$

This induces a complex structure on the space $\mathcal{A}$:

$$T^{1,0}\mathcal{A} \cong \Omega^{1,0}(\Sigma_2; \mathrm{ad}P) \tag{3.41}$$

Moreover, we can make $\mathcal{A}$ into a Kähler manifold with the compatible metric:

$$g(\alpha, \alpha) = \frac{k}{4\pi} \int_{\Sigma_2} \mathrm{tr}\,(\alpha^{1,0} \alpha^{0,1}) \tag{3.42}$$

where $\alpha \in \Omega^1(\Sigma_2; \mathrm{ad}P)$ is a globally defined adjoint-valued 1-form. The pre-quantum line bundle $\mathcal{L}_k \to \mathcal{A}$ has Kähler potential:

$$K = \frac{k}{4\pi} \int_{\Sigma_2} \mathrm{tr}\,(A^{1,0} A^{0,1}). \tag{3.43}$$

Now, gauge transformations act symplectically and we can attempt to perform Kähler quantization. If we impose the constraints classically we are led to the Kähler quantization of the moduli space of flat gauge fields. This quickly leads one into the world of algebraic geometry. See 3.4 for some brief remarks. We will instead impose the constraints quantum-mechanically.

If $\Psi \in \Gamma[\mathcal{L}_k \to \mathcal{A}]$ we can view it, formally, as a holomorphic function $\Psi[A_z]$. We will need to impose the Gauss law

$$\mu(\epsilon)\Psi = 0 \tag{3.44}$$

and, using the quantization rule (3.40) we have a first order (functional) differential equation in A_z. We view

$$\frac{\delta}{\delta A_z(x)}\Psi \in T_x^{1,0} \otimes \mathfrak{g} \otimes \mathcal{L}_{k,A} \tag{3.45}$$

and then the operator equation:

$$F\Psi = 0 \tag{3.46}$$

becomes the first order differential equation

$$D_{A,z}\frac{\delta}{\delta A_z(x)}\Psi = -\frac{k}{\pi}\bar{\partial}_{\bar{z}} A_z(x)\Psi \tag{3.47}$$

As we will see, this differential equation will require Ψ to live in a finite-dimensional vector space. We will also interpret it as an equation for an anomaly in current conservation. ♣NEED TO COME BACK AND EXPLAIN WHY THIS IS A STATEMENT OF AN ANOMALY IN GLOBAL SYMMETRY IN THE WZW MODEL ♣

On the Hilbert space of "L^2-sections of $\mathcal{L}_k$" we will have a unitary operator representation of the group of gauge transformations, $\mathrm{Aut}(P)$. When P is trivializable this is just the group of maps $g : \Sigma_2 \to G$ (subject to suitable regularity conditions). We want to impose

$$U(g)\Psi = \Psi \tag{3.48}$$

where

$$(U(g) \cdot \Psi)(A_z) = e^{f(A_z,g)}\Psi((A^g)_z) \tag{3.49}$$

Note that, since $d + A^g = g^{-1}(d + A)g$ and this is a right-action on the space of connections we will have a left-action on the Hilbert space of sections.

A priori there could be a projective representation:

$$U(g_1)U(g_2) = c(g_1, g_2)U(g_1 g_2) \tag{3.50}$$

where there is some nontrivial cocycle $c(g_1, g_2)$, however, we want to impose (3.48) and therefore the cocycle should be trivial.

Now the differential equation (3.47) makes is plain that we cannot take $f(A_z; g) = 0$ in (3.49), for we can use it to compute the infinitesimal gauge variation: ♣There is a normalization ξ fixed by above conventions. Work it out. ♣

$$\begin{aligned} \Psi[A_z + D_z\epsilon] - \Psi[A_z] &= \xi \int_{\Sigma_2} \mathrm{tr}\left(D_{A,z}\frac{\delta}{\delta A_z(x)}\Psi\right)\frac{\mathrm{i}}{2}dzd\bar{z} \\ &= -\xi\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\left(\epsilon\bar{\partial}_{\bar{z}}A_z\right)\frac{\mathrm{i}}{2}dzd\bar{z}\Psi \\ &= \xi\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\left(\bar{\partial}_{\bar{z}}\epsilon A_z\right)\frac{\mathrm{i}}{2}dzd\bar{z}\Psi \end{aligned} \tag{3.51}$$

Assuming $\mathrm{tr}\,(A) = 0$ and $g = e^{\epsilon}$ we have:

$$\Psi[A_z^g] = \exp[\frac{\mathrm{i}\xi}{2}\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}g^{-1}\bar{\partial}g)]\Psi[A_z] \tag{3.52}$$

although at this order we could equally well have written

$$\Psi[A_z^g] = \exp[\frac{\mathrm{i}\xi}{2}\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}\bar{\partial}gg^{-1})]\Psi[A_z] \tag{3.53}$$

Combining the group law (3.50) (with trivial cocycle) with the group action (3.49) gives a consistency condition:

$$f(A_z, g_1) + f(A_z^{g_1}, g_2) = f(A_z, g_1g_2)\mathrm{mod}2\pi\mathrm{i}\mathbb{Z} \tag{3.54}$$

and moreover we know that for $g = e^{\epsilon}$

$$f(A_z, g) = \frac{\mathrm{i}\xi}{2}\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}g^{-1}\bar{\partial}g) + \mathcal{O}(\epsilon^2) \tag{3.55}$$

or

$$f(A_z, g) = \frac{\mathrm{i}\xi}{2}\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}\bar{\partial}gg^{-1}) + \mathcal{O}(\epsilon^2) \tag{3.56}$$

There is a unique solution to these equations that is affine-linear in A_z, that is, even when g is not close to $g = 1$ we say:

$$f(A_z, g) = \frac{\mathrm{i}\xi}{2}\frac{k}{\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}g^{-1}\bar{\partial}g) + S(g) \tag{3.57}$$

One derives a functional equation for $S(g)$ which can then be integrated, at least for g in the connected component of the identity of $Map(\Sigma_2, G)$. The result is known as the WZW action and it takes the form:

$$f^+(A, g) = -\mathrm{i}\xi k S^+(g) + \mathrm{i}\xi\frac{k}{2\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}g^{-1}\bar{\partial}g) \tag{3.58}$$

$$f^-(A, g) = \mathrm{i}\xi k S^-(g) + \mathrm{i}\xi\frac{k}{2\pi}\int_{\Sigma_2} \mathrm{tr}\,(A^{1,0}\bar{\partial}gg^{-1}) \tag{3.59}$$

where $S^{\pm}$ are the WZW actions:

$$S^{\pm} = \frac{1}{4\pi}\int_{\Sigma_2} \mathrm{tr}\,\left(g^{-1}\partial gg^{-1}\bar{\partial}g\right) \pm \frac{1}{12\pi}\int_{\mathcal{B}(g)} \mathrm{tr}\,(g^{-1}dg)^3 \tag{3.60}$$

The second term in (3.60) is the renowned WZ term. Here $\mathcal{B}(g) \subset G$ is a 3-chain in G such that

$$\partial\mathcal{B}(g) = g(\Sigma_2) \tag{3.61}$$

Despite appearances, this term leads to a local quantum field theory of the map $g : \Sigma_2 \to G$. Of course, the choice of $\mathcal{B}(g)$ is ambiguous, but [40, 11] for all compact, simple, connected, and simply connected groups G:

$$x_3 = [\frac{1}{48\pi^2 h^{\vee}}\mathrm{Tr}_{adj}(g^{-1}dg)^3] \tag{3.62}$$

generates the integral cohomology lattice in $H^3_{DR}(G)$, where $h^\vee$ is the dual Coxeter number. In particular, for $SU(2)$ we can take

$$x_3 = [\frac{1}{24\pi^2}\mathrm{Tr}_2(g^{-1}dg)^3] \tag{3.63}$$

and more generally for $SU(N)$ we can take

$$x_3 = [\frac{1}{24\pi^2}\mathrm{Tr}_N(g^{-1}dg)^3] \tag{3.64}$$

to generate the integral cohomoloy lattice in $H^3_{DR}(G)$. Therefore, by a standard argument, the choice of $\mathcal{B}(g)$ does not matter in defining the exponentiated action, provided k is an integer.

♣Our conventions had better give $\xi = \pm 1$. ♣

Finally, it follows from the above that we can write the projection operator Π onto gauge invariant states when applied to an arbitrary section $\Psi_0[A_z]$ is

$$(\Pi\Psi_0)[A_z] := \int_{Map(\Sigma_2 \to G)} e^{f^\pm(A_z,g)}\Psi_0[A_z^g] \tag{3.65}$$

Our projection operator is a path integral of a <u>two-dimensional</u> quantum field theory known as the WZW theory for the group G. We will explain more about this theory in section 4.

A particularly useful testfunction for the case of f^- is to take

$$\Psi_0[A_z] = \exp\left[\mathrm{i}\xi k\frac{1}{2\pi}\int \mathrm{tr}\,(J^{0,1}A^{1,0})\right] \tag{3.66}$$

where we choose some external $(0,1)$ form. Then

$$(\Pi\Psi_0)[A_z] := e^{-\mathrm{i}\xi k\frac{1}{2\pi}\int \mathrm{tr}\,(A^{1,0}J^{0,1})}\int_{Map(\Sigma_2\to G)} e^{\mathrm{i}\xi k S^-[g;A^{1,0},J^{0,1}]} \tag{3.67}$$

where $S^-[g;A,\bar{A}]$ is the gauged WZW action:

$$\begin{aligned} S^-[g;A,\bar{A}] &:= S^-(g)+ \\ &+\frac{1}{2\pi}\int_{\Sigma_2}\left[\mathrm{tr}\,(A\bar{\partial}gg^{-1}+\bar{A}g^{-1}\partial g - g^{-1}Ag\bar{A}+A\bar{A})\right] \end{aligned} \tag{3.68}$$

where A is the $(1,0)$ part of the connection and $\bar{A}$ is the $(0,1)$ part of the connection.

♣PUZZLE: USING f^+ DOESN'T COMBINE NICELY INTO GAUGED WZW. WHY NOT? ♣

Remarks

1. If G has a unitary structure, so that $g^\dagger = g^{-1}$ (e.g. if $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{C})$ is a Lie subalgebra of anti-Hermitian matrices) then, with our conventions

$$\mathrm{i}\int_{\Sigma_2}\mathrm{tr}\,\left(g^{-1}\partial g g^{-1}\bar{\partial}g\right) \tag{3.69}$$

is <u>negative definite</u>. Therefore if we want the path integral over maps $g:\Sigma_2\to G$ in (3.65) to be formally convergent we should choose f^+ for $\xi k<0$ and f^- for $\xi k>0$.

2. As we will see, the possible gauge invariant functionals $\Psi[A_z]$ we obtain from (3.65) form a finite-dimensional vector space. It is canonically the vector space of (vacuum) conformal blocks of the WZW model on Σ_2. This space depends on the choice of complex structure on Σ_2, but, again thanks to the Sugawara form of the energy-momentum tensor there is a projectively flat connection on moduli space. The above argument was first give in [17]. A more rigorous version was spelled out in [7].

3.3.2 Quantization On Punctured Surfaces

We now include Wilson lines stretching along the time direction in $M_3 = \Sigma_2 \times \mathbb{R}$. They are located at points $P_i \in \Sigma_2$ in irreducible representations of G, denoted $V(\lambda_i)$ where λ_i is a highest weight associated with the line at P_i.

****

EXPLAIN HOW TO REPRESENT THE WILSON LINE IN TERMS OF QUANTUM MECHANICS FOR COADJOINT ORBIT. COULD GET THIS FROM CHERN-SIMONS ACTION BY SINGULAR GAUGE TRANSFORMATION - OR BY OTHER MEANS

****

We consider the phase space

$$\mathcal{P}(E; (\lambda_i, P_i)) \equiv \mathcal{A}(E \to \Sigma_2) \times \prod_i Map(P_i, \mathcal{O}(\lambda_i)) \tag{3.70}$$

where $E \to \Sigma_2$ is a complex hermitian vector bundle and $\mathcal{A}(E \to \Sigma_2)$ is the space of unitary connections on E, P_i are points on Σ_2, and $\mathcal{O}(\lambda_i)$ are coadjoint orbits:

$$\mathcal{O}(\lambda_i) = \{\phi = g\lambda_i g^{-1}\} \subset \mathfrak{g}^* \tag{3.71}$$

$\mathcal{P}(E; (\lambda_i, P_i))$ is symplectic, and the gauge group $\mathcal{G}$ acts symplectically. The moment map for the gauge group action is

$$\mu = k\left[\frac{1}{2\pi}F(A) - \sum \mu^{(i)}(\phi(P_i))\eta(P_i)\right] \tag{3.72}$$

where $\mu^{(i)}$ are the moment maps for the orbits:

$$\langle T^A, \mu^{(i)}(g)\rangle = \mathrm{Tr} T^A g\lambda_i g^{-1} = \mathrm{Tr} T^A \phi \tag{3.73}$$

and $\eta(P_i)$ is a delta-function representative of the PD to P_i.

Definition 1: The space of conformal blocks,

$$\mathcal{H}(\mathfrak{g}, k; (P_i, \lambda_i)) \tag{3.74}$$

is the quantum Hilbert space obtained from quantization of the symplectic reduction: $\mu^{-1}(0)/\mathcal{G}$.

Remark: Note that if $(A; \phi_i) \in \mu^{-1}(0)$ then

$$Pexp \oint_{\gamma(P_i)} A = \phi_i(P_i) \tag{3.75}$$

where $\gamma(P_i)$ is a small loop around P_i. In particular, the conjugacy class of the holonomies around P_i are fixed by the data $\lambda_i \in \mathfrak{g}^*$.

<u>Kähler Quantization</u>

Now choose a complex structure on Σ_2 and suppose $k\lambda_i$ are in the weight lattice of $\mathfrak{g}$. Then $\mathcal{P}(E;(\lambda_i,P_i))$ becomes a Kähler manifold. It is better to represent it as

♣Careful: Do you want k or k+h ??? ♣

$$\mathcal{A}(E\to\Sigma_2)\times Map(P_i,\mathcal{O}_{k\lambda_i}) \tag{3.76}$$

The complex structure on $\mathcal{A}$ comes from the splitting into $(1,0)$ and $(0,1)$ pieces, where $\mathcal{A}^{(0,1)}$ denotes the space of $(0,1)$ parts of the unitary connection.

As is well-known, the coadjoint orbits are isomorphic as manifolds to homogeneous spaces. Moreover, they can be represented as a quotient of compact or of complex groups:

$$\mathcal{O}_{k\lambda_i}\cong G/H\cong G_{\mathbb{C}}/P(k\lambda_i) \tag{3.77}$$

where $P(k\lambda_i)$ is the parabolic subgroup associated to the weight $k\lambda_i$,

From this representation it is clear that the complexified gauge group $\mathcal{G}_{\mathbb{C}}$ acts. Let $\bar{A}$ be the $(0,1)$ part of the unitary connection. Then the $\mathcal{G}_{\mathbb{C}}$ action is

$$(\bar{A};\phi_i)\to(\bar{A}^h;h(P_i)\phi_i(P_i)h(P_i)^{-1}) \tag{3.78}$$

The description of this space makes a connection with the WZW theory. The standard approach first quantizes the coadjoint orbits. Then one is left with a holomorphic wavefunction valued in the representation spaces:

♣We need to switch conventions in next few paragraph: Wavefunction is a function of A_z. ♣

$$\Psi[\bar{A}]\in\prod Map(P_i,V(\lambda_i)) \tag{3.79}$$

where $V(k\lambda_i)$ is the representation of $\mathfrak{g}_{\mathbb{C}}$ associated to the weight. Moreover, in order to descend to the quotient and be a section of the line bundle we must have

$$\Psi[\bar{A}^{h^{-1}}]=e^{kS[h;0,\bar{A}]}\otimes\rho_{\lambda_i}(h(P_i))\Psi[\bar{A}] \tag{3.80}$$

From the symplectic point of view, this is the statement of the Gauss law $\mu=0$.

The solution of (3.80) is provided, as always, by writing the projection operator onto gauge invariant states. As for the vacuum conformal blocks above we get the WZW path integral with fields g in representations $V(\lambda_i)$ at the points P_i inserted:

$$\Psi(\bar{A})=\int[dg]e^{S_k[g]-\frac{ik}{4\pi}\int_{\Sigma_2}\mathrm{Tr}[g^{-1}\partial g\bar{A}]}\prod_i\rho_{\lambda_i}(g(P_i))\Psi_0[\bar{A}^g] \tag{3.81}$$

where Ψ_0 is any "test function."

3.3.3 Quantization On $D_2\times\mathbb{R}$, With D_2 A Disk: Nonabelian Edge States

We now consider the nonabelian analog of the edge state argument of section 2.3.1.

Boundary conditions must be such that

$$\int_{\partial D\times\mathbb{R}_t}\mathrm{tr}\,(\delta A\wedge A)=0 \tag{3.82}$$

We separate out the spatial and time differential $d = d_s + \partial_0 dx^0$ and similarly we separate out $A = A_s + A_0 dx^0$. As in the Abelian case A_0 multiplies the Gauss law so integrating over A_0, considered as a Lagrange multiplier, we enforce $F(A_s) = d_s A_s + A_s^2 = 0$. On the disk we can solve this equation

$$A_s = U^{-1} d_s U \tag{3.83}$$

where $U : D \times \mathbb{R} \to G$. Note that U can still depend on time. It is now very instructive to plug this back into the remainder of Chern-Simons action:

$$\exp[\mathrm{i}\frac{k}{4\pi}\int_{D\times\mathbb{R}} \mathrm{tr}\,(A_s \partial_0 A_s) dx^0] \tag{3.84}$$

We need the simple identity

$$\mathrm{tr}\,\left[(U^{-1}d_sU)\partial_0(U^{-1}d_sU)\right] dx^0 = -d\left[\mathrm{tr}\,(U^{-1}d_sU)(U^{-1}\partial_0 U)dx^0\right] + \frac{1}{3}\mathrm{tr}\,(U^{-1}dU)^3 \tag{3.85}$$

and hence the remainder of the Chern-Simons action, evaluated on the flat gauge field $A = U^{-1}dU$ is

$$\exp\left\{\mathrm{i}\left[-\frac{k}{4\pi}\int_{\partial D\times\mathbb{R}_t} \mathrm{tr}\,(U^{-1}\partial_\phi U U^{-1}\partial_0 U) d\phi\wedge dx^0 + \frac{k}{12\pi}\int_{D\times\mathbb{R}} \mathrm{tr}\,(U^{-1}dU)^3\right]\right\} \tag{3.86}$$

So we have recovered the “chiral” WZW action - note that it is first order in time derivatives, and hence, as usual, should be considered an action for paths in phase space.

****

GENERALIZE THIS TO INCLUDE A WILSON LINE IN THE CENTER THUS GETTING NONTRIVIAL MONODROMY

GET SYMPLECTIC FORM ON LG/T

****

3.4 Quantization Of The Moduli Space Of Flat Gauge Fields And Algebraic Geometry

It is also possible to impose the constraints first, that is, descend to the finite-dimensional moduli space of flat connections - view that as a symplectic space - and quantize that. This approach leads to some fairly nontrivial mathematics.

Again, it is most useful to give Σ_2 a complex structure. This induces a complex structure on $\mathcal{M}_{\mathrm{flat}}$. The way this works is that we can decompose the covariant derivative D into $D = D^{1,0} + D^{0,1}$. Because we are in one complex dimension $(D^{0,1})^2 = 0$. The tangent space to the space of flat connections can be identified with the kernel of $D^{0,1}$ so we have

$$T^{1,0}\mathcal{M}_{\mathrm{flat}} \cong H^{0,1}(E) \cong \ker D^{0,1}/guage \tag{3.87}$$

Then the prequantum line bundle is $\mathcal{L} = \mathrm{DET}(D^{0,1})$, the determinant line bundle.

By an important theorem of Narasimhan-Seshadri the holomorphic space $\mathcal{M}_{\mathrm{flat}}$ can be identified with a moduli space of holomorphic vector bundles.

For $G = SU(N)$ we look at holomorphic rank N "semi-stable" vector bundles with fixed determinant. *************** NEED TO EXPLAIN *********************

$\mathcal{M}_{holvb}$ is birational to a projective space ***** EXPLAIN **** and hence $Pic(\mathcal{M}_{holvb}) \cong \mathbb{Z}$. The generator can be nicely described as follows: Choose an element of the Θ-divisor of the Riemann surface, that is $L \in Pic^{g-1}(\Sigma_2)$ such that $h^0(L) \neq 0$. Then define

$$\Theta_L \subset \mathcal{M}_{holvb} \tag{3.88}$$

to be the set of bundles E so that $h^0(E \otimes L) \neq 0$. One shows [REFERENCE???] that for two choices L, L' in the Θ-divisor of Σ_2 we have $\Theta_L \sim \Theta_{L'}$, so we get, canonically, a divisor in $\mathcal{M}_{holvb}$.

In fact, the generator of $Pic(\mathcal{M}_{holvb})$ is isomorphic to the determinant line bundle

BASIC IDEA: Give E a holomorphic connection then we are looking for bundles with nontrivial solution to $\bar{\partial}_E s = 0$, but this is just the divisor of $\mathrm{DET}(\bar{\partial}_E)$.

The space of conformal blocks, to an algebraic geometer is then

$$H^0(\mathcal{M}_{hol.v.b.}; \mathcal{L}^k) \tag{3.89}$$

where $\mathcal{L}$ generates the Picard group.

As we will see, by thinking about conformal field theory one is rather naturally led to a remarkable formula for the dimension of the space of conformal blocks known as the *Verlinde formula.* For the case of $G = SU(2)$ and level k at genus g it has the explicit form:

$$\dim H^0(\mathcal{M}_{hol.v.b.}; \mathcal{L}^k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{n=1}^{k+1} \left(\sin\left(\frac{n}{k+2}\pi\right)\right)^{2-2g} \tag{3.90}$$

****

COMMENT ON $k \to \infty$ LIMIT. SEMICLASSICAL SO GET SYMPLECTIC VOLUME OF PHASE SPACE GET THIS FROM 2D YANG-MILLS: Scale $g = e^{\phi/k}$ get $\int \mathrm{Tr}\phi F$.

****

PROJECTIVELY FLAT CONNECTION OVER MODULI SPACE OF CURVES: In this context, known as "Hitchin connection" – CITE HITCHIN'S PAPER. EXPLAIN?

3.4.1 Generalization To Conformal Blocks With Vertex Operator Insertions

The equivariance condition (3.80) shows that the wavefunction Ψ defines a section of a holomorphic line bundle over $[\mathcal{P}(E; (\lambda_i, P_i))]^{stable}/\mathcal{G}_{\mathbb{C}}$. The superscript "stable" refers to the fact that we must restrict the space of gauge fields on which $\mathcal{G}_{\mathbb{C}}$ acts to get a good quotient – something familiar in geometric invariant theory.

A statement analogous to the Narasimhan-Seshadri theorem is the Mehta-Seshadri theorem which identifies the quotient

$$\mu^{-1}(0)/\mathcal{G} \cong [\mathcal{P}(E; (\lambda_i, P_i))]^{stable}/\mathcal{G}_{\mathbb{C}} \tag{3.91}$$

with the moduli space of "stable bundles with parabolic structure." We indicate briefly what this refers to.

According to the Borel-Weil-Bott theorem we may associate highest weight representations λ with flags in G_C. Accordingly we can consider the moduli space of holomorphic bundles $\mathcal{E} \to X_2$ with specified flags inside the vector bundle:

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n \tag{3.92}$$

at the points P_i. Holomorphic vector bundles with such structures are called quasi-parabolic. We now associate a sequence of real numbers, called "weights" α_i with multiplicities governed by the flag to the data (3.92). (These numbers will correspond to the conjugacy class of the holonomies (3.75) in the real setting). The data of (3.92) together with the weights defines a parabolic vector bundle.

In order to define a good moduli problem one needs to provide a definition of stability of a parabolic vector bundle by introducing the normalized degree

$$par - deg(\mathcal{E}) = \mu(\mathcal{E}) = \frac{c_1(E)}{rank(\mathcal{E})} + \sum WEIGHTS \tag{3.93}$$

Roughly speaking, the definition of stability forbids the existence of holomorphic subbundles with larger μ. This condition forbids the existence of holomorphic self-automorphisms, and guarantees a good quotient in (3.91).

We now describe the space of conformal blocks using the holomorphic quantization of the RHS of (3.91). Thus, the space of conformal blocks is just the space of holomorphic sections of a line bundle:

$$\mathcal{H}(\mathfrak{g}, k; (P_i, \lambda_i)) = H^0 \Biggl\{ [\mathcal{P}(E; (\lambda_i, P_i))]^{stable} / \mathcal{G}_{\mathbb{C}}; \mathcal{L}^{\otimes k} \Biggr\} \tag{3.94}$$

4. WZW Model And RCFT

The WZW model is a very important example of a two-dimensional rational conformal field theory. It is the subject of a large and rich literature, and there are still many interesting questions that remain to be understood much better. For example, the theory of LG for G a noncompact semisimple Lie group is far from completely understood. Among the applications of the 2d quantum WZW model are:

1. Nontrivial exactly solvable quantum (conformal) field theories.
2. According to a conjecture of Moore + Seiberg, they are the basis for all rational conformal field theories.
3. Numerous uses in string theory: the gauge symmetry of the heterotic string, exactly solvable string backgrounds, solitonic five-branes, explicit nontrivial D-brane states, exactly solvable examples of AdS/CFT, ...
4. Numerous uses in condensed matter physics: Two-dimensional critical phenomena, the Kondo problem, the quantum Hall effect, applications to quantum computing.

♣COMMENTS ON THIS CONJECTURE AND ITS CURRENT STATUS GO ELSEWHERE ♣

5. Mathematics: As explained in these notes - they give an explicit and important class of examples of modular tensor categories, thereby giving relations to knot invariants, three-dimensional Chern-Simons theory, moduli spaces of flat connections on surfaces. In addition they play a role in some very striking results on K-theory (Freed-Hopkins-Teleman). They also play an important role in the Geometric Langlands program, where level $k = -h$ algebras play a special role.

4.1 The Principal Chiral Model And The WZ Term

4.1.1 The Wess-Zumino Term

Consider a sigma model of maps $g : \mathcal{S}_d \to G$ where G is a Lie group and $\mathcal{S}_d$ is a d-dimensional (pseudo-) Riemannian spacetime. The standard sigma model action for this theory is

$$\frac{f^2}{4}\int_{\mathcal{S}_d} \mathrm{Tr}(g^{-1}dg)\wedge *(g^{-1}dg) \tag{4.1}$$

where f is a coupling constant and $*$ is the Hodge star operator.

Consideration of anomalies in gauge theories led Wess and Zumino to introduce a very interesting term in the sigma model action [63] in the case of the four-dimensional sigma model. Its proper conceptual formulation and physical consequences were subsequently beautifully clarified in a series of papers by Witten [64, 65]. We will write it here for arbitrary even spacetime dimension $d = 2n$.

Let $\Theta = g^{-1}dg$ be the Maurer-Cartan form on G. Then $\mathrm{Tr}\Theta^{2n+1}$ is closed. If the rank of G is suitably larger than n (our main application is $n = 1$ and this will always be true) then it represents a nonzero cohomology class and for suitable normalization c_n

$$x_{2n+1} = [c_n \mathrm{Tr}\Theta^{2n+1}] \tag{4.2}$$

is a DeRham cohomology class that generates the integral lattice in $H^{2n+1}_{DR}(G)$.

Let $g : \mathcal{S}_{2n} \to G$ be a sigma-model field, and let us consider a *closed* spacetime so that $\partial\mathcal{S}_{2n} = 0$. Physically this is relevant even for fields on $\mathbb{R}^{2n}$ if we require that the fields approach 1 at spatial and temporal infinity. In that case, we can consider the the field to be defined on S^{2n}. [30]

There are several slightly different approaches one can take to define the Wess-Zumino term. One way to do it is to note that the image

$$g(\mathcal{S}_{2n}) \subset G \tag{4.3}$$

is a $(2n)$-cycle inside G which varies continuously with G. Now, if $H_{2n}(G;\mathbb{Z}) = 0$ (as is often the case[31]) we can fill in the image of spacetime with an oriented chain $\mathcal{B}_{2n+1}(g)$:

$$\partial\mathcal{B}_{2n+1}(g) = g(\mathcal{S}_{2n}). \tag{4.4}$$

[30] The generalization to the case when spacetime has a boundary is very interesting. In that case $\exp[\mathrm{i}WZ]$ should be regarded as a section of a line bundle.

[31] For example $H_2(G;\mathbb{Z}) = 0$ always for a compact simple simply connected Lie group. $\pi_4(G) = 0$ for all compact simple simply connected groups except $\pi_4(USp(2n)) = \mathbb{Z}_2$. LIST $H_4(G;\mathbb{Z})$

This chain also varies continuously with g. Note however that there can be different choices of the chain $\mathcal{B}_{2n+1}(g)$.

Example $\mathcal{S} = S^2$, $G = SU(2) \cong S^3$, the map g takes $\mathcal{S}$ to the equator. Then we can use the upper hemisphere D_+.

We define the Wess-Zumino term to be:

$$WZ(g) := 2\pi k \int_{\mathcal{B}_{2n+1}(g)} \omega_{2n+1} \tag{4.5}$$

where k is a real, coupling constant with dimensions of $\hbar$. The WZW (Wess-Zumino-Witten) theory is the nonlinear sigma model with Minkowski-space action

$$\frac{f^2}{4} \int_{\mathcal{S}_d} \mathrm{Tr}(g^{-1}dg) \wedge *(g^{-1}dg) + WZ(g) \tag{4.6}$$

Now, at first the definition (4.5) seems absurd. There are two immediate problems:

• It appears to be an action for field configurations in $2n+1$ dimensions.

• It appears to depend on the choice of bounding chain $\mathcal{B}_{2n+1}$, and the constraint (4.4) leaves infinitely many choices for $\mathcal{B}_{2n+1}$.

Let us first address point (1.) Although the definition of the WZ term uses a $2n+1$ dimensional field configuration, the *variation* of the action only depends on the fields on the $2n$ dimensional boundary ∂B_{2n+1}, and hence, the action is in fact local! See the exercise below for some details on how to vary the action and derive the equations of motion.

Therefore, we find for the variation of the WZ term:

$$\delta WZ(g) = 2\pi k c_n (2n+1) \int_{\mathcal{S}_{2n}} \mathrm{Tr}(g^{-1}\delta g)(g^{-1}dg)^{2n} \tag{4.7}$$

Remarkably, even though the definition of the WZ action involves an extension into one higher dimension, this is a local action in the sense that its variation under local changes in the field $g(x)$ is a local density on spacetime! It's value might depend on subtle topological questions, but the variation is local.

Therefore, the equations of motion of the WZW theory are local partial differential equations:

$$-\frac{f^2}{2} d\big(*g^{-1}dg\big) + 2\pi k c_n (2n+1)(g^{-1}dg)^{2n} = 0 \tag{4.8}$$

Now let us address the second point - the dependence on the choice of bounding chain $\mathcal{B}_{2n+1}(g)$. For a fixed g we can of course smoothly deform the chain to get a second chain as in 11. The difference $\mathcal{B}' - \mathcal{B}$ is a small closed $2n+1$ cycle in G which is, moreover, homologous to zero, so $\mathcal{B}' - \mathcal{B} = \partial \mathcal{Z}$ where $\mathcal{Z}$ is a $(2n+2)$-chain. But now, by Stokes' theorem:

Figure 11: Two slightly different $(2n+1)$-chains $\mathcal{B}$ and $\mathcal{B}'$ in G bounding the same $2n$-cycle $g(\mathcal{S}_{2n})$.

Figure 12: Two different $(2n+1)$-chains in G bounding the same $2n$-cycle $g(\mathcal{S}_{2n})$.

$$\int_{\mathcal{B}'} \omega_{2n+1} - \int_{\mathcal{B}} \omega_{2n+1} = \int_{\mathcal{Z}} d\omega_{2n+1} = 0 \tag{4.9}$$

Thus $WZ(g)$ does not change under small deformations of $\mathcal{B}$.

However, it can happen that $\mathcal{B}'$ and $\mathcal{B}$ are not small deformations of each other as in 12. In general if $\mathcal{B}, \mathcal{B}'$ are two oriented chains with

$$\partial \mathcal{B} = \Sigma_{2n} \tag{4.10}$$

and

$$\partial \mathcal{B}' = \Sigma_{2n} \tag{4.11}$$

then

$$\mathcal{B} \cup -\mathcal{B}' = \Xi_{2n+1} \tag{4.12}$$

is a closed oriented $(2n+1)$-cycle. Therefore,

$$\int_{\mathcal{B}(g)} \omega_{2n+1} = \int_{\mathcal{B}'(g)} \omega_{2n+1} + \int_{\Xi_{2n+1}} \omega_{2n+1} \tag{4.13}$$

and hence, if the periods $\int_{\Xi_{2n+1}} \omega_{2n+1}$ are nonzero then *the expression* $WZ(g)$ *is not well-defined as a real number!*

This might seem disturbing, but, the cycle Ξ_{2n+1} defines an integral homology class, and hence the periods of ω_{2n+1} are quantized. Therefore, the ambiguity in the definition of $WZ(g)$ is an additive *quantized* shift of the form $2\pi k N$ where N is an integer. Put differently,

$$WZ(g) \mathrm{mod} 2\pi k \mathbb{Z} \tag{4.14}$$

is well-defined. The quantized ambiguity cannot vary under small variations of g. Thus, $WZ(g)$ is still a local action, and the equations of motion are still local.

Note that the situation here is very similar to our discussion of the action for general quantization of a symplectic manifold when the symplectic form has nontrivial periods.

The situation in quantum mechanics is a little more subtle, since in quantum mechanics one works directly with the action, and not just the equations of motion. However, in quantum mechanics the action only enters through $\exp[\frac{i}{\hbar}S]$, and therefore all that must really be well-defined is the expression

$$\exp[\frac{i}{\hbar}WZ(g)] \tag{4.15}$$

What is the ambiguity in (4.15) ? We see that it is just

$$\exp[2\pi i\frac{k}{\hbar}\int_{\Xi_{2n+1}}\omega_{2n+1}] \tag{4.16}$$

Therefore, if $k=\kappa\hbar$, where κ is an integer, then the

$$\exp[\frac{i}{\hbar}WZ(g)] \tag{4.17}$$

in the path integral is a well-defined $U(1)$-valued function on the space of fields $Map[\mathcal{S}_{2n},G]$. Assuming we have a well-defined measure on the space of fields, there is no harm including this expression in the measure.

Thus, the coupling constant k must be quantized for a mathematically well-defined measure in the quantum mechanical path integral. This is one of the most beautiful examples of a topological quantization of a coupling constant.

We will usually set $\hbar=1$. Thus, large k corresponds to the semiclassical limit.

Remarks:

1. *Integral normalization.* Here are some relevant facts. It can be shown [32] that for all compact, simple, connected, and simply connected groups G:

$$x_3=[\frac{1}{48\pi^2h^\vee}\mathrm{Tr}_{adj}(g^{-1}dg)^3] \tag{4.18}$$

generates the integral cohomology lattice in $H^3_{DR}(G)$, where h is the dual Coxeter number. In particular, for $SU(2)$ we can take

$$x_3=[\frac{1}{24\pi^2}\mathrm{Tr}_2(g^{-1}dg)^3] \tag{4.19}$$

It follows that for $SU(N)$ we can take

$$x_3=[\frac{1}{24\pi^2}\mathrm{Tr}_N(g^{-1}dg)^3] \tag{4.20}$$

to generate the integral cohomoloy lattice in $H^3_{DR}(G)$.

[32] Ref: Mimura and Toda, *Topology of Lie Groups*, Translations of Math. Monographs **91**; R. Bott, Bull. Soc. Math. France **84**(1956) 251.

2. Here is another way to define the Wess-Zumino term. For each connected component $\mathcal{C}_\alpha$ of the fieldspace $Map(\mathcal{S}_{2n}, G) = \amalg_\alpha \mathcal{C}_\alpha$ we choose a "basepoint" field configuration $g_0^{(\alpha)} : \mathcal{S}_{2n} \to G$. If $\mathcal{S}_{2n}$ is contractible there is only one component and we can choose g_0 to be the constant map (say with image $1 \in G$). In general for field configurations $g \in \mathcal{C}_\alpha$ we choose a smooth homotopy $g(x,s)$, $0 \le s \le 1$ from $g_0^{(\alpha)}(x)$ at $s = 0$ to $g(x)$ at $s = 1$. Now we view the interpolation as a field in $2n+1$ dimensions, that is, as a map of the cylinder $\widehat{g} : I \times \mathcal{S}_{2n} \to G$. We can then define

$$WZ(g; g_0) := 2\pi k \int_{I \times \mathcal{S}_{2n}} \widehat{g}^*(\omega_{2n+1}) \tag{4.21}$$

3. The value of $WZ(g; g_0)$ depends on the choice of g_0 and on the interpolation, but only "locally," in the following sense: Suppose we have a continuous family of maps $\tilde{g}^\tau : \mathcal{S}_{2n} \to G$ in the connected component $\mathcal{C}_\alpha$. Then we find a continuous family of extensions $\tilde{g}^\tau_s : B_{2n+1} \to G$ such that $g^\tau_0(x) = g_0(x)$ for all τ. Then, letting $B = I \times \mathcal{S}_{2n}$ we have:

$$\boxed{\frac{\partial}{\partial \tau} \mathrm{Tr}(\tilde{g}^{-1} d_B \tilde{g})^{2n+1} = d_B \left[(2n+1) \mathrm{Tr} \tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial \tau} (\tilde{g}^{-1} d_B \tilde{g})^{2n} \right]} \tag{4.22}$$

Proof: We know that the Maurer-Cartan form pulled back to $I \times B_{2n+1}$ is closed, so

$$(d_B + \delta) \mathrm{Tr}(\tilde{g}^{-1} (d_B + \delta) \tilde{g})^{2n+1} = 0 \tag{4.23}$$

where $\delta = d\tau \frac{\partial}{\partial \tau}$. Now forms on the product space can be decomposed into type (a, b) with a-forms along I and b-forms along B_{2n+1}. The component of (4.23) of type $(1, 2n+1)$ is

$$\delta \mathrm{Tr}(\tilde{g}^{-1} d_B \tilde{g})^{2n+1} + (2n+1) d_B \left[\mathrm{Tr}(\tilde{g}^{-1} \delta \tilde{g})(\tilde{g}^{-1} d_B \tilde{g})^{2n} \right] = 0 \tag{4.24}$$

pulling out the $d\tau$ gives our identity. It is now an easy matter to show that the variation of the WZ term defined as in (4.21) only depends on the variation g^τ_s at $s = 1$.

Exercise

a.) Calculate $\mathrm{Tr}(g^{-1}dg)^3$ for $SU(2)$ in terms of Euler angles for the group, using the fundamental representation:

$$\mathrm{Tr}_{\mathbf{2}}(g^{-1}dg)^3 = -\frac{3}{2} d\psi \wedge \sin\theta d\theta \wedge d\phi \tag{4.25}$$

b.) Write this differential form as a locally exact form.

c.) Show that

$$\int_{SU(2)} \frac{1}{24\pi^2} \mathrm{Tr}_{\mathbf{2}}(g^{-1}dg)^3 = -1 \tag{4.26}$$

and thus conclude that the form is not globally exact. Compare with the general normalizations above.

d.) Now show that x_3 in equation (4.20) defines a nontrivial cohomology class for all $SU(N)$.

Exercise *The Polyakov-Wiegman formula*

Consider the WZ term in two spacetime dimensions.

a.) Show that

$$\mathrm{Tr}\big((g_1g_2)^{-1}d(g_1g_2)\big)^3 = \mathrm{Tr}\big(g_1^{-1}dg_1\big)^3 + \mathrm{Tr}\big(g_2^{-1}dg_2\big)^3 + 3d\Big[\mathrm{Tr}(dg_2g_2^{-1})(g_1^{-1}dg_1)\Big] \tag{4.27}$$

b.) Conclude that the WZ term satisfies:

$$WZ(g_1g_2) = WZ(g_1) + WZ(g_2) + 6\pi k c_1 \int \mathrm{Tr}(dg_2g_2^{-1})(g_1^{-1}dg_1) \tag{4.28}$$

Exercise *Variation Of The WZ Term*

Using the variational formula

$$\delta\big(g^{-1}\partial_\mu g\big) = \partial_\mu(g^{-1}\delta g) + \big[g^{-1}\partial_\mu g, g^{-1}\delta g\big] \tag{4.29}$$

We compute:

$$\frac{\partial}{\partial s}\mathrm{Tr}(g^{-1}dg)^{2n+1} = (2n+1)\mathrm{Tr}d\bigg(g^{-1}\frac{\partial g}{\partial s}\bigg)(g^{-1}dg)^{2n} \tag{4.30}$$

The second term, involving the commutator drops out.

Now we compare with the RHS

$$d\bigg[\mathrm{Tr}g^{-1}\frac{\partial g}{\partial s}(g^{-1}dg)^{2n}\bigg] = \mathrm{Tr}d\bigg(g^{-1}\frac{\partial g}{\partial s}\bigg)(g^{-1}dg)^{2n} + \mathrm{Tr}g^{-1}\frac{\partial g}{\partial s}\bigg[d\Theta\Theta^{2n-1} - \Theta d\Theta\Theta^{2n-2} \pm \cdots\bigg] \tag{4.31}$$

and using the Maurer-Cartan equation we find that the second group of terms cancel in pairs. ♠

4.1.2 The Case Of Two Dimensions And The Conformal Point

Above two dimensions the WZ term leads to a higher-derivative correction to the equations of motion. We must interpret this in terms of an effective field theory. In two dimensions the situation is different and extremely beautiful: For special values of the coupling constant f we have infinite-dimensional symmetries of the equations of motion.

We urge you to consult the two beautiful papers where the quantum theory is solved:

1. E. Witten, "Nonabelian bosonization in two dimensions," Commun.Math.Phys.92:455-472,1984.

2. V.G. Knizhnik and A.B. Zamolodchikov, "Current Algebra and Wess-Zumino Model in Two-Dimensions," Nucl.Phys.B247:83-103,1984.

When $f^2 = 12\pi k c_1$ the action becomes $S = kS^{+}_{WZW}$ where

$$S^{+}_{WZW} = 6\pi c_1 \int \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) dx^+ \wedge dx^- + 2\pi c_1 \int_{\mathcal{B}} \mathrm{Tr}(g^{-1}dg)^3 \tag{4.32}$$

while if $f^2 = -12\pi k c_1$ the action is $S = -kS^{-}_{WZW}$ with

$$S^{-}_{WZW} = 6\pi c_1 \int \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) dx^+ \wedge dx^- - 2\pi c_1 \int_{\mathcal{B}} \mathrm{Tr}(g^{-1}dg)^3 \tag{4.33}$$

These actions satisfy some nice properties: See the exercise on the Polyakov-Wiegmann formula below. These identities have significant physical implications.

In two dimensions the equations of motion are:

$$-\frac{f^2}{2} d\big(*g^{-1}dg\big) + 6\pi k c_1 (g^{-1}dg)^2 = 0 \tag{4.34}$$

Using the MC equation on the second term we can write this as:

$$d\left(\frac{f^2}{2} * g^{-1}dg + 6\pi k c_1 g^{-1}dg\right) = 0 \tag{4.35}$$

$$d\left(*g^{-1}dg + \frac{12\pi k c_1}{f^2} g^{-1}dg\right) = 0 \tag{4.36}$$

Now consider Minkowski space with metric $ds^2 = -dx^+dx^- = -dt^2 + dx^2$. We may write

$$g^{-1}dg = (g^{-1}\partial_+ g)dx^+ + (g^{-1}\partial_- g)dx^- \tag{4.37}$$

Then using the orientation $dt \wedge dx$ the self dual forms are

$$\begin{aligned} *(dt + dx) &= *dx^+ = -dx^+ \\ *(dt - dx) &= *dx^- = dx^- \end{aligned} \tag{4.38}$$

so

$$*g^{-1}dg = -(g^{-1}\partial_+ g)dx^+ + (g^{-1}\partial_- g)dx^- \tag{4.39}$$

It thus follows that

- If $f^2 = 12\pi k c_1$ then the equation of motion is

$$\partial_+\big(g^{-1}\partial_- g\big) = 0 \tag{4.40}$$

- If $f^2 = -12\pi k c_1$ then the equation of motion is

$$\partial_-\big(g^{-1}\partial_+ g\big) = 0 \tag{4.41}$$

We now see a remarkable property of the theory. Let us focus on the case $f^2 = 12\pi k c_1$ for definiteness. We can define the infinite-dimensional group

$$\mathcal{G} := \mathrm{Map}(\mathbb{R}, G) \tag{4.42}$$

Then there is a left-action of $\mathcal{G}_L \times \mathcal{G}_R$ on the set of equations of motion!

Indeed, if $g(x^+, x^-)$ solves (4.40) then so does

$$h_l(x^+) g(x^+, x^-) h_r(x^-)^{-1} \tag{4.43}$$

A closely related fact is that the equations of motion imply that there are two *separately* conserved currents. Define

$$J_- := g^{-1}\partial_- g \tag{4.44}$$

$$J_+ := \partial_+ g g^{-1} \tag{4.45}$$

Then (4.40) together with the identity:

$$\partial_+\big(g^{-1}\partial_- g\big) = g^{-1}\partial_-\big(\partial_+ g g^{-1}\big) g \tag{4.46}$$

shows that both $\partial_+ J_- = 0$ and $\partial_- J_+ = 0$.

So when $f^2 = 12\pi k c_1$ there are two separately conserved currents,

Let us use this to work out the general solution of the equations of motion. We must choose some boundary conditions.

If we are working on $\mathbb{R}^{1,1}$ then as $x^+ \to -\infty$, $J_+(x^+) \to 0$ fast enough so that $\int_{-\infty}^{x^+} J_+$ converges, and similarly for J_-. We may then regard $J_\pm$ as given, and integrate the equations (4.44)(4.45) to get [33]

$$g(x^+, x^-) = \left(P\exp\int_{-\infty}^{x^+} J_+(s)ds \right) \cdot g_0 \cdot \left(P\exp - \int_{-\infty}^{x^-} J_-(s)ds \right)^{-1} \tag{4.47}$$

In other words, on $\mathbb{R}^{1,1}$, the general solution of the equation of motion is a transformation of $g = 1$ by the group $\mathcal{G}_L \times \mathcal{G}_R$.

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[33] Our conventions for path ordering put the later times to the left.

NEED TO DISCUSS EUCLIDEAN CONTINUATION - HOLOMORPHIC AND ANTI-HOLOMORPHIC CURRENTS
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Exercise
Work out the analogous equations for the case $f^2 = -12\pi k c_1$.

Exercise *A Consequence Of the PW Formula*
Using the PW formula show that

$$S^{+}_{WZW}(g_1 g_2) = S^{+}_{WZW}(g_1) + S^{+}_{WZW}(g_2) + 12\pi c_1 \int \mathrm{Tr}\big(\partial_+ g_2 g_2^{-1} g_1^{-1} \partial_- g_1\big) \tag{4.48}$$

$$S^{-}_{WZW}(g_1 g_2) = S^{-}_{WZW}(g_1) + S^{-}_{WZW}(g_2) + 12\pi c_1 \int \mathrm{Tr}\big(\partial_- g_2 g_2^{-1} g_1^{-1} \partial_+ g_1\big) \tag{4.49}$$

****
NEED TO COMPARE WITH DISCUSSION OF GAUSS LAW ABOVE WHERE WE GAUGE A_z TO ZERO
****

Exercise
Is k quantized for the 2-dimensional $SL(2,\mathbb{R})$ WZW model?

Remarks

• Notice that there are *two* senses in which J_+ is a "left symmetry." It describes left-moving waves because it is a function of $x^+ = t + x$ so as time goes forward the wave moves to negative x. It also accounts for multiplication of $g(x^+, x^-)$ on the left.

• *Generalization to Euclidean signature.* The theory can be continued to Euclidean signature $x^+ \to z$, $x^- \to \bar{z}$. Then the symmetries are holomorphic/antiholomorphic current algebras. Because of conformal invariance the theory can be formulated on an arbitrary Riemann surface.

• We have shown that the coupling constant k is topologically quantized. The question arises: "What is the minimal value of k? " This depends on the representation and the normalization c_n. For 2D we have the WZ term for $SU(N)$:

$$\exp[ik\frac{1}{12\pi}\int_{B_3}\mathrm{Tr}_N(g^{-1}dg)^3] \tag{4.50}$$

$k \in \mathbb{Z}$, and all integers can occur. Thus, for $G = SU(N)$ using the trace in the N we have $c_1 = \frac{1}{24\pi^2}$.

• *Spectral flow and interesting solutions of the equations of motion.* Long strings in $SL(2,\mathbb{R})$ etc.

4.2 Quantization: Virasoro And Affine Lie Algebra Symmetry

We have seen that the particle on the group manifold has $G_L \times G_R$ symmetry. In the classical two-dimensional theory with space taken to be S^1, at the special values of the coupling constant $f^2 = 12\pi c_1 k$, this is promoted to the infinite dimensional $(LG)_L \times (LG)_R$ symmetry.

In this section we show that the *quantum* 2D WZW theory on a circle has $\widetilde{LG}_L \times \widetilde{LG}_R$ symmetry, where $\widetilde{LG}$ is a central extension of the loop group $LG = Map(S^1, G)$.

The Lie algebra of LG is $Lie(LG) = L\mathfrak{g}$, the vector space of maps of S^1 into $\mathfrak{g}$, and it has a central extension by $\mathbb{R}$. We will decompose our loop in the Lie algebra with respect to a basis T_a as:

$$J^a(x) = \sum_{n\in\mathbb{Z}} J^a_n e^{2\pi i n x} \tag{4.51}$$

where we choose an ON basis T_a of the Lie algebra. We are now taking space to be a circle and $x \sim x+1$.

Our conventions are that $su(N)$ is the Lie algebra of *antihermitian* matrices, $so(N)$ is the Lie algebra of *antisymmetric* matrices etc. However, we wish to keep the standard Hermiticity relations $(J^a_n)^\dagger = +J^a_{-n}$. So our currents will be valued in the space of hermitian matrices, etc.

The structure constants defining $\widetilde{L\mathfrak{g}}$ are that k is central and

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + k\delta^{ab} n \delta_{n+m,0} \tag{4.52}$$

This is known as an *affine Lie algebra.*

Note well that, setting $n = 0$, the generators J^a_0 span a finite dimensional subalgebra isomorphic to $\mathfrak{g}$.

We will sketch three ways to derive this symmetry in the WZW model. Let us choose $S = kS^+_{WZW}$ for definiteness.

First way:

The Noether current under $g \to g e^{\epsilon}$ is, by PW

$$J^R = J_- = i(12\pi c_1 k) g^{-1}\partial_- g \tag{4.53}$$

and similarly for $g \to e^{-\epsilon} g$

$$J^L = J_+ = -i(12\pi c_1 k)\partial_+ g g^{-1} \tag{4.54}$$

the factor of i arises because of our conventions explained above.

Because of the equations of motion $\partial_+ J^R_- = 0$ and $\partial_- J^L_+ = 0$ there is in fact an infinite-dimensional symmetry

$$\delta^R_\epsilon : g \to g e^{\epsilon(x^-)} \tag{4.55}$$

$$\delta^L_\epsilon : g \to e^{\epsilon(x^+)} g \tag{4.56}$$

We have already studied this at the level of the equations of motion. In the quantum theory these transformations are generated by

$$\delta^R_\epsilon \mathcal{O} = [\int dx^- \epsilon^a(x^-) J^a_-(x^-), \mathcal{O}] \tag{4.57}$$

for any operator $\mathcal{O}$. There is a similar expression for the left-symmetry.

The transformation of the currents themselves is easily computed:

$$\begin{aligned} &\delta^R_\epsilon J^R = -[\epsilon(x^-), J^R(x^-)] + i(12\pi c_1 k)\epsilon'(x^-) \\ &\delta^R_\epsilon J^L = 0 \end{aligned} \tag{4.58}$$

Combining this with (4.57) we derive:

$$[J^a(x^-), J^b(y^-)] = i f^{abc} J^c(y^-)\delta(x^- - y^-) + i(12\pi c_1 k)\delta^{ab}\frac{d}{dx^-}\delta(x^- - y^-) \tag{4.59}$$

When we define our current as in (4.51) and use $c_1 = 1/(24\pi^2)$ we see that (4.59) is precisely equivalent to (4.52).

Second way:

A second way to quantization is to use the lightcone formalism. Let us regard x^+ as time and x^- as space. Thus, in lightcone formalism, the action S^+_{WZW} is *first order* in "time" (i.e. x^+) derivatives. Therefore it is already in Hamiltonian form. We now apply once again our general remark about actions that are first order in time derivatives.

Varying the action we get:

$$\delta(kS^+_{WZW}) = -12\pi c_1 k \int dx^+ \left[\int dx^- \mathrm{Tr}(g^{-1}\delta g)\partial_+(g^{-1}\partial_- g)\right] \tag{4.60}$$

comparing with (**??**) we get the symplectic form

$$\Omega = \int dx^- \mathrm{Tr}(g^{-1}\delta g)\partial_-(g^{-1}\delta g) - g^{-1}\partial_- g(g^{-1}\delta g)^2 \tag{4.61}$$

From this we can compute the Poisson brackets of the currents. This is what is done in Witten's paper, which you can consult for further details.

****

Comment on disadvantages of light cone Hamiltonian quantization: You miss the waves travelling along $x^+ = constant$. These are "spatial slices" in this formalism, but nonetheless left-moving degrees of freedom, and hence dynamically important in a different time slice.

****

Exercise

Write (4.61) as a total derivative under δ

<u>Third way</u>:

Here it is good to recall the discussion of the Hamiltonian formalism for a particle on the group manifold G.

We can go to Hamiltonian formalism without going to lightcone gauge, but it is a little subtle. Let δ denote differentiation on the space of field configurations $LG = Map(S^1, G)$ at fixed time.

The tangent space to LG at a loop $g(\theta)$ is isomorphic to $L\mathfrak{g}$:

♣Be consistent about whether spatial coordinate along S^1 is x or θ. ♣

$$T_g LG \cong L\mathfrak{g} \tag{4.62}$$

The dual space to $L\mathfrak{g}$ is $\Omega^1(S^1; \mathfrak{g}^*)$, so the canonical momentum L which is a coordinate on the fiber of T^*LG is in

$$L \in \Omega^1(S^1; \mathfrak{g}^*) \tag{4.63}$$

with the pairing

$$\langle L, \epsilon \rangle = \oint d\theta \langle L(\theta), \epsilon(\theta) \rangle \tag{4.64}$$

We use an invariant form to identify $\mathfrak{g} \cong \mathfrak{g}^*$ so that we can write the symplectic form on T^*LG as

$$\Omega = \oint_{S^1} \mathrm{Tr}\delta L g^{-1}\delta g + \mathrm{Tr}(L + \xi g^{-1}dg)(g^{-1}\delta g)^2 \tag{4.65}$$

where $\xi = -6\pi c_1 k$. Here $g = g(\theta)$ is a loop in G, while $g^{-1}\delta g$ is a one-form on LG valued in $L\mathfrak{g}^* \cong L\mathfrak{g}$ and is a 0-form on S^1. L is now regarded as a one form on S^1 valued in $L\mathfrak{g}$. Thus (4.65) is a two-form on T^*LG. It is a good exercise to show that it is closed under δ.

Now, again, to write the Legendre transform of the Hamiltonian formalism we need to find a one-form on T^*G so that $\Omega = \delta\Theta$. To do that we need to extend g to a map of the disk $\tilde{g} \in Map(D^2, G)$. Then we can write:

$$\Theta = \oint_{S^1} Tr(Lg^{-1}\delta g) - \xi \int_{D^2} Tr(\tilde{g}^{-1}d\tilde{g})^2(\tilde{g}^{-1}\delta\tilde{g}) \tag{4.66}$$

Then one can check that

$$S = \int_{\gamma} \Theta - \int dt \ \ 6\pi c_1 k \left[\oint_{S^1} \frac{1}{2} Tr L^2 + \frac{1}{2} Tr (g^{-1}\partial_x g)^2 \right] \tag{4.67}$$

indeed gives the WZW action.

Indeed, this is precisely the form we obtained from quantizing Chern-Simons theory on the disk! See (3.86) above.

Let us compute the Poisson brackets. The discussion closely follows the case of a point particle on a group manifold. See equations (**??**) above. Written in an ON basis for the Lie algebra we have

$$\Omega = \frac{1}{2} \oint dx \oint dy \left(\delta L_a(x) \ \delta\epsilon^a(x) \right) \begin{pmatrix} 0 & \delta^a{}_b \delta(x-y) \\ -\delta_a{}^b \delta(x-y) & -f^c_{ab}(L_c + \xi g^{-1}\partial_y g)\delta(x-y) \end{pmatrix} \begin{pmatrix} \delta L_b(y) \\ \delta\epsilon^b(y) \end{pmatrix} \tag{4.68}$$

so that the inverse matrix is:

$$\Omega^{AB} = \begin{pmatrix} -f^c_{ab}\left(L_c + \xi(g^{-1}\partial_y g)_c\right)\delta(x-y) & -\delta_a{}^b \delta(x-y) \\ \delta^a{}_b(x-y) & 0 \end{pmatrix} \tag{4.69}$$

The Poisson brackets that follow from (4.69) are

$$\begin{aligned} \{g(x), g(y)\} &= 0 \\ \{L_a(x), g(y)\} &= -g(y) T_a \delta(x-y) \\ \{L_a(x), L_b(y)\} &= -f^c_{ab}\left(L_c(y) + \xi(g^{-1}g)_c(y)\right)\delta(x-y) \end{aligned} \tag{4.70}$$

Variation of the action with respect to L_a gives $L_a = 6\pi c_1 k (g^{-1}\dot{g})$ under time evolution, and hence

$$J^a(x) = L^a(x) - 6\pi c_1 k (g^{-1}\partial_x g)^a \tag{4.71}$$

is a right-moving current.

Now, using (4.70) we easily compute the Poisson brackets of $J^a(x)$ to be:

$$\{J^a(x), J^b(y)\} = -f^{abc} J^c(y)\delta(x-y) - \frac{k}{2\pi}\delta^{ab}\partial_x \delta(x-y) \tag{4.72}$$

Thus recovering the current algebra.

Remark: Using the angular momenta conjugate to the left-symmetry we can similarly derive the current algebra of the left-moving currents.

Exercise

Show that indeed $\Omega = \delta\Theta$.

Hint: Explain why

$$\delta \mathrm{Tr}(g^{-1}dg)^2(g^{-1}\delta g) + d\mathrm{Tr}(g^{-1}\delta g)^2(g^{-1}dg) = 0 \tag{4.73}$$

4.2.1 Stress-Energy Tensor

In general the coupling to gravity defines the energy-momentum tensor:

$$T_{\mu\nu} := \frac{1}{2\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S \tag{4.74}$$

Let us continue to work in light-cone coordinates. Classical conformal invariance shows that $T_{+-} = 0$. Under a variation δh^{--} we have:

$$\begin{aligned} \partial_- &\to \partial_- \\ \partial_+ &\to \partial_+ + \delta h^{--} \partial_- \end{aligned} \tag{4.75}$$

Since the WZ term doesn't depend on the metric we easily compute

$$\begin{aligned} \delta S &= -6\pi c_1 k \mathrm{Tr}(g^{-1}\partial_- g)^2 \delta h^{--} \\ T_{--} &= \frac{\pi}{k} \mathrm{Tr}(J_-)^2 \end{aligned} \tag{4.76}$$

and similarly for T_{++}.

The Hamiltonian is

$$H \sim \oint dx (T_{++} + T_{--}) \tag{4.77}$$

It is convenient to decompose into Fourier modes

$$T_{--}(x) = const. \sum_{n\in\mathbb{Z}} L_n e^{2\pi i n(t-x)} \tag{4.78}$$

$$T_{++}(x) = const. \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{2\pi i n(t+x)} \tag{4.79}$$

where the constant is such that

$$L_n = \frac{1}{2k} \sum_{m\in\mathbb{Z}} J^a_{n-m} J^a_m \tag{4.80}$$

In the quantum theory we will define a "highest weight" vacuum so that $J^a_n|vac\rangle = 0$ for $n > 0$. Therefore, we need to normal-order the expression (4.80) with J^a_n for $n > 0$ to the right.

We should allow for a multiplicative renormalization of the expression in terms of the product of two currents due to the short distance singularities when $J(x)$ multiplies $J(y)$ for x near y.

Now, L_0 generates scale transformations and J^a_- is a conserved current. Therefore, J^a_- must have scaling dimension exactly equal to one in the quantum theory. This means we must have

$$[L_0, J^a_n] = -nJ^a_n \tag{4.81}$$

On the other hand, we can compute:

$$[\sum_{m\in\mathbb{Z}} : J^b_{-m} J^b_m :, J^a_n] = -2n(k+h^\vee) J^a_n \tag{4.82}$$

where we use the fact that in an ON basis

$$\sum_{bc} f^{abc} f^{dbc} = 2h^\vee \delta^{ad} \tag{4.83}$$

where h is the dual Coxeter number.

Therefore, we learn that quantum effects have lead to a renormalization

$$\frac{1}{k} \to \frac{1}{k+h^\vee} \tag{4.84}$$

Recall that the semiclassical expansion parameter is $1/k$.

Therefore, provided $k+h\neq 0$, the quantum stress-energy tensor is [34]

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} : J^a_m J^a_{n-m} : \tag{4.85}$$

One can now compute straightforwardly (if somewhat tediously) that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{4.86}$$

with

$$c = \frac{k}{k+h^\vee} \dim \mathfrak{g} \tag{4.87}$$

Note that it approaches $\dim \mathfrak{g}$ as $k\to\infty$.

Remarks:

• Since we can represent the Virasoro algebra on the Hilbert space we have a conformal field theory. Therefore, one can maintain $T_{+-}=0$ in the quantum theory. The value $f^2 = \pm 12\pi c_1 k$ of the coupling constant defines an exact zero of the beta function of the theory. If f^2 is not at this value the model is asymptotically free.

• Since the central charge is nonzero, the left- and right-moving sectors of the theory have a *gravitational anomaly.* Just the way the classical $L\mathfrak{g}$ symmetry of the theory turns out to be anomalous and is replaced by $\widetilde{L\mathfrak{g}}$ in the quantum theory, so too the classical diff(S^1) symmetry of the theory is replaced by the Virasoro algebra.

• While there is an anomaly in the left- and right-sector the generators of diffeomorphisms of the circle are $L_n - \tilde{L}_{-n}$ and these in fact have no anomaly. Therefore, we can consistently couple the model to gravity.

• The commutators above can be evaluated straightforwardly, but there is a more elegant formalism based on the operator product expansion of holomorphic fields to obtain the same answers.

[34] When $k+h^\vee=0$ there are new operators that commute with the current algebra and the symmetry is even larger. This case is relevant in the Geometric Langlands Program.

• The central charge can be interpreted as measuring the dimension of the target space of the nonlinear sigma model. If r is the rank of the simple Lie algebra then

$$r \leq c \leq \dim \mathfrak{g} \qquad\qquad 1 \leq k < \infty \tag{4.88}$$

In the semiclassical limit $k \to \infty$ the target space is of dimension $\mathfrak{g}$: Already the zeromode sector of the model becomes the quantum mechanics of a particle on a group manifold. However, for finite k one must use a notion of "quantum geometry." This is especially dramatic for $k = 1$. In this case $c = r$ for the simply laced groups ADE. [35] In this case there is in fact an equivalent sigma model whose target space is just the Cartan torus! This is part of "nonabelian bosonization" and the Frenkel-Kac-Segal construction.

• The Sugawara construction of L_0 makes it clear that it is the quadaratic Casimir for $L\mathfrak{g}$. Put differently, it is the Laplacian on the infinite-dimensional loop group.

Exercise

Show that

$$[L_n, J^a_m] = -m J^a_{n+m} \tag{4.89}$$

4.3 The Central Extension Of The Loop Group And The WZ Term

We will give a very beautiful geometrical construction of the cocycle for LG which defines the centrally-extended loop group.

The trick is to consider the group of maps DG from the *disk* to the group G, i.e. we introduce $DG = Map(D, G)$ where D is the disk. Note that the subgroup D_1G of maps such that $g|_{\partial D} = 1$ is a normal subgroup and $DG/D_1G \cong LG$, and explicit isomorphism being given by the restriction map.

Now, in contrast to LG, it is easy to write a central extension $\widetilde{DG}$ of the group DG:

$$(g_1, \lambda_1) \cdot (g_2, \lambda_2) = (g_1 g_2, \lambda_1 \lambda_2 f(g_1, g_2)) \qquad g_i \in DG \tag{4.90}$$

where

$$f(g_1, g_2) = \exp\left[2\pi i (6\pi c_1 k) \int_D \mathrm{Tr}(dg_2 g_2^{-1})(g_1^{-1} dg_1)\right] \tag{4.91}$$

Note that we have written our Ad-invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$ in terms of a definite trace Tr in some representation. For $SU(N)$ with the trace in the N dimensional representation $c_1 = 1/(24\pi^2)$.

Exercise

[35] For all simple groups $d/(g+1) = r$ where g is the Coxeter number.

a.) Check that (4.91) is indeed a group cocycle.

b.) Compute the corresponding central extension on the Lie algebra $D\mathfrak{g}$ and show that it is trivial when one of the elements vanishes on the boundary. Indeed, show that it is

$$24\pi^2 i c_1 k \oint_{S^1} \mathrm{Tr}\epsilon_1 d\epsilon_2 = ik \oint \mathrm{Tr}\epsilon_1 d\epsilon_2 \tag{4.92}$$

for $c_1 = 1/(24\pi^2)$.

Now, the beautiful observation is that, when g_1 and g_2 are equal to 1 on the boundary ∂D, we can consider them to define maps from $S^2 \to G$, and therefore we can define the WZ term. But, because of the identity we proved above:

$$WZ(g_1 g_2) = WZ(g_1) + WZ(g_2) + 6\pi k c_1 \int \mathrm{Tr}(dg_2 g_2^{-1})(g_1^{-1} dg_1) \tag{4.93}$$

the cocycle becomes a coboundary when restricted to the subgroup $D_1 G$. Therefore, the extension

$$1 \to U(1) \to \widetilde{DG} \to DG \to 1 \tag{4.94}$$

splits over the normal subgroup $D_1 G$, that is:

$$\psi : g \to (g, e^{iWZ(g)}) \qquad g \in D_1 G \tag{4.95}$$

is a group homomorphism from $D_1 G$ to $\widetilde{DG}$, and hence we can take a quotient

$$1 \to U(1) \to \widetilde{DG}/\psi(D_1 G) \to DG/D_1 G = LG \to 1 \tag{4.96}$$

to construct the loop group $\widetilde{LG} := \widetilde{DG}/\psi(D_1 G)$.

Finally, if we include L_0 then note that

$$\exp[\mathrm{i}\theta_0 L_0] g(\theta) \exp[-\mathrm{i}\theta_0 L_0] = g(\theta + \theta_0) \tag{4.97}$$

so L_0 generates rigid rotations of loops.

Remarks

1. The above construction of the central extension is due to J. Mickelsson. For a generalization to $Map(X, G)$ for arbitrary manifolds X see [37] and references therein.

2. At the Lie group level one can construct a semidirect product with the Virasoro group - the centrally extended diffeomorphism group of the circle.

3. The above presentation of the centrally extended loop group is very convenient for quantizing three-dimensional Chern-Simons theory on $D \times \mathbb{R}$. The group of gauge transformations is $D_1 G$. The flat gauge fields on the disk are parametrized by DG.

4.4 Relation To Anomalies

4.4.1 Coupling To External Gauge Fields

When we gauge the global $LG_{hol} \times LG_{anti-hol}$ symmetry we obtain the gauged WZW action (3.68)

♣OR THE ONE FOR $S+$ ♣

For simplicity put $A^{0,1} = 0$. Then, identifying the effective action in WZW theory with the Chern-Simons wavefunction

$$Z = \Psi[A_z] \tag{4.98}$$

we identify $\frac{\delta}{\delta A_z}\Psi$ with the one-point function of the current $\langle J^{0,1}(x)\rangle$ we can (3.47) as an anomaly equation:

$$D_{A,z}\langle J_{\bar{z}}\rangle = -\frac{k}{\pi}\bar{\partial}_{\bar{z}} A_z \tag{4.99}$$

4.4.2 Anomalies Of Weyl Fermions

The WZ term summarizes the anomalous variation of quantum actions. Let us sketch that briefly here in the two-dimensional case.

Let ψ_+ be a complex chiral fermion field in $1+1$ dimensions. Let us first explain the anomaly using elementary quantum field theory.

Let us couple the fermion to a $U(1)$ gauge field and define the effective action:

$$e^{-\Gamma(A_-)} := \int d\psi_+ d\bar{\psi}_+ e^{-\int \bar{\psi}_+(\partial_- + A_-)\psi_+} \tag{4.100}$$

Formally, we have

$$\Gamma(A_-) = -\log\det(\partial_- + A_-) \tag{4.101}$$

and formally the effective action is gauge invariant under the transformation (ϵ is real):

$$A_- \to A_- + id\epsilon(x^-) \tag{4.102}$$

because we can make the compensating transformation

$$\begin{aligned} \psi_+ &\to e^{-i\epsilon}\psi_+ \\ \bar{\psi}_+ &\to e^{i\epsilon}\bar{\psi}_+ \end{aligned} \tag{4.103}$$

to get an invariance of the action. However, 1-loop quantum effects spoil this formal gauge invariance. The noninvariance of the expression $\Gamma(A)$ is called the anomaly.

Let us derive an expression for the anomaly using the Feynman diagram expansion as in 13: That is, we write

$$\Gamma(A_-) = -\mathrm{Tr}Log(1 + \frac{1}{\partial_-}A_-) \tag{4.104}$$

Figure 13: Feynman graphs in the expansion of $\Gamma(A_-)$ in a power series in A_-.

Figure 14: Momentum q flows into the two point graph.

and expand in A. The quadratic term is the graph 14. The propagator for the fermion is

$$\langle \bar{\psi}_+(p)\psi_+(-p)\rangle = \frac{1}{p_- + i\epsilon \mathrm{sign}(p_+)} \tag{4.105}$$

Note the tricky sign on the iϵ term. One can understand this from the $m \to 0$ limit of the scalar propagator

$$\frac{1}{p^2 + m^2 + \mathrm{i}\epsilon}|_{m\to 0} = \frac{1}{p_+\big(p_- + \mathrm{i}\epsilon \mathrm{sign}(p_+)\big)} \tag{4.106}$$

Thus the Feynman graph is

$$\int dp_+ dp_- \frac{1}{(p_- + i\epsilon \mathrm{sign}(p_+))(p_- - q_- + \mathrm{i}\epsilon \mathrm{sign}(p_+ - q_+))} \tag{4.107}$$

Note that if the sign of both iϵ terms is positive then we can close the p_- integral in the other half-plane and get zero. Therefore, they must have opposite sign to get a nonzero result. Suppose $q_+ > 0$. Then we must have $0 < p_+ < q_+$ and the evaluation of the integral is straightforward and gives q_+/q_-. In this way we get

$$\Gamma(A_-) \sim \int d^2 q A_-(q) A_-(-q) \frac{q_+}{q_-} + \mathcal{O}(A^3) \tag{4.108}$$

It turns out that the higher graphs are gauge invariant, so under a gauge transformation

$$A_-(q) \to A_-(q) + iq_-\epsilon(q) \tag{4.109}$$

we have

$$\delta\Gamma(A_-) \sim \int d^2 q (iq_+ A_-)\epsilon(-q) = \int d^2 q F(q)\epsilon(-q) = \int d^2 x F(x)\epsilon(x) \tag{4.110}$$

This is the anomaly.

Now let us consider the nonabelian case with a collection of Fermi fields ψ_+^i transforming in the n of $SU(n)$. Then, coupling to an external $SU(n)$ gauge field the effective action becomes

$$e^{-\Gamma(A_-)} := \int d\psi_+ d\bar{\psi}_+ e^{-\int \bar{\psi}_{+i}(\delta^i{}_j \partial_- + (A_-)^i{}_j)\psi_+^j} \tag{4.111}$$

Formally the path integral is invariant under

$$\begin{aligned} \psi &\to g^{-1}\psi \\ \bar{\psi} &\to \bar{\psi} g \\ A &\to g^{-1} A g + g^{-1} dg \end{aligned} \tag{4.112}$$

However, the result of an analogous computation to the above gives:

$$\Gamma(A_-^g) - \Gamma(A_-) = \frac{i}{2\pi} \int \mathrm{Tr}(\partial_+ g g^{-1}) A_- + F(g) \tag{4.113}$$

where $F(g)$ is independent of A_-.
****

EXPLAIN THAT $F(g)$ IS THE WZ TERM:

Now, in two-dimensions we can write $A_- = h^{-1}\partial_- h$ for some group element h and hence we conclude

$$\Gamma(A_-) \sim i S^+_{WZW}(h) \tag{4.114}$$

Thus, the WZW Lagrangian summarizes the effective of quantum anomalies in a classical action.

WHAT ABOUT PATH DEPENDENCE WHEN $\partial_+ A_- \neq 0$?

GAUGED WZW ACTION

QUESTIONS OF ANALYTIC CONTINUATION

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4.5 Highest Weight Representations Of Loop Groups

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NEED SEPARATE SECTION ON Borel-Weil-Bott especially for LG/T.

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4.5.1 Integrable highest weight representations of $\widetilde{L\mathfrak{g}}$

♣This section does not clearly distinguish when you are working in the irrep and when you are working in the Verma module. ♣

When the target space group G is compact the representations we are interested in are those for which the energy can be bounded below. Because of the Sugawara form of the energy momentum tensor these are the *highest weight representations*.

To define these we introduce the notion of a *Verma module*, which is a certain kind of representation of a Lie algebra $\mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. For any vector $\lambda \in \mathfrak{t}^*$ (not necessarily in the weight lattice) we construct a representation $M(\lambda)$, called a Verma module, as follows.

Choose a splitting into positive and negative roots. Then, a Verma module $M(\lambda)$ is a cyclic module generated by a vector v satisfying

$$\begin{aligned} E^{\alpha} v &= 0 \qquad \forall \alpha \in \Phi^+ \\ H v &= \lambda(H) v \end{aligned} \tag{4.115}$$

One builds up the module by taking the span of $\prod_i X_i v$ with X_i negative step operators. Then, one has a key

Theorem $M(\lambda)$ has a maximal proper submodule $S(\lambda)$. The quotient

$$V(\lambda) := M(\lambda)/S(\lambda) \tag{4.116}$$

is irreducible.

Vectors in $S(\lambda)$ are called "null vectors": $S(\lambda)$ is a sum of highest weight Verma modules and the highest weight vectors are called primitive null vectors.

Example: $\mathfrak{g} = sl(2)$. We use the basis

$$\begin{aligned} [E^+, E^-] &= H \\ [H, E^{\pm}] &= \pm 2 E^{\pm} \end{aligned} \tag{4.117}$$

The translation to physics notation is $H = 2J^3, E^{\pm} = J^{\pm} = J^1 \pm iJ^2$. (With the convention $[J^a, J^b] = \mathrm{i}\epsilon^{abc} J^c$, for $a, b, c = 1, 2, 3$ and $\epsilon^{123} = +1$.)

The highest weight vector is usually denoted $|j\rangle$ with

$$\begin{aligned} E^+ |j\rangle &= 0 \\ H|j\rangle &= 2j|j\rangle \end{aligned} \tag{4.118}$$

We will also denote it by v_0.

The Verma module is spanned by $v_m := (E^-)^m |j\rangle$ for $m \geq 0$. It is infinite dimensional. A simple computation shows

$$H(E^-)^m |j\rangle = (2j - 2m)(E^-)^m |j\rangle \tag{4.119}$$

Let us search for a highest weight submodule. We need to compute:

$$\begin{aligned} E^+ (E^-)^m |j\rangle &= \sum_{i=0}^{m-1} (E^-)^{m-1-i} [E^+, E^-] (E^-)^i |j\rangle \\ &= m\Big(2j - (m-1)\Big)(E^-)^{m-1} |j\rangle \end{aligned} \tag{4.120}$$

Thus, there is a nontrivial submodule iff $2j \in \mathbb{Z}_{\geq 0}$. If $2j \notin \mathbb{Z}_{\geq 0}$ then $M(2j) = V(j)$ is an infinite dimensional irreducible representation. If $2j \in \mathbb{Z}_{\geq 0}$, then

$$v_{2j+1} := (E^-)^{2j+1}|j\rangle \tag{4.121}$$

generates the maximal submodule, isomorphic to $M(-2j-2)$. The quotient

$$V(j) := M(2j)/M(-2j-2) \tag{4.122}$$

is isomorphic to the finite-dimensional irreducible module of dimension $2j+1$.

Remarks:

1. By considering $E^{\pm} = J^1 \pm \mathrm{i}J^2$ we can consider these representations to be representations of $su(2)$. We can then introduce a Hermitian form on the Verma module so that $J^- = (J^+)^\dagger$. From what we have said we see that

$$\langle j|(J^+)^{2j+1}(J^-)^{2j+1}|j\rangle = 0 \tag{4.123}$$

On the other hand, this is the norm square

$$\| \, (J^-)^{2j+1}|j\rangle \, \|^2 \tag{4.124}$$

and hence in a unitary representation it must vanish. For this reason the vector v_{2j+1} is known as a *null vector*.

2. Only for those representations with $2j+1$ a positive integer can the representation of the Lie algebra be exponentiated to a representation of the Lie group. In fact, much more is true. $SU(2)/U(1) = SL(2,C)/B = CP^1$ is a complex manifold, and the representation is naturally identified with the holomorphic sections of a holomorphic line bundle over $SU(2)/U(1)$. This is a beautiful example of geometric quantization: For a compact simple Lie group G with maximal torus T the flag manifold G/T is a compact Kahler manifold, hence symplectic, and quantization of the phase space gives Hilbert spaces which are the irreducible representations of G. This is the Borel-Weil-Bott theorem. A quantum-mechanical approach uses the phase space path integral $\int \langle \lambda, g^{-1}\dot{g}\rangle - H(g)$. The path integral can be done exactly using localization yielding the Kirillov character theorem and the Weyl character formula. For an account of this see:

   1. A. Alekseev, L.D. Faddeev, Samson L. Shatashvili, "Quantization of symplectic orbits of compact Lie groups by means of the functional integral," J.Geom.Phys.5:391-406,1988.

   2. Richard J. Szabo, Equivariant localization of path integrals. e-Print: hep-th/9608068

   ****

   SOME OF ABOVE MATERIAL MOVES TO APPENDIX B.3

Now, for affine Lie algebras $\widetilde{L\mathfrak{g}}$ based on finite-dimensional semisimple Lie algebras $\mathfrak{g}$ one can develop the entire theory of Cartan subalgebras, roots, and weights. It turns out to be very useful to extend the algebra $\widetilde{L\mathfrak{g}}$ by including L_0. Then, choosing a Cartan-Weyl decomposition of $\mathfrak{g}$, a natural Cartan subalgebra of the resulting algebra is

$$H_0^i, K, L_0 \tag{4.125}$$

The inclusion of L_0 allows one to define a nondegenerate inner produce on the root space. Once again one can construct Verma modules: A highest weight vector generating a Verma module has

$$H_0^i|v\rangle = \lambda^i|v\rangle \tag{4.126}$$

and because of the Sugawara construction

$$L_0|v\rangle = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}|v\rangle \tag{4.127}$$

Then the highest weight condition is

♣Use either v or $|v\rangle$ or $|\lambda\rangle$ consistently. It is all muddled below. ♣

$$E_0^\alpha|v\rangle = 0 \qquad \alpha \in \Phi^+ \tag{4.128}$$

$$J_n^a|v\rangle = 0 \qquad n > 0, \forall a \tag{4.129}$$

Once again one can form a Verma module and then take a quotient to get the irreducible module.

One can easily see that the irreducible representation will be infinite dimensional because already the subalgebra generated by H_n^i is a Heisenberg algebra

$$[H_n^i, H_m^j] = kn\delta^{ij}\delta_{n+m,0} \tag{4.130}$$

and acting with H_{-n}^i on the vacuum with $n > 0$ will not produce any null vectors.

At this point one restricts attention to the *integrable* highest weight representations. To construct these representations we note that for each root α of $\mathfrak{g}$ and integer n we can form an $sl(2)$ subalgebra generated by $J^+ \sim E_n^\alpha, J^- \sim E_{-n}^{-\alpha}$.

For simplicity, let us assume that $\mathfrak{g}$ is of A-D-E type, so we can normalize the roots to $\alpha^2 = 2$. Then

$$[E_n^\alpha, E_{-n}^{-\alpha}] = \alpha^i H_0^i + nK \tag{4.131}$$

Thus we have an isomorphism with the standard basis of $sl(2)$:

$$\begin{aligned} H &\to \alpha^i H_0^i + nK \\ E^+ &\to E_n^\alpha \\ E^- &\to E_{-n}^{-\alpha} \end{aligned} \tag{4.132}$$

Let us call this $sl(2)_{n,\alpha}$

Definition: An *integrable* highest weight representation $V(\lambda)$ is one such that $V(\lambda)$ can be decomposed into sums of *finite dimensional* representations of the $sl(2)$ subalgebras $sl(2)_{n,\alpha}$.

In other words, once we diagonalize the Cartan subalgebra (H_0^i, K, L_0) to produce weight vectors each state of definite weight vector must generate a finite-dimensional representation under $sl(2)_{n,\alpha}$.

♣NEED TO EXPLAIN SOME TERMS AND MORE DETAILS HERE ♣

Considering this for $sl(2)_{0,\alpha}$ we find that λ must be a dominant integral weight of $\mathfrak{g}$. Then we learn that $nk + \alpha \cdot \lambda \in \mathbb{Z}_{\geq 0}$ for $n \geq 0$ and all α. The strongest constraint comes from $n = 1$ with $\alpha = -\theta$, where θ is the highest root. So k must be a nonnegative integer and the weights of the integrable representations are constrained by

$$k \geq \lambda \cdot \theta \tag{4.133}$$

In physics this is often derived from the constraint of *unitarity*. The basic computation is:

♣Here $v = |v\rangle = |\lambda\rangle$. Use a single notation! ♣

$$\begin{aligned} 0 &\leq \parallel E_{-1}^{\alpha}|\lambda\rangle \parallel^2 \\ &= \langle\lambda|E_1^{-\alpha}E_{-1}^{\alpha}|\lambda\rangle \\ &= \langle\lambda|[E_1^{-\alpha}, E_{-1}^{\alpha}]|\lambda\rangle \\ &= (k - (\alpha, \lambda)) \parallel |\lambda\rangle \parallel^2 \\ &= [(k - (\theta, \lambda)) + (\theta - \alpha, \lambda)] \parallel |\lambda\rangle \parallel^2 \end{aligned} \tag{4.134}$$

Here θ is the highest root. Thus, the constraint of unitarity is satisfied if λ should is a dominant weight, and $k - (\theta, \lambda) \geq 0$.

Let us consider the case of $\widetilde{Lsu(2)}$. Let us choose a highest weight vector $|j\rangle$ defined by:

$$\begin{aligned} J_0^3|j\rangle &= j|j\rangle \\ K|j\rangle &= k|j\rangle \\ J_0^+|j\rangle &= 0 \\ J_n^a|j\rangle &= 0 \qquad n > 0, \forall a \end{aligned} \tag{4.135}$$

From the Sugawara formula we compute

$$L_0|j\rangle = \frac{j(j+1)}{k+2}|j\rangle \tag{4.136}$$

We begin by acting with J_0^- to fill out a spin j multiplet of $sl(2)_{0,+}$:

$$|j\rangle, J_0^-|j\rangle, (J_0^-)^2|j\rangle, \ldots, (J_0^-)^{2j}|j\rangle \tag{4.137}$$

Figure 15: Schematic drawing of points with nonzero multiplicity in the highest weight representation of $\widetilde{Lsu(2)}_k$.

Figure 16: Degeneracies of $\widetilde{Lsu(2)}_k$ highest weight representations for $k = 1, j = 0$ and $k = 3, j = 0$. From I. Affleck, et. al., J. Phys. A Math. Gen. **22**(1989)511

Notice that these states all have the same L_0 eigenvalue.

Next we act with J^+_{-1} on the highest weight vector $J^+_{-1}|j\rangle$. We can now act with J^-_0 multiple times. In general we will get the states in the Clebsch-Gordon series $(j) \otimes (1) = (j+1) \oplus (j-1)$.

One the other hand, we can also act with $sl(2)_{1,-}$ generated by

$$\begin{aligned} H &\to k - 2J^3_0 \\ E^+ &\to J^-_1 \\ E^- &\to J^+_{-1} \end{aligned} \tag{4.138}$$

Since $H|j\rangle = (k-2j)|j\rangle$ for this algebra we have a spin $k/2 - j$ representation and we learn that for an integrable representation:

$$0 \le j \le \frac{k}{2} \tag{4.139}$$

and moreover

$$(J^+_{-1})^{k+1-2j}|j\rangle \sim 0 \tag{4.140}$$

is a null vector.

Exercise

Check that

$$
\begin{aligned}
H &\to k - 2J_0^3 \\
E^+ &\to J_1^- \\
E^- &\to J_{-1}^+
\end{aligned}
\tag{4.141}
$$

$$
\begin{aligned}
H &\to k + 2J_0^3 \\
E^+ &\to J_1^+ \\
E^- &\to J_{-1}^-
\end{aligned}
\tag{4.142}
$$

Are both isomorphisms with $sl(2)$. Describe some null vectors associated with the second algebra.

Exercise

Work out the first few levels of the highest weight representations for affine $su(2)$ at level $k = 1$.

****

ABOVE SECTION REQUIRES MUCH IMPROVEMENT: LIST THE FULL STRUCTURE OF NULL VECTORS - CONSTRUCT SOME BY HAND AND THEN IMPOSE THE AFFINE WEYL GROUP. IN THIS WAY YOU GET THE BGG RESOLUTION IN THIS CASE. THEN CITE THE GENERAL BGG RESOLUTION. Nice ref: Pressley-Segal, Loop Groups.

Null vectors:

Note that

$$
(E_0^-)^{2j+1}|j\rangle \tag{4.143}
$$

is standard from finite-dimensional representation theory and has spin $2J_0^3 = -j - 1$ so generates a Verma of that type.

Now

$$
(E_{-1}^+)^{k+1-2j}|j\rangle \tag{4.144}
$$

is annihilated by E_0^+ but has spin

$$
2J_0^3 = (2k+2) - 2j \tag{4.145}
$$

(obtained by flipping $j \to (k+2) - (j+1)$ on *******)

So, now we consider

$$
(E_0^-)^{2k-2j+1}(E_{-1}^+)^{k+1-2j}|j\rangle \tag{4.146}
$$

which has spin $2J_0^3 = * * * * * * * * * * * * * * * * * *$, and so forth.

****

****

4.5.2 Integrable Highest Weight Representations Of The Group

****

Remarks:

• The representations are called integrable because these are the representations which extend to representations of $\widetilde{LG}$. Indeed, they are unitary representations of $\widetilde{LG}$. Moreover the Borel-Weil theory extends to this infinite dimensional case: $\widetilde{LG}/T$ is an infinite-dimensional complex manifold $\widetilde{LG}/T \cong \widetilde{LG}_C/B$ where B are the loops which extend holomorphically into the disk. LG/T is also an infinite-dimensional phase space, and geometric quantization of this phase space leads to the irreps of $\widetilde{LG}$. The Borel-Weil theory for loop groups is developed from the mathematical point of view in Pressley-Segal. There is again a path integral treatment in the physics literature. See ????

****

Let G be simple and compact. The centrally-extended loop group constructed above will be denoted simply G_k for $k \in \mathbb{Z}_+$. [36]

What can we say about the representations of G_k? Clearly there are many. For example, G_k has a homomorphic image LG so, choosing any representation $\rho : G \to \mathrm{Aut}(V)$ of G (for example, a finite-dimensional irreducible representation of G) and a point z_0 on the circle we can define the evaluation representation:

$$\widehat{\rho} : G_k \to \rho(g(z_0)) \tag{4.147}$$

with carrier space V. Note that these representations do <u>not</u> interact well with L_0, since L_0 translates $z_0 \to z_0 e^{\mathrm{i}\theta}$.

It turns out that G_k has a <u>finite</u> set of irreducible representations with L_0 bounded below. These representations are naturally constructed as highest weight representations of the Lie algebra and are known as *integrable representations* because they extend from representations of the Lie algebra to the Lie group.

The integrable representations are graded by L_0 and the spectrum is bounded below. The lowest weight space under L_0 is itself an irreducible representation of G, and has a highest weight λ corresponding to an element of

$$\Lambda_{wt}/W \tag{4.148}$$

where W is the Weyl group and Λ_{wt} is the weight lattice. After making a choice of simple roots for $\mathfrak{g}$ the highest weight of an irreducible representation of G can be labeled by a dominant highest weight $\lambda \in \Lambda_{wt}$. Recall this means that

$$\lambda = \sum n_i \lambda^{(i)} \tag{4.149}$$

where $n_i \geq 0$ and $\lambda^{(i)}$ is a basis of fundamental weights dual to the simple roots.

[36] In general for a compact group it can be shown that the central charge should be regarded as an element of $H^4(BG;\mathbb{Z})$. For G simple, compact, and connected this cohomology group is isomorphic to $\mathbb{Z}$ and we can consider the central extension to be an integer.

Now for the case of G_k the irreducible representations are labeled by the quotient

$$\Lambda_{wt}/\widehat{W}^{(k)} \tag{4.150}$$

where $\widehat{W}^{(k)}$ is the level k affine Weyl group. It is a discrete crystallographic subgroup of the group of affine transformations of $\mathfrak{t}^\vee$. As a group it is isomorphic to the semidirect product of the Weyl group W with the translation group by the coroot lattice Λ_{crt} but we denote it by $\widehat{W}^{(k)}$ because the translations act by

$$\{\sigma|v\} : \lambda \mapsto \sigma(\lambda) + kv \qquad \sigma \in W, v \in \Lambda_{crt} \tag{4.151}$$

It is useful to know that this is a Coxeter group, generated by reflections. These include the Weyl reflections and the reflection in the hyperplane $(\lambda, \theta) = k$, where $k \in \mathbb{Z}_+$, θ is the highest root and we use a normalization of the Killing form so that $(\theta, \theta) = 2$. A fundamental chamber for this action in Λ_{wt} is the finite set of dominant weights satisfying:

$$(\lambda, \theta) \leq k \tag{4.152}$$

This condition is usually derived in conformal field theory by using unitarity and a null-vector.

Example 1 $G = SU(2)$. Then $\theta = \alpha$ and $\lambda = j\alpha$ where $j \in \frac{1}{2}\mathbb{Z}_+$ is known in physics as the spin. (Mathematicians normally would write $\lambda = n_1\lambda^{(1)}$ where $\lambda^{(1)} = \frac{1}{2}\alpha_1$ is the fundamental weight. Thus $n_1 = 2j$ is twice the spin.) The irreducible representation has of $SU(2)$ with weight $\lambda = j\alpha_1$ has dimension $2j + 1$. The lattice Λ_{wt} is isomorphic to $\mathbb{Z}$. The hyperplane in $\mathfrak{t}^\vee \cong \mathbb{R}$ is $(x\alpha, \alpha) = k$ or $x = k/2$. So reflection in the hyperplane takes $j \to \frac{k}{2} - j$. Therefore a fundamental domain for the affine Weyl group is:

$$0 \leq j \leq \frac{k}{2} \tag{4.153}$$

Note that $\widehat{W}^k$ in this case is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$, the infinite dihedral group.

Example 2 $G = SU(3)$. For $SU(3)$ we can choose two simple roots. The standard choice is

$$\begin{aligned} \alpha_1 &= (\sqrt{2}, 0) \\ \alpha_2 &= (-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}) \end{aligned} \tag{4.154}$$

$$\begin{aligned} \lambda^1 &= \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \\ \lambda^2 &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 \end{aligned} \tag{4.155}$$

Now $\theta = \alpha_1 + \alpha_2$. The integrable weights at level k are $n_1\lambda^1 + n_2\lambda^2$ with $n_i \in \mathbb{Z}_+$ and $n_1 + n_2 \leq k$.

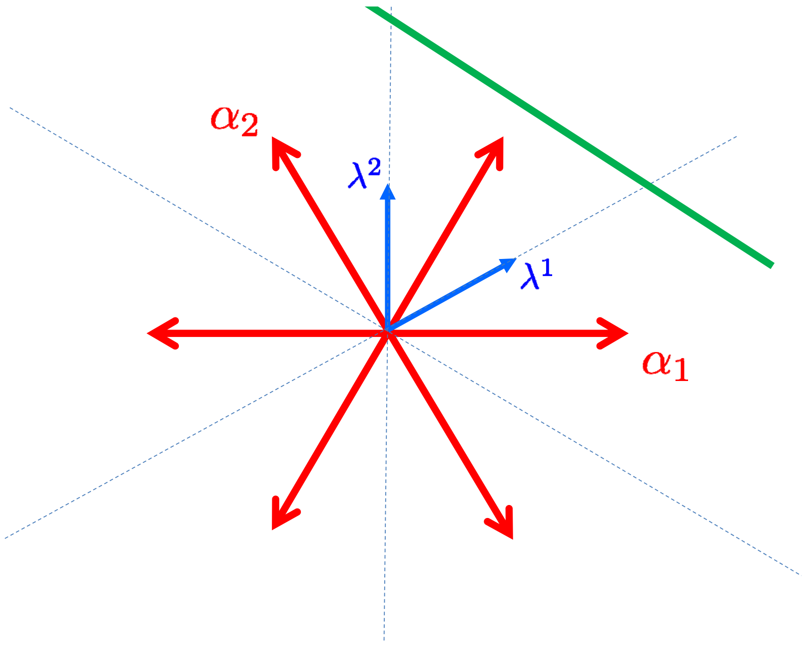

Figure 17: The root and weight lattice for $SU(3)$. A standard set of simple roots α_1, α_2 is shown along with fundamental weights λ^1, λ^2. The fundamental Weyl chamber is the positive cone spanned by these two weights. The highest root is $\theta = \alpha_1 + \alpha_2$. The heavy green line is the line $(\lambda, \theta) = k$ for some positive integer k. The affine Weyl chamber is the region $(\lambda, \theta) \leq k$ and the integrable weights at level k is the intersection of the weight lattice with the fundamental Weyl chamber.

4.5.3 Characters of integrable highest weight representations

The *characters* of the representations $V(\lambda)$ are

$$\mathrm{Tr}_{V(\lambda)} q^{L_0} e^{2\pi i \zeta \cdot H_0} \tag{4.156}$$

where $\zeta \in \mathfrak{t}^* \otimes \mathbb{C}$ and $q = e^{2\pi i \tau}$ with τ in the upper half-plane.

By bounding the dimensions of weight spaces one can learn that this is an entire function of ζ and of τ. Note that it is crucial to include L_0 to get a finite expression.

There are four ways to derive explicit formulae for characters:

1. Algebraic approach based on the Verma module construction.

2. Differential equations following from recursion relations for weight multiplicities.

3. Applying fixed point formula and the Borel-Weil-Bott theorem.

4. Writing a phase space path integral with action $S \sim \int \langle \lambda, g^{-1}\dot{g} \rangle - \int dt H(g(t))$ for a particle moving on G/T and using the stationary phase method - which turns out to be exact. The sum over the Weyl group is a sum over classical solutions of the equations of motion.

We'll sketch approach (1). The character of a Verma module is straightforward, because it is essentially a Fock space. Abstractly it can be written as

$$\frac{e^{\lambda}}{\prod_{\alpha>0}(1-e^{-\alpha})} \tag{4.157}$$

For example, for $M(2j)$ for $SU(2)$ we have

$$\mathrm{Tr}_{M(2j)}u^{2J_3} = \frac{u^{2j}}{1-u^{-2}} \tag{4.158}$$

For $2j \in \mathbb{Z}_{\geq 0}$ the irreducible representation is a quotient $M(2j)/M(-2j-2)$ and hence the $SU(2)$ the characters are:

$$\chi_j(u) := \mathrm{Tr}_{V(j)}u^{2J_3} = \mathrm{Tr}_{M(2j))}u^{2J_3} - \mathrm{Tr}_{M(-2j-2)}u^{2J_3} = \frac{u^{2j}}{(1-u^{-2})} - \frac{u^{-2j-2}}{(1-u^{-2})} = \frac{u^{2j+1}-u^{-2j-1}}{u-u^{-1}} \tag{4.159}$$

In general, the characters of irreducible representations of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ for $\mathfrak{g}$ semisimple are linear combinations of those of the Verma modules. Using the Weyl group invariance one proves the Weyl-Kac character formula:

$$\chi_{\lambda} = \frac{\sum_{w\in W}\epsilon(w)e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})} \tag{4.160}$$

For example, for $\widetilde{Lsu(2)}$ we have

$$\mathrm{Tr}_{M(j)}q^{L_0}e^{2\pi i\zeta H_0} = \frac{q^{\frac{(j+\frac{1}{2})^2-\frac{1}{4}}{k+2}}e^{2\pi i\zeta(2j)}}{(1-e^{-4\pi i\zeta})\prod_{n=1}^{\infty}(1-q^n)(1-e^{4\pi i\zeta}q^n)(1-e^{-4\pi i\zeta}q^n)} \tag{4.161}$$

The factors in the denominator account for the action with operators J_0^-, H_{-n}, J_{-n}^+, J_{-n}^- on the highest weight vector, respectively.

Now, the affine Weyl group is in general $\Lambda_{weight} \ltimes W$ where W is the Weyl group of $\mathfrak{g}$. For $\mathfrak{g} = su(2)$ this is just $\mathbb{Z} \ltimes \mathbb{Z}_2$ and it acts on the weights as

$$j \to j + m(k+2) \qquad m \in \mathbb{Z} \tag{4.162}$$

for the even transformations and

$$j \to -1 - j - m(k+2) \qquad m \in \mathbb{Z} \tag{4.163}$$

for the odd transformations. For example: $(J_0^-)^{2j+1}|j\rangle$ corresponds to the odd Weyl reflection with $m = 0$ and $(J_{-1}^+)^{k+1-2j}|j\rangle$ corresponds to the odd Weyl reflection with $m = -1$, etc.

In this way we derive

$$\begin{aligned} &Tr_{V(j)}[q^{L_0}e^{i2\pi z 2J_0^3}] = \\ &= \frac{\sum_m q^{\frac{1}{k+2}[(j+\frac{1}{2}+m(k+2))^2-1/4]}\left(e^{i4\pi z(j+\frac{1}{2}+m(k+2))} - e^{-i4\pi z(j+\frac{1}{2}+m(k+2))}\right)}{(e^{2\pi iz}-e^{-2\pi iz})\prod(1-e^{i4\pi z}q^n)(1-e^{-i4\pi z}q^n)(1-q^n)} \end{aligned} \tag{4.164}$$

for the loop group $\widehat{SU(2)}_k$.

Now comes a beautiful trick: For $j = 0, k = 0$ the only unitary irreducible representation is the trivial representation. Therefore, the numerator equals the denominator.

Introduce the level k theta function defined by

$$\Theta_{\mu,k}(z,\tau) := \sum_{n\in\mathbb{Z}} q^{k(n+\mu/(2k))^2} y^{(\mu+2kn)} = \sum_{\ell=\mu \mathrm{mod} 2k} q^{\ell^2/(4k)} y^{\ell} \tag{4.165}$$

with $y = e^{2\pi i z}$. The Riemann theta function we introduced for the particle on a circle corresponds to level 1/2. Note:

$$\begin{aligned} \Theta_{\mu,k}(\omega,\tau) &= \Theta_{\mu+2ka,k}(\omega,\tau) \qquad a \in \mathbb{Z} \\ \Theta_{\mu,k}(-\omega,\tau) &= \Theta_{2k-\mu,k}(\omega,\tau) = \Theta_{-\mu,k}(\omega,\tau) \end{aligned} \tag{4.166}$$

We recognize the numerator for $j = 0, k = 0$ as $\Theta_{1,2}(z,\tau) - \Theta_{-1,2}(z,\tau)$, and hence

$$\Theta_{1,2}(z,\tau) - \Theta_{-1,2}(z,\tau) = (e^{2\pi i z} - e^{-2\pi i z}) \prod (1 - e^{i4\pi z} q^n)(1 - e^{-i4\pi z} q^n)(1 - q^n) = -i\vartheta_1(2z,\tau) \tag{4.167}$$

Next comes a conceptually important step. The characters turn out to have very beautiful mathematical properties if we modify the definition of the character slightly and define:

$$\chi_j^k(z,\tau) := \mathrm{Tr}_{V(j)} q^{L_0 - c/24} e^{2\pi i z (2J_0^3)} \tag{4.168}$$

This might look unnatural from the point of view of Lie algebra theory, but it is well-motivated by physics: We are subtracting the groundstate energy.

Now, we can write the elegant formula:

$$\begin{aligned} \chi_j^k(z,\tau) &:= \mathrm{tr}\, q^{L_0 - c/24} e^{2\pi i z (2J_0^3)} \\ &= \frac{\Theta_{2j+1,k+2}(z,\tau) - \Theta_{-2j-1,k+2}(z,\tau)}{\Theta_{1,2}(z,\tau) - \Theta_{-1,2}(z,\tau)} \\ &= q^{(\ell+1)^2/(4(k+2)) - 1/8} \chi_j(y) + \cdots \end{aligned} \tag{4.169}$$

where $0 \le j \le k/2$ and j is half-integral.

4.5.4 Modular properties

The characters satisfy many beautiful mathematical properties. They define examples of *Jacobi forms*. When specialized to $\zeta = 0$ they form a vector of *modular forms* under $PSL(2,\mathbb{Z})$.

Recall that $SL(2,\mathbb{Z})$ is the subgroup of $SL(2,\mathbb{R})$ with integral matrix elements. It is generated by S, T defined in Appendix C in equations (C.2) and (C.3) respectively. ♣Redundant with equations in Appendix D. ♣

$SL(2,\mathbb{Z})$ acts on (τ, z) by

$$(\tau, z) \to \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \tag{4.170}$$

Note that the action on τ factors through $PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\langle -1\rangle$. Note that $T : \tau \to \tau+1, S : \tau \to -1/\tau$.

By using the Poisson summation formula one checks the transformation laws of level k theta functions under the generators of $SL(2,\mathbb{Z})$:

$$\Theta_{\mu,k}(\omega,\tau+1) = e^{2\pi i \frac{\mu^2}{4k}}\Theta_{\mu,k}(\omega,\tau) \tag{4.171}$$

$$\Theta_{\mu,k}(-\omega/\tau,-1/\tau) = (-i\tau)^{1/2}e^{2\pi i k\omega^2/\tau}\sum_{\nu=0}^{2k-1}\frac{1}{\sqrt{2k}}e^{2\pi i\frac{\mu\nu}{2k}}\Theta_{\nu,k}(\omega,\tau) \tag{4.172}$$

From (4.172) we deduce that

$$(\Theta_{1,2}-\Theta_{-1,2})(-z/\tau,-1/\tau) = i(-i\tau)^{1/2}e^{4\pi i z^2/\tau}(\Theta_{1,2}-\Theta_{-1,2})(z,\tau) \tag{4.173}$$

and then a short computation yields:

$$\chi_j^{su(2)}(-1/\tau,-z/\tau) = e^{2\pi i k z^2/\tau}\sum_{j'=0}^{k/2}S_{jj'}\chi_{j'}^{su(2)}(\tau,z) \tag{4.174}$$

with

$$S_{jj'} = \sqrt{\frac{2}{k+2}}\sin\frac{\pi(2j+1)(2j'+1)}{k+2}. \tag{4.175}$$

The action of T is easily computed to be:

$$\chi_j^k(\tau+1,z) = e^{2\pi i\left(\frac{(\ell+1)^2}{4(k+2)}-\frac{1}{8}\right)}\chi_j^k(\tau,z) = e^{2\pi i\frac{2\ell(\ell+2)-k}{8(k+2)}}\chi_j^k(\tau,z) \tag{4.176}$$

where $\ell = 2j$. These two transformations generate a unitary representation of the modular group $SL(2,\mathbb{Z})$.

Remarks

1. In general, the characters are given by the Weyl-Kac character formula. Just as the Weyl character formula can be written

$$\frac{\sum_{w\in W} e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W} e^{w(\rho)-\rho}} \tag{4.177}$$

the Weyl-Kac character formula can be written in the identical form, where we replace the sum over the Weyl group by the sum over the affine Weyl group and λ,ρ are replaced by suitable affine weights. We recognize the structure of the sum over the affine Weyl group in equation (**??**): The theta functions come from the sum over $\mathbb{Z}$ and the difference of the theta functions comes from the nontrivial reflection in the Weyl group of $SU(2)$.

2. In general one can put $\lambda = 0$ and get the denominator identity:

$$\sum_{w\in W}\epsilon(w)e^{w(\rho)} = \prod_{\alpha>0}(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}) \tag{4.178}$$

3. There are very interesting generalizations of (4.178) to generalized Kac-Moody algebras. [NEED CITATIONS HERE!]

4. It turns out that the representation theory of G_k is closely related to that of the corresponding quantum group when q is a suitable root of unity:

$$q = \exp(\frac{2\pi \mathrm{i}}{k+h}) \tag{4.179}$$

See the book by Fuchs for a detailed exposition.

5. There is a generalization of the above story to a much wider class of two-dimensional conformal field theories known as "rational conformal field theories."

Exercise

Explain carefully why the irreducible representation for $j = k = 0$ is the trivial representation.

Show that $M(j = 0, k = 0)/S(j = 0, k = 0)$ is one-dimensional.

Exercise

Find the integrable level k representations for $\widehat{su}(3)$, and explain how they are distributed on the weight lattice.

4.6 The Hilbert Space Of WZW And The One-Loop Partition Function

Hilbert space and one-loop partition function.

Note: Large k limit: Particle moves on group G. Truncation of $L^2(G)$ decomposition.

Conformal blocks on torus. S and T matrix.

In the case of a compact group G the Hilbert space has a beautiful structure. It turns out that the decomposition of the Hilbert space with respect to the quantum $\widetilde{LG}_L \times \widetilde{LG}_R$ symmetry encodes each of the unitary irreducible representations precisely once. Thus, it is a perfect analog of the Peter-Weyl theorem for compact groups.

We have

$$\mathcal{H} = \oplus_{0 \leq \lambda \cdot \theta \leq k} \overline{V_\lambda} \otimes V_\lambda \tag{4.180}$$

where

$$0 \leq \lambda \cdot \theta \leq k \tag{4.181}$$

and θ is the highest root of $\mathfrak{g}$. For the $SU(2)$ level k model we have in particular:

$$\mathcal{H} = \oplus_{0 \le j \le k/2} \overline{V_j} \otimes V_j \tag{4.182}$$

The modularity of the characters now receives a beautiful explanation. Consider the partition function:

$$Z = \mathrm{Tr}_{\mathcal{H}} e^{-2\pi\beta H + 2\pi i\theta P} = \mathrm{Tr}_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - c/24} \tag{4.183}$$

where

$$H = L_0 + \tilde{L}_0 - c/12 \tag{4.184}$$

$$P = L_0 - \tilde{L}_0 \tag{4.185}$$

and

$$\tau = \theta + i\beta \tag{4.186}$$

so that

$$Z = \sum_\lambda \overline{\chi_\lambda(\tau,0)} \chi_\lambda(\tau,0) \tag{4.187}$$

♣REALLY NEED TO ADD COUPLING TO FLAT GAUGE FIELDS HERE!!! ♣

Figure 18: Identifications in the $z = \sigma + it$ plane induced by taking a trace. We propagate in Euclidean time β and then twist by a shift $2\pi\theta$ in the σ direction before identifying states.

The partition function (4.183) has the interpretation of being the path integral on a flat torus with modular parameter τ as shown in 18. As we will see in section 2.2.12 modular transformations on τ are equivalent to the action of nontrivial diffeomorphisms on the torus.

On the other hand, the partition function must be diffeomorphism invariant. The effect of the Weyl scaling on the partition function is zero because of the conformal anomaly and because the background metric is flat. Therefore, the partition function must be modular invariant. This is achieved since the χ_λ transform in a unitary representation of the modular group.

Remark: There are different modular invariant combinations of characters. That is, it is possible to find nonnegative integers $N_{j,\bar{j}}$ not proportional to the unit matrix so that

$$\sum_{j,\bar{j}} N_{j\bar{j}} \chi_j \overline{\chi_{\bar{j}}} \tag{4.188}$$

is modular invariant. These correspond to other theories.

For $G = SU(2)$ the modular invariants have an ADE classification. as shown in a beautiful paper of Capelli, Itzkyson, Zuber.

For $G = SU(3)$ the classification was done by Terry Gannon (hep-th/9212060, hep-th/9404185). The paper

hep-th/9604104: "Comments on the Links between su(3) Modular Invariants, Simple Factors in the Jacobian of Fermat Curves, and Rational Triangular Billiards" by M. Bauer, A. Coste, C. Itzykson, P. Ruelle

finds fascinating links to various branches of mathematics.

For higher rank groups the complete classification is not known(?)

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MATHEMATICIANS HAVE SOME PROBLEM WITH THE NAIVE DIAGONAL MODULAR INVARIANT. I learned of this from A. Neitzke. (May 8, 2019)

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4.7 Primary Fields

State-operator correspondence: Primary vertex operators

Holomorphic factorization: The chiral vertex operators.

COMMENT: VERTEX OPERATOR ALGEBRAS and the purely algebraic approach. (Borcherds, FLM, Lepowsky-Huang, etc.)

OPE - fusion rules: F-matrix. Consistency: Pentagon.

Surprise! S diagonalizes the fusion rules! Verlinde formula.

4.7.1 Semisimple Algebras

Remark: "Verlinde formula" for a general semisimple algebra. Give example of finite group ring Exercise: Fusion rules and Verlinde formula in finite group theory.

In general this is all there is to it. However, one can say a little more when the algebra $(\mathcal{C}, \theta)$ is *semisimple*.

The most useful characterization of semisimplicity is the following. The structure constants [37]

$$\phi_\mu \phi_\nu = N_{\mu\nu}^{\lambda} \phi_\lambda \tag{4.189}$$

defines a set of matrices via the left-regular representation, $L(\phi_\mu)$, with matrix elements $N_{\mu\nu}^{\lambda}$. Since $\mathcal{C}$ is commutative these are commuting matrices. Then:

Definition : $\mathcal{C}$ is semi-simple iff the matrices $L(\phi_\mu)$ are simultaneously diagonalizable.

Thus in the semisimple case we can find a matrix S such that:

$$N_{\mu\nu}{}^{\lambda} = \sum_x S_\nu{}^x \Lambda_x^{(\mu)} (S^{-1})_x{}^{\lambda} \tag{4.190}$$

[37] The structure constants $N_{\mu\nu}^{\lambda}$ need not be integral. But in many interesting examples there is a basis for the algebra in which they are in fact integral.

where $\Lambda_x^{(\mu)}$ are the different eigenvalues of $L(\phi_\mu)$.

Now choose a basis such with the index μ running over values $\mu = 0,\dots,n$, and take $\phi_0 = 1_{\mathcal{C}}$, the multiplicative identity. Putting $\mu = 0$ in equation (4.190) leads to a trivial identity, but putting $\nu = 0$ and using $N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda$ so that $N_{\mu 0}^\lambda = \delta_\mu{}^\lambda$ we see that $S_0{}^x\Lambda_x^{(\mu)} = S_\mu{}^x$ since this is the matrix inverse of S^{-1}. So:

$$S_0{}^x\Lambda_x^{(\mu)} = S_\mu{}^x \qquad \text{no sum on x} \tag{4.191}$$

Plugging back into (4.190) we get:

$$N_{\mu\nu}{}^\lambda = \sum_x \frac{S_\nu{}^x S_\mu{}^x (S^{-1})_x{}^\lambda}{S_0{}^x} \tag{4.192}$$

Note that $\theta(\phi_\mu\phi_\nu\phi_\lambda) = N_{\mu\nu}^{\lambda'}Q_{\lambda'\lambda} := N_{\mu\nu\lambda}$ is totally symmetric on μ,ν,λ. Suppose we further restrict attention to a basis $\{\phi_\mu\}$ so that $\theta(\phi_\mu) = \delta_{\mu,0}$. Then taking the trace of (4.189) we learn that $Q_{\mu\nu} = N_{\mu\nu}^0$ and then (4.192) gives

$$Q_{\mu\nu} = \sum_x S_\nu{}^x S_\mu{}^x \frac{(S^{-1})_x{}^0}{S_0{}^x} \tag{4.193}$$

so that

$$N_{\mu\nu\lambda} = \sum_x S_\mu{}^x S_\nu{}^x S_\lambda{}^x \frac{(S^{-1})_x{}^0}{(S_0{}^x)^2} \tag{4.194}$$

If we form the linear combinations

$$\epsilon_x = \sum_\mu S_0{}^x (S^{-1})_x{}^\mu \phi_\mu \tag{4.195}$$

then the ϵ_x serve as a set of basic idempotents, that is,

$$\begin{aligned} \mathcal{C} &= \oplus_x \mathbb{C}\epsilon_x \\ \epsilon_x\epsilon_y &= \delta_{x,y}\epsilon_y \end{aligned} \tag{4.196}$$

Moreover, if we choose the natural normalization $\theta(\phi_\mu) = \delta_{\mu,0}$ then

$$\theta_x := \theta_{\mathcal{C}}(\epsilon_x) = S_0{}^x (S^{-1})_x{}^0 \tag{4.197}$$

Here θ_i are some nonzero complex numbers. The unordered set $\{\theta_x\}$ is the only invariant of a finite dimensional commutative semisimple Frobenius algebra.

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SOME OF THE FOLLOWING MATERIAL SHOULD BE MOVED TO CHAPTER OF TOPOLOGICAL FIELD THEORY: EXAMPLE OF 2D TFT:

Note that in this case the characteristic element is simply

$$H = \sum_x \frac{1}{\theta_x} \epsilon_x \tag{4.198}$$

and hence the vacuum amplitude on a genus $g \geq 0$ surface is

$$Z_g = \sum_x \theta_x^{1-g} \tag{4.199}$$

Remarks:

1. If we consider the sum over all genera then the sum only converges when $|\theta_x| > 1$ (with conditional convergence on the unit circle), in which case it is:

$$Z_{\text{string}} = \sum_x \frac{\theta_x}{1 - \theta_x^{-1}} \tag{4.200}$$

2. Note that nothing has fixed the overall normalization of the matrix S at this point. In some cases S will be unitary so that $(S^{-1})_x{}^0 = (S_0{}^x)^*$. Moreover, if the matrix elements $S_0{}^x$ can be taken to be real then we have a nice simplification of (4.194):

$$N_{\mu\nu\lambda} = \sum_x \frac{S_\mu{}^x S_\nu{}^x S_\lambda{}^x}{S_0{}^x} \tag{4.201}$$

   This is how the Verlinde formula is usually stated.

Exercise
Show that the eigenvalues $\Lambda_x^{(\rho)}$ satisfy the algebra

$$\Lambda_x^{(\mu)} \Lambda_x^{(\nu)} = \sum_\lambda N_{\mu\nu}^\lambda \Lambda_x^{(\lambda)} \tag{4.202}$$

Exercise
A natural question in field theory is whether the vacuum amplitudes of a theory completely determine all the amplitudes in the theory.
Investigate this for the case of a semisimple 2d TFT.

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4.7.2 An Example Of Semisimple Fusion Rules: Finite Group Theory

Let G be a finite group. The space of complex-valued functions $C[G]$ is a C^* algebra (see below) with the obvious product given by pointwise multiplication

$$f_1 \cdot f_2(g) := f_1(g) f_2(g) \tag{4.203}$$

Let $\mathcal{C}$ be the subspace of class functions, that is, functions such that

$$f(hgh^{-1}) = f(g) \qquad \forall g, h \in G \tag{4.204}$$

This is the space of functions on the the (finite) set of conjugacy classes of G.

There are two natural bases of functions for $\mathcal{C}$. One makes it clear that $\mathcal{C}$ is a Frobenius algebra in a natural way, and the other makes it clear that this Frobenius algebra is semisimple.

The first natural basis for the space of class functions is given by the characters of the distinct irreps χ_μ, μ labels the distinct irreps of G.

Under the pointwise product

$$\chi_\mu \chi_\nu = \sum_\lambda N_{\mu\nu}^{\lambda} \chi_\lambda \tag{4.205}$$

where $N_{\mu\nu}^{\lambda}$ are the *fusion coefficients*, (they are also known as "Littlewood-Richardson coefficients"). They are determined by the Clebsch-Gordon series

$$T^\mu \otimes T^\nu = \oplus_\lambda N_{\mu\nu}^{\lambda} T^\lambda \tag{4.206}$$

and are nonnegative integers. The natural trace is

$$\theta(\chi_\mu) = \delta_{\mu,0} \tag{4.207}$$

where $\chi_0 = 1$ corresponds to the identity representation. Since for every rep μ there is a rep μ^* with $\chi_\mu \chi_{\mu^*} = \chi_0 + \cdots$, we conclude $\langle f, g\rangle = \theta(fg)$ is nondegenerate, and hence that $\mathcal{C}$ is indeed a Frobenius algebra.

Another natural basis of class functions are the delta functions on conjugacy classes:

$$\begin{aligned} \delta_C(g) &= 1 \qquad g \in C \\ &= 0 \qquad g \notin C \end{aligned} \tag{4.208}$$

where C is a conjugacy class. Note that in this basis the pointwise product is diagonal. Thus it is clear that $\mathcal{C}$ is semi-simple.

We can of course expand one basis in terms of another:

$$\chi_\mu = \sum_i \chi_\mu(C_i) \delta_{C_i} \tag{4.209}$$

Now recall a standard result from group representation theory: the orthogonality relations on the characters of the irreducible representations:

$$\frac{1}{|G|} \sum_g \chi_\mu(g) \chi_\nu(g^{-1}) = \delta_{\mu,\nu} \tag{4.210}$$

Since G is finite we can, WLOG, assume the representation T^μ is unitary. There fore the matrix

$$S_{i\mu} = \sqrt{\frac{m_i}{|G|}}\chi_\mu(C_i) \tag{4.211}$$

where m_i is the order of the class $|C_i|$, is a unitary matrix.

Now we have:

$$\chi_\mu = \sum_i \sqrt{\frac{|G|}{m_i}} S_{i\mu}\delta_{C_i} \tag{4.212}$$

and therefore since multiplication is diagonal in the basis δ_{C_i}, $S_{i\mu}$ is the matrix which diagonalizes the fusion rules in the character basis.

Now, using (4.197) we compute

$$\theta_x = |S_{0x}|^2 = \frac{(\dim V_x)^2}{|G|} \tag{4.213}$$

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MOVE THIS TO CHAPTER ON TOPOLOGICAL FIELD THEORY -2D EXAMPLE.

and hence the partition function on a compact Riemann surface of genus g is

$$Z_g = |G|^{1-g} \sum_x \frac{1}{(\dim V_x)^{2g-2}} \tag{4.214}$$

where the sum runs over irreducible representations of G. The first factor is relatively uninteresting (it can be absorbed in the scale of the string coupling) but the second is interesting.

What geometrical object is the sum in (4.214) *counting?*

We will answer this question in a few lectures.

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Exercise

a.) Show that the center of the group algebra $C[G]$ with the convolution product is $\mathcal{C}$, the space of class functions.

b.) Show that the matrix $S_{i\mu}$ is a kind of Fourier transform between these two product structures on $\mathcal{C}$. Note that the basis of characters of irreps χ_μ diagonalize the convolution product:

$$\chi_\mu * \chi_\nu = \frac{\delta_{\mu\nu}}{n_\nu}\chi_\nu \tag{4.215}$$

c.) Show that the invariants θ_x of this Frobenius algebra are given by

$$\theta(\epsilon_\mu) = \frac{(\dim V_\mu)^2}{|G|} \tag{4.216}$$

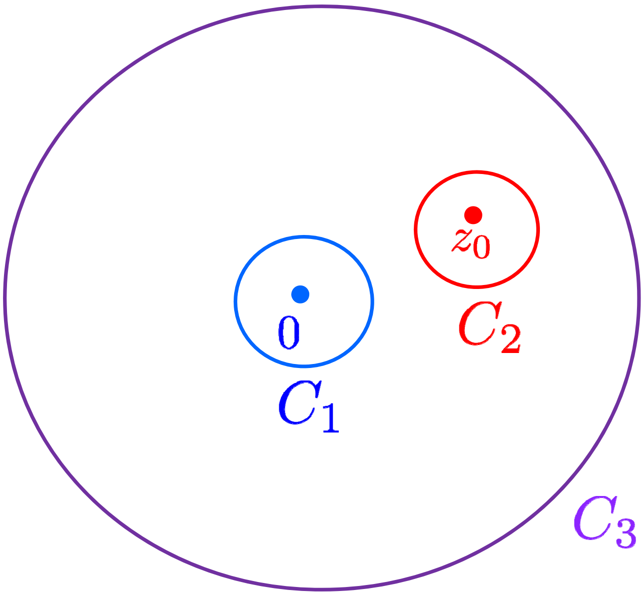

Figure 19: Three CFT state spaces are associated with the circles C_1, C_2, and C_3 and are associated with radial quantization around $z = 0, z_0, 0$, respectively.

4.8 Tensor Product Of Highest Weight Representations

The integrable highest weight representations $L(\lambda)$ turn out to be objects in a tensor category. The tensor product is <u>not</u> symmetric. Note that it is <u>not</u> obvious how to take a tensor product of two representations $L(\lambda)$ and $L(\lambda')$ to get a representation of the KM algebra with the <u>same</u> value of k. [38] The way to do this is to use conformal field theory.

The tensor product can be thought of as follows. (We follow the description from [42], equation (2.5). Rigorous descriptions of the tensor product using vertex operator algebra theory are given in [29, 30].)

We can form a current:

$$J^a(z) = \sum T^a_n z^{-n-1} dz \tag{4.217}$$

where we now analytically continue z to the complex plane - regarded as the Euclidean worldsheet of a 2d Euclidean QFT for the WZW model. Note that

$$T^a_n = \oint z^n J^a(z) \tag{4.218}$$

[38] For the same reason one does not want to multiply characters.

There is a state-operator correspondence: The insertion of a local operator $\Phi(z)$ at a point z on the plane produces a state in the Hilbert space of radial quantization centered on that point.

To give a tensor product we need a comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A}$ is the algebra of local observables.

We imagine one Hilbert space of states on a small circle C_1 centered at $z = 0$, a second circle C_2 centered at $z = z_0$, and a third on a larger circle C_3 centered at $z = 0$ but encircling z_0. See Figure 19.

If local operators creating states in representations L_λ and L_μ are inserted at $z = 0$ and $z = z_0$ then the resulting state on the circle C_3 will have an action of the current with modes

$$\begin{aligned} \Delta_{0,z_0}(T^a_n) &= \oint_{C_3} z^n J^a(z) \\ &= \left(\oint_{C_1} z^n J^a(z)\right) \otimes 1 + 1 \otimes \left(\oint_{C_2} z^n J^a(z)\right) \end{aligned} \tag{4.219}$$

In the first line we have written an operator acting on the space of states on the circle C_3. (Think of it as the outgoing state space in a pair of pants diagram.) The next line is a contour deformation (since $J^a(z)$ is a holomorphic current) to give an action on the space of states on the circles C_1 and C_2. Since there are two ingoing states on the pair of pants we have a tensor product of state spaces. The interesting term is $\oint_{C_2} z^n J^a(z) dz$. When acting on the Hilbert space obtained by radial quantization centered at z_0 we should expand the current as

$$J^a(z) = \sum_{m\in\mathbb{Z}} (z - z_0)^{-m-1} J^a_m(z_0) d(z - z_0) \tag{4.220}$$

but

$$\oint_{C_2} z^n (z - z_0)^{-m-1} dz = \begin{cases} 0 & m \le -1 \\ \binom{n}{m} z_0^{n-m} & m \ge 0 \end{cases} \tag{4.221}$$

and hence

$$\Delta_{0,z_0}(T^a_n) = T^a_n \otimes 1 + 1 \otimes \left(\sum_{k=0}^{\infty} \binom{n}{k} z_0^{n-k} T^a_k(z_0)\right) \tag{4.222}$$

Now, the fusion rules for multiplication, with this tensor product, of the the simple objects (that is, the irreducible representations of G_k) turn out to define a semisimple Frobenius algebra: [39]

$$L(\mu) \otimes_{0,z_0} L(\nu) \cong \oplus_\lambda N^\lambda_{\mu\nu} L(\lambda). \tag{4.223}$$

Therefore, there is a matrix S that diagonalizes these rules.

[39]The conceptual reason for this is that one can gauge the G symmetry of the WZW model to produce the G/G model. This is a 2d TFT.

4.8.1 Fusion Rules For The WZW Model

In general, labeling the irreducible highest weight representations of G_k by the dominant weight of the representation of G at the lowest eigenvalue of L_0 we have the eigenvalues:

$$\Lambda_\nu^{(\mu)} = \frac{S_\mu^{\ \nu}}{S_0^{\ \nu}} = \mathrm{ch}_\mu\left(2\pi\frac{\nu+\rho}{k+h}\right) \tag{4.224}$$

where μ, ν are dominant weights, ρ is the Weyl vector, [40] h is the dual coxeter number, and we have used the Killing form, normalized so that $(\theta,\theta) = 2$ to identify $\mathfrak{t}^\vee \cong \mathfrak{t}$ and thereby regard $\nu + \rho$ as an element of $\mathfrak{t}$.

♣Give the specialization of this formula to $SU(2)_k$. ♣

Using equation (4.224) it is possible to express the CFT fusion rules $N_{\mu\nu}^\lambda$ in terms of the Littlewood-Richardson coefficients $\bar{N}_{\mu\nu}^\lambda$ of the finite-dimensional group:

$$\mathrm{ch}_\mu \mathrm{ch}_\nu = \sum_{\lambda\in\Lambda_{wt}^+} \bar{N}_{\mu\nu}^\lambda \mathrm{ch}_\lambda \tag{4.225}$$

We know that, in general

$$\Lambda_x^{(\mu)}\Lambda_x^{(\nu)} = \sum_\lambda N_{\mu\nu}^\lambda \Lambda_x^{(\lambda)} \tag{4.226}$$

for semisimple Frobenius algebras. Evaluating (4.225) on the special conjugacy classes $2\pi(\lambda+\rho)/(k+h)$ and using some simple manipulations [41] one obtains:

$$N_{\mu\nu}^\lambda = \sum_{w\in\widehat{W}^k, w\cdot\lambda\in\Lambda_{wt}^+} \mathrm{sign}(w)\bar{N}_{\mu\nu}^{w\cdot\lambda} \tag{4.227}$$

(The sign of w is defined since $\widehat{W}^k$ is a Coxeter group. It is ±1 according to whether the group element is a product of an even/odd number of reflections.)

For example, for $SU(2)_k$ the ordinary Clebsch-Gordon rules $\bar{N}_{jj'}^{j''}$ give

$$[j]\otimes[j'] = [|j-j'|]\oplus[|j-j'|+1]\oplus\cdots\oplus[j+j'] \tag{4.228}$$

However, if $j + j' > k/2$ then there will be an affine Weyl reflection around $j = k/2$. Each weight larger than $k - j - j'$ will have a reflected image larger than $k/2$ and these will cancel in pairs. In this way we get:

$$N_{jj'}^{j''} = \begin{cases} 1 & |j-j'| \le j'' \le \min\{j+j', k-j-j'\} \& j+j'+j'' \in \mathbb{Z} \\ 0 & \text{else} \end{cases} \tag{4.229}$$

♣Say something about tetrahedra ♣

Exercise

a.) Find the invariants θ_x for the Frobenius algebra defined by $N_{jj'}^{j''}$.

[40] The Weyl vector is half the sum of positive roots. It is equal to the sum of fundamental weights.

[41] See Di Francesco et. al. Section 16.2.1 or Fuchs, Section 5.5

b.) Note that since $N_{jj'}^{\ \ \ j''}$ are integers, $Z(\Sigma_g)$ is an integer, a surprising fact when viewed as (4.199). This is a special case of the famous Verlinde formula.

Exercise

a.) Show that the product of theta functions of level k and k', as functions of z can be expanded in terms of theta functions of level $k + k'$. Thus, taking a direct sum of the span of the level k theta function defines a graded ring.

b.) Show that the characters $\chi_j^k(z, \tau)$ can be expanded in level k theta functions.

This is another way to see that the standard tensor products of representations of $SU(2)_k$ will not produce a representation of $SU(2)_k$.

4.9 Conformal Blocks And Monodromy

Monodromy of conformal blocks: Braiding matrix B: Consistency: Braid Relations.

Formula for dimension of space of conformal blocks.

Conformal blocks in higher genus: Projectively flat connection over moduli space of Riemann surfaces. Sugawara and parallel transport.

4.10 Open String WZW Model: Symmetry Preserving Boundary Conditions

Open string and branes: Conjugacy classes. Symmetry-preserving D-branes (boundary conditions)

When the spacetime $\mathcal{S}_{2n}$ is not closed but is a manifold with boundary the nature of the WZ term becomes much more subtle. We will comment on this below.

However, we can discuss now the solutions of the equations of motion without entering into the subtleties of formulating the action. Let us illustrate this in the 2-dimensional case:

Our field is defined for $g(\sigma, t)$ with $\sigma \in [0, \pi]$.

Let us choose the action kS^+_{WZW} so the conserved currents are

$$\begin{aligned} \partial_+ J_R(x^-) &= 0 \qquad\qquad & J_R &= g^{-1}\partial_- g \\ \partial_- J_L(x^+) &= 0 \qquad\qquad & J_L &= \partial_+ g g^{-1} \end{aligned} \tag{4.230}$$

where $x^{\pm} = t \pm \sigma$.

We have seen that the general solution of the equations of motion is:

$$g(t, \sigma) = g_L(x^+) g_R(x^-) \tag{4.231}$$

But now we should choose boundary conditions. A natural choice of such boundary conditions are the "symmetry-preserving" boundary condition at $\sigma = 0$ which says that the total J_σ current should not leak out the side:

$$(J^{(L)} + J^{(R)})_\sigma = 0 \ \Rightarrow \ g_L'(t) g_L^{-1}(t) = -g_R^{-1}(t) g_R'(t) \tag{4.232}$$

Therefore, there is some $\mathfrak{g}$-valued function of time, $A(t)$ such that:

$$\begin{aligned} g_L'(t) &= A(t) g_L(t) \\ g_R'(t) &= -g_R(t) A(t) \end{aligned} \tag{4.233}$$

Integrating using the path-ordered exponential we have

$$A(t) = \partial_t U U^{-1} = -U \partial_t (U^{-1}) \tag{4.234}$$

The general solution is therefore

$$\begin{aligned} g_L(t) &= U(t) g_L(0) \\ g_R(t) &= g_R(0) U^{-1}(t) \end{aligned} \tag{4.235}$$

and hence

$$g(t) = g_L(t) g_R(t) = U(t) g_L(0) g_R(0) U^{-1}(t) = U(t) g(0) U^{-1}(t) \tag{4.236}$$

Thus, *the ends of the the open string move along a conjugacy class in the group*

FIGURE FOR SU2

In the quantum theory (for compact groups) it turns out that the allowed conjugacy classes themselves are quantized. For some references on the open string WZW model see:

1. Volker Schomerus, " Lectures on branes in curved backgrounds," Class.Quant.Grav.19:5781-5847,2002. e-Print: hep-th/0209241

2. G. Moore, "K theory from a physical perspective," e-Print: hep-th/0304018

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COMMENT ON FREED-HOPKINS-TELEMAN THEOREM

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5. Topological Field Theories And Categories

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DO JUST ENOUGH SO THAT YOU CAN RETURN TO CS THEORY AND GO THROUGH THE SURGERY ARGUMENTS AND THE OLYMPIC PROOF OF THE VERLINDE FORMULA

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5.1 Basic Ideas

Topological field theory is an excellent pedagogical tool for introducing both some basic ideas of physics along with some beautiful mathematical ideas.

The idea of TFT arose from both the study of two-dimensional conformal field theories and from Witten's work on the relation of Donaldson theory to N=2 supersymmetric field theory and Witten's work on the Jones polynomial and three-dimensional quantum field

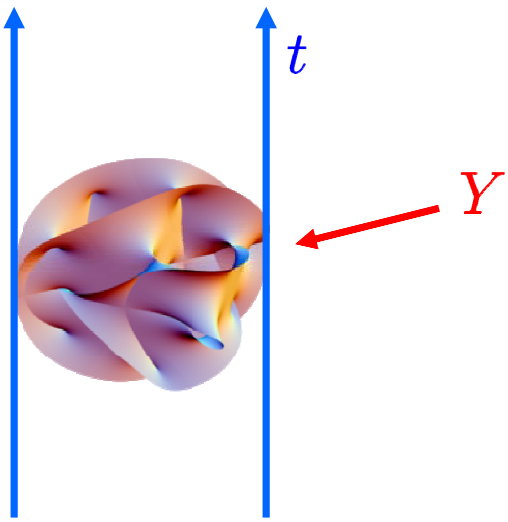

Figure 20: A spacetime $X_d = Y \times \mathbb{R}$. Y is $(d-1)$-dimensional space, possibly with nontrivial topology.

theories. In conformal field theory, Graeme Segal stated a number of axioms for the definition of a CFT. These were adapted to define a notion of a TFT by Atiyah.

TFT might be viewed as a basic framework for physics. It assigns Hilbert spaces, states, and transition amplitudes to topological spaces in a way that captures the most primitive notions of locality. By stripping away the many complications of "real physics" one is left with a very simple structure. Nevertheless, the resulting structure is elegant, it is related to beautiful algebraic structures which, at least in two dimensions, which have surprisingly useful consequences. This is one case where one can truly "solve the theory."

Of course, we are interested in more complicated theories. But the basic framework here can be adapted to any field theory. What changes is the geometric category under consideration. Thus, it offers one approach to the general question of "What is a quantum field theory?"

It is possible to speak of physics in 0-dimensional spacetime. From the functional integral viewpoint this is quite natural: Path integrals become ordinary integrals. It is also very fruitful to consider string theories whose target spaces are 0-dimensional spacetimes. Nevertheless, in the vast majority of physical problems we work with systems in d spacetime dimensions with $d > 0$. We will henceforth assume $d > 0$.

What are the most primitive things we want from a physical theory in d spacetime dimensions? In a physical theory one often decomposes spacetime into space and time as in (20). If space is a $(d-1)$-dimensional manifold Y then, in quantum mechanics, we associate to it a vector space of states $\mathcal{H}(Y_{d-1})$.

Of course, in quantum mechanics $\mathcal{H}(Y_{d-1})$ usually has more structure - it is a Hilbert space. But in the spirit of developing just the most primitive aspects we will not incorporate that for the moment. (The notion of a *unitary TFT* captures the Hilbert space, as described below.) Moreover, in a generic physical theory there are natural operators acting on this Hilbert space such as the Hamiltonian. The spectrum of the Hamiltonian and other physical observables depends on a great deal of data. Certainly they depend on the metric on spacetime since a nonzero energy defines a length scale

$$L = \frac{\hbar c}{E}.$$

In topological field theory one ignores most of this structure, and focuses on the dependence of $\mathcal{H}(Y)$ on the topology of Y. For simplicity, we will initially assume Y is compact without boundary.

So: In topological field theory we want to have an association:

$(d-1)$-manifolds Y to vector spaces: $Y \to \mathcal{H}(Y)$, such that "$\mathcal{H}(Y)$ is the same for homeomorphic vector spaces." What this means is that if there is a homeomorphism

$$\varphi : Y \to Y' \tag{5.1}$$

then there is a corresponding isomorphism of vector spaces:

$$\varphi_* : \mathcal{H}(Y) \to \mathcal{H}(Y') \tag{5.2}$$

so that composition of homeomorphisms corresponds to composition of vector space isomorphisms. In particular, self-homeomorphisms of Y act as automorphisms of $\mathcal{H}(Y)$: It therefore provides a (possibly trivial) representation of the diffeomorphism group.

Now, we also want to incorporate some form of locality, at the most primitive level. Thus, if we take disjoint unions

$$\mathcal{H}(Y_1 \amalg Y_2) = \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2) \tag{5.3}$$

Note that (5.3) implies that we should assign to $\mathcal{H}(\emptyset)$ the field of definition of our vector space. For simplicity we will take $\mathcal{H}(\emptyset) = \mathbb{C}$, although one could use other ground fields.

Remark: In algebraic topology it is quite common to assign an abelian group or vector space to a topological space. This is what the cohomology groups do, for example. But here we see a big difference from the standard algebraic topology examples. In those examples the spaces <u>add</u> under disjoint union. In quantum mechanics the spaces <u>multiply</u>. This is the fundamental reason why many topologists refer to the topological invariants arising from topological field theories as "quantum invariants."

Finally, there is an obvious homeomorphism

$$Y \amalg Y' \cong Y' \amalg Y \tag{5.4}$$

and hence there must be an isomorphism

$$\Omega : \mathcal{H}(Y) \otimes \mathcal{H}(Y') \to \mathcal{H}(Y') \otimes \mathcal{H}(Y) \tag{5.5}$$

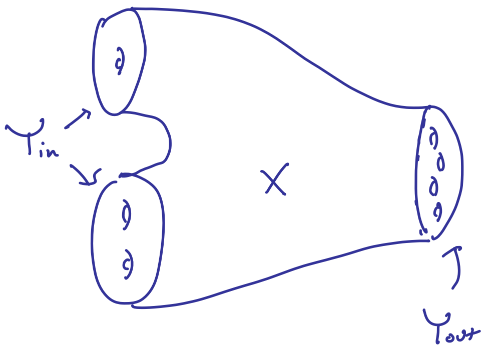

Figure 21: Generalizing the product structure, a d-dimensional bordism X can include topology change between the initial $(d-1)$-dimensional spatial slices Y_{in} and the final spatial slice Y_{out}. The amplitude $F(X)$ determined by a path integral on this bordism is a linear map $\mathcal{H}(Y_{\text{in}}) \to \mathcal{H}(Y_{\text{out}})$.

In addition, in physics we want to speak of transition amplitudes. If there is a spacetime X_d interpolating between two time-slices, then mathematically, we say there is a bordism between Y and Y'. That is, a *bordism* from Y to Y' is a d-manifold with boundary and a disjoint partition of its boundary into two sets the "in-boundary" and the "out-boundary"

$$\partial X_d = (\partial X_d)_{\text{in}} \cup (\partial X_d)_{\text{out}}$$

so that there is a homeomorphism $(\partial X_d)_{\text{in}} \cong Y$ and $(\partial X_d)_{\text{out}} \cong Y'$. We will say this a bit more precisely, and discuss some variants, in Section **** below.

If X_d is a bordism from Y to Y' then the Feynman path integral assigns a linear transformation

$$F(X_d) : \mathcal{H}(Y) \to \mathcal{H}(Y').$$

Again, in the general case, the amplitudes depend on much more than just the topology of X_d, but in topological field theory they are supposed only to depend on the topology.

More precisely, if $X_d \cong X'_d$ are homeomorphic by a homeomorphism $= 1$ on the boundary of the bordism, then

$$F(X_d) = F(X'_d)$$

One key aspect of the path integral - in quantum mechanics, or functional integral - in quantum field theory, we want to capture - again a consequence of locality - is the idea of summing over a complete set of intermediate states. In the path integral formalism we can formulate the sum over all paths of field configurations from t_0 to t_2 by composing the amplitude for all paths from t_0 to t_1 and then from t_1 to t_2, where $t_0 < t_1 < t_2$, and then summing over all intermediate field configurations at t_1. We refer to this property as the "gluing property." The gluing property is particularly obvious in the functional integral formulation of field theories.

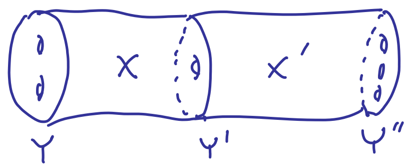

Figure 22: Gluing two bordisms to produce a third bordism.

In topological field theory this is formalized as:
If X is a bordism from Y to Y' with

$$(\partial X)_{\text{in}} = Y \qquad\qquad (\partial X)_{\text{out}} = Y'$$

and X' is another oriented bordism from Y' to Y''

$$(\partial X')_{\rm in} = Y' \qquad\qquad (\partial X)_{\rm out} = Y''$$

then we can compose $X' \circ X$ as in (**??**) to get a bordism from Y to Y''.

Naturally enough we want the associated linear maps to compose:

$$F(X' \circ X) = F(X') \circ F(X) : \mathcal{H}(Y) \rightarrow \mathcal{H}(Y'')$$

What we are describing, in mathematical terms, is a functor between categories. See appendix K for background material on categories. After describing a few variations on the above theme, we will explain the categorical picture in some more detail.

5.1.1 More Structure

We can regard the above picture as a basic framework for building up more interesting theories by enriching the topological and geometric data associated with the spaces X and Y.

For example, we might be able to endow X and Y with

1. Orientations, spin, spin-c, pin structures, etc. (for certain X's and Y's).
2. Complex structures, conformal structures, causal structures, Riemannian structures.
3. Other fields - Principal G-bundles with connection, sections of associated bundles etc.

One of the motivating examples was two-dimensional conformal field theory. In this case, Segal's axioms were based on two-dimensional bordisms endowed with conformal structure.

Two important complications that will arise when considering nontopological theories are:

1. The notion of scale and renormalization becomes important.
2. The Hilbert space is actually not defined for a $(d-1)$-dimensional manifold but rather for a germ of d-manifolds around a $(d-1)$-dimensional manifold, and then the operator algebra can depend on extra structure, such as the second fundamental form. [CITE SEGAL for example].

5.2 The Definition Of Topological Field Theory

See Appendix **??** for basic math backgound on bordism theory. This will be assumed known in the following.

Let S be a structure on the tangent bundle and C any symmetric monoidal category. Then

Definition A d-dimensional topological field theory of S-manifolds is a symmetric tensor functor from the tensor category $\mathrm{Bord}^S_{\langle d-1,d\rangle}$ to some symmetric tensor category C.

The example we started out with is the case where S is empty and the target category is VECT_κ for some field κ, so a topological field theory is a tensor functor from $\mathrm{Bord}_{\langle d-1,d\rangle}$ to VECT_κ.

For examples of this more general notion:

1. Use the identity functor! This gives what Michael Freedman calls the "lazy TFT" and it leads to a pairing of manifolds with very interesting positivity properties. See [**?**].

2. We can generalize this as follows: Let K be a closed manifold of dimension k. Then Cartesian product with K defines a symmetric tensor functor t_K

$$t_K : \mathrm{Bord}_{\langle d-1,d\rangle} \to \mathrm{Bord}_{\langle d+k-1,d+k\rangle} \tag{5.6}$$

   where $t_K(Y) = Y \times K$, etc. If F is a $(d+k)$-dimensional TFT then we can compose $F \circ t_K$ to obtain a d-dimensional TFT denoted F^{KK}. This is the topological field theory analog of "Kaluza-Klein compactification". For example the state space on $(d-1)$-manifolds is

$$F^{KK}(Y) := F(Y \times K) \tag{5.7}$$

3. If there are sufficiently natural constructions of quantum field theories depending on some geometric category then one can define a TFT whose values are moduli spaces of vacua of the quantum field theory. This is done for the case of a target category of holomorphic symplectic varieties in [**?**].

5.2.1 Some General Properties

Let us deduce some simple general facts following from the above simple remarks.

For the moment take the target category to be SVECT_κ, the category of super-vector spaces over the field κ. (If one prefers, just ignore the signs and work with the category of vector spaces.)

First note that if X is closed then it can be regarded as a bordism from $\emptyset$ to $\emptyset$. Therefore $F(X)$ must be a linear map from κ to κ. But any linear map $T \in \mathrm{Hom}(\kappa,\kappa)$ must be of the form

$$T(z) = tz \tag{5.8}$$

for some scalar $t \in \kappa$. That is, any linear map $\kappa \to \kappa$ is canonically associated to an element of the ground field. For the case of $F(X) : \kappa \to \kappa$ we call that number the *partition function* of X, and denote it $Z(X)$.

There is one bordism which is distinguished, namely $[0,1] \times Y$. This corresponds to a linear map $P : \mathcal{H}(Y) \to \mathcal{H}(Y)$. In Euclidean field theory the amplitude one would associate to a cylindrical spacetime $[0,1] \times Y$ is just

$$\exp[-TH]$$

where H is the Hamiltonian, and T is the Euclidean time interval. Notice that this requires a metric. A change of the length of the cylinder leads to a change in T.

Evidently, by the axioms of topological field theory, $P^2 = P$ and therefore we can decompose

$$\mathcal{H}(Y) = P\mathcal{H}(Y) \oplus (1 - P)\mathcal{H}(Y) \tag{5.9}$$

All possible transitions are zero on the second summand since, topologically, we can always insert such a cylinder. It follows that it is natural to assume that

$$F(Y \times [0,1]) = Id_{\mathcal{H}(Y)} \tag{5.10}$$

One can think of this as the statement that the Hamiltonian is zero. Note that this renders the amplitude independent of the length of the cylinder.

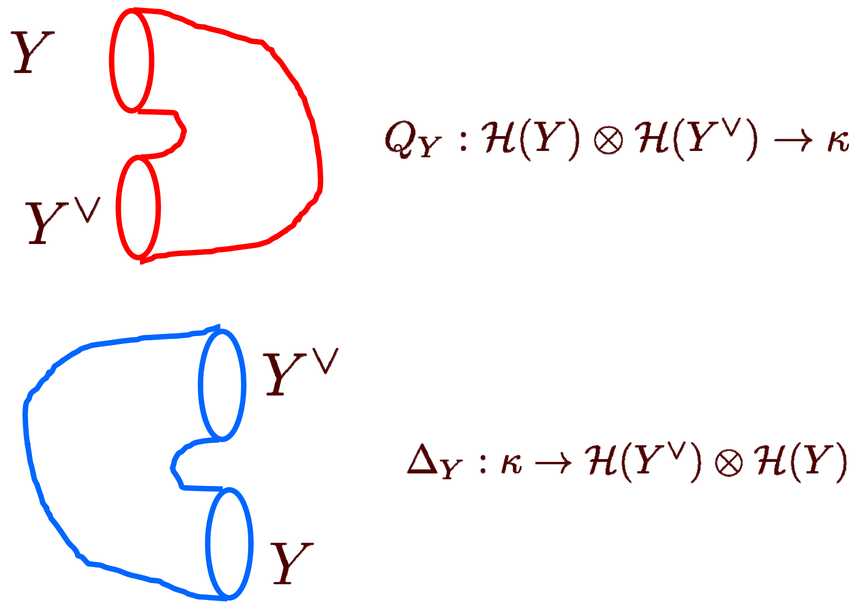

Figure 23: Bending the cylinder to define Δ_Y and Q_Y.

Now, let us consider the oriented bordism category, so Y is oriented. Let $Y^\vee$ denote Y with the opposite orientation. The bordism (5.10) is closely related to the bordism $\emptyset \to Y^\vee \amalg Y$ thus defining a map

$$\Delta_Y : \kappa \to \mathcal{H}(Y^\vee) \otimes \mathcal{H}(Y) \tag{5.11}$$

and also to a bordism $Y \amalg Y^\vee \to \emptyset$ thus defining a quadratic form:

$$Q_Y : \mathcal{H}(Y) \otimes \mathcal{H}(Y^\vee) \to \kappa \tag{5.12}$$

Let us now compose these bordisms we get the identity map as in 24. It then follows from some linear algebra that Q is a *nondegenerate* pairing, so we have an isomorphism to the linear dual space:

$$\mathcal{H}(Y^\vee) \cong \mathcal{H}(Y)^\vee,$$

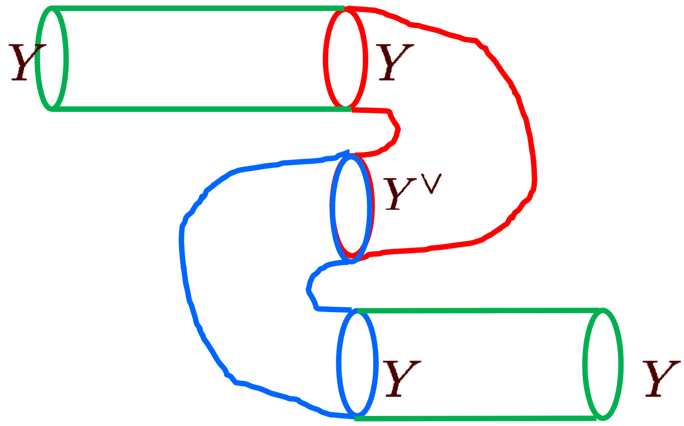

Figure 24: Composing $\Delta \otimes 1$ and $1 \otimes Q$ in a way that gives P.

under which Q is just the dual pairing. (On the left $Y^\vee$ is Y is the reversal of orientation, and on the right $\mathcal{H}(Y)^\vee$ is the linear dual space.)

To prove this choose a basis $\{\phi_i\}$ for $\mathcal{H}(Y)$ and a basis $\{\psi_a\}$ for $\mathcal{H}(Y^\vee)$. Then we must have

$$\Delta_Y(1) = \sum_{i,a} \Delta^{ai} \psi_a \otimes \phi_i \tag{5.13}$$

The S-diagram shows that

$$\phi \to \sum_{i,a} \Delta^{ai} Q(\phi, \psi_a)\phi_i \tag{5.14}$$

must be the identity map, so, choosing $\phi = \phi_j$ and defining $Q_Y(\phi_j, \psi_a) := Q_{ja}$ we must have

$$\sum_a \Delta^{ai} Q_{ja} = \delta^i{}_j \tag{5.15}$$

In addition to this we can exchange the roles of Y and $Y^\vee$. Including signs for the $\mathbb{Z}_2$-graded case (with a homogeneous basis) we get

$$\sum_i \Delta^{ai} (-1)^{(|a|+|b|)|i|} Q_{ib} = \delta^a{}_b \tag{5.16}$$

It follows that Q is invertible, hence the pairing is nondegenerate. This implies hence there is an isomorphism $\mathcal{H}(Y^\vee) \cong \mathcal{H}(Y)^\vee$ as asserted above. Moreover, choosing an isomorphism so that $Q_{i,a} = \delta_{i,a}$, now labeling the dual basis by an index i and changing notation to $\psi_i \to \psi^i$ in this basis we have simply

$$\Delta_Y(1) = \sum_i \psi^i \otimes \phi_i \tag{5.17}$$

Now, the result (5.17) brings up an important point. It is not obvious that (5.17) will converge if $\mathcal{H}(Y)$ is infinite dimensional. In fact, even if $\mathcal{H}(Y)$ is a normed vector space, or a Hilbert space, so that convergence of infinite sums of vectors does make sense, since ϕ_i and ϕ^i are dual bases the sum will not converge if $\mathcal{H}(Y)$ is infinite dimensional. *Therefore, the space of states $\mathcal{H}(Y)$ must be finite-dimensional!*

There are many examples of interesting "topological field theories" where $\mathcal{H}(Y)$ is decidedly infinite-dimensional. We will comment on this below.

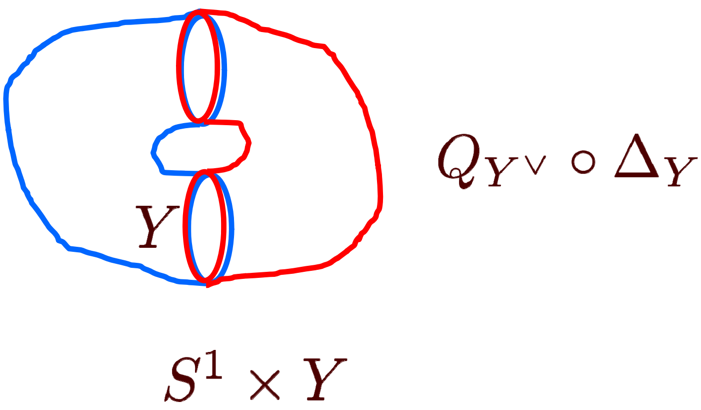

Figure 25: Composing $Q_{Y^\vee}$ with Δ_Y gives the super dimension of $\mathcal{H}(Y)$ in the $\mathbb{Z}_2$-graded case, and $\dim \mathcal{H}(Y) = Z(Y \times S^1)$ in the ungraded case.

Now consider the diagram in 25. On the one hand this is just the partition function $Z(Y \times S^1)$. On the other hand, the linear map $\kappa \to \kappa$ must be the composition $Q_{Y^\vee}\Delta_Y$, or, equivalently, $Q_Y \circ \Omega \circ \Delta_Y : \kappa \to \kappa$. From our formula for $\Delta_Y(1)$ above we see that the value $Z(Y \times S^1)$ is just the dimension $\dim \mathcal{H}(Y)$, or, in the $\mathbb{Z}_2$-graded case, the superdimension

$$\mathrm{sdim}\mathcal{H}(Y) = \dim \mathcal{H}(Y)_0 - \dim \mathcal{H}(Y)_1 \tag{5.18}$$

Remarks

1. Note that if we change the category to the category of manifolds with Riemannian structure and we take the product Riemannian structure on $Y \times S^1$ then

$$Z(Y \times S^1) = \mathrm{Tr} e^{-\beta H} \tag{5.19}$$

where β is the radius of the circle and H is the Hamiltonian.

2. There are important examples of "topological field theories" of interest in the physics literature where this condition is violated. One example is Chern-Simons theory with noncompact gauge group. Another example is two-dimensional Yang-Mills theory with zero area element. These are "partially defined" topological field theories. They are only defined on a subset of objects in the bordism category. ♣Say more. ♣

3. The S-diagram argument above points the way to a definition of a dual object in a symmetric monoidal category. A dual object $x \in \mathrm{Obj}(C)$ is one such that there exists an object $x^\vee \in \mathrm{Obj}(C)$ and morphisms $\delta_x : 1_C \to x \otimes x^\vee$ and $q_x : x^\vee \otimes x \to 1_C$ such that

$$x \xrightarrow{\iota_L(x)^{-1}} 1_C \otimes x \xrightarrow{\delta_x \otimes 1_x} x \otimes x^\vee \otimes x \xrightarrow{1_x \otimes q_x} x \otimes 1_C \xrightarrow{\iota_R(x)} x \tag{5.20}$$

and (omitting the isomorphisms with multiplication by the tensor unit, for simplicity)

$$x^\vee \xrightarrow{1_{x^\vee} \otimes \delta_x} x^\vee \otimes x \otimes x^\vee \xrightarrow{q_x \otimes 1_{x^\vee}} x^\vee \tag{5.21}$$

are the identity morphisms.

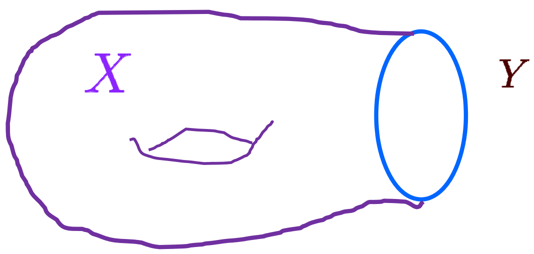

Figure 26: A state created by a bordism of $\emptyset$ to Y.

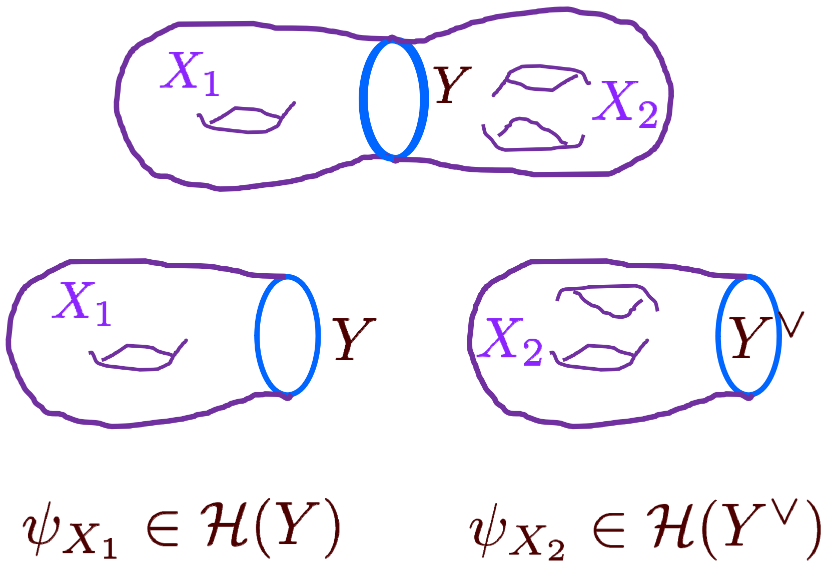

Figure 27: If a closed manifold X is cut along a codimension one submanifold Y that divides X into two pieces X_1 and X_2 then there are two associated states $\psi_{X_1} \in \mathcal{H}(Y)$ and $\psi_{X_2} \in \mathcal{H}(Y^\vee)$, and the value of the partition function $Z(X)$ may be viewed as the natural contraction of these states using the nondegenerate pairing Q_Y.

Exercise *Mapping cylinders and characters of the diffeomorphism group*

Let $f \in \mathrm{Diff}(Y)$ and consider the mapping cylinder $M_f(Y) = ([0,1] \times Y)/\sim$ where we identify $(0, y)$ with $(1, f(y))$. Recall that $\mathcal{H}(Y)$ has a representation $\rho(f)$ of the diffeomorphism group.

Show that

$$Z(M_f(Y)) = \mathrm{Tr}_{\mathcal{H}(Y)}\rho(f) \tag{5.22}$$

is a character of the diffeomorphism group.

In fact, $\rho(f)$ only depends on the image of f in the mapping class group Γ_Y: This is defined as follows: The diffeomorphisms isotopic to the identity form a normal subgroup $\mathrm{Diff}_0(Y)$ of the full diffeomorphism group and $\Gamma_Y := \mathrm{Diff}(Y)/\mathrm{Diff}_0(Y)$.

♣Explain in detail how the independence of $\rho(f)$ under isotopy of f follows from the axioms. ♣

Exercise *Hartle-Hawking sates and partition functions as inner products*

a.) Show that any bordism of $X : \emptyset \to Y$ defines a *state* in the space $\mathcal{H}(Y)$. (See Figure 26.) The functor of the topological field theory defines a map $F(X) : \kappa \to \mathcal{H}(Y)$, and we can define,

$$\psi_X := F(X)(1) \in \mathcal{H}(Y) \tag{5.23}$$

This simple observation is very important in physics.

The state, of course, depends on the (topological) details of the bordism. For example, any Riemann surface with a single hole defines a bordism of the circle to zero and there are many such topologies. This is a primitive version of the notion of the "Hartle-Hawking" state in quantum gravity. It is also related to the state/operator correspondence in conformal field theory.

b.) Show that, in the oriented bordism category, by exchanging in and out boundaries (but not the orientation of X) the same manifold defines a bordism $X^\vee : -Y \to \emptyset$, and hence a linear functional on $\mathcal{H}(Y^\vee)$.

c.) Show that applying this linear functional to $\Delta_Y(1)$ gives back the original vector in $\mathcal{H}(Y)$ associated to X.

d.) Show that if a closed manifold X is cut along an oriented manifold Y to produce X_1 and X_2 then $Z(X)$ can be interpreted as a contraction of a state $\psi_{X_1} \in \mathcal{H}(Y)$ and $\psi^\vee_{X_2} \in \mathcal{H}(Y^\vee)$:

$$Z(X) = \langle \psi^\vee_{X_2}, \psi_{X_1} \rangle \tag{5.24}$$

See Figure 27.

5.2.2 Unitarity

In unitary theories, and certainly in the axioms of quantum mechanics, one wants the state space to be a complex Hilbert space, and $F(X)$ for a bordism X should be a unitary operator.

Now, in general, a sesquilinear form on a complex vector space V is a linear map $V \to \bar{V}^\vee$. Therefore, in a unitary theory changing orientation of Y complex conjugates the Hilbert space

$$\mathcal{H}(Y^\vee) \cong \bar{\mathcal{H}}(Y) \tag{5.25}$$

Moreover, in physical unitary theories there is a positivity condition on Q_Y. If $X : Y_1 \to Y_2$ is a bordism then, if we change the orientation of X and take the dual we get a bordism

$$\bar{X}^\vee : Y_2 \to Y_1 \tag{5.26}$$

It is natural to add a condition that

$$F(\bar{X}^\vee) = F(X)^\dagger \tag{5.27}$$

In particular, changing orientation of the manifold invariant $Z(X)$ for a closed manifold complex conjugates the invariant.

5.3 One Dimensional Field Theories

Consider the oriented case. Then the objects in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$ are disjoint unions of points $\mathrm{pt}_\pm$ with $+$ and $-$ orientation.

The topological field theory with symmetric monoidal category C gives two objects $y_\pm = F(\mathrm{pt}_\pm)$ with data δ and q as described above. The general object is a disjoint union of $n_\pm$ points of type $\mathrm{pt}_\pm$. The diffeomorphism group of this manifold is just $S_{n_+} \times S_{n_-}$ and it acts in a natural way on the "state space" $y_+^{\otimes n_+} \otimes y_-^{\otimes n_-}$.

Specializing to VECT_κ, we get a pair of finite-dimensional vector spaces $V_\pm$ together with the data mentioned above: A nondegenerate pairing $Q : V_+ \otimes V_- \to \kappa$ and the "inverse" $\Delta : \kappa \to V_+ \otimes V_-$. As mentioned above, these constitute duality data for $V_- = V_+^\vee$.

A good example of a physical origin of such a topological field theory is to consider quantization of a compact symplectic manifold (M, ω).

A useful concrete example to keep in mind is $M = S^2$ with a symplectic form

$$\omega = \frac{1}{2\hbar} \sin\theta d\theta d\phi \tag{5.28}$$

where here $\hbar$ is some dimensionless normalization of the form.

In the Hamiltonian formulation of the path integral we consider paths in phase space M. We form a path integral of the form

$$\int_{\mathcal{P}} [d\gamma] \exp[\mathrm{i}S] \tag{5.29}$$

where $\mathcal{P}$ is a space of paths in M, $[d\gamma]$ is an induced measure on the space of paths from the symplectic form, and S is an action. There are many issues to settle in making sense of this expressions. We will just touch on a few of them here.

If the symplectic form ω is globally exact then we can write $\omega = d\lambda$ where, in terms of local Darboux coordinates

$$\lambda = \frac{1}{\hbar} p dq \tag{5.30}$$

A good example of this is the case $M = T^*X$ for some manifold X. Note that the Hamiltonian associated with the action principle:

$$S[\gamma] = \frac{1}{\hbar} \int_\gamma p dq \tag{5.31}$$

is zero.

But what if ω is not exact? (As in our above example with $M = S^2$.) Let us suppose first that M is simply connected. Then, if γ is a closed path we can attempt to define the action by choosing a disk $\Sigma \subset M$ such that $\partial\Sigma = \gamma$ and then take

$$S_\Sigma[\gamma] := \int_\Sigma \omega \tag{5.32}$$

If ω is exact this reduces to the previous definition.

Now there is a problem because there can be more than one disk bounding γ. If Σ_1, Σ_2 both bound γ then $\Sigma_{12} := \Sigma_1 \cup_\gamma \Sigma_2^\vee$ is a closed 2-cycle and the ambiguity in the action is

$$S_{\Sigma_1}[\gamma] - S_{\Sigma_2}[\gamma] = \int_{\Sigma_{12}} \omega \tag{5.33}$$

So the action is not well-defined. However, all we need for the quantum path integral is that the weight

$$\exp[\mathrm{i}S] = \exp[\mathrm{i} \int_\Sigma \omega] \tag{5.34}$$

should be well-defined. The ambiguity in the exponentiated action is:

$$\exp[\mathrm{i}S_1]/\exp[\mathrm{i}S_2] = \exp[\mathrm{i}\int_{\Sigma_{12}}\omega] \tag{5.35}$$

The LHS will be one - and there will be an unambiguous weight in the path integral - if the periods of ω are integral multiples of 2π. Notice that this quantizes "$1/\hbar$" to be an integer.

Now, suppose that γ is not closed. Let us consider a path space

$$\mathcal{P} = \{\gamma : [0,1] \to M | \gamma(0) = x_0 \quad \& \quad \gamma(1) = x_1\} \tag{5.36}$$

(We assume x_0, x_1 are in the same path-connected component of M.) Choose a basepoint path γ_0 in $\mathcal{P}$. Then any other path homotopic to γ_0 will be such that $\gamma_0^{-1} * \gamma$ bounds a disk Σ. We then use this data to define an action as

$$S_{\gamma_0,\Sigma}[\gamma] := \int_\Sigma \omega. \tag{5.37}$$

For a fixed basepath the exponentiated action will be independent of the choice of Σ if the periods of ω are in $2\pi\mathbb{Z}$. If we change the basepath γ_0 to another one in the same homotopy class then the action only shifts by a constant, and in fact with ω quantized as above, the choice will again not matter in the exponentiated action.

If M is not simply connected further considerations are needed because there will be paths in $\mathcal{P}$ not in the path-component of γ_0 even when x_0, x_1 are in the same path component of M. One way to deal with this is to work on the universal cover $\widetilde{M}$. It is best to couple the theory to a flat connection on M to keep track of the fundamental group. ♣EXPLAIN MORE! ♣

An important special case of the quantization above is the case of coadjoint orbits of a compact simple Lie group defined by integral weights $\lambda \in \mathfrak{g}^*$. There is a natural integrally-quantized symplectic form - the Kirillov-Kostant symplectic form, and quantization gives a representation with dominant weight vector a suitable Weyl rotation of λ. Pursuing this line of thought leads to a path integral interpretation of the Borel-Weil-Bott theorem. In ♣Do it later? ♣ the topological field theory the space V_+ is the representation with dominant weight λ and the space V_- is the conjugate representation with anti-dominant weight $-\lambda$. The duality data Q is the standard pairing of a representation and its conjugate to form the singlet while Δ is the embedding of the singlet into $R \otimes R^\vee$.

****

SHOULD ADD MATERIAL ON STOLZ-TEICHNER VIEWPOINT HERE

****

5.4 Two-Dimensional Topological Field Theories

Explain:

1. Sewing theorem and commutative Frobenius
2. How to compute all amplitudes

3. Semisimple case: Spacetime interpretation.

4. Open-closed theorem and boundary conditions as objects in a category.

5. Semi-simple: Spacetime interpretation.

6. Making the theory equivariant.

FOR ALL THE ABOVE: THOROUGH DISCUSSION IS IN GMP2015: [47]

5.5 Extended Topological Field Theories

****

CAN ALREADY ILLUSTRATE THE IDEA WITH 2D THEORIES
Follow discussion in [47].
****

5.6 Three-Dimensional Topological Field Theories

5. Tensor categories

6. Modular tensor categories. Theorem: MTC gives 3d unitary TFT.
This might go after we return to 3d nonabelian chern simons?

5.7 Some Source Material For Topological Field Theories

There is an enormous literature on topological field theory. One of the key early papers is

1. M.F. Atiyah, "Topological quantum field theories." Inst. Hautes Etudes Sci. Publ. Math. No. 68 (1988), 175–186 (1989).

This paper was inspired by Witten's work together with Graeme Segal's axiomatization of conformal field theory, now available as:

2. G. Segal, "The Definition Of Conformal Field Theory," in *Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal*

For an introduction written more from a physicist's perspective see

D. Birmingham, M. Blau, M. Rakowski,m and G. Thompson, "Topological Field Theory," Phys.Rept. 209 (1991) 129-340

We have also relied on lecture notes

3. Segal, Stanford notes, http://www.cgtp.duke.edu/ITP99/segal/

as well as the expository article:

4. G. Segal and G. Moore, "D-branes and K-theory in 2D topological field theory" hep-th/0609042. See also chapter 2 in *Mirror Symmetry II*, Clay Mathematics Institute Monograph, by P. Aspinwall et. al.

For a very meticulous can careful discussion of topological field theory in general see

5. V.G. Turaev, *Quantum Invariants Of Knots And 3-Manifolds*, De Gruyter, 2nd edition 2010.

The special case of d=3 TQFT and modular tensor categories was first described in detail in

G. Moore and N. Seiberg, "Lectures on RCFT," in *Superstrings '89*, World Scientific, pp. 1-129.

The subsequent book

Bakalov and Kirillov

gives a more careful treatment.

We have used some material from the very nice lecture notes of Dan Freed :

10. D. Freed, "Bordism Old And New,"

https://www.ma.utexas.edu/users/dafr/M392C-2012/index.html

For recent developments in higher category theory and locality see

11. A. Kapustin, "Topological Field Theory, Higher Categories, and Their Applications," arXiv:1004.2307 [math.QA]

12. J. Lurie, "On the classification of topological field theories," arXiv:0905.0465 [math.AT]

13. D. Freed, "The cobordism hypothesis" BAMS

14. D. Freed, "Remarks On Chern-Simons Theory," Bull. AMS, **46** (2009), pp. 221-254

6. Computing Nonabelian Chern-Simons Correlators Using RCFT

6.1 3D Interpretation Of Braiding And Fusing Matrices

6.2 Surgery Manipulations

3D Interpretation Of Modular S-Matrix

We observed, experimentally, in Figure 10 that the modular transformation S-matrix, expressed in the canonical basis of corresponding to irreps of the chiral algebra is equal to the value of the path integral for linked circles in S^3 carrying line defects corresponding to those representations. In fact, this relation is quite general, as we can now explain.

We need two key facts:

1. Take two copies of the handlebody $D^2 \times S^1$. The boundary is $S^1 \times S^1$, where the first S^1 factor is the boundary of D^2. If we glue the two handlebodies together so that the two copies of D^2 are glued to form a sphere then the resulting closed 3-manifold is $S^2 \times S^1$. On the other hand, if we glue the two T^2 factors using an S-transformation that exchanges the two circles then the resulting 3-manifold is S^3. One quite way to see this is to observe that the resulting 3-manifold is simply connected, since the a, b cycles in the torus can be continuously shrunk to a point in one or the other handlebodies. One then invokes the Poincaé conjecture. A more low-level proof is....

♣GIVE A PROOF THAT DOES NOT RELY ON POINCARE ♣

2. Now consider the path integral with line defects inserted along the core of each of the above two handlebodies, one for representation i and one for representation j. These create corresponding states χ_i, χ_j in $\mathcal{H}(T^2)$. There is a bilinear form on this space in which these states are orthonormal On the other hand, when we glue the boundaries ♣NEED TO EXPLAIN!! ♣

using the modular transformation we must apply S to one of the two states before the gluing. The general result follows.

♣EXPLAIN THIS BETTER. ♣

Olympic proof of the Verlinde formula: Follow Moore-Seiberg Lectures on RCFT exercise 7.13.

6.3 The Jones polynomial and WRT Invariants

1. Knot invariants and Reidemeister moves

2. Skein Relations

3. Jones polynomial

4. Generalizations: Colored Jones polynomial according to Witten.

5. The path integral: WRT invariant

6. Computation using data of braiding, fusing and S-matrices.

6.4 Beyond knot polynomials: The idea of categorification and Knot homology.

7. Finite Group Chern-Simons Theory

7.1 The General Compact Group G

Three-dimensional Chern-Simons theory can be defined for any compact Lie group. See Appendix J.1 for some idea of what the general compact Lie group looks like.

7.1.1 Chern-Simons Theory When The Gauge Group Is A Finite Group

group cohomology.

group cohomology and BG.

general home for the level $H^4(BG;\mathbb{Z})$.

cite: Dijgraaf-Witten. Freed.

8. p-form Generalizations And Differential Cohomology

Review of differential cohomology: Follow chapter in FK lectures.

stress Abelian CS action is just a quadratic refinement of the Cheeger-Simons multiply-and-integrate bilinear form.

9. More About Topological Field Theories

Invertible topological field theories and anomaly cancellation.

Invertible topological field theory, bordism theory, and SPT phases.

10. Anyons, nonabelions, and quantum computation

10.1 Anyons in 2+1 dimensions

Let us now consider what happens when charged particles are constrained to live in two spatial dimensions. See

1. http://en.wikipedia.org/wiki/2DEG

2. S. Girvin, "The Quantum Hall Effect: Novel Excitations and Broken Symmetries," arXiv:cond-mat/9907002

for a description of some experimental realizations approximating this idealized situation.

Now, it is interesting if our plane also has thin solenoids of flux Φ piercing it. We can imagine a situation in which the flux cannot spread out, so they behave like particles in 2+1 dimensions as well. A good example of how this can happen is in a superconductor. A nice way to understand superconductivity is that it is a theory of electromagnetism where the $U(1)$ gauge theory symmetry is spontaneously broken by the vacuum expectation value of a charge two field representing the Cooper pairs. The flux tubes are regions of normal phase, where the photon is massless. The superconductor is a region where the photon gets a mass. The flux cannot spread out.

Now imagine that - for some unspecified reason - a particle of charge q binds to such a solenoid-particle. We label the boundstate by (q, Φ). These $2+1$ dimensional analogs of dyons have some very curious properties.

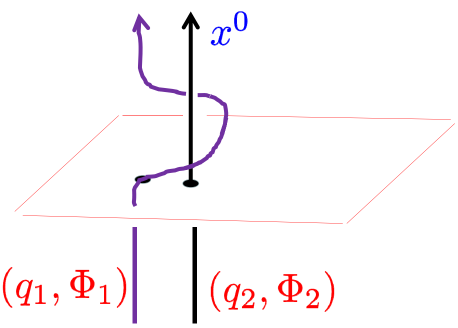

Figure 28: A boundstate (q_1, Φ_1) moves very slowly counterclockwise around a boundstate (q_2, Φ_2). Only the topology of the path matters in computing the change of phase of the wavefunction. Do not confuse the vertical direction with the z-axis. The vertical direction now represents the time direction.

Let us move a particle (q_1, Φ_1) very slowly around a particle (q_2, Φ_2) as in Figure 28 Applying the formula (**??**) the wavefunction picks up a phase $\exp[\frac{i}{\hbar c}(q_2\Phi_1)]$. Note that this does not depend on the exact shape of the trajectories, only that one particle circles around the other. At the same time, there is a phase change because particle (q_2, Φ_1) loops around the flux Φ_1. Indeed we could deform Figure 28 to Figure 29. Altogether then, the

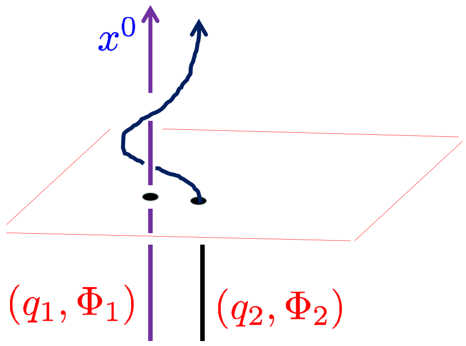

Figure 29: A topologically equivalent formulation of the path in Figure 7. This makes it clear that the boundstate (q_2, Φ_2) also moves counterclockwise around (q_1, Φ_1).

wavefunction of the pair of particles changes by

$$\exp[\frac{\mathrm{i}}{\hbar}(q_2\Phi_1 + q_2\Phi_2)] \tag{10.1}$$

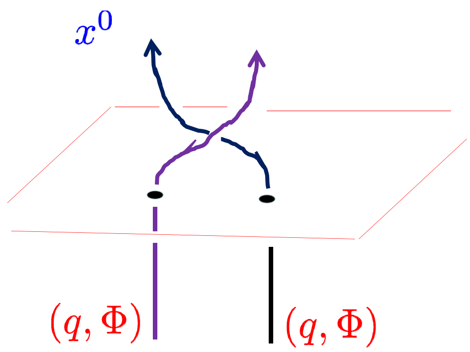

Figure 30: A pair of identical particles (q, Φ) are exchanged.

Now let us consider a pair of identical particles which are exchanged as in Figure 30. The net phase change is just $\exp[\mathrm{i}\frac{q\Phi}{\hbar}]$. But since we have exchanged identical particles we can interpret this as a statistics phase. Unlike the case of particles in $3+1$ dimensions, in the present case the statistics phase can be *any* phase. Such particles are called *anyons.* [42]

It is interesting to check the relation between spin and statistics.

We now apply the general formula (**??**) to the angular momentum in $d = 2+1$ dimensions. Here there is just the one generator $J = J_{12}$.

[42]The possible existence of anyons was pointed out by Leinaas and Myrheim in 1977. The term "anyon" was invented in F. Wilczek, "Quantum Mechanics of Fractional-Spin Particles". Physical Review Letters 49 (14): 957959.

In 2+1 dimensions the solenoid contributes $F_{ij} = \epsilon_{ij}\Phi\delta^2(x)$, (here i, j run from 1 to 2 and $\epsilon_{12} = +1$). From (**??**) the electric particle at $\vec{R}$ contributes an electric field

$$\vec{E} = 2q\frac{\vec{x} - \vec{R}}{|\vec{x} - \vec{R}|^2} \tag{10.2}$$

so $F_{0j} = -2q(x - R)^j/|\vec{x} - \vec{R}|^2$. Therefore the momentum density is

$$T_{0i} = F_{0k}F_{ik} = \frac{q\Phi}{2\pi}\epsilon_{ij}\frac{R_j}{R^2}\delta^{(2)}(\vec{x}) \tag{10.3}$$

Thanks to the δ-function in the integral is easily done and we find

$$J_{12} = \frac{q\Phi}{2\pi}\frac{\vec{a}\cdot\vec{R}}{\vec{R}\cdot\vec{R}} \tag{10.4}$$

It is now amusing to check the relation of spin and statistics:

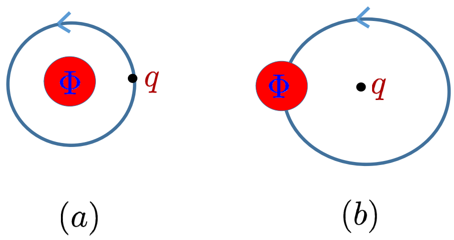

Figure 31: In (a) the charge q is slowly moved around the fluxon Φ and the wavefunction acquires an Aharonov-Bohm phase. In (b) we perform a rotation by 2π centered on q and the wavefunction of the electromagnetic field acquires a phase. These two phases are the same.

1. If we slowly rotate the particle around the flux in a counterclockwise fashion then the wavefunction picks up a phase $\exp[\mathrm{i}q\Phi/\hbar]$.

2. On the other hand, if we rotate the flux around the particle then the wavefunction should change by $\exp[2\pi\mathrm{i}J/\hbar]$. Taking $\vec{a} = \vec{R}$ in (10.4) we get the same phase:

$$\exp[2\pi\mathrm{i}J/\hbar] = \exp[\mathrm{i}q\Phi/\hbar] \tag{10.5}$$

Remark *Spin-statistics theorem*: The important property used in proving the spin-statistics theorem is the existence of an analytic continuation to Euclidean space.

♣Need to elaborate much more on that. ♣

Here are some sources for more material about anyons:

1. There are some nice lecture notes by John Preskill, which discuss the potential relation to quantum computation and quantum information theory: http://www.theory.caltech.edu/~preski

2. For a reasonably up-to-date review see A. Stern, "Anyons and the quantum Hall effectA pedagogical review". Annals of Physics 323: 204; arXiv:0711.4697v1.

3. A. Lerda, *Anyons: Quantum mechanics of particles with fractional statistics* Lect.Notes Phys. M14 (1992) 1-138

4. Other refs include: A. Khare, *Fractional Statistics and Quantum Theory*, and G. Dunne, *Self-Dual Chern-Simons Theories.*

OTHER REFS:
S. Buron, "A Short Guide to Anyons and Modular Functors,"
From categories to anyons: a travelogue
Preskill
Rowell and Wang: [52] BAMS

11. A Survey Of Further Generalizations And Applications

11.1 Applications To Supergravity

1. RR fields and self-duality
2. Anomaly cancellation.
3. M-theory Chern-Simons term.
4. Anomaly cancellation for M-theory on a manifold with boundary and E_8 gauge theory.

11.2 Noncompact Groups

3d quantum gravity:
Brown-Henneaux
Witten's approaches to 3d quantum gravity.
Analytic continuation in k: Complex groups.

11.3 Applications to Holography

SL(2,R)
Singletons

11.4 Many Other Applications And Extensions

1. Lattice Fermions (David Kaplan)

2. Arithmetic Chern-Simons (Minhyong Kim)

3. Chern-Simons for supergroups

4. Chern-Simons and topological strings. See review by Marino [39].

5. Holomorphic Chern-Simons on CY 3-folds: The B-model open topological string

6. String Field Theory

A. Symplectic Geometry

Recall that a phase space is a symplectic manifold: A manifold with a nondegenerate two-form

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu \tag{A.1}$$

from which we can define Poisson brackets of two functions on phase space:

$$\{f_1, f_2\} = \omega^{\mu\nu} \frac{\partial f_1}{\partial x^\mu} \frac{\partial f_2}{\partial x^\nu} \tag{A.2}$$

The first main theorem of symplectic geometry is that there are no nontrivial local invariants of a symplectic form. This is the Darboux theorem: It says that there are always local coordinates (p_i, q^i) on $\mathcal{P}$ so that ♣Not hard to prove. Give it? ♣

$$\omega = \sum_i dp_i \wedge dq^i \tag{A.3}$$

Phase space is the proper arena in which to discuss classical mechanics, and is also a good starting point for quantization.

A.1 Kahler quotients: A lightning review

A.1.1 The moment map

Let (M, ω) be a phase space, that is, a symplectic manifold.

Recall that given a function h on M one can produce a corresponding Hamiltonian vector field by $\iota(V)\omega = dh$, or, in local coordinates

$$V^i = \omega^{ij}\partial_j h. \tag{A.4}$$

The vector field V generates a group of diffeomorphisms which preserve the symplectic form.

Conversely, suppose that (M, ω) is a symplectic manifold and a Lie group G acts symplectically:

$$g^*(\omega) = \omega \qquad\qquad \forall g \in G \tag{A.5}$$

Applying this to a one-parameter subgroups generated by $T \in \mathfrak{g}$ we get vector fields $\xi = \xi(T)$ such that

$$\begin{aligned} 0 &= \mathcal{L}_\xi \omega \\ &= (d\iota(\xi) + \iota(\xi)d)\omega \\ &= d\big(\iota(\xi)\omega\ \big) \end{aligned} \tag{A.6}$$

Therefore, if $H^1_{DR}(M) = 0$, then we can conclude

$$\iota(\xi)\omega = d\mu \tag{A.7}$$

for some globally well-defined function $\mu : M \to \mathbb{R}$. Note that μ is only defined up to a constant. It depends on T so we denote its value at a point by $\mu(p; T)$, $p \in M, T \in \mathfrak{g}$.

In Hamiltonian dynamics, μ is the *charge* associated to the symmetry generated by the vector field ξ. It generates the transformations associated with the symmetry:

$$\begin{aligned}\{\mu, f\} &= \omega^{ij}\partial_i\mu\partial_j f \\ &= \xi^i\partial_i f \\ &= \mathcal{L}_\xi(f)\end{aligned} \tag{A.8}$$

where in the last line we are using the Lie derivative wrt ξ.

We have a conserved charge $\mu(\cdot; T)$ for each element $T \in \mathfrak{g}$. Since $T \to \xi(T)$ is linear in T so is $T \to \mu(\cdot; T)$. To give a function $M \to \mathbb{R}$ for each $T \in \mathfrak{g}$ in a way that depends linearly on T is equivalent to giving a single function

$$\mu : M \to \mathfrak{g}^* \tag{A.9}$$

This function is known as the *moment map*. It satisfies:

$$\langle T, \mu(p)\rangle := \mu(p; T) \tag{A.10}$$

all $T \in \mathfrak{g}$, by definition, and also

$$\mu(g \cdot p) = \mathrm{Ad}^*(g)(\mu(p)) \tag{A.11}$$

where on the right hand side we have the coadjoint action.

Choosing a basis T^A for $\mathfrak{g}$ we may write:

$$\mu = \sum_A \mu^A T_A^* \tag{A.12}$$

where $\mu^A = \mu(T^A)$ is an ordinary function on M.

The word “moment” is short for “momentum.” It could be linear or angular momentum as the following examples show:

Example 1: Let $M = T^*\mathbb{R}^n$ with Darboux coordinates $(\vec{q}, \vec{p}) = (\vec{q}^i, \vec{p}_i)$ so that

$$\omega = \sum_i dp_i \wedge dq^i \tag{A.13}$$

Let $G = \mathbb{R}^n$ acting by translation:

$$\vec{a} \cdot (\vec{q}, \vec{p}) = (\vec{q} + \vec{a}, \vec{p}) \tag{A.14}$$

We also have $\mathfrak{g} \cong \mathbb{R}^n$ as a vector space, with Lie bracket given by 0 and infinitesimal action by $g(t) = e^{t\vec{v}}$:

$$g(t) : (\vec{q}, \vec{p}) \to (\vec{q} + t\vec{v}, \vec{p}) \tag{A.15}$$

Then

$$\xi(\vec{v}) = v^i \frac{\partial}{\partial q^i} \tag{A.16}$$

and

$$\iota(\xi(\vec{v}))\omega = -v^i dp_i = d\mu(\vec{v}) \tag{A.17}$$

with

$$\mu(q^i, p_i; \vec{v}) = -v^i p_i \tag{A.18}$$

Example 2 Let us turn to our main example: Suppose G acts linearly on a vector space $V = \{(x^1, \dots, x^N)\}$ with symplectic form $\omega_{ij} dx^i dx^j$ where ω_{ij} is a constant antisymmetric matrix. Let $\rho(T^A)^i{}_j$ be the $N \times N$ matrices representing the Lie algebra. Then

$$\xi^A := \xi(T^A) = x^j \rho(T^A)^i{}_j \frac{\partial}{\partial x^i} \tag{A.19}$$

Then the moment map is:

$$\mu^A(x) = \frac{1}{2} x^j \rho(T^A)^i{}_j \omega_{ik} x^k \tag{A.20}$$

(The group acts symplectically iff the matrix $\rho(T^A)^i{}_j \omega_{ik}$ is symmetric in jk.)

Specializing to the case $M = T^*\mathbb{R}^n$ with standard Darboux coordinates we can consider the embedding $GL(n, \mathbb{R}) \hookrightarrow Sp(2n, \mathbb{R})$ with

$$\begin{pmatrix} A & \\ & A^{tr,-1} \end{pmatrix} \tag{A.21}$$

For the group elements $g(t) = 1 + te_{ij} \in GL(n, \mathbb{R})$ we have

$$g(t) : (\vec{q}, \vec{p}) \to (\vec{q} + tq_j \vec{e}_i, \vec{p} - tp_i \vec{e}_j) \tag{A.22}$$

so

$$\xi(e_{ij}) = q^j \frac{\partial}{\partial q^i} - p_i \frac{\partial}{\partial p_j} \tag{A.23}$$

and

$$\mu(q^i, p_i; e_{ij}) = -q^i p_j \tag{A.24}$$

If we further specialize to $SO(n) \subset GL(n, \mathbb{R})$ with basis $T_{ij} = e_{ij} - e_{ji}$ we get $\mu(T_{ij}) = q^j p_i - q^i p_j$ and finally, specializing to $T^*\mathbb{R}^3$ with $T_i = \frac{1}{2}\epsilon_{ijk} T_{jk}$ we find that $\mu(T_i) = L_i$ is the standard formula for angular momentum:

$$\mu(q^i, p_i; T_i) = \epsilon_{ijk} q^j p_k \tag{A.25}$$

A.1.2 Symplectic Quotient

If $H^1_{DR}(M) = 0$ one can show that

$$\{\mu^A, \mu^B\} = f^{AB}_C \mu^C + c^{AB} \tag{A.26}$$

for some constants c^{AB}. The Jacobi relation on Poisson brackets shows that the c^{AB} in fact define a Lie algebra 2-cocycle. Shifting μ^A by constants redefines the cocycle by a coboundary. If the 2-cocycle vanishes then by we can simultaneously impose the constraints $\mu^A = 0$, as in constrained Hamiltonian dynamics. If the 2-cocycle defines a nontrivial cohomology class then there is an anomaly attempting to impose these constraints.

Let us assume that we can put to zero any cocycles c^{AB} via a shift with a coboundary. If $\zeta \in Z(\mathfrak{g})^*$ is in the (dual of) the center then the constraints $\phi = \mu - \zeta$ are first class constraints. That is, the set of constraints

$$\phi^A = \mu(T^A) - \langle T^A, \zeta \rangle = 0 \tag{A.27}$$

form a set of first class constraints.

Now equation (A.26) with $c^{AB} = 0$ implies

$$\mathcal{L}_{V^A}(\mu^B) = f_C^{AB} \mu^C \tag{A.28}$$

and hence, if G is connected we can conclude that the constrained space

$$\mathcal{S} := \{p | \phi^A(p) = 0\} \subset M \tag{A.29}$$

in phase space defined by (A.27) is G-invariant. We can therefore take the quotient by the G-action.

Definition The symplectic quotient – or Hamiltonian reduction – of M by G is the manifold:

$$M//_\zeta G = \mu^{-1}(\zeta)/G \tag{A.30}$$

A.1.3 Kähler quotient

Recall that a Kähler manifold is a Riemannian manifold with a compatible complex structure I, i.e. $g(IX, IY) = g(X, Y)$, $X, Y \in TM$ which is parallel $\nabla I = 0$ wrt the Levi-Civita connection. Equivalently, the two-form defined by $\omega(X, Y) = g(X, IY)$ is a closed 2-form.

In coordinate terms: There exist local complex coordinates z^i so that $ds^2 = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + cc$ and then

$$\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \tag{A.31}$$

is a closed 2-form.

Of course, a Kahler manifold is a symplectic manifold with symplectic form ω.

Theorem: Suppose M is Kähler and G acts symplecticaly without fixed points. Then the symplectic quotient is a smooth Kahler manifold with Kahler structure inherited from M.

For a proof see the references.

An important special case of this is the following. Take $V = \mathbb{C}^N$ to be a complex vector space with symplectic form

$$\omega = \frac{i}{2} \sum_{a=1}^{N} dz^a \wedge d\bar{z}_a = \sum_a dx^a \wedge dy^a \tag{A.32}$$

where $z^a = x^a + \sqrt{-1}y^a$. Clearly this is preserved by the unitary group $U(N)$ acting on z^a. An element of the Lie algebra is an antihermitian matrix

$$(T^a_{\ b})^* = -T^b_{\ a} \tag{A.33}$$

the corresponding vector field is

$$\xi(T) = z^b T^a_{\ b} \frac{\partial}{\partial z^a} - \bar{z}_b (T)^b_{\ a} \frac{\partial}{\partial \bar{z}^a} \tag{A.34}$$

and the corresponding momentum is

$$\mu(T) = i z^b T^a_{\ b} \bar{z}_a \tag{A.35}$$

up to a constant.

Example 1 $V = \mathbb{C}^N$, with symplectic form (A.32) with $G = U(1)$ acting by

$$e^{i\theta} : z^a \to e^{iQ^a\theta} z^a, \qquad a = 1, \dots, N \tag{A.36}$$

The moment map is

$$\mu = \sum Q^a |z^a|^2 \tag{A.37}$$

The nature of the symplectic quotient depends very strongly on the Q^a and the level ζ of the moment map:

a.) Unless the Q^a are commensurate the $U(1)$ action is ergodic. Let us suppose the Q^a are commensurate. Then we can write $Q^a = p^a/L$ where L is some integer and $g.c.d(p^1, \dots, p^N) = 1$. The nature of the quotient depends on the integers p^i. If they are pairwise mutually prime, then the quotient space is smooth. However, suppose that there is a subset $S \subset \{1, \dots, N\}$ on which we have $p^a = rq^a$ for some common factor r. If $z^b = 0$ for $b \notin S$ and we take $\theta = 2\pi/r$ then the group has a fixed point. Thus, we have a Z_r orbifold singularity.

b.) If all the Q^a are positive and rational and $\zeta > 0$ then the constrained space $\mu = \zeta$ is compact and nonempty. It is a deformed sphere.

c.) If the Q^a have different signs then the constrained space is noncompact, and very different phenomena can happen.

This example also illustrates how topology of Kähler quotients can change as we change the level ζ. Suppose there are n_+ values $Q_a = +1$ and n_- values with $Q_a = -1$. Then for $\zeta > 0$ the symplectic quotient is the total space of the bundle $\mathcal{O}(1)^{\oplus n_-} \to \mathbb{C}P^{n_+-1}$ and for $\zeta < 0$ it is $\mathcal{O}(1)^{\oplus n_+} \to \mathbb{C}P^{n_--1}$.

Example 2 *D-terms in supersymmetric field theories.* Suppose we have a supersymmetric field theory in four dimensions with four supersymmetries with a set of chiral superfields Φ^a transforming in some unitary representation of a gauge group with generators T^A. Suppose the chiral superfields take values in a linear space and have the standard Kahler potential $\sum |\Phi^a|^2$. Then the D-terms in the potential energy D^A are precisely the moment maps for the G-action on the target space. The levels ζ^A of the moment maps are associated with $U(1)$ factors in G and are called Fayet-Iliopoulos terms.

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B. Lightning Review Of Kähler Quantization

B.1 The Bargmann Representation

The Bargmann representation is an alternative way of quantizing the standard phase space $\mathcal{P} = \mathbb{R}^2$ with coordinates (q, p) and symplectic form

$$\omega = \frac{dp \wedge dq}{2\pi\hbar} \tag{B.1}$$

leading to the standard $[\hat{p}, \hat{q}] = -\mathrm{i}\hbar$. ♣SIGN??? ♣

In the usual quantization we regard $\mathcal{P} = T^*\mathbb{R}$ with coordinate q on the real line so $\mathcal{H} = L^2(\mathbb{R})$, wavefunctions are normalizable functions $\psi(q)$ and $\hat{p} = -\frac{\mathrm{i}}{\hbar}\frac{d}{dq}$, etc. One must choose an operator ordering prescription to define $Q^{(k)}(f)$ for general functions $f(p, q)$ and this can be done in several ways to satisfy the above criteria (2.75)- (2.78).

In the Bargmann representation instead the Hilbert space $\mathcal{H} = \mathrm{Hol}(\mathbb{C})$ is the space of entire functions on the complex plane that are normalizable in the inner product defined by:

$$\langle\psi_1, \psi_2\rangle = \int_{\mathbb{C}} e^{-\kappa|z|^2}\psi_1(z)^*\psi_2(z)\omega \tag{B.2}$$

Here $\kappa > 0$ will play the role of $1/\hbar$ and

$$\omega = \frac{\mathrm{i}dz \wedge d\bar{z}}{2\pi} = \frac{dxdy}{\pi} = \frac{d\phi}{2\pi}d(r^2) \tag{B.3}$$

The quantization $Q^{(k)}(f)$ of an integrable function $f(z, \bar{z})$ (<u>not</u> necessarily holomorphic or anti-holomorphic) is defined by

$$\langle\psi_{j_1}|Q^{(\kappa)}(f)|\psi_{j_2}\rangle = \int_{\mathbb{C}} e^{-\kappa|z|^2}\psi^*_{j_1}(\bar{z})f(z, \bar{z})\psi_{j_2}(z)\frac{dxdy}{\pi} \tag{B.4}$$

In particular if we define

$$a = \sqrt{\kappa}Q^{(\kappa)}(\bar{z}) \qquad a^\dagger = \sqrt{\kappa}Q^{(\kappa)}(z) \tag{B.5}$$

then we can verify the usual relations

$$[a, a^\dagger] = 1 \tag{B.6}$$

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SOME REMARKS ABOUT COHERENT STATES. WHY THEY ARE USEFUL FOR SEMICLASSICAL COMPUTATION.

BEREZIN TRANSFORM.

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B.2 Generalization: Berezin-Toeplitz Quantization

There is a very nice quantization procedure, which is extremely useful in Chern-Simons theory, that is available when the phase space carries the extra structure of a Kähler manifold. See, for example, [53] and references therein for a description of Berezin-Toeplitz quantization.

Recall that symplectic manifold is Kähler if it has a Riemannian metric

$$ds^2 = g_{\mu\nu} du^\mu du^\nu \qquad\qquad \mu, \nu = 1, \dots, 2n. \tag{B.7}$$

with a metric-compatible complex structure such that the symplectic form is a positive $(1,1)$ form so that

$$\omega(v_1, Jv_2) = g(v_1, v_2) \tag{B.8}$$

Written out in local coordinates, this means the symplectic form in general in local coordinates has the form

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu \tag{B.9}$$

Poisson brackets:

$$\{f_1, f_2\} = \omega^{\mu\nu} \frac{\partial f_1}{\partial x^\mu} \frac{\partial f_2}{\partial x^\nu} \tag{B.10}$$

and

$$\omega_{\mu\lambda} J_\nu{}^\lambda = g_{\mu\nu} \tag{B.11}$$

where the complex structure is

$$J(\frac{\partial}{\partial u^\mu}) = J_\mu{}^\nu \frac{\partial}{\partial u^\nu} \tag{B.12}$$

There are local complex coordinates z^i so that we can write

$$ds^2 = \frac{1}{2} g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}} + cplx.conj. \qquad i, j = 1, \dots, n. \tag{B.13}$$

where $g_{i\bar{j}}$ is an Hermitian metric

$$(g_{i\bar{j}})^* = g_{j\bar{i}} \tag{B.14}$$

and in these coordinates the symplectic form is

$$\omega = \frac{\mathrm{i}}{2} g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \tag{B.15}$$

One can check that it is a real form. A (choice of) Kähler potential is a locally-defined function so that

$$\omega = \mathrm{i}\partial\bar{\partial}K \tag{B.16}$$

♣MISSING NORMALIZAATION FACTOR HERE!!!! ♣

We now assume that there is a holomorphic line bundle $L \to \mathcal{P}$ with an Hermitian metric h and a metric-compatible connection ∇ so that the curvature $F(\nabla) = \omega$. Then we can define

$$\mathcal{H}_\kappa := \ker[\bar{\partial} : \Omega^{0,0}(L^{\otimes\kappa}) \to \Omega^{0,1}(L^{\otimes\kappa})] \tag{B.17}$$

Here $\Omega^{p,q}(L^{\otimes\kappa})$ is the inner-product space of globally defined $\mathcal{C}^\infty$ sections of $L^{\otimes\kappa}$ with values in (p,q)-forms on $\mathcal{P}$. We regard $\mathcal{H}_\kappa$ as a subspace of a the Hilbert space completion of $\Omega^{0,0}(L^{\otimes\kappa})$. The Berezin-Toeplitz quantization of the algebra $\mathcal{C}(X)$ is $f \mapsto Q^{(\kappa)}(f)$ where

$$Q^{(\kappa)}(f) := \Pi \circ M_f \circ \iota \tag{B.18}$$

Here $\iota : \mathcal{H}_\kappa \hookrightarrow \Omega^{0,0}(L^{\otimes\kappa})$ is inclusion, M_f is the multiplication operator on the Hilbert space $\Omega^{0,0}(L^{\otimes\kappa})$ and Π is the orthogonal projection onto the closed subspace $\mathcal{H}_\kappa$. The inner product on two holomorphic sections is

$$\langle s_1, s_2\rangle = \int_X h^\kappa(s_1,s_2)\frac{\omega^n}{n!} = \int_X e^{-\kappa K} s_1(x)^* s_2(x) \prod_i \frac{d\bar{z}_i dz_i}{2\pi \mathrm{i}} \tag{B.19}$$

In terms of this inner product we can define the quantization (B.18) of a function is the operator with matrix elements: ♣NORMALIZATION FACTORS: FACTORS OF π ♣

$$\langle s_1, Q^{(\kappa)}(f) s_2\rangle = \int_X e^{-\kappa K} s_1(x)^* f(x) s_2(x) \prod_i \frac{d\bar{z}_i dz_i}{2\pi \mathrm{i}} \tag{B.20}$$

GENERAL COHERENT STATES

Remarks:

1. We can think of a section s of $L^{\otimes\kappa}$ as a collection of complex-valued functions $s_\alpha(x)$ defined on patches $\mathcal{U}_\alpha$ providing an atlas for X such that

$$s_\alpha(x) = (g_{\alpha\beta}(z))^\kappa s_\beta(x) \tag{B.21}$$

when $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $g_{\alpha\beta}(z)$ are holomorphic transition functions. Then $e^{-\kappa K}|s|^2$ is a well-defined function on X so that, on patch overlaps

$$e^{-\kappa K^{(\alpha)}}|s_\alpha(x)|^2 = e^{-\kappa K^{(\beta)}}|s_\beta(x)|^2 \tag{B.22}$$

because, in the trivialization we have used, K changes by Kähler transformation:

$$K^{(\alpha)} = K^{(\beta)} + \log g_{\alpha\beta}(z) + \log \overline{g_{\alpha\beta}(z)} \tag{B.23}$$

2. At large k there is one state per $2\pi\hbar$ unit of volume. This is just the Bohr-Sommerfeld semiclassical quantization rule. In particular, if the phase space has finite symplectic volume there will be a finite-dimensional Hilbert space of states.

3. In good cases, when $\mathcal{L}$ is sufficiently positive $H^i(\mathcal{P},\mathcal{L}^{\otimes k})$ will vanish for $i>0$ and then we can compute the exact dimension of the Hilbert space using the index formula:

$$\dim_{\mathbb{C}} H^0(\mathcal{P};\mathcal{L}^{\otimes k}) = \int_{\mathcal{P}} e^{kc_1(\mathcal{L})} Td(T^{1,0}\mathcal{P}) \tag{B.24}$$

Since $c_1(\mathcal{L})$ is represented by $\frac{\omega}{2\pi}$ this can be writte, for $k \to \infty$ as

$$\dim_{\mathbb{C}} H^0(\mathcal{P};\mathcal{L}^{\otimes k}) = k^n \mathrm{vol}\,(\mathcal{P}) \left(1 + \frac{n\pi}{k}\frac{\int_{\mathcal{P}} \omega^{n-1} c_1(T^{1,0}\mathcal{P})}{\int_{\mathcal{P}}\omega^n} + \mathcal{O}(k^{-2})\right) \tag{B.25}$$

where $\mathrm{vol}\,(\mathcal{P}) := \int_{\mathcal{P}} \frac{1}{n!}\left(\frac{\omega}{2\pi}\right)^n$ and $\dim_{\mathbb{R}} \mathcal{P} = 2n$.

Recalling that $\hbar = 1/k$ so we see:

4. There is a finite series of $\hbar = 1/k$ corrections to the semiclassical result. The sign depends on the sign of $c_1(T^{1,0}\mathcal{P})$. In general, curvature alters the semiclassical expectation. At large k, positive curvature creates room for more quantum states and negative curvature eliminates quantum states.

B.2.1 Example: Quantization Of The Sphere And Representations Of $SU(2)$

The quotient $SU(2)/U(1)$ is an example of a coadjoint orbit whose quantization is required when quantizing Chern-Simons on surfaces with punctures. In general G/T can be given the structure of a Kähler manifold and Kähler quantization is well-suited to the problem.

The Hopf fibration $\pi : SU(2) \to S^2$ shows that $S^2 \cong SU(2)/U(1)$. We can also give S^2 a natural complex structure by the identification with $\mathbb{CP}^1$. It is useful to describe things explicitly using stereographic projection to the complex plane. Here are the basic formulae:

$$\begin{aligned} z &= \frac{x_1 + \mathrm{i}x_2}{1+x_3} = e^{\mathrm{i}\phi}\tan(\frac{\theta}{2}) \\ \frac{z}{1+|z|^2} &= \frac{1}{2}(x_1 + \mathrm{i}x_2) = e^{\mathrm{i}\phi}\sin\theta \\ \frac{1}{1+|z|^2} &= \cos^2(\theta/2) = \frac{1+\cos\theta}{2} \\ \frac{-\mathrm{i}dz\wedge d\bar{z}}{(1+|z|^2)^2} &= \frac{1}{2}\sin\theta d\phi\wedge d\theta \end{aligned} \tag{B.26}$$

with the Bott projector:

$$\frac{1}{2}(1+\hat{x}\cdot\vec{\sigma}) = \frac{1}{1+|z|^2}\begin{pmatrix} |z|^2 & \bar{z} \\ z & 1 \end{pmatrix} \tag{B.27}$$

The symplectic form after stereographic projection is

$$\frac{\omega}{2\pi} = \frac{\mathrm{i}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} = \frac{1}{2\pi\mathrm{i}}\partial\bar{\partial}\log(1+|z|^2) \tag{B.28}$$

Note the symplectic volume is

$$\int_{\mathbb{CP}^1}\frac{\omega}{2\pi} = +1 \tag{B.29}$$

We can take Kähler potential $K = \log(1+|z|^2)$. Note that it is not single-valued on $\mathbb{CP}^1$: If we choose the opposite stereographic projection with coordinate $w = 1/z$ then we would have used $K = \log(1+|w|^2) = \log(1+|z|^2) - \log z - \log\bar{z}$.

The Hopf line bundle has Hermitian metric so that if we view sections of L as functions on $\mathbb{C}$ then they are simply complex-valued functions on $\mathbb{C}$ with restricted growth at infinity. The Hermitian inner product on the fibers $L_x^{\otimes\kappa}$ are give by:

$$h(\psi_1(x),\psi_2(x)) := (1+|z|^2)^{-\kappa}\psi_1^*(x)\psi_2(x) = e^{-\kappa K}\psi_1^*(x)\psi_2(x) \tag{B.30}$$

so that the inner product on $\Gamma(L^{\otimes\kappa})$ is given by

$$\langle\psi_1,\psi_2\rangle = \int_{\mathbb{C}} h(\psi_1(x),\psi_2(x))\frac{\omega}{2\pi} = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\frac{1}{(1+|z|^2)^{\kappa+2}}\psi_1^*(x)\psi_2(x)dz\wedge d\bar{z} \tag{B.31}$$

Normalizable functions must have $\psi(z,\bar{z}) \sim r^{\ell}$ with $\ell < \frac{1}{2}(\kappa+1)$.

In geometric quantization the Hilbert space $\mathcal{H}_\kappa = Hol(L^{\otimes\kappa})$ is $\kappa+1$ dimensional. Note that it is finite-dimensional because the symplectic space has finite volume. Indeed, if one defines operators:

$$J^i := (j+1)Q^{(\kappa)}(x^i) \tag{B.32}$$

where $j := \kappa/2$ and x^i are the functions on S^2 given by restricting the standard coordinate functions on $\mathbb{R}^3$, viewing S^2 as the unit sphere in $\mathbb{R}^3$ then one finds that J^i satisfy the standard commutation relations for $\mathfrak{su}(2)$:

$$[J^i, J^j] = \mathrm{i}\epsilon^{ijk}J^k \tag{B.33}$$

In fact, $\mathcal{H}_\kappa$ is the irreducible representation of $\mathfrak{su}(2)$ of spin $j = \kappa/2$. Note that the symplectic volume ♣SIGN IN THIS COMMUTATOR ? ♣

$$\int_{\mathbb{CP}^1} \frac{\omega^{(\kappa)}}{2\pi} = \kappa = 2j \tag{B.34}$$

is not the exact dimension of the Hilbert space, which is $\kappa + 1 = 2j + 1$. The two only agree in the semiclassical $\kappa \to \infty$ limit.

B.3 The Borel-Weil-Bott Theorem

The previous example is a very special case of a more general and very beautiful theorem known as the Borel-Weil-Bott theorem.

$G/T = G^c/P$ both compact and Kähler. Need to discuss complex structures and Kähler form.

Good reference on geometry of flag manifolds: Alekseevsky.

$$\mathcal{O}(\lambda) \subset \mathfrak{g}^* \tag{B.35}$$

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Simply using bilinear form to identity $\mathfrak{g} \cong \mathfrak{g}*$ as vector spaces
****

Family of symplectic structures: If $\lambda \in \mathfrak{g}^*$ then

$$\omega_\lambda(X,Y) = \frac{1}{2}\langle \lambda, [X,Y]\rangle \tag{B.36}$$

Descends from 2-form on $\mathfrak{g}$ to a 2-form on $T_\lambda\mathcal{O}(\lambda)$ where it becomes nondegenerate. Using a nondegenerate bilinear form tr on $\mathfrak{g}$ we can identify with adjoint orbits, so if $t_0 \in \mathfrak{t} \subset \mathfrak{g}$ we can instead write:

$$\omega_{t_0}(X,Y) = \frac{1}{2}\mathrm{tr}\,(t_0[X,Y]) \tag{B.37}$$

This is the form that is more useful in discussions of Chern-Simons theory.

Mathematical version: Sections of holomorphic line bundles. Induced representations. etc.

****

Physics version: Phase space integral. There is also a SQM interpretation.

1. Alekseev, Faddeev, Shatashvili [1]
2. Alvarez, Singer, Windey [2][3]
3. Stone....
4. Szabo [57]

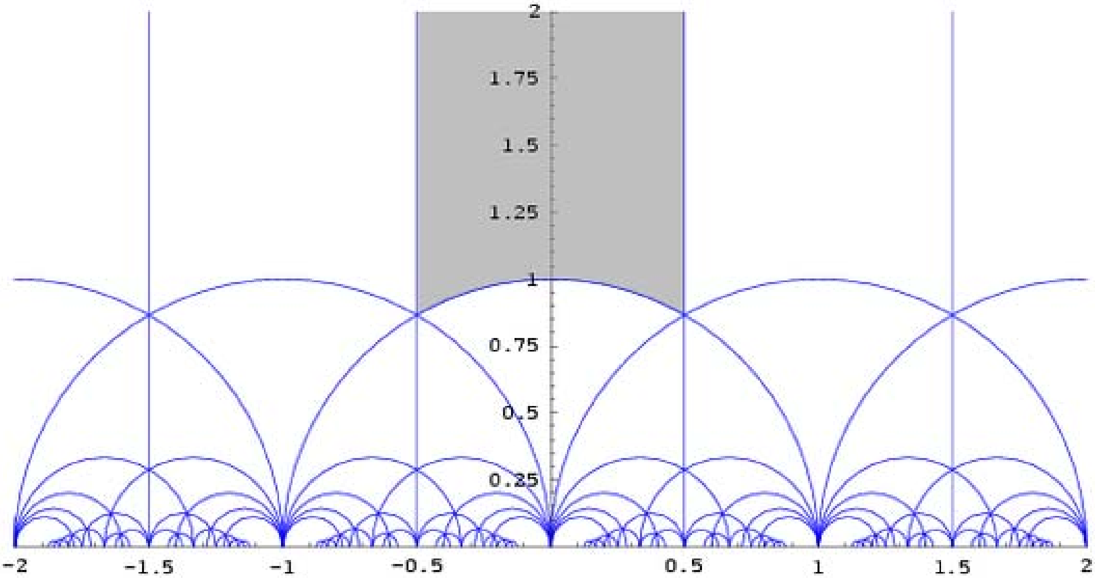

Figure 32: The keyhole region, a standard choice of fundamental domain for the action of $PSL(2,\mathbb{Z})$ on the complex upper half-plane. Figure from Wikipedia article on "Modular Group".

C. A Few Details About $SL(2,\mathbb{Z})$

C.1 Generators And A Fundamental Domain For $PSL(2,\mathbb{Z})$

The modular group is $PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\{\pm 1\}$, where $SL(2,\mathbb{Z})$ is the subgroup of $SL(2,\mathbb{R})$ of matrices all of whose matrix elements are integers. Recall that this group acts effectively on the complex upper half-plane $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{C.1}$$

We will find a fundamental domain for this group action, and in the process prove that $SL(2,\mathbb{Z})$ is generated by the group elements S and T defined by:

♣Need to check we have used a consistent sign for S throughout the notes. ♣

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.2}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{C.3}$$

These matrices satisfy relations:

$$\begin{aligned} S^2 &= -1 \\ (ST)^3 = (TS)^3 &= -1 \end{aligned} \tag{C.4}$$

Note that if we change the sign of S the $(ST)^3 = (TS)^3 = +1$, so we should take some care with the definition of the sign of S.

Denote the images of S, T in $PSL(2, \mathbb{Z})$ by $\bar{S}, \bar{T}$. Note that $\bar{S}$ does not depend on the sign of S. [43]

Let:

$$\tilde{\mathcal{F}} := \{\tau \in \mathcal{H} ||\tau| \geq 1 \quad \& \quad |\mathrm{Re}(\tau)| \leq \frac{1}{2}\} \tag{C.5}$$

This is almost, but not quite the canonical fundamental domain for the modular group. It is the famous *keyhole region* shown in Figure 32. Let $\bar{G}$ be the subgroup generated by $\bar{S}$ and $\bar{T}$. We claim that $\cup_{g \in \bar{G}} g \cdot \tilde{\mathcal{F}}$ is the entire half-plane. To prove this recall that, for any $g \in SL(2, \mathbb{Z})$,

$$\mathrm{Im}(g \cdot \tau) = \frac{\mathrm{Im}\tau}{|c\tau + d|^2} \tag{C.6}$$

Now, for any fixed $\tau \in \mathcal{H}$ the function $|c\tau + d|$ is bounded below on $SL(2, \mathbb{Z})$, and hence on G. Indeed, decomposing $\tau = x + \mathrm{i}y$ into its real and imaginary parts

$$|c\tau + d|^2 = (cx + d)^2 + c^2 y^2 \geq \begin{cases} y^2 & c \neq 0 \\ d^2 \geq 1 & c = 0 \end{cases} \tag{C.7}$$

Therefore, for any fixed τ there will exist a group element $g \in \bar{G}$ such that $\mathrm{Im}(g \cdot \tau)$ takes a maximal value as a function of g. Note that multiplying g on the left by a power of $\bar{T}$ or $\bar{T}^{-1}$ does not change this property, so there is not a unique g which maximizes $\mathrm{Im}(g \cdot \tau)$. We can fix the ambiguity by requiring $|\mathrm{Re}(g \cdot \tau)| \leq \frac{1}{2}$. Choose such a group element g. We claim that for this transformation, $\tau' = g \cdot \tau \in \tilde{\mathcal{F}}$. We need only check that $|\tau'| \geq 1$. If not, then $|\tau'| < 1$ but then $\mathrm{Im}(\bar{S} \cdot \tau') = \mathrm{Im}(\tau')/|\tau'|^2 > \mathrm{Im}(\tau')$, contradicting the definition of g. In conclusion, every element of the upper half-plane can be brought to $\tilde{\mathcal{F}}$ by a suitable element of $\bar{G}$.

Now we need two Lemmas:

Lemma 1: If $g \in SL(2, \mathbb{Z})$ and τ have the property that both $\tau \in \tilde{\mathcal{F}}$ and $g \cdot \tau \in \tilde{\mathcal{F}}$ then

1. $|\mathrm{Re}(\tau)| = \frac{1}{2}$ and $g \cdot \tau = \tau \pm 1$, <u>or</u>

[43] We are here following a very nice argument by J.-P. Serre, *A Course in Arithmetic*, Springer GTM 7, pp. 78-79.

2. $|\tau| = 1$

To prove this note that, WLOG, we may assume that $\mathrm{Im}(g \cdot \tau) \geq \mathrm{Im}\tau$. (If not replace $g \to g^{-1}$ and $\tau \to g^{-1}\tau$.) But this equation implies $1 \geq |c\tau + d|$ which in turn implies:

$$\begin{aligned}
1 &\geq |c\tau + d|^2 \\
&= (cx + d)^2 + c^2 y^2 \\
&= c^2|\tau|^2 + 2cdx + d^2 \\
&\geq c^2|\tau|^2 - |cd| + d^2 \\
&= c^2(|\tau|^2 - \frac{1}{4}) + (|d| - \frac{1}{2}|c|)^2 \\
&\geq \frac{3}{4}c^2 + (|d| - \frac{1}{2}|c|)^2
\end{aligned} \tag{C.8}$$

From (C.8) we conclude:

1. $(c = 0, d = \pm 1)$ or $(c = \pm 1, d = 0, \pm 1)$.
2. The inequalities are saturated iff $2cdx = -|cd|$ and $|\tau| = 1$.

If $c = 0$ and $d = \pm 1$ then $g \cdot \tau = \tau \pm 1$. In this case it is clear that $|\mathrm{Re}(\tau)| = \frac{1}{2}$. If $c = \pm 1$, and $d = 0, \pm 1$ then the inequality is saturated, and hence $|\tau| = 1$.

Lemma 2: If $\tau \in \tilde{\mathcal{F}}$ and $\bar{g} \cdot \tau = \tau$ with $\bar{g} \in PSL(2, \mathbb{Z})$ and $\bar{g} \neq 1$ then either

1. $\tau = \mathrm{i}$ and the stabilizer group is $\{1, \bar{S}\}$
2. $\tau = \omega = e^{2\pi \mathrm{i}/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and the stabilizer group is
$$\{1, \bar{S}\bar{T}, (\bar{S}\bar{T})^2\} \tag{C.9}$$
3. $\tau = -\omega^2 = e^{\pi \mathrm{i}/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and the stabilizer group is
$$\{1, \bar{T}\bar{S}, (\bar{T}\bar{S})^2\} \tag{C.10}$$

In particular, for all other points $\tau \in \tilde{\mathcal{F}}$, the stabilizer group is the trivial group.

Lemma 2 follows quickly from Lemma 1: If $g\tau = \tau$ then we must be in the case $c = \pm 1$. If $d = 0$ then $a - 1/\tau = \tau$ for some integer a. But we must also have $|\tau| = 1$ and hence $a = \tau + \bar{\tau}$. Since a is an integer we quickly find that $a = 0$ with $\tau = \mathrm{i}$, or $a = \pm 1$ with $\tau = \omega$ or $-\omega^2$. If $d = \pm 1$ then from the saturation condition $2cdx = -|cd|$ we get $x = -\frac{1}{2}d/|d|$ and hence $\tau = \omega$ or $= -\omega^2$.

Now we can finally prove:

Theorem: $SL(2, \mathbb{Z})$ is generated by S and T.

Proof: Let τ_0 be in the interior of $\tilde{\mathcal{F}}$. Then choose any element $g \in SL(2, \mathbb{Z})$ with $\bar{g} \neq 1$. Then there is an element $g' \in \bar{G}$ so that $g' g \cdot \tau_0 \in \tilde{\mathcal{F}}$. Moreover, this element must be in the interior of $\tilde{\mathcal{F}}$ by Lemma 1 and and hence, by Lemma 2, $g'g = 1$ in $PSL(2,\mathbb{Z})$. Therefore $\bar{g} \in \bar{G}$, which means $\bar{G} = PSL(2,\mathbb{Z})$. Moreover, $S^2 = -1$, and hence S and T generate $SL(2,\mathbb{Z})$. ♠

1. The exact fundamental domain $\mathcal{F}$ must be chosen so that no two distinct points on the boundary are G-related. So, for example, we could choose the part of the boundary with $\mathrm{Re}(\tau) \geq 0$.

2. By studying the possible fixed points, as above it follows that the only relations on S and T are:
$$S^2 = -1 \qquad (ST)^3 = -1 \tag{C.11}$$
(and all those that follow from these).

3. Given $g \in SL(2,\mathbb{Z})$ it is possible to write the word in S, T giving g by applying the Euclidean algorithm to (a, c) and interpreting the standard equations there in terms of matrices. See Chapter 1, Section 8.

4. Although the keyhole region is the standard fundamental domain there is no unique choice of fundamental domain. For example, one could equally well use any of the images shown in Figure 32 (and of course there are infinitely many such regions). Moreover, we could displace $\mathcal{F} \to \mathcal{F} + \epsilon$ and still produce a fundamental domain.

5. The action of $PSL(2,\mathbb{Z})$ is properly discontinuous on $\mathcal{H}$, but not quite free. If we consider finite-index subgroups that do not contain the stabilizer groups mentioned above then the action will be free and the quotient space will be a nice Riemann surface.

C.2 Expressing Elements Of $SL(2,\mathbb{Z})$ As Words In S And T

The group $SL(2,\mathbb{Z})$ is generated by
$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \& \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{C.12}$$
Here is an algorithm for decomposing an arbitrary element
$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{C.13}$$
as a word in S and T.

First, note the following simple

Lemma Suppose $h \in SL(2,\mathbb{Z})$ as in (C.13). Suppose moreover that $g \in SL(2,\mathbb{Z})$ satisfies:
$$g \cdot \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{C.14}$$

Then

$$gh = T^n \tag{C.15}$$

for some integer $n \in \mathbb{Z}$.

The proof is almost immediate by combining the criterion that $gh \in SL(2,\mathbb{Z})$ has determinant one and yet must have the first column $(1,0)$.

Now, suppose h is such that $A > C > 0$. Then $(A,C) = 1$ and hence we have the Euclidean algorithm to define integers q_ℓ, $\ell = 1, \dots N+1$, where $N \geq 1$, such that

$$\begin{aligned}
A &= q_1 C + r_1 \qquad 0 < r_1 < C \\
C &= q_2 r_1 + r_2 \qquad 0 < r_2 < r_1 \\
r_1 &= q_3 r_2 + r_3 \qquad 0 < r_3 < r_2 \\
\vdots & \quad \vdots \\
r_{N-2} &= q_N r_{N-1} + r_N \qquad 0 < r_N < r_{N-1} \\
r_{N-1} &= q_{N+1} r_N
\end{aligned} \tag{C.16}$$

with $r_N = (A,C) = 1$. (Note you can interpret $r_0 = C$, as is necessary if $N = 1$.) Now, write the first line in the Euclidean algorithm in matrix form as: ♣$N = 0$ here?♣

$$\begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} r_1 \\ C \end{pmatrix} \tag{C.17}$$

We would like to have the equation in a form that we can iterate the algorithm, so we need the larger integer on top. Therefore, rewrite the identity as:

$$\sigma^1 \begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} C \\ r_1 \end{pmatrix} \tag{C.18}$$

We can now iterate the procedure. So the Euclidean algorithm implies the matrix identity:

$$\tilde{g} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{C.19}$$

$$\tilde{g} = (\sigma^1 T^{-q_{N+1}}) \cdots (\sigma^1 T^{-q_1}) \tag{C.20}$$

Now, to apply the Lemma we need g to be in $SL(2,\mathbb{Z})$, but

$$\det \tilde{g} = (-1)^{N+1} \tag{C.21}$$

We can easily modify the equation to obtained a desired element g. We divide the argument into two cases:

1. Suppose first that $N + 1 = 2s$ is even. Then we group the factors of $\tilde{g}$ in pairs and write

$$\begin{aligned}
(\sigma^1 T^{-q_{2\ell}})(\sigma^1 T^{-q_{2\ell-1}}) &= (\sigma^1\sigma^3)(\sigma^3 T^{-q_{2\ell}}\sigma^3)(\sigma^3\sigma^1) T^{-q_{2\ell-1}} \\
&= -S T^{q_{2\ell}} S T^{-q_{2\ell-1}}
\end{aligned} \tag{C.22}$$

where we used that $\sigma^1\sigma^3 = -\mathrm{i}\sigma^2 = S$. Therefore, we can write

$$\tilde{g} = g = (-1)^s \prod_{\ell=1}^{s} (ST^{q_{2\ell}} ST^{-q_{2\ell-1}}) \tag{C.23}$$

2. Now suppose that $N+1 = 2s+1$ is odd. Then we rewrite the identity (C.19) as:

$$\sigma^1 \tilde{g} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{C.24}$$

so now we simply take

$$g = \sigma^1 \tilde{g} = (-1)^{s+1} (ST^{-q_{2s+1}}) \prod_{\ell=1}^{s} (ST^{q_{2\ell}} ST^{-q_{2\ell-1}}) \tag{C.25}$$

Thus we can summarize both cases by saying that

$$g = (-1)^{\lfloor \frac{N+1}{2} \rfloor} \prod_{\ell=1}^{N+1} (ST^{(-1)^\ell q_\ell}) \tag{C.26}$$

Then we can finally write

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = g^{-1} T^n \tag{C.27}$$

as a word in S and T for a suitable integer n. (Note that $S^2 = -1$.)

♣It would be good to give an algorithm for determining n. ♣

Now we need to show how to bring the general element $h \in SL(2,\mathbb{Z})$ to the form with $A > C > 0$ so we can apply the above formula. Note that

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C + mA & D + mB \end{pmatrix} \tag{C.28}$$

while

$$\begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} = ST^m S^{-1} \tag{C.29}$$

Thus, if $A > 0$ we can use this operation to shift C so that $0 \le C < A$. In case $A < 0$ we can multiply by $S^2 = -1$ to reduce to the case $A > 0$. Finally, if $A = 0$ then

$$h = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & n \end{pmatrix} \tag{C.30}$$

and we write

$$ST^n = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix} \tag{C.31}$$

♣Need to summarize the result in a useful way ♣

D. Level k Theta Functions

D.1 Definition

Let us collect a few facts about theta functions: Recall that

$$\begin{aligned}\Theta_{\mu,\kappa}(z,\tau) &:= \sum_{\ell=\mu \mathrm{mod} 2\kappa} q^{\ell^2/(4\kappa)} y^{\ell} \\ &= \sum_{n\in\mathbb{Z}} q^{\kappa(n+\mu/(2\kappa))^2} y^{(\mu+2\kappa n)} \\ &= \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi\tau(2\kappa)(n+\mu/(2\kappa))^2+2\pi\mathrm{i}(2\kappa)z(n+\mu/(2\kappa))}\end{aligned} \tag{D.1}$$

with $q = e^{2\pi \mathrm{i}\tau}$ and $y = e^{2\pi \mathrm{i} z}$. Here μ is an integer and κ is a positive half-integer (i.e. in $\frac{1}{2}\mathbb{Z}_+$). Note that if we shift $\mu \to \mu + 2\kappa s$, where s is any integer, then $\Theta_{\mu,\kappa}$ is unchanged. Often people take μ to be in the fundamental domain $-\kappa < \mu \le \kappa$, but one should generally regard μ as an element of $\mathbb{Z}/2\kappa\mathbb{Z}$. As functions of z these functions are doubly-quasiperiodic:

$$\begin{aligned}\Theta_{\mu,\kappa}(z+\nu,\tau) &= \Theta_{\mu,\kappa}(z,\tau) \\ \Theta_{\mu,\kappa}(z+\nu\tau,\tau) &= e^{-2\pi\mathrm{i}\kappa\nu^2\tau - 4\pi\mathrm{i}\kappa\nu z}\Theta_{\mu,\kappa}(z,\tau)\end{aligned} \tag{D.2}$$

Here ν is any integer. Note that the theta functions transform the same way for all $\mu \in \mathbb{Z}/2\kappa\mathbb{Z}$. We will explain more about the geometrical meaning of these theta functions in section **** below.

D.2 Modular Transformation Laws For Level k Theta Functions

♣Need to change conventions $k \to \kappa$ here ♣

We will actually find it more convenient to work with

$$f_{\mu,k}(z,\tau) := \frac{\Theta_{\mu,k}(z,\tau)}{\eta(\tau)} \tag{D.3}$$

Call the span of these functions V_k. It will be a representation of $SL(2,\mathbb{Z})$ (and <u>not</u> of $PSL(2,\mathbb{Z})$) and we aim to determine this representation. The involution $z \to -z$ corresponding to the action of $-1 \in SL(2,\mathbb{Z})$ can be diagonalized to decompose V_k into a direct sum of even and odd subspaces

$$V_k \cong V_k^+ \oplus V_k^- \tag{D.4}$$

So $\dim V_k^+ = k+1$ and $\dim V_k^- = k-1$. The transformation rule for the general level k theta function in the odd space V_k^- was derived in an impressive computation by Lisa Jeffrey [31]. Using the ideas from her paper it is possible to generalize the result to the transformation laws for all the theta functions in V_k. The computation is slightly nontrivial so we give the result here (as it does not seem to be available anywhere else).

We have a <u>left</u> action of $SL(2,\mathbb{Z})$ on (z,τ): If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{D.5}$$

then

$$A \cdot (z,\tau) := \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) \tag{D.6}$$

Define

$$\phi_k(A;(z,\tau)) = \exp 2\pi \mathrm{i} k \frac{cz^2}{c\tau+d} \tag{D.7}$$

Then

$$\phi_k(A_1; A_2 \cdot (z,\tau))\phi_k(A_2;(z,\tau)) = \phi_k(A_1 A_2;(z,\tau)) \tag{D.8}$$

Now define a LEFT action of $SL(2,\mathbb{Z})$ on V_k:

$$(\rho(A) \cdot f)(z,\tau) := (\phi_k(A^{-1};(z,\tau)))^{-1} f(A^{-1} \cdot (z,\tau)) \tag{D.9}$$

Now, with respect to the basis $f_{\mu,k}$ we have

$$\rho(A) \cdot f_{\mu,m} := \sum_\nu \rho(A)_{\nu\mu} f_\nu \tag{D.10}$$

where again μ,ν are to be regarded as integers modulo $2k$.

If $a=0$ then $A=S$ or S^{-1} where we define

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.11}$$

If $c=0$ and $a=1$ then $A=T^m$ where it suffices to compute the matrix for

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{D.12}$$

If $c=0$ and $a=-1$ it suffices to compute the action of $A=-1$.

Straightforward computation gives:

$$\rho(-1)_{\mu\nu} = \delta_{\mu+\nu,0} \tag{D.13}$$

This is the charge conjugation matrix.

$$\rho(S)_{\mu\nu} = \frac{1}{\sqrt{2k}} \exp\left(2\pi \mathrm{i} \frac{\mu\nu}{2k}\right) \tag{D.14}$$

$$\rho(T)_{\mu\nu} = \delta_{\mu,\nu} \exp\left(2\pi \mathrm{i} \left[\frac{\mu^2}{4k} - \frac{1}{24}\right]\right) \tag{D.15}$$

If $ac \neq 0$ then

$$\rho(A)_{\mu\nu} = \frac{1}{\sqrt{2k|c|}} e^{\frac{\mathrm{i}\pi}{2k}\frac{b}{a}\nu^2} e^{-\frac{2\pi \mathrm{i}}{24}\Phi(A)} \sum_{\gamma \mathrm{mod}(2kc), \gamma=\mu \mathrm{mod} 2k} \exp\left[\mathrm{i}\pi \frac{a}{2kc}\left(\gamma + (-1)^{t+1}\frac{\nu}{a}\right)^2\right] \tag{D.16}$$

Here $\Phi(A)$ is the Rademacher function, and t is the number of terms in any continued fraction expansion of A. While A can admit many continued fraction expansions, we claim that the number of terms modulo 2 is well defined, so $(-1)^t$ is a well-defined function of A.

(See section **??**.) It is this sign which constitutes the most important difference compared to the expressions found by Lisa Jeffrey for the odd theta functions.

Note that

$$\rho(A)_{\mu,\nu} = \rho(A)_{-\mu,-\nu} \tag{D.17}$$

for all $A \in SL(2,\mathbb{Z})$. This must be since the charge conjugation representing -1 must commute with all matrices. For the case (D.16) this can also be checked by changing variables in the sum $\gamma \to -\gamma$.

It remains to define t and the Rademacher function.

The Rademacher function is defined exactly as in [31]. The function $\Phi : SL(2,\mathbb{Z}) \to \mathbb{Z}$ is defined by

$$\Phi(A) := \begin{cases} \frac{a+d}{c} - 12\mathrm{sign}(c)s(d,|c|) & c \neq 0 \\ \frac{b}{d} & c = 0 \end{cases} \tag{D.18}$$

where $s(d,c)$ is the Dedekind sum, defined for $c > 0$ by:

$$s(d,c) = \frac{1}{4c}\sum_{k=1}^{c-1} \cot\frac{\pi k}{c} \cot\frac{\pi d k}{c} \tag{D.19}$$

when $d \neq 0$ and $s(0,1) = 0$. It is far from obvious that Φ is integer-valued, but this follows from the remarkable formula

$$\Phi(A_{12}) = \Phi(A_1) + \Phi(A_2) - 3\mathrm{sign}(c_1 c_2 c_{12}) \tag{D.20}$$

where $A_{12} = A_1 A_2$.

Using the theory of continued fractions it is possible to show that any $A \in SL(2,\mathbb{Z})$ is of the form

$$A = T^{m_t} S T^{m_{t-1}} S \cdots T^{m_1} S \tag{D.21}$$

where - importantly - all the $m_i \neq 0$. Because of relations in the group the integer t is only well defined modulo 2.

E. Theta Functions And Holomorphic Line Bundles On Tori

When working with the Abelian Chern-Simons theory it is useful to have a general perspective on theta functions and their relation to sections of line bundles over tori. We summarize the theory here.

For some mathematical treatments see

1. D. Mumford, *Tata Lectures on Theta*, vol. I, p.202. There is a nice description of the transformations of theta functions here.

2. D. Mumford, M. Nori, and P. Norman, Tata Lectures on Theta, vol. III. They interpret θ as a distinguished vector in a Fock space representation, and also interpret θ as a distinguished matrix element in a Heisenberg representation.

3. Another nice description within the framework of algebraic geometry can be found in Griffiths and Harris, p.300 et. seq.

4. An excellent book devoted to the subject is Birkenhake and Lange,

5. The subject is treated from various viewpoints in hundreds of physics papers. We are using [9] pp. 68 -69 and section 6 of [20].

E.1 Heisenberg Groups As Principal $U(1)$ Bundles

Suppose that Λ is a lattice of rank $2n$ with an integer-valued symplectic form on it:

$$\Omega : \Lambda \times \Lambda \to \mathbb{Z} \tag{E.1}$$

This induces a symplectic form on the real $2n$-dimensional vector space $V = \Lambda \otimes \mathbb{R}$. Using Ω we define a central extension

$$0 \to U(1) \to \tilde{V} \to V \to 0. \tag{E.2}$$

We can view $\tilde{V}$ as the set of pairs $(v, z) \in V \times U(1)$ with the product given by a cocycle

♣Would be better to change convention and put the central part as the first factor, so write elements as (z, v) rather than (v, z). That needs to be changed throughout the following. ♣

$$(v, z)(v', z') = (v + v', zz'c(v, v')) \tag{E.3}$$

such that

$$\frac{c(v, v')}{c(v', v)} = e^{2\pi i\Omega(v,v')}. \tag{E.4}$$

It is natural to take $c(v, v') = e^{i\pi\Omega(v,v')}$, corresponding to a distinguished trivialization of the principal $U(1)$ bundle $\tilde{V} \to V$.

Now we can construct a principal $U(1)$ bundle over the torus V/Λ. To do this we split the sequence (E.2) over Λ. That is, we choose a function $\epsilon : \Lambda \to U(1)$ with

♣Explain why such a splitting exists. ♣

$$\epsilon_v \epsilon_{v'} = e^{-i\pi\Omega(v,v')}\epsilon_{v+v'} \qquad \forall v, v' \in \Lambda \tag{E.5}$$

Now (E.5) is precisely the condition we need so that $v \to (v, \epsilon_v)$ defines a homomorphism $\Lambda \to \tilde{V}$ embedding Λ in the group $\tilde{V}$.

Let us call Λ_ϵ the image of this homomorphism. It now makes sense to take the quotient $\tilde{V}/\Lambda_\epsilon$. This space is a principal $U(1)$ bundle over V/Λ.

As a space our $U(1)$ bundle can be written as $(V \times U(1))/\Lambda$ where the equivalence relation is

$$(v, z) \sim (v, z) \cdot (\lambda, \epsilon_\lambda) = (v + \lambda, \epsilon_\lambda e^{i\pi\Omega(v,\lambda)} \cdot z) \tag{E.6}$$

Remarks:

1. In the above extension $z \in U(1)$ and so we constructed principal $U(1)$ bundles. The associated bundles are easily obtained in this description. Below it will be important to describe holomorphic bundles over the torus. We can define associated $\mathbb{C}^*$ or $\mathbb{C}$ bundles simply by letting z be valued in $\mathbb{C}^*$ or $\mathbb{C}$ in the above formulae! The extension by $\mathbb{C}^*$ contracts onto the extension by $U(1)$.

♣This first remark is out of place or just not necessary. ♣

2. Since Ω is integral on $\Lambda \times \Lambda$ the bilinear form $e^{i\pi\Omega(v,v')}$ is in fact a *symmetric* bilinear form. In general, given a symmetric form

$$b : A \times A \to \mathbb{R}/\mathbb{Z} \tag{E.7}$$

on an abelian group A, a *quadratic refinement* of b is a function $q : A \to \mathbb{R}/\mathbb{Z}$ such that

$$q(x+y) - q(x) - q(y) + q(0) = b(x,y) \tag{E.8}$$

Note that we may regard ϵ as a quadratic refinement of the symmetric

3. This gives a nice example of a *compact phase space*. $T(V/\Lambda)$ is a symplectic manifold. The quantization of such a phase space should produce a finite dimensional Hilbert space, which, in the WKB approximation should be of dimension

$$\dim \mathcal{H} = \int_T \frac{\Omega^n}{n!} \tag{E.9}$$

Now any integral-valued antisymmetric form on Λ can be thought of as a matrix and under change of basis of the lattice $\Omega \to A^{tr}\Omega A$, where $A \in GL(2n,\mathbb{Z})$ we can put Ω in the form:

♣Indicate the proof ♣

$$\Omega = -ie_1\sigma^2 \oplus -ie_2\sigma^2 \oplus \cdots \tag{E.10}$$

where e_I, $I = 1,\dots,n$ are integers. We can fix them uniquely by demanding that $e_1|e_2|\cdots|e_n$. This defines what is known as a *symplectic basis.* More on that below. In this case $\dim \mathcal{H} = e_1 \cdots e_n$. In geometric quantization the associated line bundle to our $U(1)$ bundle is the pre-quantum line bundle.

Exercise

Show that another representation of the principal $U(1)$ bundle can be given as follows. Construct the Heisenberg representation

$$0 \to \mathbb{R} \to Heis(\mathbb{R}\times\mathbb{R}) \to \mathbb{R}\times\mathbb{R} \to 0 \tag{E.11}$$

from the cocycle $c((x,y),(x',y')) = xy'$. This has a nice 3×3 matrix representation

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{E.12}$$

Topologically this is just $\mathbb{R}^3$, but with a noncommutative group law. Note that

$$0 \to \mathbb{Z} \to Heis(\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \times \mathbb{Z} \to 0 \tag{E.13}$$

forms a discrete subgroup, just by taking x, y, z to be integers. Then $\mathbb{R}^3/\mathbb{Z}^3$ realized as $Heis(\mathbb{R} \times \mathbb{R})/Heis(\mathbb{Z} \times \mathbb{Z})$ is a $U(1)$ bundle over a torus.

In this literature on string compactification this is referred to as the "twisted torus."

E.2 A Natural Connection

The principal $U(1)$ bundle (E.2) has a natural connection that is described as follows. Given a path

$$p_{w_0,v} = \{w_0 + tv | 0 \le t \le 1\} \tag{E.14}$$

in V from w_0 to v and an initial lift of $w_0 \in V$, say, $(w_0, z_0) \in \tilde{V}$ we define the parallel transported point in $\tilde{V}$ to be

$$U(p_{w_0,v}) : (w_0, z_0) \to (w_0 + v, c(v, w_0)z_0) \tag{E.15}$$

By computing the parallel transport around a closed path where we translate w_0 first by v_1, then by v_2, then by $-v_1$, then by $-v_2$ one checks that the curvature is in fact Ω. ♣CHECK THE SIGN! DO WE WANT $c(w_0, v)$? ♣

Now, this connection descends to a connection the principal $U(1)$ bundle $\tilde{V}/\Lambda_\epsilon$ over the torus V/Λ. The curvature is again given by Ω, now thought of as a 2-form on the torus. From this we can compute the first Chern class: $(e_1, \dots, e_n) \in H^2(V/\Lambda; \mathbb{Z})$. ♣DEMONSTRATE EXPLICITLY THAT IT DOES DESCEND ♣

Moreover, if $\lambda \in \Lambda$ then $[p_{w_0,\lambda}]$ is a closed path in the torus and the holonomy around the path is

$$e^{2\pi i \Omega(\lambda, w_0)} \epsilon_\lambda^{-1} \tag{E.16}$$

After a suitable change of cocycle (considered valued in $\mathbb{C}^*$) by a coboundary this connection will be the Chern connection on a Hermitian holomorphic line bundle below. ♣Explain this more fully below. ♣

♣Comment on the slight, but important difference when we have an affine torus. ♣

E.3 Putting A Complex Structure On The Symplectic Tori

We would now like to construct holomorphic $\mathbb{C}^*$-bundles and their associated holomorphic complex line bundles over the torus. Therefore, we need to introduce a complex structure on the torus. To do this we need to introduce a complex structure J on V:

$$J : V \to V \qquad J^2 = -1 \tag{E.17}$$

which is *compatible with* Ω:

$$\Omega(Jv, Jw) = \Omega(v, w) \tag{E.18}$$

Note that this means that

$$g(v, w) := \Omega(Jv, w) = \Omega(Jw, v) \tag{E.19}$$

is *symmetric*.

We will further assume that g is positive definite.

Now, since V has a complex structure $V \otimes_{\mathbb{R}} \mathbb{C} \cong V^+ \oplus V^-$ splits into $+i$ and $-i$ subspaces of J. We work on the $+$ subspace and define $v^+ = \frac{1}{2}(1 - J \otimes i)v$ for $v \in V$. ♣Which one we call $(1, 0)$ and which $(0, 1)$ is a convention that matters below. ♣

E.4 Lagrangian Decompositions

An *isotropic* subspace W of V is a linear subspace such that Ω vanishes when restricted to it, that is, $\forall w, w' \in W$, $\Omega(w, w') = 0$.

A Lagrangian decomposition of (V, Ω) is an internal direct sum

$$V = W_1 \oplus W_2 \tag{E.20}$$

where the W_i are *maximal isotropic subspaces.* Because Ω is nondegenerate, a Lagrangian decomposition defines a nondegenerate pairing of W_1 and W_2, thus defining an isomorphism $W_2 \cong W_1^\vee$.

Above we introduced a symplectic basis in equation (E.10). Such a basis gives a Lagrangian decomposition. It is convenient to introduce a notation for such a basis by writing α_I, $I = 1, \dots, n$ for W_1 and β^J, $J = 1, \dots, n$ for W_2 such that

$$\begin{aligned} \Omega(\alpha_I, \alpha_J) &= 0 \\ \Omega(\beta^I, \beta^J) &= 0 \\ \Omega(\alpha_I, \beta^J) &= e_I {\delta_I}^J \end{aligned} \tag{E.21}$$

♣Need integral structure of lattice otherwise you could just set $e_I = 1$. ♣

If we expand $v = q^I \alpha_I + p_I \beta^I$ then

$$\Omega = \sum_I e_I dp_I \wedge dq^I \tag{E.22}$$

♣SIGN? ♣

E.5 The Space Of Complex Structures On The Torus: The Siegel Upper Half-Plane

We now assume the invariants $e_I = 1$.

Now we choose a basis of vectors ζ_I of type $(1,0)$. We extend J $\mathbb{C}$-linearly to $V_{\mathbb{C}}$ so that, by definition

♣ $(1,0)$ vs. $(0,1)$ is all mixed up in this section. ♣

$$J \cdot \zeta_I = \mathrm{i}\zeta_I \tag{E.23}$$

♣SIGN? Opposite from convention in PiTP 2008 ♣

We can express the complex structure J in terms of the components of the period matrix. The latter is defined by choosing a basis ζ^I of vectors of type $(1,0)$ of the form:

$$\zeta_I := \alpha_I + \tau_{IJ}\beta^J \tag{E.24}$$

This gives an explicit isomorphism of $V \cong V^+$ as complex vector spaces.

Now let us note some properties of the quadratic form τ_{IJ}. Note that

$$g(\zeta_I, \zeta_J) = \Omega(J\zeta_I, \zeta_J) = \mathrm{i}\Omega(\zeta_I, \zeta_J) \tag{E.25}$$

is both symmetric and antisymmetric in IJ and therefore must vanish. Therefore we learn that τ_{IJ} is symmetric, and moreover g is of type $(1,1)$. Note that

$$g(\zeta_I, \bar{\zeta}_J) = 2\mathrm{Im}\tau_{IJ} \tag{E.26}$$

It is often useful to work with *positive definite* metrics. In this case $\mathrm{Im}\tau_{IJ}$ is a positive definite real quadratic form.

Definition: The space of complex $N \times N$ symmetric matrices with positive definite imaginary part is the *Siegel upper half plane* $\mathcal{H}_N$.

Exercise

A choice of complex structure on a real symplectic vector space with $g(v,v) > 0$ is equivalent to a choice of $\tau \in \mathcal{H}_N$.

a.) Show how to express the complex structure in terms of the period matrix as follows: The complex structure acts as:

$$\begin{aligned} J \cdot \alpha_I &= A^{I'}{}_I \alpha_{I'} + C_{I'I} \beta^{I'} \\ J \cdot \beta^I &= B^{I'I} \alpha_{I'} + D_{I'}{}^I \beta^{I'} \end{aligned} \tag{E.27}$$

We define components of vectors by

$$v = v_I^1 \alpha_I + v_2^I \beta^I = \begin{pmatrix} v_I^1 & v_2^I \end{pmatrix} \begin{pmatrix} \alpha_I \\ \beta^I \end{pmatrix} \tag{E.28}$$

so that J acts on the components as the matrix

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{E.29}$$

Compatibility of the complex structure implies that this defines a symplectic matrix.

Equating real and imaginary parts of (E.23), using the definition (E.27) we find the matrix expression of J in the basis α_I, β^I:

$$J = \begin{pmatrix} -Y^{-1}X & Y^{-1} \\ -Y - XY^{-1}X & XY^{-1} \end{pmatrix} \tag{E.30}$$

where $\tau = X + iY$ are the real and imaginary parts of τ. One can check both $J^2 = -1$ and $J^{tr}\Omega J = \Omega$.

b.) Show that the metric g in the α, β basis is:

$$g(v,w) = \begin{pmatrix} v_I^1 & v_2^J \end{pmatrix} \begin{pmatrix} XY^{-1}X + Y & -XY^{-1} \\ -Y^{-1}X & Y^{-1} \end{pmatrix} \begin{pmatrix} w_I^1 \\ w_2^J \end{pmatrix} \tag{E.31}$$

Exercise

Show that $\mathcal{H}_N = Sp(2N, \mathbb{R})/U(N)$.

Exercise

It is useful to have formulae for the transformation from the integral symplectic basis to the complex basis.

$$\begin{pmatrix} \zeta_I \\ \bar{a}\zeta_I \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ 1 & \bar{a}\tau \end{pmatrix} \begin{pmatrix} \alpha \\ \_ \end{pmatrix} \tag{E.32}$$

has inverse:

$$\begin{pmatrix} \alpha \\ \_ \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \bar{a}\tau & -\tau \\ -1 & 1 \end{pmatrix} Y^{-1}_{IJ} \begin{pmatrix} \zeta_J \\ \bar{a}\zeta_J \end{pmatrix} \tag{E.33}$$

Thus the complex projections of (E.28) are:

$$\begin{aligned} v^{(1,0)} &= -\frac{i}{2}(v_2^J - v_I^1 \bar{a}\tau_g IJ) Y^{-1,JK} \zeta_K \\ v^{(0,1)} &= \frac{i}{2}(v_2^J - v_I^1 \tau_{IJ}) Y^{-1,JK} \bar{a}\zeta_K \end{aligned} \tag{E.34}$$

Note that

$$v = v^{(1,0)} + v^{(0,1)} \tag{E.35}$$

Now, the symplectic group $Sp(2N, \mathbb{R})$ acts on the space of complex structures: $J \to hJh^{-1}$. The corresponding action on the period matrix can be given as follows.

If we work with components of vectors, then the antiholomorphic coordinates are $v_2^I - \tau^{IJ} v_J^1$. But if the components transform by a symplectic matrix

$$\begin{pmatrix} \tilde{v}^1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v^1 \\ v_2 \end{pmatrix} \tag{E.36}$$

then in the new complex structure $(Cv^1 + Dv_2) - \tau(Av^1 + Bv_2)$ are the new antiholomorphic coordinate. But then we can write these as

$$(C - \tau A)v^1 + (D - \tau B)v_2 \tag{E.37}$$

and so

$$\tilde{\tau} = -(D - \tau B)^{-1}(C - \tau A) \tag{E.38}$$

Exercise *Automorphisms of* $Sp(2N, \mathbb{R})$

There are many forms of the transformation law because we can use passive vs. active transformations and because we can compose with automorphisms.

The symplectic group is $g^{tr}\mathcal{J}g = \mathcal{J}$ and hence ♣CHECK!! ♣

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \tag{E.39}$$

is an automorphism, so composing we get

$$\tilde{\tau} = (A + \tau C)^{-1}(B + \tau D) \tag{E.40}$$

Show by further automorphisms that we can write

$$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1} \tag{E.41}$$

Prove that $\tilde{\tau}$ is symmetric.

Now, if we are interested in the complex structures on a torus compatible with a symplectic structure then we want our symplectic transformation to preserve the lattice. The space of complex structures on the covering space is given by $\mathcal{H}_N \cong Sp(2N, \mathbb{R})/U(N)$. However, we should identify complex structures which are simply related by change of basis for the lattice:

Theorem The space of complex structures on $\mathbb{R}^{2N}/\mathbb{Z}^{2N}$ is

$$Sp(2N, \mathbb{Z})\backslash\mathcal{H}_N \cong Sp(2N, \mathbb{Z})\backslash Sp(2N, \mathbb{R})/U(N) \tag{E.42}$$

Remark: Compare with the space of complex structures on $\mathbb{R}^{2N}$ compatible with a Euclidean structure. This space is $O(2N)/U(N)$.

Finally, if we make integral symplectic transformations we just change the choice of Lagrangian decomposition. But the line bundle and its space of sections is intrinsic to the torus. This implies remarkable transformation properties on theta functions.

The (integer) symplectic group is generated by 3 kinds of transformations:

$$\begin{pmatrix} A & 0 \\ 0 & A^{tr,-1} \end{pmatrix} \tag{E.43}$$

Rearranges p's and q's separately.

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \tag{E.44}$$

adds some p's to q's.

$$\begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix} \tag{E.45}$$

Exchanges p's and q's.

For a proof that these generate the integral symplectic group see D. Mumford, *Tata Lectures on Theta*, vol. I, p.202.

E.6 Lagrangian Decomposition: Group-Theoretic Interpretation

Returning to the Heisenberg group $\tilde{V}$, one way to think about $\tilde{V}$ is that it is the group of operators of the form $U(v) := \exp[\mathrm{i}(q^I \hat{\alpha}_I + p_I \hat{\beta}^I)]$ where

$$[\hat{\alpha})I, \hat{\beta}^J] = -2\pi \mathrm{i} \delta_I{}^J \tag{E.46}$$

****

NEED TO ADD SOMETHING ABOUT THE RELATION TO SvN REPRESENTATIONS OF THE HEISENBERG GROUP

****

E.7 Hermitian Holomorphic Line Bundles

We would like to define a complex analytic group isomorphic to the Heisenberg group above (where we centrally extend by $\mathbb{C}^*$). To that end we introduce:

$$H(v^+, w^+) := \Omega(Jv, w) + i\Omega(v, w) = g(v, w) + i\Omega(v, w) \tag{E.47}$$

Note that $H(iv^+, w^+) = iH(v^+, w^+)$ and $H(v^+, iw^+) = -iH(v^+, w^+)$ so H is an Hermitian form. Since H is a holomorphic function of v^+, w^+ we can define the holomorphic extension by $\mathbb{C}^*$:

$$(v^+, z)(w^+, z') = (v^+ + w^+, zz' e^{\pi H(v^+, w^+)}) \tag{E.48}$$

defining a group $\tilde{V}^+$. Note that the cocycles $e^{i\pi\Omega(v,w)}$ and $e^{\pi H(v^+, w^+)}$ differ by a coboundary since

$$e^{\pi g(v,w)} = e^{\pi\Omega(Jv,w)} = \frac{e^{\frac{\pi}{2}\Omega(J(v+w),v+w)}}{e^{\frac{\pi}{2}\Omega(Jv,v)} e^{\frac{\pi}{2}\Omega(Jw,w)}} = \frac{e^{\frac{\pi}{2}g(v+w,v+w)}}{e^{\frac{\pi}{2}g(v,v)} e^{\frac{\pi}{2}g(w,w)}} \tag{E.49}$$

Thus (E.48) is isomorphic as a group to the $\mathbb{C}^*$ extension we defined earlier.

Now, to define a holomorphic $\mathbb{C}^*$ bundle over $V/\Lambda = V^+/\Lambda^+$, as before we must choose a splitting, now defined by $\lambda \to \tilde{\epsilon}_\lambda$ satisfying

$$\tilde{\epsilon}_\lambda \tilde{\epsilon}_\mu = \tilde{\epsilon}_{\lambda+\mu} e^{-\pi H(\lambda^+, \mu^+)} \tag{E.50}$$

Then the $\mathbb{C}^*$ bundle can be described as the group quotient $\tilde{V}^+/\Lambda_{\tilde{\epsilon}}$

$$(v^+, z) \sim (v^+, z)(\lambda^+, \tilde{\epsilon}_\lambda) = (v^+ + \lambda^+, \tilde{\epsilon}_\lambda e^{\pi H(v^+, \lambda^+)} \cdot z) \tag{E.51}$$

If $z \in \mathbb{C}^*$ we have a holomorphic principal $\mathbb{C}^*$ bundle over the complex torus, and if $z \in \mathbb{C}$ we have the associated holomorphic principal line bundle.

In terms of our earlier discussion the multiplier system is defined by

$$e_\lambda(v^+) = \tilde{\epsilon}_\lambda e^{\pi H(v^+, \lambda^+)} \tag{E.52}$$

Sections can be lifted to entire functions on V^+ satisfying

$$\vartheta(v^+ + \lambda^+) = e_\lambda(v^+)\vartheta(v^+) \tag{E.53}$$

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INVARIANT NORM HERE: GET HERMITIAN METRIC ON THE HOLOMORPHIC LINE BUNDLE

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If $d_1 = d_2 = \cdots = d_n = 1$ there should be a unique such function up to a constant. In the next section we will construct it.

Exercise

a.) Show explicitly that $e_\lambda(v^+)$ satsify the cocycle constraints.

b.) Show that given ϵ_λ we can construct $\tilde{\epsilon}_\lambda$ as

$$\tilde{\epsilon}_\lambda = \epsilon_\lambda e^{\frac{\pi}{2}\Omega(J\lambda,\lambda)} = \epsilon_\lambda e^{\frac{\pi}{2}H(\lambda^+,\lambda^+)} = \epsilon_\lambda e^{\frac{\pi}{2}g(\lambda,\lambda)} \tag{E.54}$$

Exercise

Show how by introducing a simple prefactor we can solve the equation (E.53) in terms of classical theta functions.

First of all, using the Lagrangian splitting we see that we can write ϵ_λ in the form:

$$\epsilon_\lambda = e^{i\pi\lambda_2\lambda^1 + 2\pi i(\theta\cdot\lambda_2 - \phi\cdot\lambda^1)} \tag{E.55}$$

for some θ, ϕ.

Now define

$$B(v,w) = v^{tr}Y^{-1}w \tag{E.56}$$

on V^+ and compute

$$(H-B)(v,w) = -2ivw^1 \tag{E.57}$$

that is,

$$H(v,w) = v^{tr}Y^{-1}\bar{w} \tag{E.58}$$

Then

$$\tilde{\epsilon}_\lambda e^{\pi H(v^+,\lambda^+)} \frac{e^{\frac{\pi}{2}B(v^+,v^+)}}{e^{\frac{\pi}{2}B(v^+ + \lambda^+, v^+ + \lambda^+)}} \tag{E.59}$$

is given by

$$e^{-i\pi\lambda^1\tau\lambda^1 - 2\pi i v\lambda^1 + 2\pi i(\theta\lambda_2 - \phi\lambda^1)} \tag{E.60}$$

Show that this is precisely the factor of automorphy of the classical theta function.

Thus, the theta functions of eq.(E.53) are related to the classical theta function by $\exp[\frac{\pi}{2}B(v^+, v^+)]$.

E.7.1 Computing The First Chern Class

When L is a holomorphic there is yet another viewpoint: If s is a globally defined holomorphic section of L it will vanish on a holomorphic hypersurface D. Then the Poincaré dual of D will represent $c_1(L)$. To get an explicit differential form one can choose an Hermitian metric h on L: Thus, $\parallel s \parallel^2 = h(z,\bar{z})|s(z)|^2$ is globally well-defined as a C^∞ function on M. Then

$$\frac{1}{2\pi i}\partial\bar{\partial}\log \parallel s \parallel^2 \tag{E.61}$$

is a closed $(1,1)$ form which represents $c_1(L)$.

To see this, near a point on D we can choose holomorphic coordinates $(z_1,\dots,z_n)$ so that D is defined by the simple equation $z_1 = 0$.

Now recall that in one complex dimension:

$$\nabla^2 \log |z|^2 = 4\pi\delta^2(z) \tag{E.62}$$

Or, put differently, if $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ then

$$\partial_z\partial_{\bar{z}} \log |z|^2 = \pi\delta^2(z) \tag{E.63}$$

and therefore, in these local coordinates

$$\frac{1}{2\pi i}\partial\bar{\partial}\log \parallel s \parallel^2 = \delta^2(z)\frac{dz_1 \wedge d\bar{z}_1}{2i} \tag{E.64}$$

Note, however, that the LHS is globally well-defined. We have taken $h(z,\bar{z}) = 1$ locally in the coordinates. If it were different we would just change the representative of $c_1(L)$ by an exact form.

E.7.2 A Basis Of Holomorphic Sections: Level κ Theta Functions

We define our level κ theta functions to be

♣Don't use τ^{IJ} !! That looks like the inverse of τ. Need to convert these equations back to τ_{IJ}. ♣

$$\Theta_{\beta,\kappa}(\xi,\tau) = \sum_{s_I\in\mathbb{Z}} e^{2\pi i\kappa(s_I+\frac{1}{2\kappa}\beta_I)\tau^{IJ}(s_I+\frac{1}{2\kappa}\beta_I)}e^{2\pi i\xi^I(2\kappa s_I+\beta_I)} \tag{E.65}$$

(these are simply related to the Riemann theta functions discussed earlier.)

$I = 1,\dots,g$.

$\Theta_{\beta,\kappa}$ only depends on the projection of β to $\mathbb{Z}^g/(2\kappa\mathbb{Z})^g$. There are $(2\kappa)^g$ linearly independent functions of ξ.

$T = V_{\mathbb{R}}/V_{\mathbb{Z}}$ is a principally polarized variety (that is, the invariants $d_i = 1$ for the symplectic form). Moreover, as we have seen, $\Theta_{\beta,\kappa}(\xi,\tau)$ is a section of a line bundle $\mathcal{L} = L^{\otimes 2\kappa} \to T$. If μ is a vector of integers:

$$\begin{aligned}\Theta_{\beta,\kappa}(\xi+\mu,\tau) &= \Theta_{\beta,\kappa}(\xi,\tau)\\ \Theta_{\beta,\kappa}(\xi+\tau\mu,\tau) &= e^{-2\pi i\kappa\mu_I\tau^{IJ}\mu_J-4\pi i\kappa\xi^I\mu_I}\Theta_{\beta,\kappa}(\xi,\tau)\end{aligned} \tag{E.66}$$

Therefore,

$$\parallel \Theta_{\beta,\kappa}(\xi,\tau) \parallel^2 := e^{-4\pi\kappa(\mathrm{Im}\xi^I)Y^{-1}_{IJ}(\mathrm{Im}\xi^J)}\Theta_{\beta,\kappa}(\xi,\tau)\overline{\Theta_{\beta',\kappa}(\xi,\tau)} \tag{E.67}$$

is invariant - it is a smoothly defined function on the torus. Now we compute the representative of $c_1(\mathcal{L})$:

$$\omega = \frac{1}{2\pi i}\partial\bar{\partial}\log \parallel s \parallel^2 = 2\kappa dx^I dy_I \tag{E.68}$$

where $\xi^I = x^I + \tau^{IJ}y_J$, with x^I, y^I real.

For $\kappa = \frac{1}{2}$ we get the basic symplectic form on the torus with volume 1.

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E.8 Modular Transformation Laws

E.9 The Heat Equation

HEAT EQUATION

MORE MATERIAL IN:

Freed-Moore-Segal, 2006, [20], section 6.

F. Hodge Theory And Analytic Torsion

Analytic torsion, also known as Ray-Singer torsion is a topological invariant of a closed manifold M of dimension n. But it is expressed in terms of the determinants of Laplacians on differential forms.

F.1 Some Hodge Theory

On any manifold we have the DeRham complex: $d : \Omega^*(M) \to \Omega^*(M)$. We refer to the degree k of the differential forms as the "fermion number" because of the relation to supersymmetric quantum mechanics of a particle moving on M. So the fermion number operator F is $F = k$ acting on $\Omega^k(M)$.

If we give M a Riemannian metric we can speak of L^2-forms. These form the Hilbert space of the susy quantum mechanics.

If, in addition, M has a Riemannian metric we can define the Hodge star operator:

$$* : \Omega^k(M) \to \Omega^{n-k}(M) \tag{F.1}$$

Acting on $\Omega^k(M)$ the Hodge dual satisfies

$$\alpha \wedge *\beta = \frac{1}{k!}g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}\alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_k}\mathrm{vol}\,(g) \tag{F.2}$$

and this equation suffices to define it.

It is useful to bear in mind that if $\omega_1 \in \Omega^k(M_{n_1})$ and $\omega_2 \in \Omega^j(M_{n_2})$ and $g = g_1 \oplus g_2$ and the give the product manifold $M_1 \times M_2$ orientation $\mathrm{vol}\,(g_1) \wedge \mathrm{vol}\,(g_2)$ then

$$*(\omega_1 \wedge \omega_2) = (-1)^{j(n_1-k)}(*_1\omega_1) \wedge (*_2\omega_2) \tag{F.3}$$

Also note that

$$*^2 = \mathrm{sign}(\det g)(-1)^{k(n-k)} \tag{F.4}$$

on k-forms. Here we will focus on Riemannian metrics, which are positive definite. Therefore

$$*^2 = (-1)^{k(n-k)} = (-1)^F(-1)^{nF} = (-1)^{(n+1)F} = \begin{cases} (-1)^F & n = 0(2) \\ +1 & n = 1(2) \end{cases} \tag{F.5}$$

For a one-parameter family of metrics $g(s)$ we have the derivative

$$\dot{*} * + * \dot{*} = 0 \tag{F.6}$$

so

$$u = \dot{*}*^{-1} = - *^{-1} \dot{*} \tag{F.7}$$

Also

$$d^\dagger = (-1)^{nF+1} * d* = (-1)^{nF+1}\delta \tag{F.8}$$

The positive definite Hamiltonian

$$H = dd^\dagger + d^\dagger d = (-1)^{nF+1}(\delta d + (-1)^n d\delta) = \begin{cases} -(*d*d + d*d*) & n = 0(2) \\ -(-1)^F(*d*d - d*d*) & n = 1(2) \end{cases} \tag{F.9}$$

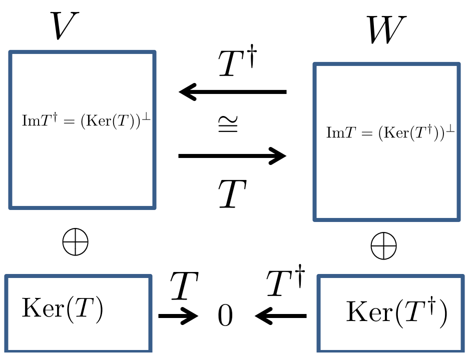

Figure 33: Orthogonal decomposition of domain and range associated to an operator T between inner product spaces.

If, V, W are inner-product spaces and $T : V \to W$ is a (reasonable) linear operator then T restricts to an isomorphism between $(\ker T)^\perp$ and $(\ker T^\dagger)^\perp$ as illustrated in 33. Applying this to the DeRham complex in three dimensions we have the picture:

EXPLAIN $*$ AS ISOMETRY AND SPECIAL CASE OF 3-MANIFOLDS:

1. d and $d^\dagger$ as supersymmetry operators.

2. Nonzero spectrum of $d^\dagger d$ on Ω^0 is spectrum of $dd^\dagger$ on Ω^3. Call it $(\lambda_n^{(0)})^2$

3. On Ω^1 we have $dd^\dagger$ with spectrum $(\lambda_n^{(0)})^2$ and we diagonalize $*d$ with $\mu_n^{(1)}$, so $d^\dagger d$ has spectrum $(\mu_n^{(1)})^2$.

4. On Ω^2 we have $dd^\dagger$ with spectrum $(\mu_n^{(1)})^2$ and $d^\dagger d$ with spectrum $(\lambda_n^{(0)})^2$.

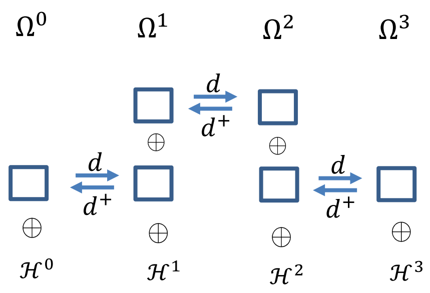

Figure 34: A picture of the Hodge decomposition of the DeRham complex on a Riemannian 3-fold. The horizontal arrows are isomorphisms between Hilbert spaces. The Hodge $*$ operator is an isometry that flips the picture about the vertical in the center of the figure. This leads to a useful simplification of the analytic torsion.

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FIGURE ILLUSTRATING THE HODGE DECOMPOSITION – check Group Theory notes

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F.2 Analytic Torsion

Analytic torsion is based on the integral

$$\tau = \int_0^\infty \frac{dt}{t} \mathrm{Tr} F(-1)^F e^{-tH} \tag{F.10}$$

where the trace is on the DR complex $\Omega^*(M)$.

The basic idea is that if λ and λ' have positive real parts then

$$\int_0^\infty \frac{dt}{t} \left(e^{-\lambda t} - e^{-\lambda' t}\right) = \log \frac{\lambda'}{\lambda} \tag{F.11}$$

So the difference of expressions (F.10) for two different metrics is the difference of

$$\sum_k (-1)^k k \log \mathrm{Det}'(\Delta_k) \tag{F.12}$$

for the two metrics. On the other hand, (F.10) makes sense by itself. As we will now show, it is metric-independent and hence defines a topological invariant.

The definition depends *a priori* on a metric. Take a family $g(s)$ and differentiate wrt s:

$$\dot{\tau} = -\int_0^\infty dt \mathrm{Tr} F(-1)^F \dot{H} e^{-tH} \tag{F.13}$$

$$\begin{aligned} \dot{H} &= -(-1)^{nF}(\dot{*} d * d + * d \dot{*} d + (-1)^n d \dot{*} d * + (-1)^n d * d \dot{*}) \\ &= -(-1)^{nF}(u\delta d - \delta u d + (-1)^n d u \delta - (-1)^n d \delta u) \end{aligned} \tag{F.14}$$

Now try to write $-\mathrm{Tr}F(-1)^F\dot{H}e^{-tH}$ in the form $\mathrm{Tr}F(-1)^{(n+1)F}ue^{-tH}(\cdots)$ by cycling d and δ when necessary. Note that $[d,H]=0$ and $[\delta,H]=0$.

Meanwhile we need

For $n=0(2)$:

$$[F(-1)^F,d]=d(-1)^{F+1}(2F+1) \tag{F.15}$$

because $((f+1)(-1)^{f+1}-f(-1)^f)=(-1)^{f+1}(2f+1)$

$$[F(-1)^F,\delta]=\delta(-1)^{F+1}(2F-1) \tag{F.16}$$

because $((f-1)(-1)^{f-1}-f(-1)^f)=(-1)^{f-1}(2f-1)$

For $n=1(2)$:

$$[F,d]=d \tag{F.17}$$

$$[F,\delta]=-\delta \tag{F.18}$$

First take $n=0(2)$. Then we have

$$\begin{aligned}-\mathrm{Tr}F(-1)^F\dot{H}e^{-tH}&=\mathrm{Tr}F(-1)^F(u\delta d-\delta ud+du\delta-d\delta u)e^{-tH}\\&=-\frac{d}{dt}\left(\mathrm{Tr}(-1)^Fue^{-tH}\right)\end{aligned} \tag{F.19}$$

Now we can do an integral by parts. Near $t=0$ we have $\mathrm{Tr}(-1)^Fue^{-\epsilon H}$. This is a regularized version of $\mathrm{Tr}(-1)^Fu$. The latter is formally zero since $*$ is an isometry of both the even and odd subcomplexes of the DeRham complex, so we are taking the trace of an antisymmetric operator.

Now consider $n=1(2)$.

$$\begin{aligned}-\mathrm{Tr}F(-1)^F\dot{H}e^{-tH}&=\mathrm{Tr}F(u\delta d-\delta ud-du\delta+d\delta u)e^{-tH}\\&=-\frac{d}{dt}\left(\mathrm{Tr}(-1)^Fue^{-tH}\right)\end{aligned} \tag{F.20}$$

We integrate by parts again. To deal with the lower limit note that u commutes with $(-1)^F$, and also takes $\Omega^k\to\Omega^k$. If it can be diagonalized then it has the same eigenvalues on Ω^k and Ω^{n-k} so $(-1)^Fu$ has opposite eigenvalues and hence, at least formally, $\mathrm{Tr}(-1)^Fu=0$.

For the case of a 3-manifold

$$\begin{aligned}\tau&=-\log\det\Delta^{(1)}+2\log\det\Delta^{(2)}-3\log\det\Delta^{(3)}\\&=\log\prod_n(\mu_n^{(1)})^2-2\log\prod_n(\lambda_n^{(0)})^2\end{aligned} \tag{F.21}$$

G. Orientation, Spin, Spin^c, $Pin^{\pm}$, And Pin^c Structures On Manifolds

G.1 Reduction Of Structure Group: General Discussion

Given two compact Lie groups G_1,G_2 and a homomorphism $\phi:G_1\to G_2$ we can define a functor F_ϕ from principal G_1 bundles on M to principal G_2 bundles on M by taking principal G_1 bundle $G_1\to P\to M$ to $(P\times_{G_1}G_2)\to M$. Recall that $(P\times_{G_1}G_2)$ is the set of pairs $(p, g) \in P \times G_2$ with equivalence relation $(ph, g) = (p, \phi(h)g)$ for $h \in G_1$, and this clearly admits a free right G_2 action.

Definition If $G_2 \to P_2 \to M$ is a principal G_2 bundle, a *reduction to* G_1 *under* $\phi : G_1 \to G_2$ is a principal G_1 bundle $G_1 \to P_1 \to M$ together with an isomorphism ψ such that we have the commutative diagram:

$$\begin{tikzcd} (P_1 \times_{G_1} G_2) \arrow[r, "\psi"] \arrow[dr] & P_2 \arrow[d] \\ & M \end{tikzcd} \tag{G.1}$$

Working through the definitions one can give a description in terms of transitions functions on patch overlaps $\mathcal{U}_{\alpha\beta}$. If $h_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to G_1$ are the transition functions of P_1 then there is a bundle isomorphism of P_2, as a principal G_2 bundle to a bundle with transition functions $\phi(h_{\alpha_\beta}) : \mathcal{U}_{\alpha\beta} \to G_2$.

Examples

1. If $\phi : H \to G$ is the inclusion of a subgroup then given a G-bundle $P \to M$, H acts freely, so we can consider $G/H \to P/H \to M$, a bundle of homogeneous spaces. In this case a section of the bundle P/H gives a reduction of P to an H bundle, which is in fact a subbundle. As a special case, take $H = \{1\}$. This is the familiar fact that a global section of a principal G bundle trivializes the bundle.

2. Take $H = O(n)$, $G = GL(n, \mathbb{R})$. A metric gives a reduction of the frame bundle to the orthonormal frame bundle $\mathcal{B}_O(M)$. Clearly the bundle of orthonormal frames is a subbundle of the frame bundle.

3. Suppose M is a Riemannian manifold, $H = SO(n), G = O(n)$, and ϕ is the inclusion. Then $\mathcal{B}_O(M)/H$ is the orientation double cover. If M is *orientable* then the orientation bundle has a section. Indeed, if M is connected $\mathcal{B}_O(M)$ has two components and there are two sections. A choice of section gives a reduction of the bundle to an $SO(n)$ bundle of oriented frames. The choice of section is the choice of orientation of M.

4. Now suppose ϕ is a covering map, i.e. $\phi : \tilde{G} \to G$ is *surjective* with kernel K. Then a "reduction" of a principal G bundle P to a principal $\tilde{G}$ bundle $\tilde{P}$ is an isomorphism

$$\begin{tikzcd} \tilde{P}/K \arrow[r, "\psi"] \arrow[dr] & P \arrow[d] \\ & M \end{tikzcd} \tag{G.2}$$

Note the word "reduction" in the general definition is misleading since $\tilde{P}$ is really a covering of P. Put differently, a "reduction of structure group" of a principal G

bundle P to $\tilde{G}$ is the same thing as a principal $\tilde{G}$ bundle $\tilde{P}$ that covers P:

$$\tilde{P} \to P \to M \tag{G.3}$$

so that the fiber above every point $m \in M$ "looks like" the covering

$$1 \to K \to \tilde{G} \to G \to 1 \tag{G.4}$$

5. A choice of *spin structure* on an oriented manifold is a special case of the previous remark, (G.2) for the case $\phi : Spin(n) \to SO(n)$. In order to classify spin structures we begin by classifying principal $\mathbb{Z}_2$ bundles over $\mathcal{B}_{SO}(M)$ by $z \in H^1(\mathcal{B}_{SO}(M); \mathbb{Z}_2)$. The spin structures are those which restrict to the fibers to the double cover $Spin(n) \to SO(n)$. The double cover $Spin(n) \to SO(n)$ corresponds to $z_s \in H^1(SO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$. Accordingly, the spin structures are the double covers of $\mathcal{B}_{SO}(M)$ which restrict to the fibers of $\mathcal{B}_{SO}(M) \to M$ to give the class z_s. Note that the difference of two spin structures $z_1 - z_2$ is therefore trivial on the fibers, and hence pulls back from a class in $H^1(M; \mathbb{Z}_2)$. Thus, the spin structures on M form a torsor for $H^1(M; \mathbb{Z}_2)$.

6. We can proceed in this way with other structures. For a manifold M we can speak of $\mathrm{Pin}^c, \mathrm{Spin}^c, \mathrm{Pin}^\pm, \mathrm{Spin}$ structures based on the above concept applied to the homomorphisms

$$\phi : \mathrm{Pin}^c \to O(n) \tag{G.5}$$
$$\phi : \mathrm{Spin}^c \to O(n) \tag{G.6}$$
$$\phi : \mathrm{Pin}^\pm \to O(n) \tag{G.7}$$
$$\phi : \mathrm{Spin} \to O(n) \tag{G.8}$$
$$\tag{G.9}$$

Note that these homomorphisms are in general neither injective nor surjective. For a BLOTZ structure on an *oriented* manifold we apply the analogous homomorphisms to $SO(n)$ for the bundle of oriented frames.

G.2 Obstructions To Spin And Pin Structures

It is worthwhile translating the above somewhat abstract description into the language of transition functions for the tangent bundle of a manifold X. Let $\{\mathcal{U}_{\alpha\beta}\}$ be a coordinate atlas for X. Using the metric we can form orthonormal frames and these will have transition functions

$$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to O(n) \tag{G.10}$$

if $\dim_{\mathbb{R}} X = n$. If we can modify these with cocycles

$$\tilde{g}_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1} \tag{G.11}$$

with $h_\alpha : \mathcal{U}_\alpha \to O(n)$ so that $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to SO(n)$ then the manifold is orientable. The only obstruction to orientability is provided by a Cech 2-cochain $\det g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to \mathbb{Z}_2$ which defines a cohomology class $w_1(X) \in H^1(X;\mathbb{Z}_2)$. Note that if X is simply connected then $H^1(X;\mathbb{Z}_2) = 0$ and hence it must be orientable.

When X is orientable we can take $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to SO(n)$. If we choose a *good cover*, meaning that all the intersections $\mathcal{U}_{\alpha\beta\cdots}$ are contractible then, on each overlap $\mathcal{U}_{\alpha\beta}$ we can choose lifts $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to Spin(n)$. The only problem is that the cocycle condition for these lifts might fail. Because we have chosen lifts and the kernel of the covering $Spin(n) \to SO(n)$ is just the group $\{\pm 1\}$ we know for sure that on $\mathcal{U}_{\alpha\beta\gamma}$

$$\tilde{g}_{\alpha\beta}(x)\tilde{g}_{\beta\gamma}(x)\tilde{g}_{\gamma\alpha}(x) := \xi_{\alpha\beta\gamma} \in \{\pm 1\} \subset Spin(n) \qquad \forall x \in \mathcal{U}_{\alpha\beta\gamma} \tag{G.12}$$

The signs $\xi_{\alpha\beta\gamma}$ define a Cech 3-cocycle and this defines a cohomology class in $H^2(X;\mathbb{Z}_2)$. It is one (very concrete) definition of w_2. Note that any modification of the lifts $g_{\alpha\beta}$ by a cocycle, or different choice of lift $\tilde{g}_{\alpha\beta}$ only changes $\{\xi_{\alpha\beta\gamma}\}$ by a coboundary. Almost by definition, this is the only obstruction to the existence of a spin structure.

1. The simplest nontrivial example is the case of spin structures on the circle. The circle is one-dimensional so we take $n = 1$. Then $SO(1)$ is the trivial group and $Spin(1) \cong \mathbb{Z}_2$. Thus, a spin structure on the circle is literally a double-cover of the circle. There are two such: The trivial one, and the nontrivial one. In physics they are called the *Neveu-Schwarz* and *Ramond* spin structures, respectively. Note that the trivial double cover extends to a double cover of the disk, but the nontrivial double cover of the circle does not extend over the disk.

   Somewhat confusingly, with respect to a natural trivialization, the spinors in the Neveu-Schwarz spin structures should be regarded as <u>anti-periodic</u> functions on the circle, while those in the Ramond spin structure should be regarded as <u>periodic</u>. One way to see this is simply to consider the Euclidean metric on the disk with orthonormal frame $e_1 = dr$ and $e_2 = r d\theta$. Then $\omega_{12} = -d\theta$ is the spin connection. Now consider the covariant spinor equation

   $$D_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab}\psi = 0 \tag{G.13}$$

   and in the spin bundle over the disk we can take the constant spinor representation

   $$\gamma^1 = \mathrm{i}\sigma^1 \qquad \gamma^2 = \mathrm{i}\sigma^2 \tag{G.14}$$

   Then the general covariant spinor is

   $$\psi = \left(c_1 e^{-\mathrm{i}\theta/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\mathrm{i}\theta/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \tag{G.15}$$

   ****

   THIS IS STILL A CONFUSING POINT SINCE THOSE WAVEFUNCTIONS DO NOT EXTEND OVER THE DISK. ABOVE SPINOR FUNCTIONS ARE RELATIVE TO A TRIVIALIZATION OF $K^{1/2}$ THAT DOES NOT EXTEND OVER THE DISK. STILL NEED TO EXPLAIN MORE FULLY.

THERE MIGHT BE SOME GOOD COMMENTS IN WITTEN'S WORK ON superstring perturbation theory.

****

♣THE FOLLOWING TWO EXAMPLES PRESUME THAT X IS FOUR DIMENSIONAL ♣

2. An example: $BSO(2) = BU(1) = \mathbb{C}P^\infty$. w_2 is the reduction mod two of c_1, which is the cochain dual to the 2-cell. So explain why c_1 mod two is an obstruction to the spin structure. On $\mathbb{C}P^2$ $c_1 = 3x$. The complexified tangent bundle admits a reduction of the $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$ structure group to $U(2) = SU(2) \times U(1)/\mathbb{Z}_2$. Restricting to a $\mathbb{C}P^1 \subset \mathbb{C}P^2$ the tangent bundle splits as $\mathcal{O}(2) \oplus \mathcal{O}(1)$ where $\mathcal{O}(2)$ is the tangent bundle of $\mathbb{C}P^1$ and $\mathcal{O}(1)$ is the normal bundle. The structure group is further reduced to $U(1) \times U(1)$. Clearly, the principal $U(1)$ bundle associated to the normal bundle $\mathcal{O}(1)$ does not admit a two-fold covering restricting to a double covering of $U(1)$ over $U(1)$.

3. The obstruction to a $Spin^c$ structure is W_3, the image of w_2 under the Bockstein map. Therefore, it vanishes when $w_2(X) \in H^2(X; \mathbb{Z}_2)$ has an integral lift. Using the fact that, for all $\sigma \in H_2(X; \mathbb{Z})$

$$\int_\sigma w_2(X) = \sigma \cdot \sigma \text{ mod } 2 \tag{G.16}$$

where $\sigma \cdot \sigma$ is the oriented <u>integral</u> intersection number one can show that indeed such an integral lift exists. See [?] for the details. The analogous statement fails in the unorientable case: $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a Pin^c structure. Moreover, there are orientable five-dimensional manifolds which are not $Spin^c$. A simple example is the space of symmetric $SU(3)$ matrices, which is diffeomorphic to $SU(3)/SO(3)$. See [?] for an explanation.

♣Also look up proof in Gompf, Stipsicz, 4-Manifolds and Kirby Calculus ♣

♣Give proof. ♣

G.3 *Spin^c* Structures On Four-Manifolds

The group $\mathrm{Spin}^c(4)$ is defined to be

$$\mathrm{Spin}^c(4) := (\mathrm{Spin}(4) \times U(1))/\mathbb{Z}_2 = (SU(2) \times SU(2) \times U(1))/\mathbb{Z}_2 \tag{G.17}$$

where we divide by the group $\mathbb{Z}_2$ embedded as $(-1, -1, -1)$. The bundle of oriented ON frames of X, $OrFr(X)$ is a principal $SO(4)$ bundle and a spin-c structure is - by definition - a reduction of structure group to a principal $\mathrm{Spin}^c(4)$ bundle defined by the obvious homomorphism $\mathrm{Spin}^c(4) \to SO(4)$. Working out the definition this means that a spin-c structure is defined by a principal $\mathrm{Spin}^c(4)$ bundle $\mathfrak{P}$ with a projection $\mathfrak{P} \to OrFr(X)$ which along the fibers looks like the exact sequence

$$1 \to U(1) \to \mathrm{Spin}^c(4) \to SO(4) \to 1 \tag{G.18}$$

A spin-c structure exists when $w_2(X)$ has an integral lift. As mentioned above, this is indeed true for every compact orientable four-manifold.

The group homomorphism $\mathrm{Spin}^c(4) \to U(2) \times U(2)$ given by $[(v_L, v_R, \zeta)] \mapsto (\zeta v_L, \zeta v_R)$ defines an isomorphism

$$\mathrm{Spin}^c(4) \cong \{(u_L, u_R) | \det u_L = \det u_R\} \subset U(2) \times U(2) \tag{G.19}$$

and the latter presentation makes it obvious that there are two inequivalent rank 2 representations $W^\pm$ of $\mathrm{Spin}^c(4)$ simply defined by the fundamental representations of each of the two $U(2)$ factors. Given a spin-c structure there are therefore two associated complex rank two bundles $W^\pm \to X$ and we can identify

$$W^\pm = S^\pm \otimes L \tag{G.20}$$

in the discussion of section **??** above. Conversely such a pair of bundles $W^\pm$ defines a spin-c structure. Consequently the space of spin-c structures is a torsor for the group of line bundles, since give a line bundle $\mathcal{L}$ we can always take

$$W^\pm \to W^\pm \otimes \mathcal{L} \tag{G.21}$$

so that

$$c_1(\det W^\pm) \to c_1(\det W^\pm) + 2c_1(\mathcal{L}) \tag{G.22}$$

Note that given an almost complex structure on X there is a canonical spin-c structure

$$W^+ = \Omega^{0,0}(X) \oplus \Omega^{0,2}(X) \qquad\qquad W^- = \Omega^{0,1}(X) \tag{G.23}$$

G.4 't Hooft Flux

At several points in the notes we used the second Stiefel-Whitney class $w_2(P)$ where P is a principal $SO(3)$ bundle. In general, an $SO(n)$ bundle P over a manifold M has a characteristic class $w_2(P) \in H^2(M; \mathbb{Z}_2)$. One way to define it follows the discussion of spin structures above: Choose a good cover $\{\mathcal{U}_\alpha\}$ of M and a trivialization of P on this cover. On patch overlaps $\mathcal{U}_{\alpha\beta}$ choose lifts $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to Spin(n)$ and measure the failure of the cocycle condition on $\mathcal{U}_{\alpha\beta\gamma}$ to define a class in $H^2(M; \mathbb{Z}_2)$. If $w_2(P)$ is nonzero then there is no reduction of structure group of P from $SO(n)$ to $Spin(n)$.

A simple example of an $SO(n)$ bundle that does not lift to a $Spin(n)$ bundle is obtained by considering M to be a two-dimensional compact surface. Choose a point $p \in M$ and a small disk around p. Define an $SO(n)$ bundle by taking the transition function around the boundary of the disk to define a nontrivial closed loop in $SO(n)$.

H. Connections, Parallel Transport, And Holonomy

****

GIVE DEFINITION OF CONNECTION IN TERMS OF PATH-LIFTING PROPERTY

Our convention for concatenating curves:

$$f_1 \star f_2(t) := \begin{cases} f_1(2t) & 0 \le t \le \frac{1}{2} \\ f_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{H.1}$$

****

In the case of principal bundles and vector bundles we can express that property in terms of a local differential equation, thereby establishing the relation to locally-defined matrix-valued one-forms.

Example 1: The most elementary example is obtained by taking $B = \mathbb{R}$ and $E = \mathbb{R} \times \mathbb{C}^n$, with $p : E \to B$ simply being projection onto the first factor, i.e. $p(x, \vec{v}) = x$. Now, given a path $\wp$ in B given by $x(t)$, $0 \le t \le 1$ we can define a path-lifting rule, i.e., a connection, by choosing a function of x valued in $n \times n$ complex matrices. Let us call it $A(x)$. Then the path-lifting rule is

1. If $\wp$ is a path from $\wp : x_0 \rightsquigarrow x_1$, choose an element of the fiber $e_0 = (x_0, \vec{v}_0)$ above the initial point.

2. Then, solve the ordinary differential equation, with boundary condition provided by the lift e_0 of the initial point:

$$\frac{d}{dt}\vec{v}(t) = A(x(t))\frac{dx}{dt}\vec{v}(t) \qquad \vec{v}(0) = \vec{v}_0. \tag{H.2}$$

The rule for the lifted path (determined by the choice $A(x)$) is then

$$\mathbb{P}(\wp)(t) := (x(t), \vec{v}(t)) \tag{H.3}$$

See Figure **??**.

The reason that this rule is indeed compatible with composition of paths is that the equation (H.2) is invariant under reparametrization $t \to f(t)$ of the time t, so long as $f(t)$ is differentiable and $f'(t) > 0$. Using this fact and the existence and uniqueness of solutions to first order linear ODE's we have

$$\mathbb{P}(\wp_1 \star \wp_2) = \mathbb{P}(\wp_1) \star \mathbb{P}(\wp_2) \tag{H.4}$$

Remarks

1. We assume here that $\wp$ is a piecewise-differentiable path, i.e. $x(t)$ is a continuous function which is differentiable on intervals. (But the derivative can be discontinuous at isolated points.)

2. We assume that $A(x)$ is nonsingular on the path.

3. Note that since the differential equation is invariant under complex conjugation, if $A(x)$ is real then if $\vec{v}_0$ is real the solution $\vec{v}(t)$ will be real.

The path ordered exponential

♣WE SHOULD REALLY CHANGE CONVENTION ON OUR PATH ORDERED EXPONENTIAL SO THAT $\mathbb{U}(\wp_1 * \wp_2) = \mathbb{U}(\wp_1)\mathbb{U}(\wp_2)$. ♣

As a matter of fact, one can write an “explicit” solution of the differential equation (H.2) and this representation can be useful. The solution is can be written as follows. Define an $n \times n$ complex matrix:

$$\mathbb{U}(\wp_t) := 1 + \int_0^t A(x(t_1))\dot{x}(t_1)dt_1 + \sum_{m=2}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m A(x(t_1))\dot{x}_1 \cdots A(x(t_m))\dot{x}_m \tag{H.5}$$

then we have

$$\vec{v}(t) = \mathbb{U}(\wp_t)\vec{v}_0. \tag{H.6}$$

To prove this, note that by explicit differentiation

$$\frac{d}{dt}\mathbb{U}(\wp_t) = A(x(t))\dot{x}(t)\mathbb{U}(\wp_t) \tag{H.7}$$

and note that $\mathbb{U}(\wp_0) = 1$.

Remarks

1. $\mathbb{U}(\wp_t)$ is an operator, independent of the choice of lift $\vec{v}_0$ of the initial point.

2. Matrix multiplication of $\mathbb{U}(\wp)$ is contravariant with respect to composition of paths:

$$\mathbb{U}(\wp_1 \star \wp_2) = \mathbb{U}(\wp_2)\mathbb{U}(\wp_1) \tag{H.8}$$

3. For the piecewise continuous path $\wp_t \star \bar{\wp}_t$ it is clear that the parallel transport takes $(x_0, \vec{v}_0) \to (x_0, \vec{v}_0)$. Therefore $\mathbb{U}(\wp_t \star \bar{\wp}_t) = 1_{n\times n}$. It follows that $\mathbb{U}(\wp_t)$ is an *invertible* matrix.

4. $\mathbb{U}(\wp_t)$ is invariant under reparametrizations of the path $x(t)$ by $t \to f(t)$ where $f'(t) > 0$. It therefore makes sense to write

$$\mathbb{U}(\wp_t) = 1 + \int_{x_0}^{x(t)} A(x_1)dx_1 + \sum_{m=2}^{\infty} \int_{x_0}^{x(t)} A(x_1)dx_1 \int_{x_0}^{x_1} dx_2 \cdots \int_{x_0}^{x_{m-1}} A(x_m)dx_m \tag{H.9}$$

This expression has a further useful representation by introducing the *(left) time ordered product* of matrices defined by

$$T_\ell\left(A(x(t_1)), \ldots, A(x(t_m))\right) := A(x(t_{\sigma(1)})) \cdots A(x(t_{\sigma(m)})) \tag{H.10}$$

where $\sigma \in S_m$ is a permutation such that

$$t_{\sigma(1)} \geq \cdots \geq t_{\sigma(m)} \tag{H.11}$$

Note that if all the times are distinct then the permutation is unique. If some times coincide then σ is not uniquely determined, but any two permutations lead to the same RHS for (H.10). Using this notation we can write:

$$\mathbb{U}(\wp_t) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_0^t dt_1 \dot{x}_1 \cdots \int_0^t dt_m \dot{x}_m T_\ell[A(x(t_1)), \cdots, A(x(t_m))] \tag{H.12}$$

which motivates the notation

$$\mathbb{U}(\wp_t) := \mathrm{Pexp} \int_0^t dt_1 \dot{x}_1 A(x(t_1)) \tag{H.13}$$

5. **Warning:** Do not confuse the path-ordered exponential with the ordinary exponential:

$$\exp \int_0^t dt_1 \dot{x}_1 A(x(t_1)) \tag{H.14}$$

If $A(x)$ is a family of *commuting* matrices then the exponential and the path-ordered exponential will be the same. In general they are different.

6. If one reversed the order of all the inequalities in (H.11) then (H.10) would define the *(right) time-ordered product* and the multiplication rule would be covariant. This is the matrix we would apply to the differential equation

$$\frac{d}{dt}\vec{v}(t) = \vec{v}(t)A(x(t))\frac{dx}{dt} \qquad\qquad \vec{v}(0) = \vec{v}_0. \tag{H.15}$$

where $\vec{v}(t)$ is a row vector.

♣Should have more detail about left- vs. right- path ordered exponentials and how they are related. And more detail on zig-zag invariance. ♣

Now we are not going to see interesting monodromy around closed paths in the above example, because $\pi_1(\mathbb{R}, x_0) = 0$, but a small modification of the above example produces interesting examples.

Example 2: We now take $B = S^1$ which we regard as both the unit circle in the complex plane and the quotient $\mathbb{R}/\mathbb{Z}$ given by identifying $x \sim x + 1$. Now we take

$$E = S^1 \times \mathbb{C}^n \tag{H.16}$$

and the projection p is simply projection onto the first factor: $p(z, \vec{v}) = z$. Now let $A(x)$ be a matrix-valued function as before but now impose the condition that it be periodic: $A(x + 1) = A(x)$. Consider a path $\wp$ on S^1 with $\wp(0) = z_0$. The fiber above z_0 is the set of points $(z_0, \vec{v})$ where $\vec{v} \in \mathbb{C}^n$. As always, to lift the path $\wp$ to a path $\mathbb{P}(\wp)$ in E we must choose a lift $e_0 = (z_0, \vec{v}_0)$ of the initial point of the path. Now, to write the differential equation we also lift the path $z(t)$ by choosing some initial point x_0 so that $z(t) = \exp[2\pi i x(t)]$, with $x(0) = x_0$. Of course, if $\wp$ is a closed path so that $z(1) = z_0$ then $x(t)$ need not be closed, but rather $x(1) = x_0 + n$, where n is the winding number of $\wp$. Now we consider exactly the same differential equation (H.2), and we produce a family $(x(t), \vec{v}(t))$. It now makes sense to pass from $x(t)$ to $z(t)$ precisely because $A(x)$ is periodic in x, so now the lifted path is:

$$\mathbb{P}(\wp)(t) = (z(t), \vec{v}(t)) \tag{H.17}$$

Even though $z(1) = z(0) = z_0$ and $A(x(1)) = A(x_0 + n) = A(x(0))$ are single-valued, there is no reason for $\vec{v}(t)$ to be single valued. Rather, the monodromy of the connection determined by A around the path $\wp$ can be thought of as an invertible linear transformation

$$\vec{v}_0 \to \mathbb{U}(\wp)\vec{v}_0. \tag{H.18}$$

To be more explicit, let us take $n = 2$ and $\wp$ given by simple path with winding number 1. Say, for simplicity, it has lift $x(t) = t$. Suppose moreover that

$$A = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{H.19}$$

is constant in x. Then the differential equation (H.2) is easily solved to give

$$\vec{v}(t) = \begin{pmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{pmatrix} \vec{v}_0 \tag{H.20}$$

where $\vec{v}_0 \in \mathbb{R}^2$. Clearly, $\vec{v}(1)$ is not $\vec{v}_0$, in general.

Remark: Note that since $A(x)$ is invariant under $x \to x + 1$ it would make sense to define $\tilde{A}(z)$ as $A(x)$ for any x such that $z = \exp[2\pi i x]$. However, this is not the most convenient definition if we want the differential equation (H.2) to look the same in terms of z. Rather, if we define $\tilde{A}(z)$ so that $\tilde{A}(z)dz = A(x)dx$, that is, so that

$$\tilde{A}(z) := \frac{1}{2\pi \mathrm{i}} e^{-2\pi \mathrm{i} x} A(x) \tag{H.21}$$

then the equation (H.2) is equivalent to

$$\frac{d}{dt}\vec{v}(t) = \tilde{A}(z(t))\frac{dz}{dt}\vec{v}(t) \qquad\qquad \vec{v}(0) = \vec{v}_0. \tag{H.22}$$

That is, A should transform under change of coordinates as a 1-form.

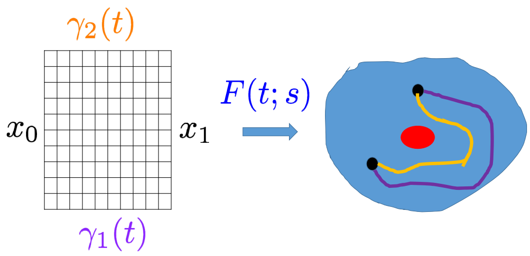

Figure 35: Illustrating the argument that for a flat connection on a domain in $\mathbb{C}$ the parallel transport only depends on the homotopy class of the curve with fixed endpoints. Using the homotopy, divide the region between the two curves into small regions by dividing the domain of the homotopy into sufficiently small squares. Then the monodromy around each small square is computed by the connection and its covariant derivatives: But these are all zero.

Example 3: Let B be an open path-connected domain in $\mathbb{C}$. For example, B might be $\mathbb{C}-\{z_1, \dots, z_m\}$, i.e., $\mathbb{C}$ with some set of points deleted. We can also view it as the extended complex plane with the point at infinity also deleted: $B = \mathbb{CP}^1 - \{z_1, \dots, z_m, \infty\}$. Thus it is most definitely not simply connected for $m \geq 1$. Again let $E = B \times \mathbb{C}^n$ for some positive integer n. A path in B can be represented by $z(t)$. Our path lifting rule will be similar to Example 1: Choose a pair of $n \times n$ matrix-valued functions on B, call it $(A_z, A_{\bar{z}})$ where each matrix in the pair is a single-valued and nonsingular function of $(z, \bar{z})$. [44] Then the lifted path will be $(z(t), \vec{v}(t))$ where the differential equation is now:

$$\frac{d}{dt}\vec{v}(t) + \left(A_z(z(t), \bar{z}(t))\frac{dz}{dt} + A_{\bar{z}}(z(t), \bar{z}(t))\frac{d\bar{z}}{dt} \right) \vec{v}(t) = 0 \qquad \vec{v}(0) = \vec{v}_0. \tag{H.23}$$

We can again "solve" the differential equation with the path-ordered exponential, and again there will be quite interesting monodromy. Note that we put both terms on the same side of the equation (thus A is related to the previous examples by a sign flip). This sign convention turns out to be more useful.

As an example of the monodromy let us consider a small loop based at z_0, written as $z(t) = z_0 + \epsilon(t)$, where for fixed t, the complex number $\epsilon(t)$ will be taken to be small. In particular, the loop will be homotopically trivial, so there is a small disk D with basepoint z_0 such that ∂D is the image of $\wp$. Then the leading nontrivial contribution to $\mathbb{U}(\wp)$ will be appear at order $\mathcal{O}(\epsilon^2)$ and one can show that

$$\boxed{\mathbb{U}(\wp) = 1 + \alpha F_{z\bar{z}}(z_0, \bar{z}_0) + \mathcal{O}(\epsilon^3)} \tag{H.24}$$

where

$$\alpha = \int_0^1 dt \dot{\epsilon}(t)\bar{\epsilon}(t) \tag{H.25}$$

is ((2i) times) the Euclidean area enclosed by the small loop at z_0 and

$$\boxed{F_{z\bar{z}} := \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]} \tag{H.26}$$

One way to see that α is proportional to the area enclosed by the loop is is to write

$$\alpha = \oint_{\wp} d\epsilon\bar{\epsilon} = \int_D d\epsilon \wedge d\bar{\epsilon} \tag{H.27}$$

and the latter integral is 2i times the Euclidean area enclosed by the loop. One can show that in the full expansion of (H.24) in powers of $\epsilon(t)$ all the terms involve products of $\mathbb{F}_{z\bar{z}}$ and its (covariant) derivatives. ♣Need to justify that more fully. ♣

The expression $F_{z\bar{z}}$ is known as the *curvature of the connection*. It is more properly regarded as a locally matrix valued 2-form $F = F_{z\bar{z}} dz \wedge d\bar{z}$.

A particularly important class of connections are the *flat connections*, defined to be the connections with zero curvature. In this case, given a flat connection, *the monodromy matrix* $\mathbb{U}(\wp)$ *for a closed path only depends on the homotopy class of* $\wp$ *in* B. This is easy to show using (H.24) and the path composition property. From (H.24) the monodromy around a small loop must be trivial. But now if $F(t; s)$ is a homotopy from the closed loop $\wp_1(s)$ to $\wp_2(s)$ then we can divide up the square I^2 into many small squares, and the monodromy around each of these must be trivial, therefore, the monodromy around the full square must be trivial. Therefore $\mathbb{U}(\wp_1 \star \bar{\wp}_2) = 1$ and hence $\mathbb{U}(\wp_1) = \mathbb{U}(\wp_2)$. ♣Need more explanation ♣

[44]We are not assuming any relation between the complex conjugate $(A_z)^*$ and $A_{\bar{z}}$.

In the special case that $A_{\bar{z}} = 0$ and A_z is a *holomorphic* function of z the property that $\mathbb{U}(\wp)$ only depends on the homotopy class can be seen more directly from the path-ordered exponential:

$$\mathbb{U}(\wp_t) = 1 + \int_{z_0}^{z(t)} A(z_1)dz_1 + \sum_{n=2}^{\infty} \int_{z_0}^{z(t)} dz_1 \int_{z_0}^{z_1} dz_2 \cdots \int_{z_0}^{z_{n-1}} dz_n A(z_1)\cdots A(z_n) \quad \text{(H.28)}$$

and now the assertion follows from Cauchy's theorem. Here dz_i is short for $\dot{z}(t_i)dt_i$ etc.

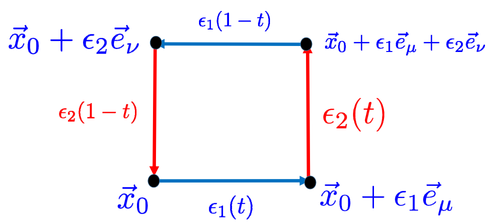

Figure 36: When ϵ_1 and ϵ_2 are small the entire contribution to the monodromy comes from the curvature of the connection, and the leading term is determined by the area of the loop times the curvature element in the plane spanned by the loop (in the tangent space) at $\vec{x}_0$.

Example 4: Now take B to be an open domain in $\mathbb{R}^m$ for any $m > 0$ and $E = B \times \mathbb{C}^N$, where m and N are in general completely unrelated. Choose coordinates x^μ, $\mu = 1, \dots, m$ on $\mathbb{C}^N$ and let $A_\mu(x)$ be a collection of m complex $N \times N$ matrix-valued functions on B. We assume they are single-valued and nonsingular. In close analogy to the above examples, this data suffices to define a connection on the fibration $p : E \to B$ given by projection on the first factor: Suppose $\wp : [0,1] \to B$ is a (piecewise differentiable) path in B from $\vec{x}_0$ to $\vec{x}_1$. Then we choose a lift $(\vec{x}_0, \vec{v}_0) \in E$ in the fiber above $\vec{x}_0$ and solve the differential equation

$$\frac{d}{dt}\vec{v}(t) + A_\mu(\vec{x}(t))\frac{dx^\mu(t)}{dt}\vec{v}(t) = 0 \qquad \vec{v}(0) = \vec{v}_0 \quad \text{(H.29)}$$

and $\mathbb{P}(\wp)(t) = (\vec{x}(t), \vec{v}(t))$ is the lift. In general there will be interesting monodromy, already for small homotopically trivial paths near any point $\vec{x}_0$. The expressions are simple generalizations of (H.24) and (H.26). Indeed, any infinitesimal curve can be thought of as sitting in some plane passing through $\vec{x}_0$ and then it is simply a matter of changing back from complex to real coordinates. Alternatively, we can consider a small path given by a composition of four open paths going around a square in the $x^\mu - x^\nu$ plane:

$$\begin{aligned}
\wp_1(t) &= \vec{x}_0 + \epsilon_1(t)\vec{e}_\mu \\
\wp_2(t) &= (\vec{x}_0 + \epsilon_1\vec{e}_\mu) + \epsilon_2(t)\vec{e}_\nu \\
\wp_3(t) &= (\vec{x}_0 + \epsilon_2\vec{e}_\nu) + \epsilon_1(1-t)\vec{e}_\mu \\
\wp_4(t) &= \vec{x}_0 + +\epsilon_2(1-t)\vec{e}_\nu
\end{aligned} \quad \text{(H.30)}$$

and then $\wp = \wp_1 \star \wp_2 \star \wp_3 \star \wp_4$. Here $\vec{e}_\mu$ is a unit vector pointing in the x^μ direction and $\epsilon_i := \epsilon_i(t = 1)$. See Figure 36. A simple computation shows that (H.24) and (H.26) are generalized to

$$\boxed{\mathbb{U}(\wp) = 1 - \epsilon_1\epsilon_2 F_{\mu\nu}(\vec{x}_0) + \cdots} \tag{H.31}$$

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]} \tag{H.32}$$

and the higher order terms in (H.31) are all of order $\epsilon_1^a \epsilon_2^b$ with $a > 0$ and $b > 0$ and $a+b > 2$.

Remark: It can be shown that the coefficients of the higher order terms in (H.31) are polynomials in $F_{\mu\nu}$ and its covariant derivatives in the μ and ν direction. In general, the covariant derivative of any matrix-valued function $\Phi(x)$ in the λ direction is

$$D_\lambda \Phi := \partial_\lambda \Phi + [A_\lambda, \Phi] \tag{H.33}$$

♣Is there a simple proof of that? Or a way to generate the series? I am guessing it is $e^{\epsilon_1 D_\mu} e^{\epsilon_2 D_\nu} e^{-\epsilon_1 D_\mu} e^{-}$ ♣

Exercise

Solve (H.23) for $B = \mathbb{C}^*$, $N = 1$, and $A_z = \frac{\mu}{z}$ and $A_{\bar{z}} = 0$, where μ is a complex number. Compute the monodromy of this connection around some simple closed curves in B.

Exercise

Give a careful derivation of equations (H.24),(H.26), (H.31), and (H.32).

Exercise

Show that if the matrices $A_\mu(x)$ are anti-hermitian, i.e. $(A_\mu(x))^\dagger = -A_\mu(x)$ then

$$\mathrm{Pexp} \int_0^t A_\mu(x(t_1)) \frac{dx^\mu}{dt_1} dt_1 \tag{H.34}$$

is unitary.

Exercise *Gauge transformations*

Let $A(x)$ be a matrix-valued $n \times n$ complex matrix on $\mathbb{R}$ and and $x \mapsto g(x)$ a differentiable map from $\mathbb{R}$ to $GL(n, \mathbb{C})$. Define a new matrix-valued function $\tilde{A}(x)$ by

$$A(x) = g(x)^{-1} \tilde{A}(x) g(x) + g(x)^{-1} \frac{d}{dx} g(x) \tag{H.35}$$

a.) Show that

$$d + \tilde{A} = g(x)(d + A)g(x)^{-1} \tag{H.36}$$

where $d = dx^\mu \frac{\partial}{\partial x^\mu} 1_{N\times N}$ is a first order differential operator and $A = dx^\mu A_\mu$.

b.) Show that, for any piecewise-differentiable path $x(t)$ from x_0 to x_1 we have

$$\mathrm{Pexp}\left[-\int_0^1 \tilde{A}(x(t))\dot{x}(t)dt\right] = g(x_1)\mathrm{Pexp}\left[-\int_0^1 A(x(t))\dot{x}(t)dt\right]g(x_0)^{-1} \tag{H.37}$$

c.) Show by direct computation that, if $\tilde{F}_{\mu\nu}$ is computed from $\tilde{A}_\mu$ then

$$F_{\mu\nu}(x) = g(x)^{-1}\tilde{F}_{\mu\nu}(x)g(x) \tag{H.38}$$

d.) Show that the commutator of matrix-valued first order differential operators gives the curvature:

$$[D_\mu, D_\nu] = F_{\mu\nu} \tag{H.39}$$

Use this to give another proof of the gauge transformation rule of part (c).

e.) Suppose $\Phi(x)$ and $\tilde{\Phi}(x)$ are matrix valued functions of x^μ related by $\tilde{\Phi}(x) = g(x)\Phi(x)g(x)^{-1}$. Show by direct computation that

$$\tilde{D}_\lambda\tilde{\Phi}(x) = g(x)D_\lambda\Phi(x)g(x)^{-1} \tag{H.40}$$

where $\tilde{D}_\lambda$ is the covariant derivative computed with $\tilde{A}_\lambda$.

f.) Show that $[D_\mu, \Phi] = D_\mu\Phi$. Use this to give another proof of the gauge transformation rule in (e).

I. Chern-Weil Theory And Chern-Simons Forms

I.1 Characteristic classes

We do not have space here to do justice to this important subject. Here is the telegraphic version:

Roughly, characteristic classes are a way of measuring the twisting of vector bundles and principal bundles.

A characteristic class is, by definition, a rule for assigning cohomology classes to (isomorphism classes of) principal bundles. When $\pi : P \to M$ is a principal bundle with structure group G then the characteristic classes are integral cohomology classes on M. So a characteristic class is a map from the set of isomorphism classes of principal bundles over M to $H^*(M;\mathbb{Z})$. If $P \to M$ is a principal bundle we let $\xi(P)$ denote the corresponding cohomology class. Moreover, this map should be *natural*, that is:

$$f^*(\xi(P)) = \xi(f^*(P)) \tag{I.1}$$

Note that such characteristic classes form a ring.

Because of the property (I.1) characteristic classes can be defined in terms of the cohomology classes of the classifying space BG. If $\theta \in H^*(BG;\mathbb{Z})$ and $f : M \to BG$ is a classifying map for P so that $P = f^*(EG)$ then we can take $\xi(P) = f^*(\theta)$. Moreover, all characteristic classes arise in this way.

When the bundle P is endowed with a connection we can construct DeRham representatives of the characteristic classes. Although we lose torsion information, the construction is very useful, and occurs very frequently in physics. It leads to topological terms in Yang-Mills actions and, via the secondary characteristic classes, known as Chern-Simons forms, to many of the most interesting and important examples of the interactions between geometry and physics.

I.2 Basic forms

Definition A *basic form* on a bundle $P \to M$ is a form ω which is a pullback from a form $\tilde{\omega}$ from the base: $\omega = \pi^*(\tilde{\omega})$.

Proposition. Suppose P is a principal G bundle for a *compact and connected* group G. Then $\omega \in \Omega^*(P)$ is basic iff it satisfies:

a.) $\iota(\xi(X))(\omega) = 0$

b.) $\mathcal{L}(\xi(X))(\omega) = 0$

for all $X \in \mathfrak{g}$

Proof: Work locally, with coordinates (x, g). Then we can expand all forms on the total space in terms of dx^μ, and e^a, where $\Theta = e^a T_a$.

The verticality equation $\iota(\xi(X))(\omega) = 0$ means that ω has no vertical components. That is, in local coordinates it has the form:

$$\omega = \omega_{\mu_1 \ldots \mu_k}(x, g) dx^{\mu_1 \cdots \mu_k} \tag{I.2}$$

Next, the Lie derivative condition shows that the coefficient functions must be independent of g. ♠

Remark: More generally, if G is not connected, condition (b) above should be formulated as the condition that

$$R_g^*(\omega) = \omega \qquad \forall g \in G \tag{I.3}$$

I.3 Invariant polynomials on the Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An polynomial of degree k on $\mathfrak{g}$ is a totally symmetric multilinear function:

$$\wp : \mathfrak{g}^{\otimes k} \to \mathbb{F} \tag{I.4}$$

The symmetric polynomials are elements of $\mathrm{Sym}^k \mathfrak{g}^*$, and together $\oplus_k \mathrm{Sym}^k \mathfrak{g}^*$ form an algebra. The product of two polynomials $\wp_1$ and $\wp_2$ of degrees k_1, k_2 is

$$(\wp_1 \cdot \wp_2)(v_1, \ldots, v_{k_1+k_2}) := \frac{1}{(k_1+k_2)!} \sum_{\sigma \in S_{k_1+k_2}} \wp_1(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)}) \wp_2(v_{\sigma(k_1+1)}, \ldots, v_{\sigma(k_1+k_2)}) \tag{I.5}$$

An *invariant polynomial* is a polynomial which is $Ad(G)$ invariant. The invariant polynomials form a subring of $S^*(\mathfrak{g}^*)$ denoted $I(G)^*$ or $\mathrm{Inv}_G S^*(\mathfrak{g})$

Remark: An invariant polynomial is completely determined by its values on the diagonal $\wp(v,v,\dots,v)$. Proof: Substitute $v = \sum v_i x_i$ in terms of a basis with indeterminates x_i. We will abuse notation and refer to this diagonal value as $\wp(v)$. Thus $\wp(gvg^{-1}) = \wp(v)$.

Theorem If G is a compact simple Lie group the ring of invariant polynomials is a polynomial ring on the Casimirs

$$\mathrm{Inv}_G S^*(\mathfrak{g}) \cong \mathbb{R}[C_1, C_2, \dots, C_\ell] \tag{I.6}$$

where ℓ is the rank of G and C_i are of degree k_i, the exponents of G.

Proof: An invariant polynomial is determined by its value on the diagonal $\wp(v)$, and by Ad-invariance it is determined by its value on a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Conversely, any W-invariant polynomial on $\mathfrak{t}$ defines an invariant polynomial on $\mathfrak{g}$. Here W is the Weyl group. Since W is a finite group we can always average a polynomial on $\mathfrak{t}$ to get one which is W-invariant. It turns out that the ring of W-invariant polynomials is in fact itself a polynomial ring: See C. Chevalley, "Invariants of finite groups generated by reflections," Amer. J. Math. **77**(1955)778. ♠

I.3.1 Examples

Example 1: $Lie(GL(n,\mathbb{C})) = Mat_n(\mathbb{C})$. The ring of invariant polynomials is generated by the Chern polynomials C_k defined by

$$\det(xI + \frac{1}{2\pi i}A) = \sum_j C_j(A,\dots,A)x^{n-j} \tag{I.7}$$

Example 2: GL(n,R)

Example 3: U(N)

Example 4: $G = SO(2r)$....

For $G = SO(2r)$ an important new invariant polynomial appears - the Pfaffian:

$$\mathrm{Pfaff}(m) = \frac{1}{2^r r!} \sum_{\sigma \in S_{2r}} \mathrm{sign}(\sigma) m_{\sigma(1)\sigma(2)} m_{\sigma(3)\sigma(4)} \cdots m_{\sigma(2r-1)\sigma(2r)} \tag{I.8}$$

for an antisymmetric matrix is a polynomial in the matrix elements such that $(\mathrm{Pfaff}(m))^2 = \det m$.

Example 5: $G = SO(2r+1)$.

Example 6: $G = USp(2r)$.

Exercise

Show that $\mathrm{Pfaff}(gmg^{-1}) = \det(g)\mathrm{Pfaff}(m)$ for any $g \in O(2r)$.

I.4 The Chern-Weil homomorphism

Now suppose that $\pi : P \to M$ is a principal G-bundle with connection Θ and fieldstrength $\bar{F} \in \Omega^2(P;\mathfrak{g})$. Let $\wp$ be an invariant polynomial on $\mathfrak{g}$.

Then we may consider

$$\wp(\bar{F}) \tag{I.9}$$

If $\wp$ is homogeneous of degree k, $\wp(\bar{F})$ is a differential form of degree $2k$ on P.

A typical example to keep in mind is the trace in some representation ρ:

$$\wp_{k,\rho}(\bar{F}) := \mathrm{Tr}_\rho \rho(\bar{F})^k \tag{I.10}$$

(we will often drop the ρ in this notation).

Proposition: The form $\wp(\bar{F}) = \pi^*(\bar{\wp})$ is a basic form.

Proof Clearly, $\wp(\bar{F})$ is horizontal, i.e. $\iota(\xi(X))\wp(\bar{F}) = 0$, since $\iota(\xi(X))(\bar{F}) = 0$. Moreover,

$$R_g^*(\wp(\bar{F})) = \wp(R_g^*(\bar{F})) = \wp(g\bar{F}g^{-1}) = \wp(\bar{F}) \tag{I.11}$$

and thus it does not vary along the fiber ♠

Proposition: $\bar{\wp}(\bar{F})$ is a closed differential form on the base M. Moreover it is natural.

Proof Note that:

$$d\wp(\bar{F}, \cdots, \bar{F}) = k\wp(d\bar{F}, \bar{F}, \cdots, \bar{F}) \tag{I.12}$$

But now we use the Bianchi identity

$$D\bar{F} = d\bar{F} + [\Theta, \bar{F}] = 0 \tag{I.13}$$

Now note that $\wp([\Theta, \bar{F}], \bar{F}, \cdots, \bar{F}) = 0$ because $\wp$ is Ad-invariant. (Recall that Θ is valued in $\mathfrak{g}$, and use the infinitesimal version of Ad-invariance.) Then π^* has no kernel (on the DeRham complex!) so $d\bar{\wp}(\bar{F}) = 0$. ♠

To prove naturality:

Note that if $f : P_1 \to P_2$ is a bundle map covering $\bar{f} : M_1 \to M_2$, and $\Theta_1 = f^*(\Theta_2)$ is the pulled-back connection then $\wp(\bar{F}_1) = f^*(\wp(\bar{F}_2))$ so $\bar{\wp}(\bar{F}_1) = f^*(\bar{\wp}(\bar{F}_2))$. In particular, the cohomology class of $\bar{\wp}(\bar{F})$ only depends on the isomorphism class of (P, Θ). ♠

Finally note that if $\wp$ is of degree k then $\bar{\wp}(\bar{F}) \in \Omega^{2k}(M)$ is of degree $2k$. Moreover, the product of symmetric polynomials maps to the product of cohomology classes. Thus, we have the definition:

Definition: The Chern-Weil homomorphism is the homomorphism of graded rings:

$$I^*(G) \to H^*_{DR}(M) \tag{I.14}$$

taking $\wp$ to $[\bar{\wp}(\bar{F})]$.

Remarks:

1. Consider the natural connection on the Hopf fibration. Since $u(1)$ is Abelian, any polynomial on the Lie algebra is invariant. The fieldstrength $\bar{F}$ is a basic form, which had a nontrivial integral on the S^2 base. ♣NEED TO EXPLAIN ♣

2. Another way to understand the fact the form is basic is to work purely downstairs, with local patches. In each patch $\mathcal{U}_\alpha$, $\mathrm{Tr}F_\alpha^k$ is gauge invariant. Therefore, comparing connections across path boundaries $F_\alpha = g_{\alpha\beta}F_\beta g_{\alpha\beta}^{-1}$, and hence $\mathrm{Tr}F_\alpha^k$ is in fact a globally well-defined $2k$-form on M. In our work below we will sometimes work directly in terms of F defined on the base. Recall that in local coordinates $\Theta = g^{-1}dg + g^{-1}Ag$ and $\bar{F} = g^{-1}Fg$.

3. More generally, any gauge invariant quantity, e.g. the Yang-Mills action density, $\mathrm{Tr}F * F$ made out of the connection will be globally well-defined on the base.

Exercise
Let α, β be degree j, k differential forms valued in $\mathfrak{g}$. Show that

$$\boxed{d\mathrm{Tr}(\alpha \wedge \beta) = \mathrm{Tr}\left[D_A(\alpha) \wedge \beta + (-1)^k \alpha \wedge D_A(\beta)\right]} \tag{I.15}$$

where

$$\begin{aligned} D_A\alpha &= d\alpha + A\alpha - (-1)^k \alpha A \\ D_A\beta &= d\beta + A\beta - (-1)^j \beta A \end{aligned} \tag{I.16}$$

Using this and the Bianchi identity verify directly that

$$d\mathrm{Tr}F^k = k\mathrm{Tr}(D_A F)F^{k-1} = 0 \tag{I.17}$$

I.5 Dependence on connection: Construction of characteristic classes

Now, a crucial property of the Weil homomorphism is that the image in fact does not depend on the choice of connection on P:

Now we show that the cohomology classes $[\bar{\wp}(\bar{F})]$ are in fact independent of the choice of connection on P.

Let us begin with the main special case, and work locally on the base. Thus, we consider the dependence of the basic forms $\mathrm{Tr}F^n$ on the choice of connection. Recall that under an infinitesimal variation:

$$\delta F = D_A(\delta A) \tag{I.18}$$

and therefore

$$\begin{aligned}\delta \mathrm{Tr} F^n &= n \mathrm{Tr}(\delta F) F^{n-1} \\ &= n \mathrm{Tr}\Bigl(D_A(\delta A) F^{n-1} \Bigr) \\ &= n d \Bigl(\mathrm{Tr} \delta A F^{n-1} \Bigr)\end{aligned} \tag{I.19}$$

where we have used the Bianchi identity $D_A F = 0$.

More formally, if we have a *linear* path of connections: $A(t) = (1-t)A_0 + tA_1 = A_0 + t\alpha$, $\alpha \in \Omega^1(M; \mathrm{ad}P)$, then

$$\begin{aligned}\mathrm{Tr} F_1^n - \mathrm{Tr} F_0^n &= \int_0^1 dt \frac{d}{dt} \mathrm{Tr} F(A(t))^n \\ &= n \int_0^1 dt \mathrm{Tr} D_{A(t)}(\alpha) F(A(t))^{n-1} \\ &= d \left[n \int_0^1 dt \mathrm{Tr} \alpha F(A(t))^{n-1} \right] \\ &= d \left[n \int_0^1 dt \mathrm{Tr} \alpha \Bigl(F_0 + t D_{A_0} \alpha + t^2 \alpha^2 \Bigr)^{n-1} \right]\end{aligned} \tag{I.20}$$

Note that the expression

$$n \int_0^1 dt \mathrm{Tr} \alpha \Bigl(F_0 + t D_{A_0} \alpha + t^2 \alpha^2 \Bigr)^{n-1} \tag{I.21}$$

is *globally well defined*, since it only involves tensorial quantities. Therefore, the DeRham cohomology class of the closed globally well-defined form $\mathrm{Tr} F(A)^n$ on M is independent of connection.

The same manipulations work for an arbitrary invariant polynomial of degree n:

$$\begin{aligned}\frac{d}{dt} \wp(F(A(t)) &= n \wp(D_{A(t)} \alpha, F(A(t)), \dots, F(A(t))) \\ &= n d \Bigl(\wp(\alpha, F(A(t)), \dots, F(A(t))) \Bigr)\end{aligned} \tag{I.22}$$

and integrating gives

$$\wp(F(A_1)) - \wp(F(A_0)) = d \left[n \int_0^1 \wp(\alpha, F(A(t)), \dots, F(A(t))) \right] \tag{I.23}$$

and hence the cohomology class is unchanged.

As we have shown, $[\wp(F)]$ is natural, and hence, given an invariant polynomial $\wp$ we can produce a characteristic class by the rule:

$$\xi : P \to [\bar{\wp}(\bar{F})] \tag{I.24}$$

where we can use any connection on P. That is, the Chern-Weil homomorphism is independent of connection.

Remark: It follows that integrals over closed cycles (including the whole manifold, if M is closed) are "topological invariants." This is the source of many topological terms in Yang-Mills actions.

I.6 The Borel Theorem

Now we can close the loop of ideas by tying together the invariant polynomials with the cohomology of the classifying spaces, through the Borel theorem:

Theorem. Let G be a compact connected Lie group. Then

$$H^*(BG;\Lambda) = (H^*(BT,\Lambda))^W \tag{I.25}$$

where W is the Weyl group, T is a maximal torus, and Λ is any ring in which $|W|$ is invertible.

The superscript W means we take the invariants under the action of the Weyl group.

If $\Lambda = \mathbb{R}$ then we can construct this ring using Ad-invariant polynomials on the Lie algebra:

$$I^*(G) = S^*(\mathfrak{g}^*)^W \tag{I.26}$$

Then

Theorem: If G is a compact Lie group

$$I^*(G) \cong H^*(BG;\mathbb{R}) \tag{I.27}$$

Reference for Borel's theorem:

1. A. Borel, "Topology of Lie Groups and Characteristic Classes," Bull. Amer. Math. Soc. **61**(1955) 397

2. J. Dupont, *Curvature and Characteristic Classes*, Lecture Notes in Mathematics **640** Springer 1978

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Discuss the question: If all the characteristic classes are trivial is the bundle trivial?

PROPER INTEGRAL NORMALIZATIONS OF THE CLASSES!!!!!

****

I.7 Secondary characteristic classes: The Chern-Simons forms

Let us return to the expression:

$$\wp(F(A_1)) - \wp(F(A_0)) = d\left[n\int_0^1 dt \wp(\alpha, F(A(t)), \dots, F(A(t)))\right] \tag{I.28}$$

giving the dependence of the Chern-Weil form on the choice of connection. We will now examine the forms which trivialize the difference in greater detail. For simplicity we will consider $\wp$ to be the trace in some representation of $\mathfrak{g}$.

The *secondary characteristic class* is defined by the form which trivializes the difference:

Definition. The *relative Chern-Simons form* between two connections is the form:

$$CS_{2n-1}(A_1, A_0) := n \int_0^1 dt \mathrm{Tr}\alpha\Big(F_0 + tD_{A_0}\alpha + t^2\alpha^2\Big)^{n-1} \tag{I.29}$$

where $A_1 = A_0 + \alpha$.

Note that

$$dCS_{2n-1}(A_1, A_0) = \mathrm{Tr}F_1^n - \mathrm{Tr}F_0^n \tag{I.30}$$

Examples

$$CS_1(A_1, A_0) = \mathrm{Tr}\alpha \tag{I.31}$$

$$CS_3(A_1, A_0) = \mathrm{Tr}(2\alpha F_0 + \alpha D_{A_0}\alpha + \frac{2}{3}\alpha^3) \tag{I.32}$$

$$\begin{aligned} CS_5(A_1, A_0) = \mathrm{Tr}\Bigg(& 3\alpha F_0^2 + \frac{3}{2}\alpha\{F_0, D_{A_0}\alpha\} + 2F_0\alpha^3 \\ & + \alpha(D_{A_0}\alpha)^2 + \frac{3}{2}\alpha^3 D_{A_0}\alpha + \frac{3}{5}\alpha^5\Bigg) \end{aligned} \tag{I.33}$$

Remarks

1. Often Chern-Simons forms are expressed in terms of a single connection A. We can make sense of this if A is a connection on a trivial bundle. Then we may take $A_0 = 0$, and $A_1 = A$ in all the above formulae. Much of the physics literature is very sloppy about this point, writing $CS(A)$ when A is a nontrivial connection. Such sloppiness can lead to mistakes. Beware!

2. Having said that, there is a way to define the Chern-Simons form $CS(A)$ for a single connection as a well-defined form on the total space of the principal bundle $P \to M$, because the connection form is globally well-defined on the total space of P. For a more elaborate discussion see Appendix A of [22].

3. We can then define the Chern-Simons action by choosing a section $s : M \to P$ away from a set of measure zero and defining $\int_M s^*CS(A)$. This will be ambiguous because we can choose different sections and therefore $\int_M s^*CS(A)$ is only defined modulo $\mathbb{Z}$.

♣Well. If you expontiate, then you CAN define for a single connection, and this is important to the rest of the lectures !!!! ♣

♣Why does Dan use this more complicated argument? ♣

4. Warning: For many applications, the Chern-Simons form is considered only to be defined modulo exact forms. Note that $CS \to CS + d(*)$ leaves the key property (I.20) unchanged.

Exercise
Note that when M is not closed $\int_M \mathrm{Tr} F^n$ depends on connection

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Exercise
Show that:

$$CS_3(A+\alpha+\beta, A) = CS_3(A+\alpha, A) + CS_3(A+\alpha+\beta, A+\alpha) - d\big(\mathrm{Tr}\alpha\beta\big) \tag{I.34}$$

In particular:

$$\frac{d}{dt} CS(A+\alpha(t), A) = 2\mathrm{Tr}\dot{\alpha}(F(A+\alpha)) - d\big(\mathrm{Tr}\alpha\dot{\alpha}\big) \tag{I.35}$$

This is often written as:

$$\delta CS(A) = 2\mathrm{Tr}\delta A \wedge F(A) - d(\mathrm{Tr}(A\delta A)) \tag{I.36}$$

in the case we have a trivial bundle.

In particular, consider a principal G-bundle on a 3-manifold M. What is the variation of the "Chern-Simons action"

$$\int_M \mathrm{Tr} A dA + \frac{2}{3} A^3 \tag{I.37}$$

Exercise
Compute the variation of $CS_3(A_\alpha) - CS_3(A_\beta)$ across patch boundaries.

Exercise
Show that $CS(A_1^g, A_0^g) = CS(A_1, A_0)$

Let us now consider the A-dependence of the Chern-Simons forms themselves.

The following general remark is useful: Supoose we have a family $\mathcal{S}$ of connections $A(s,x)$. It is useful to introduce a bigrading on $\Omega^{p,q}(M \times \mathcal{S})$ so that (p,q) forms are in $\Omega^p(M)\otimes\Omega^q(\mathcal{S})$. We can form a connection $\mathcal{A}(s,x)$ on $\pi^*P \to M\times\mathcal{S}$ such that the covariant derivative is

$$d_M + d_{\mathcal{S}} + A(s,x) \tag{I.38}$$

Note that the connection form only has components of type $(1,0)$. This is an example of what is called the "universal connection." (For more on this, see below.) Let $\mathcal{F}$ the the curvature of this universal connection. It has components of type $(2,0)$ and $(1,1)$. Specifically

$$\mathcal{F} = F(\mathcal{A}) = F(A) + d_{\mathcal{S}}A(s,x) \tag{I.39}$$

Now $\mathrm{Tr}\mathcal{F}^n$ is closed on the total space,

$$(d_M + d_{\mathcal{S}})\mathrm{Tr}\mathcal{F}^n = 0, \tag{I.40}$$

so if we decompose into (p,q) types:

$$\mathrm{Tr}\mathcal{F}^n = \omega^{n,0} + \omega^{n-1,1} + \cdots + \omega^{0,n} \tag{I.41}$$

and set $d = d_M$, $\delta = d_{\mathcal{S}}$ then we get the "descent equations:"

$$\begin{aligned} d\omega^{n,0} &= 0 \\ d\omega^{n-1,1} + \delta\omega^{n,0} &= 0 \\ d\omega^{n-2,2} + \delta\omega^{n-1,1} &= 0 \\ \vdots \qquad & \quad \vdots \end{aligned} \tag{I.42}$$

Figure 37: A triangle of connections in $\mathcal{A}(P)$.

Let us now consider a simplex $\triangle$ of connections:

$$A_0 + t_1\alpha_1 + t_2\alpha_2 \tag{I.43}$$

with $0 \le t_2 \le t_1 \le 1$, as in 37.

Let $A_1 = A_0 + \alpha_1, A_2 = A_1 + \alpha_2$. We form the universal connection on $\pi^*(P) \to M\times\triangle$ and compute the curvature:

$$F(\mathcal{A}) = F(A_0) + t_1 D_{A_0}\alpha_1 + t_2 D_{A_0}\alpha_2 + (t_1\alpha_1 + t_2\alpha_2)^2 + dt_1\alpha_1 + dt_2\alpha_2 \tag{I.44}$$

Using the descent equations we can write

$$\int_{\triangle} \delta\omega^{n-1,1} = -d\int_{\triangle} \omega^{n-2,2} = -d\int_{\triangle} \mathrm{Tr}F(\mathcal{A})^n \tag{I.45}$$

On the other hand, we can apply Stokes' theorem to the triangle to write

$$\int_{\triangle} \delta\omega^{n-1,1} = \int_{\partial\triangle} \omega^{n-1,1} = \int_{\partial\triangle} \mathrm{Tr}F(\mathcal{A})^n = CS_{2n-1}(A_1,A_0)+CS_{2n-1}(A_2,A_1)+CS_{2n-1}(A_0,A_2) \tag{I.46}$$

Putting these together we obtain the result:

$$\begin{aligned} &CS_{2n-1}(A_0,A_1) + CS_{2n-1}(A_1,A_2) + CS_{2n-1}(A_2,A_0) \\ &= d_M\left(\int_{\Delta} \mathrm{SymTr}(dt_1\alpha_1, dt_2\alpha_2, (F(A_0) + t_1 D_{A_0}\alpha_1 + t_2 D_{A_0}\alpha_2 + (t_1\alpha_1 + t_2\alpha_2)^2)^{n-2}\right) \end{aligned} \tag{I.47}$$

So, for example:

$$CS_3(A_1,A_0) + CS_3(A_2,A_1) + CS_3(A_0,A_2) = d\big(\mathrm{Tr}\alpha_1\alpha_2\big) \tag{I.48}$$

with $A_1 = A_0 + \alpha_1, A_2 = A_1 + \alpha_2$.

Similarly:

$$\begin{aligned} &CS_5(A_1,A_0) + CS_5(A_2,A_1) + CS_5(A_0,A_2) = \\ &3d\left[\int_{\triangle} \mathrm{Tr}\left(F + t_1 D\alpha_1 + t_2 D\alpha_2 + (t_1\alpha_1 + t_2\alpha_2)^2\right)(\alpha_1\alpha_2 - \alpha_2\alpha_1)\right] \end{aligned} \tag{I.49}$$

Figure 38: Triangle of connections in $\mathcal{A}(P)$ used to compute the gauge-dependence of the Chern-Simons form $CS(A,0)$.

As an application suppose A is a connection on a trivializable bundle so that there is a basepoint connection $A_0 = 0$. Then it makes sense to define $CS(A) := CS(A,0)$. Although $dCS(A)$ is gauge invariant, $CS(A^g)$ will not be gauge invariant.

We can take $A_1 = g^{-1}dg$ and $A_2 = A^g$, for some globally defined connection 1-form A, in the above formula. Note that this means we have a linear path $A(t) = g^{-1}dg + tg^{-1}Ag$, from A_1 to A_2. Since, quite generally $CS(A_1^g, A_0^g) = CS(A_1,A_0)$ for any pair of connections, we have

$$CS_{2n-1}(A^g, g^{-1}dg) = CS_{2n-1}(A,0) \tag{I.50}$$

Therefore, our identity becomes:

$$CS_{2n-1}(A^g) - CS_{2n-1}(A) = CS_{2n-1}(g^{-1}dg) - d_M(\alpha_{2n}) \tag{I.51}$$

with

$$\alpha_{2n} = \int_{\triangle} \mathrm{SymTr}\left[dt_1\alpha_1, dt_2\alpha_2, \left((t_1^2 - t_1)\alpha_1^2 + (t_2^2 - t_2)\alpha_2^2 + (t_1t_2 - t_2)[\alpha_1, \alpha_2]_+ + t_2 g^{-1}F(A)g\right)^{n-2}\right] \tag{I.52}$$

where $\alpha_1 = g^{-1}dg$ and $\alpha_2 = g^{-1}Ag$ and

$$CS_{2n-1}(g^{-1}dg) = (-1)^{n-1}\frac{n!(n-1)!}{(2n-1)!}\mathrm{Tr}(g^{-1}dg)^{2n-1} \tag{I.53}$$

This formula is useful in discussing physical actions with Chern-Simons terms and topics such as the descent formalism for anomalies, and "anomaly inflow." See below.

One can also derive this using the formalism of the "Cartan homotopy operator." See

1. Nakahara, sec. 11.5.3
2. B. Zumino, Les Houches lectures
3. Alvarez-Gaumé and Ginsparg, Ann. Phys.

Exercise

Show that if A is a flat connection on a bundle P, not necessarily trivializable, then

$$CS(A^g, A) = (-1)^{n-1}\frac{n!(n-1)!}{(2n-1)!}\mathrm{Tr}(g^{-1}D_Ag)^{2n-1} \tag{I.54}$$

where $g^{-1}D_Ag = g^{-1}(dg + Ag - gA) = A^g - A$.

Exercise *Generalizing Polyakov-Wiegmann*

Put $A = 0^h$ in (I.51) and derive the Polyakov-Wiegmann formula and its generalizations.

Exercise

Let $E = M \times \mathbb{C}^N$ be a trivial vector bundle and let $\mathcal{P}$ and $\mathcal{Q}$ be orthogonal projection operators to possibly nontrivial subbundles.

Compute $CS(\mathcal{P}d \oplus \mathcal{Q}d, d)$ where d is the trivial connection on E.

I.7.1 Other paths to Chern-Simons

A more conceptual description of Chern-Simons forms is the following. Consider an arbitrary path γ of connections, $A(t)$. Using the pullback bundle from $M \times I \to M$ we get a bundle $\pi^*(P) \to M \times I$, and the family of connections defines a connection:

$$d_M + dt\frac{\partial}{\partial t} + A(t) \tag{I.55}$$

on $\pi^*(P)$. It is useful to introduce a bigrading on $\Omega^{p,q}(M \times I)$ so that (p,q) forms are in $\Omega^p(M) \otimes \Omega^q(I)$. Locally, our connection form $A(t)$ is of type $(1,0)$. Let us call the connection corresponding to the covariant derivative (I.55) $\mathcal{A}$. Because of the t-dependence in $A(t)$, the curvature $F(\mathcal{A})$ has components of type $(2,0)$ as well as $(1,1)$:

$$F(\mathcal{A}) = F(A(t)) + dt \wedge \frac{\partial A}{\partial t} \tag{I.56}$$

(note that $\frac{\partial A}{\partial t} \in \Omega^1(M; \mathrm{ad}P)$.) Now, we can define a more general Chern-Simons form:

$$CS_\gamma := \int_I \mathrm{Tr}F(\mathcal{A})^n \tag{I.57}$$

for any path γ. If $\gamma(0) = A_0$ and $\gamma(1) = A_1$ then this satisfies

$$d_M CS_\gamma = \mathrm{Tr}F(A_1)^n - \mathrm{Tr}F(A_0)^n \tag{I.58}$$

The Chern-Simons forms above were defined by choosing the linear path between A_0 and A_1.

An example where this is important arises when one considers a path of gauge transformations $g(t)$, $0 \le t \le 1$ defining a path from A to A^g, with $g = g(1)$. Note that this path differs from the linear path.

Suppose we have two paths $A_0 + \alpha_1(t)$ and $A_0 + \alpha_2(t)$. Since the space of connections is an affine space we can find a homotopy $\alpha(s,t)$ between these paths.

Thus: $\alpha(0,t) = \alpha_1(t), \alpha(1,t) = \alpha_2(t)$, and $\alpha_1(1) = \alpha_2(1) = \alpha$. Then

$$CS_{\gamma_1} - CS_{\gamma_2} = d\int_{I \times I} \mathrm{Tr}F(\mathcal{A})^n \tag{I.59}$$

where the integral over $I \times I$ is over a surface in the infinite-dimensional vector space $\Gamma(\Omega(M; adP))$ spanning the two paths, oriented appropriately.

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STILL NEEDED:

1. VALUE OF $\int_{M_3} CS(A)$ AS ELEMENT OF $\mathbb{R}/2\pi\mathbb{Z}$.

2. CAREFUL NORMALIZATIONS OF INVARIANT FORM ON LIE ALGEBRAS.

3. CHERN-SIMONS AS QUADRATIC REFINEMENT: GOES IN DIFFL COHO SECTION.

J. Some Material On Lie Groups And Lie Algebras

J.1 The Structure Of The General Compact Lie Group

If G is a Lie group then the connected component of the identity, denoted G_1 is a normal subgroup. [45] Therefore G/G_1 is a group, but this is just the group of connected components of G so we have

$$1 \to G_1 \to G \to \pi_0(G) \to 1 \tag{J.1}$$

In general G is not a direct product: G is not isomorphic, as a group, to $G_1 \times \pi_0(G)$. As a simple example: $O(2) = SO(2) \rtimes \mathbb{Z}_2$. As a manifold it is a product of two circles, but the $\mathbb{Z}_2$ factor acts by the nontrivial automorphism $R(\theta) \to R(-\theta)$. ♣Give example where it does not split ♣

Now $\pi_0(G)$ can be any finite group, whatsoever (after all, we could take G to be that finite group!).

Now let us focus on G_1. It is a connected compact Lie group. The universal cover $\widetilde{G_1}$ is also a connected, Lie group, although it might be noncompact. One can show that π_1 of any Lie group whatsoever is Abelian [46] and therefore, assuming π_1 is finitely generated, it is, noncanonically, a product of a finite group and a lattice $\mathbb{Z}^d$. The lattice must act on $\mathbb{R}^d$ to give $U(1)^d$ and "so" (we are not giving a completely rigorous argument here, but the conclusion is true) G_1 is of the form

$$G_1 \cong \left(G^{ss} \times U(1)^r\right)/Z \tag{J.2}$$

where G^{ss} is compact, connected, semisimple, and simply connected Lie group and $Z \subset G^{ss} \times U(1)^r$ is a subgroup of the center, and is a finite Abelian group. In turn G^{ss} has the form

$$G^{ss} = \prod_i G_i^{ss} \tag{J.3}$$

where G_i^{ss} are compact, connected, simple, and simply connected Lie groups. The list of these groups, together with their centers and outer automorphism groups is given in

[45] *Proof*: If $g \in G_1$ let $g(t)$ be a path connecting g to 1_G. If $h \in G$ then $h^{-1}g(t)h$ connects $h^{-1}gh$ to 1_G.

[46] *Proof*: Given two paths $g_1(t), g_2(t)$ consider the continuous function $F(t,s) = g_1(t)g_2(s)$.

$\mathfrak{g}$	$\tilde{G}(\mathfrak{g})$	$Z(\mathfrak{g})$
A_r	$SU(r+1)$	$\mathbb{Z}_{r+1}$
B_r	$Spin(2r+1)$	$\mathbb{Z}_2$
C_r	$USp(2r)$	$\mathbb{Z}_2$
D_{2s+1}	$Spin(4s+2)$	$\mathbb{Z}_4$
D_{2s}	$Spin(4s)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
E_6	E_6	$\mathbb{Z}_3$
E_7	E_7	$\mathbb{Z}_2$
E_8	E_8	1
F_4	F_4	1
G_2	G_2	1

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ALSO GIVE DYNKIN DIAGRAMS AND AUTOMORPHISM GROUP OF THOSE AND EXPLAIN ITS RELATION TO THE OUTER AUTOMORPHISMS OF G.

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WHAT CAN WE SAY ABOUT THE POSSIBLE EXTENSIONS?

If our group has positive dimension then the existence of an extension automatically implies a canonical homomorphism $\bar{\omega} : \pi_0(G) \to \mathrm{Out}(G_1)$.

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J.2 Dual Coxeter Number

The dual Coxeter number of the simple Lie algebras is given by [47]

[47]Warning: For nonsimply laced algebras there is also a Coxeter number, and it differs from the dual Coxeter number.

$\mathfrak{g}$	$\tilde{G}(\mathfrak{g})$	$h(\mathfrak{g})$
A_r	$SU(r+1)$	$r+1$
B_r	$Spin(2r+1)$	$2r-1$
C_r	$USp(2r)$	$r+1$
D_r	$Spin(2r)$	$2r-2$
E_6	E_6	12
E_7	E_7	18
E_8	E_8	30
F_4	F_4	9
G_2	G_2	4

K. Categories: Groups and Groupoids

A rather abstract notion, which nevertheless has found recent application in string theory and conformal field theory is the language of categories. Many physicists object to the high level of abstraction entailed in the category language. Some mathematicians even refer to the subject as "abstract nonsense." (Others take it very seriously.) However, it seems to be of increasing utility in the further formal development of string theory and supersymmetric gauge theory. It is also essential for reading any of the literature on topological field theory.

We briefly illustrate some of that language here. Our main point here is to introduce a different viewpoint on what groups are that leads to a significant generalization: groupoids. Moreover, this point of view also provides some very interesting insight into the meaning of group cohomology. Related constructions have been popular in condensed matter physics and topological field theory.

Definition A *category* $\mathcal{C}$ consists of

a.) A set $Ob(\mathcal{C})$ of "objects"

b.) A collection $Mor(\mathcal{C})$ of sets $\hom(X,Y)$, defined for any two objects $X,Y \in Ob(\mathcal{C})$. The elements of $\hom(X,Y)$ are called the "morphisms from X to Y." They are often denoted as arrows:

$$X \quad \stackrel{\phi}{\to} \quad Y \tag{K.1}$$

c.) A composition law:

$$\hom(X,Y) \times \hom(Y,Z) \to \hom(X,Z) \tag{K.2}$$

$$(\psi_1,\psi_2) \mapsto \psi_2 \circ \psi_1 \tag{K.3}$$

Such that

1. A morphism ϕ uniquely determines its source X and target Y. That is, $\hom(X,Y)$ are disjoint for distinct ordered pairs (X,Y).

2. $\forall X \in Ob(\mathcal{C})$ there is a distinguished morphism, denoted $1_X \in \hom(X,X)$ or $\mathrm{Id}_X \in \hom(X,X)$, which satisfies:

$$1_X \circ \phi = \phi \qquad \psi \circ 1_X = \psi \tag{K.4}$$

for all morphisms $\phi \in \hom(Y,X)$ and $\psi \in \hom(X,Y)$ for all $Y \in Ob(\mathcal{C})$. [48]

3. Composition of morphisms is associative:

$$(\psi_1 \circ \psi_2) \circ \psi_3 = \psi_1 \circ (\psi_2 \circ \psi_3) \tag{K.5}$$

An alternative definition one sometimes finds is that a category is defined by two sets $\mathcal{C}_0$ (the objects) and $\mathcal{C}_1$ (the morphisms) with two maps $p_0 : \mathcal{C}_1 \to \mathcal{C}_0$ and $p_1 : \mathcal{C}_1 \to \mathcal{C}_0$. The map $p_0(f) = x_1 \in \mathcal{C}_0$ is the *range* map and $p_1(f) = x_0 \in \mathcal{C}_0$ is the *domain* map. In this alternative definition a category is then defined by a composition law on the set of *composable morphisms*

$$\mathcal{C}_2 = \{(f,g) \in \mathcal{C}_1 \times \mathcal{C}_1 | p_0(f) = p_1(g)\} \tag{K.6}$$

which is sometimes denoted $\mathcal{C}_1{}_{p_1} \times_{p_0} \mathcal{C}_1$ and called the *fiber product*. The composition law takes $\mathcal{C}_2 \to \mathcal{C}_1$ and may be pictured as the composition of arrows. If $f : x_0 \to x_1$ and $g : x_1 \to x_2$ then the composed arrow will be denoted $g \circ f : x_0 \to x_2$. The composition law satisfies the axioms

1. For all $x \in X_0$ there is an identity morphism in X_1, denoted 1_x, or Id_x, such that $1_x f = f$ and $g 1_x = g$ for all suitably composable morphisms f, g.
2. The composition law is associative. If f, g, h are 3-composable morphisms then $(hg)f = h(gf)$.

Remarks:

1. When defining composition of arrows one needs to make an important notational decision. If $f : x_0 \to x_1$ and $g : x_1 \to x_2$ then the composed arrow is an arrow $x_0 \to x_2$. We will write $g \circ f$ when we want to think of f, g as functions and fg when we think of them as arrows.

♣Is this dual notation really a good idea?? ♣

2. It is possible to endow the data X_0, X_1 and p_0, p_1 with additional structures, such as topologies, and demand that p_0, p_1 have continuity or other properties.
3. A morphism $\phi \in \hom(X,Y)$ is said to be *invertible* if there is a morphism $\psi \in \hom(Y,X)$ such that $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$. If X and Y are objects with an invertible morphism between them then they are called *isomorphic objects*. One key reason to use the language of categories is that objects can have nontrivial automorphisms. That is, $\hom(X,X)$ can have invertible elements other than just 1_X in it. When this is true then it is tricky to speak of "equality" of objects, and the language of categories becomes very helpful. As a concrete example you might ponder the following question: "Are all real vector spaces of dimension *n the same*?"

[48] As an exercise, show that these conditions uniquely determine the morphism 1_X.

Here are some simple examples of categories:

1. **SET**: The category of sets and maps of sets. [49]
2. **TOP**: The category of topological spaces and continuous maps.
3. **TOPH**: The category of topological spaces and homotopy classes of continuous maps.
4. **MANIFOLD**: The category of manifolds and suitable maps. We could take topological manifolds and continuous maps of manifolds. Or we could take smooth manifolds and smooth maps as morphisms. The two choices lead to two (very different!) categories.
5. **BORD**(n): The bordism category of n-dimensional manifolds. Roughly speaking, the objects are n-dimensional manifolds without boundary and the morphisms are bordisms. A bordism Y from an n-manifold M_1 to and n-manifold M_2 is an $(n+1)$-dimensional manifold with a decomposition of its boundary $\partial Y = (\partial Y)_{in} \amalg (\partial Y)_{out}$ together with diffeomorphisms $\theta_1 : (\partial Y)_{in} \to M_1$ and $\theta_2 : (\partial Y)_{out} \to M_2$.
6. **GROUP**: the category of groups and homomorphisms of groups. Note that here if we took our morphisms to be isomorphisms instead of homomorphisms then we would get a very different category. All the pairs of objects in the category with nontrivial morphism spaces between them would be pairs of isomorphic groups.
7. **AB**: The (sub) category of abelian groups.
8. Fix a group G and let **G-SET** be the category of G-sets, that is, sets X with a G-action. For simplicity let us just write the G-action $\Phi(g,x)$ as $g \cdot x$ for x a point in a G-set X. We take the morphisms $f : X_1 \to X_2$ to satisfy satisfy $f(g \cdot x_1) = g \cdot f(x_1)$.
9. $\mathbf{VECT}_\kappa$: The category of finite-dimensional vector spaces over a field κ with morphisms the linear transformations.

One use of categories is that they provide a language for describing precisely notions of "similar structures" in different mathematical contexts. When discussed in this way it is important to introduce the notion of "functors" and "natural transformations" to speak of interesting relationships between categories.

In order to state a relation between categories one needs a "map of categories." This is what is known as a functor:

Definition A *functor* between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a pair of maps $F_{\rm obj} : Obj(\mathcal{C}_1) \to Obj(\mathcal{C}_2)$ and $F_{\rm mor} : Mor(\mathcal{C}_1) \to Mor(\mathcal{C}_2)$ so that if

$$x \xrightarrow{f} y \ \in \hom(x,y) \tag{K.7}$$

[49]We take an appropriate collection of sets and maps to avoid the annoying paradoxes of set theory.

then

$$F_{\mathrm{obj}}(x) \xrightarrow{F_{\mathrm{mor}}(f)} F_{\mathrm{obj}}(y) \ \in \hom(F_{\mathrm{obj}}(x), F_{\mathrm{obj}}(y)) \tag{K.8}$$

and moreover we require that F_{mor} should be compatible with composition of morphisms: There are two ways this can happen. If f_1, f_2 are composable morphisms then we say F is a *covariant functor* if

$$F_{\mathrm{mor}}(f_1 \circ f_2) = F_{\mathrm{mor}}(f_1) \circ F_{\mathrm{mor}}(f_2) \tag{K.9}$$

and we say that F is a *contravariant functor* if

$$F_{\mathrm{mor}}(f_1 \circ f_2) = F_{\mathrm{mor}}(f_2) \circ F_{\mathrm{mor}}(f_1) \tag{K.10}$$

In both cases we also require [50]

$$F_{\mathrm{mor}}(\mathrm{Id}_X) = \mathrm{Id}_{F(X)} \tag{K.11}$$

We usually drop the subscript on F since it is clear what is meant from context.

Exercise

Using the alternative definition of a category in terms of data $p_{0,1} : X_1 \to X_0$ define the notion of a functor writing out the relevant commutative diagrams.

Exercise *Opposite Category*

If $\mathcal{C}$ is a category then the *opposite category* $\mathcal{C}^{\mathrm{opp}}$ is defined by just reversing all arrows. More formally: The set of objects is the same and

$$\hom_{\mathcal{C}^{\mathrm{opp}}}(X,Y) := \hom_{\mathcal{C}}(Y,X) \tag{K.12}$$

so for every morphism $f \in \hom_{\mathcal{C}}(Y,X)$ we associate $f^{\mathrm{opp}} \in \hom_{\mathcal{C}^{\mathrm{opp}}}(X,Y)$ such that

$$f_1 \circ_{\mathcal{C}^{\mathrm{opp}}} f_2 = (f_2 \circ_{\mathcal{C}} f_1)^{\mathrm{opp}} \tag{K.13}$$

a.) Show that if $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor then one can define in a natural way a covariant functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathcal{D}$.

b.) Show that if $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor then we can naturally define another covariant functor $F^{\mathrm{opp}} : \mathcal{C}^{\mathrm{opp}} \to \mathcal{D}^{\mathrm{opp}}$

Example 1: Every category has a canonical functor to itself, called the identity functor $Id_{\mathcal{C}}$.

[50] Although we do have $F_{\mathrm{mor}}(\mathrm{Id}_X) \circ F_{\mathrm{mor}}(f) = F_{\mathrm{mor}}(f)$ for all $f \in \hom(Y,X)$ and $F_{\mathrm{mor}}(f) \circ F_{\mathrm{mor}}(\mathrm{Id}_X) = F_{\mathrm{mor}}(f)$ for all $f \in \hom(X,Y)$ this is not the same as the statement that $F_{\mathrm{mor}}(\mathrm{Id}_X) \circ \phi = \phi$ for all $\phi \in \hom(F(Y), F(X))$, so we need to impose this extra axiom.

Example 2: There is an obvious functor, the *forgetful functor* from **GROUP** to **SET**. This idea extends to many other situations where we "forget" some mathematical structure and map to a category of more primitive objects.

Example 3: Since **AB** is a subcategory of **GROUP** there is an obvious functor $\mathcal{F}$: **AB** $\to$ **GROUP**.

Example 4: In an exercise below you are asked to show that the abelianization of a group defines a functor $\mathcal{G}$: **GROUP** $\to$ **AB**.

Example 5: Fix a group G. Then in the notes above we have on several occasions used the functor

$$F_G : \mathbf{SET} \to \mathbf{GROUP} \tag{K.14}$$

by observing that if X is a set, then $F_G(X) = Maps[X \to G]$ is a group. Check this is a contravariant functor: If $f : X_1 \to X_2$ is a map of sets then

$$F_G(X_1) \xleftarrow{F_G(f)} F_G(X_2) \tag{K.15}$$

The map $F_G(f)$ is usually denoted f^* and is known as the *pull-back*. To be quite explicit: If Ψ is a map of $X_2 \to G$ then $f^*(\Psi) := \Psi \circ f$ is a map $X_1 \to G$.

This functor is used in the construction of certain *nonlinear sigma models* which are quantum field theories where the target space is a group G. The viewpoint that we are studying the representation theory of an infinite-dimensional group of maps to G has been extremely successful in a particular case of the *Wess-Zumino-Witten* model, a certain two dimensional quantum field theory that enjoys conformal invariance (and more).

Example 6: Now let us return to the category **G-SET**. Now fix any set Y. Then in the notes above we have on several occasions used the functor

$$F_{G,Y} : \mathbf{G\text{-}SET} \to \mathbf{G\text{-}SET} \tag{K.16}$$

by observing that if X is a G-set, then $F_Y(X) = Maps[X \to Y]$ is also a G-set. To check this is a contravariant functor we write:

$$\begin{aligned}
[g \cdot (f^*\Psi)](x_1) &= (f^*\Psi)(g^{-1} \cdot x_1) \\
&= \Psi(f(g^{-1} \cdot x_1)) \\
&= \Psi(g^{-1} \cdot (f(x_1))) \\
&= (g \cdot \Psi)(f(x_1)) \\
&= (f^*(g \cdot \Psi))(x_1)
\end{aligned} \tag{K.17}$$

and hence $\Psi \to g \cdot \Psi$ is a morphism of G-sets.

This functor is ubiquitous in quantum field theory: If a spacetime enjoys some symmetry (for example rotational or Poincaré symmetry) then the same group will act on the space of fields defined on that spacetime.

Example 7: Fix a nonnegative integer n and a group G. Then the group cohomology we discussed above (take the trivial twisting $\omega_g = \mathrm{Id}_A$ for all g) defines a covariant functor

$$H^n(G, \bullet) : \mathbf{AB} \to \mathbf{AB} \tag{K.18}$$

To check this is really a functor we need to observe the following: If $\varphi : A_1 \to A_2$ is a homomorphism of Abelian groups then there is an induced homomorphim, usually denoted

$$\varphi_* : H^n(G, A_1) \to H^n(G, A_2) \tag{K.19}$$

You have to check that $\mathrm{Id}_* = \mathrm{Id}$ and

$$(\varphi_1 \circ \varphi_2)_* = (\varphi_1)_* \circ (\varphi_2)_* \tag{K.20}$$

Strictly speaking we should denote φ_* by $H^n(G, \varphi)$, but this is too fastidious for the present author.

Example 8: Fix a nonnegative integer n and any group A. Then the group cohomology we discussed above (take the trivial twisting $\omega_g = \mathrm{Id}_A$ for all g) defines a contravariant functor

$$H^n(\bullet, A) : \mathbf{GROUP} \to \mathbf{AB} \tag{K.21}$$

To check this is really a functor we need to observe the following: If $\varphi : G_1 \to G_2$ is a homomorphism of Abelian groups then there is an induced homomorphim, usually denoted φ^*

$$\varphi^* : H^n(G_2, A) \to H^n(G_1, A) \tag{K.22}$$

Example 9: *Topological Field Theory.* The very definition of topological field theory is that it is a functor from a bordism category of manifolds to the category of vector spaces and linear transformations. See chapter 5 for some discussion. For much more about this one can consult a number of papers. Two online resources are:

http://www.physics.rutgers.edu/∼gmoore/695Fall2015/TopologicalFieldTheory.pdf (very similar to the present notes).

https://www.ma.utexas.edu/users/dafr/bordism.pdf

Note that in example 2 there is no obvious functor going the reverse direction. When there are functors both ways between two categories we might ask whether they might be, in some sense, "the same." But saying precisely what is meant by "the same" requires some care.

Definition If $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories and $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_1 \to \mathcal{C}_2$ are two functors then a *natural transformation* $\tau : F_1 \to F_2$ is a rule which, for every $X \in Obj(\mathcal{C}_1)$ assigns an arrow $\tau_X : F_1(X) \to F_2(X)$ so that, for all $X, Y \in Obj(\mathcal{C}_1)$ and all $f \in \mathrm{hom}(X, Y)$,

$$\tau_Y \circ F_1(f) = F_2(f) \circ \tau_X \tag{K.23}$$

Or, in terms of diagrams.

$$\begin{tikzcd} F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\tau_X"] & F_1(Y) \arrow[d, "\tau_Y"] \\ F_2(X) \arrow[r, "F_2(f)"] & F_2(Y) \end{tikzcd} \tag{K.24}$$

Example 1: *The evaluation map.* Here is another tautological construction which nevertheless can be useful. Let S be any set and define a functor

$$F_S : \mathbf{SET} \to \mathbf{SET} \tag{K.25}$$

by saying that on objects we have

$$F_S(X) := Map[S \to X] \times S \tag{K.26}$$

and if $\varphi : X_1 \to X_2$ is a map of sets then

$$F_S(\varphi) : Map[S \to X_1] \times S \to Map[S \to X_2] \times S \tag{K.27}$$

is defined by $F_S(\varphi) : (f, s) \mapsto (\varphi \circ f, s)$. Then we claim there is a natural transformation to the identity functor. For every set X we have

$$\tau_X : F_S(X) = Map[S \to X] \times S \to \mathrm{Id}(X) = X \tag{K.28}$$

It is defined by $\tau_X(f, s) := f(s)$. This is known as the "evaluation map." Then we need to check

$$\begin{tikzcd} F_S(X) \arrow[r, "\tau_X"] \arrow[d, "F_S(\varphi)"] & X \arrow[d, "\varphi"] \\ F_S(Y) \arrow[r, "\tau_Y"] & Y \end{tikzcd} \tag{K.29}$$

commutes. If you work it out, it is just a tautology.

Example 2: *The determinant.* [51] Let **COMMRING** be the category of commutative rings with morphisms the ring morphisms. (So, $\varphi : R_1 \to R_2$ is a homomorphism of Abelian groups and moreover $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$.) Let us consider two functors

$$\mathbf{COMMRING} \to \mathbf{GROUP} \tag{K.30}$$

The first functor F_1 maps a ring R to the multiplicative group $U(R)$ of multiplicatively invertible elements. This is often called the group of units in R. If φ is a morphism of rings and $r \in U(R_1)$ then $\varphi(r) \in U(R_2)$ and the map $\varphi_* : U(R_1) \to U(R_2)$ defined by

$$\varphi_* : r \mapsto \varphi(r) \tag{K.31}$$

is a group homomorphism. So F_1 is a functor. The second functor F_2 maps a ring R to the matrix group $GL(n, R)$ of $n \times n$ matrices such that there exists an inverse matrix

[51] This example uses some terms from linear algebra which can be found in the "User's Manual," Chapter 2 below.

with values in R. Again, if $\varphi : R_1 \to R_2$ is a morphism then applying φ to each matrix element defines a group homomorphism $\varphi_* : GL(n, R_1) \to GL(n, R_2)$. Now consider the determinant of a matrix $g \in GL(n, R)$. The usual formula

$$\det(g) := \sum_{\sigma \in S_n} \epsilon(\sigma) g_{1,\sigma(1)} \cdot g_{2,\sigma(2)} \cdots g_{n,\sigma(n)} \tag{K.32}$$

makes perfect sense for $g \in GL(n, R)$. Moreover,

$$\det(g_1 g_2) = \det(g_1) \det(g_2) \tag{K.33}$$

Now we claim that the determinant defines a natural transformation $\tau : F_1 \to F_2$. For each object $R \in Ob(\mathbf{COMMRING})$ we assign the morphism

$$\tau_R : GL(n, R) \to U(R) \tag{K.34}$$

defined by $\tau_R(g) := \det(g)$. Thanks to (K.33) this is indeed a morphism in the category **GROUP**, that is, it is a group homomorphism. Moreover, it satisfies the required commutative diagram because if $\varphi : R_1 \to R_2$ is a morphism of rings then

$$\varphi_*(\det(g)) = \det(\varphi_*(g)). \tag{K.35}$$

Example 3: *Natural transformations in cohomology theory.* Cohomology groups provide natural examples of functors, as we have stressed above. There are a number of interesting natural transformations between these different cohomology-group functors.

♣Can we explain an elementary example with group cohomology as developed so far??? ♣

Definition Two categories are said to be *equivalent* if there are functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ together with isomorphisms (via natural transformations) $FG \cong Id_{\mathcal{C}_2}$ and $GF \cong Id_{\mathcal{C}_1}$. (Note that FG and $Id_{\mathcal{C}_2}$ are both objects in the category of functors FUNCT$(\mathcal{C}_2, \mathcal{C}_2)$ so it makes sense to say that they are isomorphic.)

♣Should explain example showing category of finite-dimensional vector spaces over a field is equivalent to the catetgory of nonnegative integers. ♣

Many important theorems in mathematics can be given an elegant and concise formulation by saying that two seemingly different categories are in fact equivalent. Here is a (very selective) list: [52]

Example 1: Consider the category with one object for each nonnegative integer n and the morphism space $GL(n, \kappa)$ of invertible $n \times n$ matrices over the field κ. These categories are equivalent. That is one way of saying that the only invariant of a finite-dimensional vector space is its dimension.

Example 2: The basic relation between Lie groups and Lie algebras the statement that the functor which takes a Lie group G to its tangent space at the identity, $T_1 G$ is an equivalence of the category of connected and simply-connected Lie groups with the category of finite-dimensional Lie algebras. One of the nontrivial theorems in the theory is the existence of a functor from the category of finite-dimensional Lie algebras to the category of connected

[52] I thank G. Segal for a nice discussion that helped prepare this list.

and simply-connected Lie groups. Intuitively, it is given by exponentiating the elements of the Lie algebra.

Example 3: Covering space theory is about an equivalence of categories. On the one hand we have the category of coverings of a pointed space (X, x_0) and on the other hand the category of topological spaces with an action of the group $\pi_1(X, x_0)$. Closely related to this, Galois theory can be viewed as an equivalence of categories.

Example 4: The category of unital commutative C^*-algebras is equivalent to the category of compact Hausdorff topological spaces. This is known as Gelfand's theorem.

Example 5: Similar to the previous example, an important point in algebraic geometry is that there is an equivalence of categories of commutative algebras over a field κ (with no nilpotent elements) and the category of affine algebraic varieties.

Example 6: Pontryagin duality is a nontrivial self-equivalence of the category of locally compact abelian groups (and continuous homomorphisms) with itself.

Example 7: A generalization of Pontryagin duality is Tannaka-Krein duality between the category of compact groups and a certain category of linear tensor categories. (The idea is that, given an abstract tensor category satisfying certain conditions one can construct a group, and if that tensor category is the category of representations of a compact group, one recovers that group.)

Example 8: The Riemann-Hilbert correspondence can be viewed as an equivalence of categories of flat connections (a.k.a. linear differential equations, a.k.a. D-modules) with their monodromy representations.

♣This needs a lot more explanation. ♣

In physics, the statement of "dualities" between different physical theories can sometimes be formulated precisely as an equivalence of categories. One important example of this is mirror symmetry, which asserts an equivalence of (A_∞)-) categories of the derived category of holomorphic bundles on X and the Fukaya category of Lagrangians on $X^\vee$. But more generally, nontrivial duality symmetries in string theory and field theory have a strong flavor of an equivalence of categories.

Exercise *Playing with natural transformations*

a.) Given two categories $\mathcal{C}_1, \mathcal{C}_2$ show that the natural transformations allow one to define a category $\mathrm{FUNCT}(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ and whose morphisms are natural transformations. For this reason natural transformations are often called "morphisms of functors."

b.) Write out the meaning of a natural transformation of the identity functor $Id_{\mathcal{C}}$ to itself. Show that $End(Id_{\mathcal{C}})$, the set of all natural transformations of the identity functor to itself is a monoid.

Exercise *Freyd's theorem*

A "practical" way to tell if two categories are equivalent is the following:

By definition, a *fully faithful functor* is a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ where $F_{\rm mor}$ is a bijection on all the hom-sets. That is, for all $X, Y \in Obj(\mathcal{C}_1)$ the map

$$F_{\rm mor} : \hom(X, Y) \to \hom(F_{\rm obj}(X), F_{\rm obj}(Y)) \tag{K.36}$$

is a bijection.

Show that $\mathcal{C}_1$ is equivalent to $\mathcal{C}_2$ iff there is a fully faithful functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ so that any object $\alpha \in Obj(\mathcal{C}_2)$ is isomorphic to an object of the form $F(X)$ for some $X \in Obj(\mathcal{C}_1)$.

Exercise

As we noted above, there is a functor $\mathbf{AB} \to \mathbf{GROUP}$ just given by inclusion.

a.) Show that the abelianization map $G \to G/[G, G]$ defines a functor $\mathbf{GROUP} \to \mathbf{AB}$.

b.) Show that the existence of nontrivial perfect groups, such as A_5, implies that this functor cannot be an equivalence of categories.

In addition to the very abstract view of categories we have just sketched, very concrete objects, like groups, manifolds, and orbifolds can profitably be viewed as categories.

One may always picture a category with the objects constituting points and the morphisms directed arrows between the points as shown in Figure 39.

As an extreme example of this let us consider a category with only *one object*, but we allow the possibility that there are several morphisms. For such a category let us look carefully at the structure on morphisms $f \in Mor(\mathcal{C})$. We know that there is a binary operation, with an identity 1 which is associative.

But this is just the definition of a monoid!

If we have in addition inverses then we get a group. Hence:

Definition A *group* is a category with one object, all of whose morphisms are invertible.

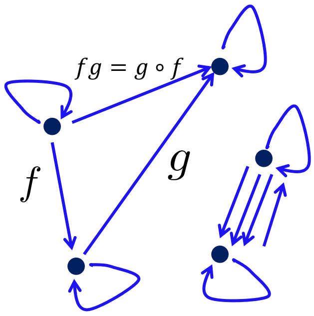

Figure 39: Pictorial illustration of a category. The objects are the black dots. The arrows are shown, and one must give a rule for composing each arrow and identifying with one of the other arrows. For example, given the arrows denoted f and g it follows that there must be an arrow of the type denoted $f \circ g$. Note that every object x has at least one arrow, the identity arrow in $Hom(x, x)$.

To see that this is equivalent to our previous notion of a group we associate to each morphism a group element. Composition of morphisms is the group operation. The invertibility of morphisms is the existence of inverses.

We will briefly describe an important and far-reaching generalization of a group afforded by this viewpoint. Then we will show that this viewpoint leads to a nice geometrical construction making the formulae of group cohomology a little bit more intuitive.

K.1 Groupoids

Definition A *groupoid* is a category all of whose morphisms are invertible.

Note that for any object x in a groupoid, $\hom(x, x)$ is a group. It is called the *automorphism group* of the object x.

Example 1. Any equivalence relation on a set X defines a groupoid. The objects are the elements of X. The set $\mathrm{Hom}(a, b)$ has one element if $a \sim b$ and is empty otherwise. The composition law on morphisms then means that $a \sim b$ with $b \sim c$ implies $a \sim c$. Clearly, every morphism is invertible.

Example 2. Consider time evolution in quantum mechanics with a time-dependent Hamiltonian. There is no sense to time evolution $U(t)$. Rather one must speak of unitary evolution $U(t_1, t_2)$ such that $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$. Given a solution of the Schrodinger equation $\Psi(t)$ we may consider the state vectors $\Psi(t)$ as objects and $U(t_1, t_2)$ as morphisms. In this way a solution of the Schrodinger equation defines a groupoid. ♣Clarify this remark. ♣

Example 3. Let X be a topological space. The fundamental groupoid $\pi_{\leq 1}(X)$ is the category whose objects are points $x \in X$, and whose morphisms are homotopy classes of paths $f : x \to x'$. These compose in a natural way. Note that the automorphism group of a point $x \in X$, namely, $\hom(x, x)$ is the fundamental group of X based at x, $\pi_1(X, x)$.

Example 4. Gauge theory: Objects = connections on a principal bundle. Morphisms = gauge transformations. This is the right point of view for thinking about some more exotic (abelian) gauge theories of higher degree forms which arise in supergravity and string theories.

Example 5. In the theory of string theory orbifolds and orientifolds spacetime must be considered to be a groupoid. Suppose we have a right action of G on a set X, so we have a map

$$\Phi : X \times G \to X \tag{K.37}$$

such that

$$\Phi(\Phi(x, g_1), g_2) = \Phi(x, g_1 g_2) \tag{K.38}$$

$$\Phi(x, 1_G) = x \tag{K.39}$$

for all $x \in X$ and $g_1, g_2 \in G$. We can just write $\Phi(x, g) := x \cdot g$ for short. We can then form the category $X//G$ with

$$\begin{aligned} Ob(X//G) &= X \\ Mor(X//G) &= X \times G \end{aligned} \tag{K.40}$$

We should think of a morphism as an arrow, labeled by g, connecting the point x to the point $x \cdot g$. The target and source maps are: ♣FIGURE NEEDED HERE! ♣

$$p_0((x, g)) := x \cdot g \qquad\qquad p_1((x, g)) := x \tag{K.41}$$

The composition of morphisms is defined by

$$(xg_1, g_2) \circ (x, g_1) := (x, g_1 g_2) \tag{K.42}$$

or, in the other notation (better suited to a right-action):

$$(x, g_1)(xg_1, g_2) := (x, g_1 g_2) \tag{K.43}$$

Note that $(x, 1_G) \in \hom(x, x)$ is the identity morphism, and the composition of morphisms makes sense because we have a group action. Also note that $pt//G$ where G has the trivial action on a point realizes the group G as a category, as sketched above.

Example 6. In the theory of string theory orbifolds and orientifolds spacetime must be considered to be a groupoid. (This is closely related to the previous example.)

Exercise

For a group G let us define a groupoid denoted $G//G$ (for reasons explained later) whose objects are group elements $Obj(G//G) = G$ and whose morphisms are arrows defined by

$$g_1 \xrightarrow{h} g_2 \tag{K.44}$$

iff $g_2 = h^{-1} g_1 h$. This is the groupoid of principal G-bundles on the circle.

Draw the groupoid corresponding to S_3.

K.2 Tensor Categories

To define a TFT we need the further notion of a tensor category. Note that given a category C, the Cartesian products $C \times C$, $C \times C \times C$, ... are also categories in a natural way.

Definition A *tensor category* (also known as a *monoidal category*) is a category with a functor $\otimes : C \times C \to C$ such that there is an isomorphism $\mathcal{A}$ of the two functors $\otimes \circ \otimes_{12} : C \times C \times C \to C$ and $\otimes \circ \otimes_{23} : C \times C \times C \to C$ satisfying the pentagon identity, and such that there is an identity object 1_C together with natural transformations of functors $C \to C$:

$$\iota_L : 1_C \otimes \cdot \to \mathrm{Id} \tag{K.45}$$

$$\iota_R : \cdot \otimes 1_C \to \mathrm{Id} \tag{K.46}$$

These data are subject to a number of natural compatibility conditions: ♣Do we require the existence of a dual object? ♣

To give an example of the compatibility conditions we consider the the first condition on the natural transformation $\mathcal{A}$: for all objects x, x', x'' in C_0 we have an isomorphism:

$$\mathcal{A}_{x,x',x''} : (x \otimes x') \otimes x'' \to x \otimes (x' \otimes x'') \tag{K.47}$$

which satisfies the pentagon identity: ♣FIX xy matrix ♣

$$\begin{tikzcd}
& ((x_1 \otimes x_2) \otimes x_3) \otimes x_4 \arrow[r] \arrow[dl] & (x_1 \otimes x_2) \otimes (x_3 \otimes x_4) \arrow[dr] & \\
(x_1 \otimes (x_2 \otimes x_3)) \otimes x_4 \arrow[drr] & & & x_1 \otimes (x_2 \otimes (x_3 \otimes x_4)) \\
& & x_1 \otimes ((x_2 \otimes x_3) \otimes x_4) \arrow[ur] &
\end{tikzcd} \tag{K.48}$$

It is then a theorem (the "coherence theorem") that $x_0 \otimes x_1 \cdots \otimes x_n$ is well-defined up to isomorphism no matter how one brackets the products. The conditions on the natural transformations ι_L and ι_R are fairly obvious.

Example The category $\mathbf{VECT}_\kappa$ is a tensor category. What is the tensor unit $1_{\mathbf{VECT}_\kappa}$?

Let $\sigma : C \times C \to C \times C$ be the exchange functor that switches factors on objects and morphisms.

Definition A *symmetric monoidal category* is a monoidal category with an isomorphism Ω of $\otimes \circ \sigma$ with $\otimes$ which squares to one. Again, there are many rather obvious compatibility conditions with $\mathcal{A}$, ι_L and ι_R.

Again, this means that for all objects x, y we have an isomorphism

$$\Omega_{x,y} : x \otimes y \to y \otimes x \tag{K.49}$$

so that $\Omega_{y,x} \circ \Omega_{x,y} = 1_{x \otimes y}$.

Remark: An important generalization for conformal field theory and for quasiparticle statistics in 2+1 dimensions is the notion of a *braided tensor category* where there is an isomorphism Ω, but it does not square to 1.

Finally, we need the notation of a (symmetric) tensor functor. This is a functor $F : C \to D$ between symmetric tensor categories together with an isomorphism $1_D \to F(1_C)$ and an isomorphism of the two functors $C \times C \to D$ given by $F \circ \otimes$ and $\otimes \circ F \times F$.

K.2.1 $\mathbb{Z}_2$-graded vector spaces

A $\mathbb{Z}_2$ graded vector space is a vector space with a decomposition $V = V_0 \oplus V_1$, where the subscripts are understood as elements of $\mathbb{Z}_2$. In the category of $\mathbb{Z}_2$-graded vector spaces we can introduce two different kinds of tensor categories. For $\mathbb{Z}_2$ graded vector spaces we can and will use the graded tensor product. Then there is an isomorphism

$$\Omega : V \otimes W \to W \otimes V \tag{K.50}$$

but we must be careful to apply the *Koszul sign rule*: If v, w are homogeneous elements then

$$\Omega(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v \tag{K.51}$$

This rule has the important consequence that if we have any collection $(V_\alpha)_{\alpha \in I}$ of supervector spaces (where the subscript α denotes different supervector spaces and should not be confused with the $\mathbb{Z}_2$ grading) then there is a single canonical tensor product

$$\otimes_\alpha V_\alpha$$

without the need to specify any ordering.

K.2.2 Category Of Representations Of A Group

Let G be a group. Then then there is a category whose objects are representations and morphisms are intertwiners of representations (i.e. maps between representations that commute with the G action.

Now let G be a compact group and restrict to the subcategory of finite-dimensional representations. Call this $\mathrm{Rep}(G)$. This is a tensor category. Moreover, there is a set of "simple" objects, the irreducible representations V_λ such that all objects are isomorphic to direct sums of simple objects. The tensor functor is determined by the "fusion rules"

$$V_\lambda \otimes V_\mu \cong D^\rho_{\lambda\mu} \otimes V_\rho \tag{K.52}$$

where $D^\rho_{\lambda\mu}$ is a finite-dimensional real vector space of degeneracies.

L. Bordism

This mathematical topic is important background for topological field theories. There is a classic book on the subject:

Milnor and Stasheff, *Characteristic Classes*, PUP

More technical details can be found in the book of R. Stong.

In addition we have used some material from the very nice lecture notes of Dan Freed
:

D. Freed, "Bordism Old And New,"

https://www.ma.utexas.edu/users/dafr/M392C-2012/index.html

L.1 Unoriented Bordism: Definition And Examples

Here we give the official definition of a bordism:

Definition Let Y_0, Y_1 be two closed $(d-1)$-dimensional manifolds. A *bordism* from Y_0 to Y_1 is

1. A d-manifold X together with a disjoint partition of its boundary:

$$\partial X = (\partial X)_0 \amalg (\partial X)_1 \tag{L.1}$$

2. A pair of embeddings $\theta_0 : [0,1) \times Y_0 \to X$ and $\theta_1 : (-1,0] \times Y_1 \to X$, which are diffeomorphisms onto their images such that the restrictions $\theta_0 : \{0\} \times Y_0 \to (\partial X)_{\rm in}$ and $\theta_1 : \{0\} \times Y_1 \to (\partial X)_{\rm out}$ are homeomorphisms.

The reason for the extra level of complexity in this definition compared to what we said earlier is that this extra data facilitates the gluing of bordisms to produce a new bordism.

It is easy to see that bordism is an equivalence relation and that disjoint union defines an abelian group structure on the space of bordism equivalence classes Ω_n of n-manifolds. The zero element of the abelian group is the equivalence class of the empty set $\emptyset^n$ and any closed n-manifold X is its own inverse since $[0,1] \times X$ can be considered as a bordism of $X \amalg X$ with $\emptyset$. So $2[X] = 0$ in Ω_n.

Examples

1. There is only one nontrivial zero-dimensional manifold, the point, and we have just seen that the disjoint union of two points is null-bordant, hence $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$. Note that if we dropped the manifold condition on X then the letter Y would define a bordism of two points (equivalent to zero) with one point, and hence the bordism group would be trivial. Thus, the manifold condition is important.

2. $\Omega_1 = 0$, because the only closed connected one-manifold is the circle, and this clearly bounds a disk.

3. One can show that $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ with generator $[\mathbb{RP}^2]$. Here is the argument (taken from D. Freed's notes "Bordism Old And New," on his homepage). The classification of compact surfaces shows that they are characterized by two invariants: Orientability and the Euler character. Oriented surfaces are clearly bordant to zero. Note well! The Eulcer character is <u>not</u> a bordism invariant! Unorientable surfaces are all obtained by connected sums with $\mathbb{RP}^2$. The connected sum of two copies of $\mathbb{RP}^2$ is a circle bundle over the circle. Take $[0,1] \times S^1$ and quotient by $\{(0,z)\} \sim \{(1,\bar{z})\}$ (where we view S^1 as the unit complex numbers). Note that we can replace the S^1 by the disk D^2 and use the same identification $\{(0,z)\} \sim \{(1,\bar{z})\}$ to produce a bordism of the Klein bottle to zero. Next we claim that $\mathbb{RP}^2$ descends to a nontrivial bordism class. For, if it had a bordism to zero $\partial X = \mathbb{RP}^2$ then triangulation of X gives a triangulation of the double $X \cup_{\mathbb{RP}^2} X$ with Euler character $2\chi(X) - 1$. On the other hand, the Euler character of a closed 3-fold is zero. Now, the general connected unorientable surface is a connected sum of n copies of $\mathbb{RP}^2$. Separate these in pairs and choose a bordism of the pairs to zero to identify the bordism class with an element $n \mathrm{mod} 2$ of $\mathbb{Z}/2\mathbb{Z}$.

4. To describe all bordism groups Ω_d it is useful to note that Cartesion product of manifolds is compatible with the bordism equivalence relation and this makes $\Omega_* \cong \amalg_{d\geq 0}\Omega_d$ into a $\mathbb{Z}$-graded ring, with the grading given by the dimension. Thom proved that
$$\Omega_* \cong R[x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{16}, x_{17}, ...] \tag{L.2}$$
where $R = \mathbb{Z}/2\mathbb{Z}$ and there is precisely one generator x_k of degree k so long as k is <u>not</u> of the form $2^j - 1$. The even degree generators are the bordism classes of $\mathbb{RP}^k$ and the odd ones are a quotient of $(S^m \times \mathbb{CP}^\ell)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as (antipodal map, complex conjugation).

5. Moreover, to any manifold there is a series of cohomology classes $w_i(Y) \in H^i(Y; \mathbb{Z}/2\mathbb{Z})$ known as Stiefel-Whitney classes. They are associated with the twisting of the tangent bundle. (For example, $w_1(Y)$ measures whether Y is orientable or not.) The *Stiefel numbers* of a manifold is the sequence of elements of $\mathbb{Z}/2\mathbb{Z}$:
$$\langle w_{i_1}(Y) \cup \cdots w_{i_k}(Y), [Y] \rangle \tag{L.3}$$
and two manifolds are bordant iff all their Stiefel numbers agree. For the last two items see the excellent book by Milnor and Stasheff, *Characteristic Classes.* [53]

L.2 The Bordism Category $\mathrm{Bord}_{\langle d-1,d\rangle}$

Now, we can define a bordism <u>category</u> $\mathrm{Bord}_{\langle d-1,d\rangle}$.

1. Objects: Closed $(d-1)$-manifolds, usually denoted Y.

2. $\mathrm{hom}(Y_0, Y_1)$ is the set of homeomorphism classes of bordisms $X : Y_0 \to Y_1$. A homeomorphism of bordisms X, X' is a homeomorphism of manifolds with boundaries which takes $(\partial X)_{\mathrm{in}} \to (\partial X')_{\mathrm{in}}$ and commutes with the collars θ_0, θ_1.

[53] If we cover the chapter on characteristic classes we will prove these two results.

The composition of morphisms in the bordism category is by gluing. Since we identify bordisms by homeomorphism the bordism $X = [0,1] \times Y$ from $Y \to Y$ is the identity morphism 1_Y. The category $\mathrm{Bord}_{\langle d-1,d\rangle}$ is a symmetric tensor category: The tensor product is disjoint union, and the empty manifold $\emptyset^{d-1}$ is the tensor unit.

L.3 The Oriented Bordism Category $\mathrm{Bord}^{\mathrm{SO}}_{\langle d-1,d\rangle}$

We are often interested in *oriented bordism.*

To define an oriented bordism we modify the definition of bordism slightly. Now, Y_0, Y_1, X are all oriented. The embeddings θ_0 and θ_1 are required to be orientation preserving and we identify bordisms X and X' by oriented diffeomorphisms.

The condition that θ_0 and θ_1 are orientation preserving must be treated with care. Note that if we are given a sum of oriented real vector spaces there is no natural orientation on the direct sum. However, if we are given an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0 \tag{L.4}$$

Then there is a canonical isomorphism $\mathrm{DET}V_3 \cong \mathrm{DET}V_1 \to \mathrm{DET}V_2$ so if two of the three spaces are oriented, we can determine an orientation on the third by requiring this canonical isomorphism to be orientation preserving. In particular, an orientation on a submanifold and the ambient manifold determines an orientation on the normal bundle. When defining θ_0, θ_1 we orient $[0,+1)$ and $(-1,0]$ with the standard orientation on $\mathbb{R}$, $+\frac{\partial}{\partial x}$ and then we take the product orientation on $[0,+1) \times Y$ and $(-1,0] \times Y$.

Definition To every oriented bordism $X : Y_0 \to Y_1$ there is a *dual oriented bordism* $X^\vee : Y_1^\vee \to Y_0^\vee$. Let us write it out carefully, since it can cause confusion. $Y^\vee$ denotes Y with the opposite orientation. $X^\vee$ is the manifold with the <u>same</u> orientation. However, we exchange ingoing and outgoing boundaries. Moreover,

$$\theta_0^\vee(t, y_1) = \theta_1(-t, y_1) \qquad \forall t \in [0,+1) \qquad \& \qquad y_1 \in Y_1 \tag{L.5}$$

$$\theta_1^\vee(t, y_0) = \theta_0(-t, y_0) \qquad \forall t \in (-1,0] \qquad \& \qquad y_0 \in Y_0 \tag{L.6}$$

Note that the relation between $\theta_0^\vee$ and θ_1 involves an orientation-reversing transformation $t \to -t$ and hence we require orientation reversal on Y since $X^\vee$ has the same orientation as X. Forgetting about orientations we also obtain a notion of dual bordism for the unoriented case.

Once again we can define oriented bordism groups Ω_n^{SO}, for $n \geq 0$, the oriented bordism ring Ω_*^{SO} and the oriented bordism category $\mathrm{Bord}^{\mathrm{SO}}_{\langle d-1,d\rangle}$.

Example 1 Let us consider the oriented bordism group Ω_0^{SO}. There are two kinds of points pt_+ and pt_-, and five basic connected oriented bordisms, shown in figure 40. Accordingly, $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$. The isomorphism takes the difference of the number of $+$ and $-$ points.

Example 2 In dimensions 1 and 2 we again have zero bordism groups.

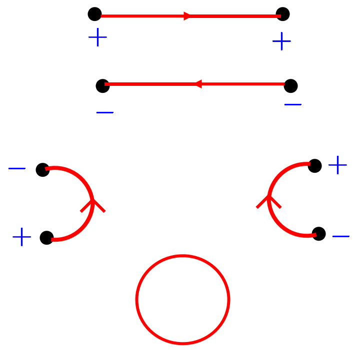

Figure 40: Five connected bordisms in the oriented bordism category. Ingoing boundaries are on the left and outgoing boundaries are on the right.

A summary of the main factors on the oriented bordism ring $\Omega_*^{\rm SO}$ is the following. (See Milnor and Stasheff. Several further references are provided in Freed's notes, near Theorem 2.24.) [54]

Theorem

1. All torsion elements in $\Omega_*^{\rm SO}$ have order two.

2. $\Omega_*^{\rm SO}$/torsion is a ring with one generator in degrees $4k$, $k \geq 1$.

3. There is an isomorphism

$$\Omega_*^{\rm SO} \otimes \mathbb{Q} \cong \mathbb{Q}[y_4, y_8, \cdots] \tag{L.7}$$

   under which y_{4k} corresponds to the oriented bordism class of $\mathbb{CP}^{2k}$.

4. There are characteristic classes of the tangent bundle of Y, the Stiefel-Whitney classes $w_i(Y) \in H^i(Y;\mathbb{Z}_2)$ and the Pontryagin classes $p_i(Y) \in H^{4i}(Y;\mathbb{Z})$ (the latter depending on the orientation of Y) such that Y_1 and Y_2 and bordant iff all the Stiefel-Whitney and Pontryagin numbers are the same. We defined the Stiefel-Whitney classes above and the Pontryagin numbers are similarly the collection

$$\langle p_{i_1}(Y) \cup \cdots \cup p_{i_k}(Y), [Y] \rangle \in \mathbb{Z} \tag{L.8}$$

L.4 Other Bordism Categories

We can go on an consider other forms of bordism:

1. Framed bordism. (Closely related to the stable homotopy of spheres, by the Pontryagin-Thom construction.)
2. Spin and Pin$^{\pm}$ bordism.
3. Riemannian bordism.

[54] If we cover the chapter on characteristic classes we will prove some of these results.

Accordingly, there are generalizations of the bordism cateogry. In general, if we take into account a structure $\mathcal{S}$ we denote the bordism category by $\mathrm{Bord}^{\mathcal{S}}_{\langle d-1,d\rangle}$, where it is understood that the bordisms are identified by homeomorphisms preserving the structure $\mathcal{S}$. Thus, the oriented bordism category is denoted by $\mathrm{Bord}^{\mathrm{SO}}_{\langle d-1,d\rangle}$ (because the structure group of the tangent bundle is $SO(d-1)$ and $SO(d)$, respectively). Similarly we can define a Riemannian bordism category $\mathrm{Bord}^{\mathrm{Riem}}_{\langle d-1,d\rangle}$, and so on.

M. Some Sources From Course Notes

Material on theta functions and Heisenberg algebras:

GMP7-06, section 6

GMP-Ch8-AssociatedBundles-2010, section 11.

Invariant norm, first Chern class for level k theta functions: GMP9-06, section 3.3.

Also group cohomology. Some Dijkgraaf-Witten theory. GRP-THEORY-Lect1

Topological Field Theory and Categories:

GMP 2015.

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