Summing Over Bordisms In TQFT

Gregory Moore Rutgers

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Work with
Anindya Banerjee

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Comments On Summing Over Bordisms In TQFT

Anindya Banerjee and Gregory W. Moore[a]

[a] *NHETC and Department of Physics and Astronomy, Rutgers Univer*
126 Frelinghuysen Rd., Piscataway NJ 08855, USA

E-mail: ab1702@scarletmail.rutgers.edu, gwmoore@physics.

I. Motivation

A longstanding problem in quantum gravity:

Probability amplitudes are computed by ``summing''
(as in a path integral) over metrics on some spacetime Y

$$\exp\{ -\frac{G_N}{16\pi^2} \int_Y \mathcal{R}(g) vol(g) + \dots \}$$

If we sum over metrics, should we
also sum over topologies?

Puzzles In AdS/CFT

There are hyperbolic Y where ∂Y has multiple connected components.

$\Rightarrow$ Puzzling aspects of the AdS/CFT correspondence - the ``factorization problem'' [Yau & Witten 1999; Maldacena & Maoz 2004]

Saad-Shenker-Stanford [1903.11115] identifies sum of topologies in``JT gravity'' with a matrix model:
Raises conceptual questions about whether string theory should be dual to an ***ensemble*** of QFTs.

Motivated by these issues, and the recent vigorous discussion in the quantum gravity community, D. Marolf and H. Maxfield recently [2002.08950] considered a curious ``topological model of 2d gravity.''

An essential part of their discussion involved summing over topologies with disconnected components.

My project with Anindya Banerjee was motivated by the desire to understand the MM model in terms of standard TQFT.

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3. Summed & Total Amplitudes: Splitting Property
4. Example: $d = 1$
5. Example: $d = 2$, closed
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II. Reminders On TQFT

Definition of a ``bordism''

Let X_{in}, X_{out} be smooth, compact manifolds of dimension $d-1$.

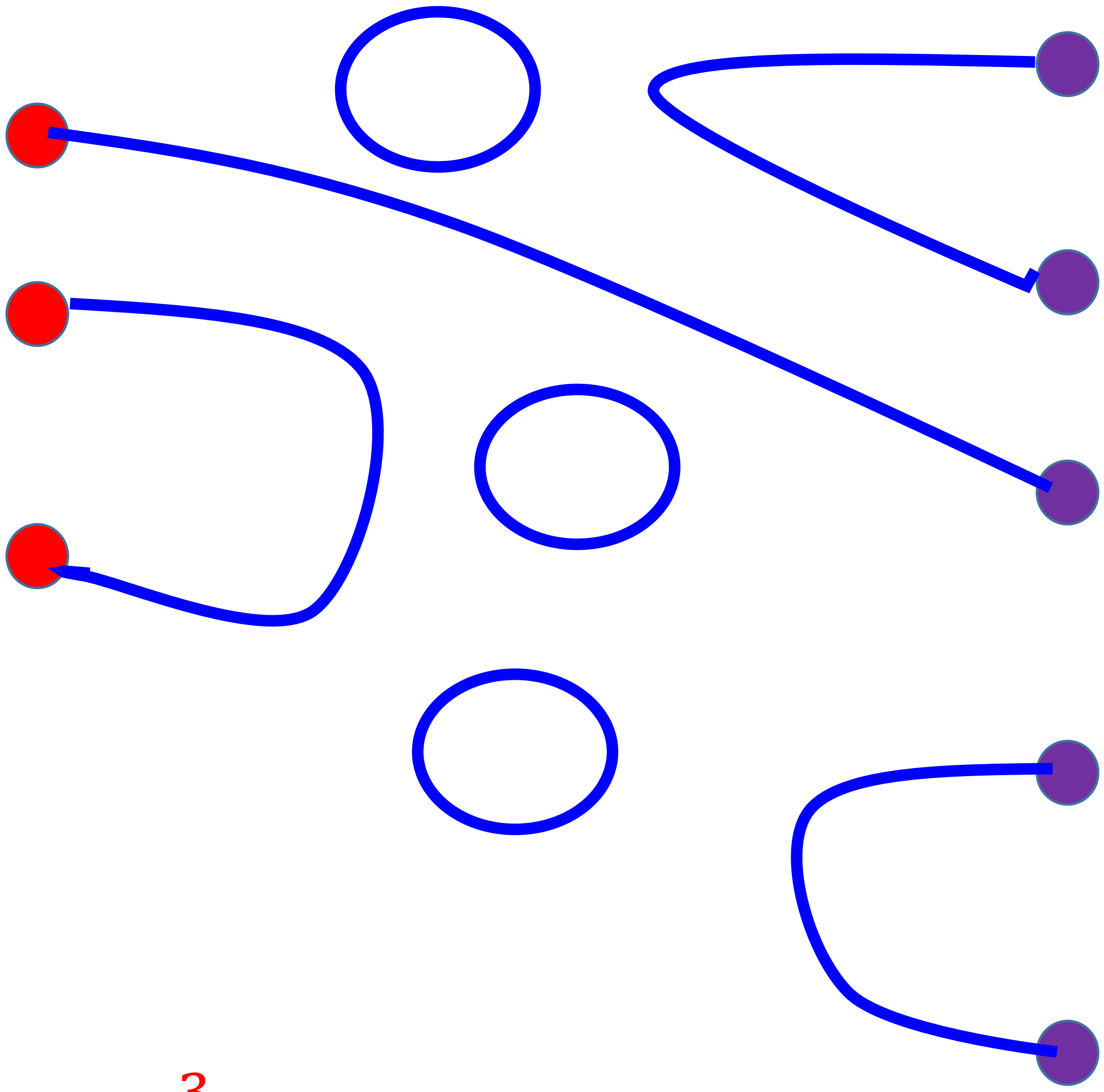
A ***bordism*** $Y: X_{in} \to X_{out}$ is:

A d-manifold Y together with a disjoint decomposition $\partial Y = (\partial Y)_{in} \coprod (\partial Y)_{out}$

Diffeomorphisms $(\partial Y)_{in} \cong X_{in}$ & $(\partial Y)_{out} \cong X_{out}$

Embeddings $X_{in} \times [0,1) \to Y$ & $X_{out} \times (-1,0] \to Y$

which reduce to the specified diffeos on the boundary of Y



There are 105 such bordisms.

+ infinitely many more including disjoint unions with circles….

$$Y: \quad X_{in} = \coprod_{1}^{3} pt \quad \rightarrow \quad X_{out} = \coprod_{1}^{5} pt$$

Bordisms are morphisms in a category $\mathfrak{Bord}_{\langle d,d-1\rangle}$

A TQFT (in this talk) is a monoidal functor $\mathcal{Z}$ to the category $VECT_{\kappa}$ of vector spaces over a field κ

$\mathcal{Z}(X)$: Vector space of ``states'' for spatial manifold X

$$\mathcal{Z}(X_1 \coprod X_2) \cong \mathcal{Z}(X_1) \otimes \mathcal{Z}(X_2)$$

$$Y: X_{in} \rightarrow X_{out}$$

$$\mathcal{Z}(Y) \in Hom\big(\mathcal{Z}(X_{in}), \mathcal{Z}(X_{out})\big)$$

$$\mathcal{Z}(Y_1 \circ Y_2) = \mathcal{Z}(Y_1) \circ \mathcal{Z}(Y_2)$$

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III. Summed & Total Amplitudes: Splitting Property

Recall we can have different bordisms between fixed X_{in} and X_{out}

Given a TQFT $\mathcal{Z}$ (the ``seed TQFT'') define the ``summed amplitude''

$$\mathcal{A}(X_{in}, X_{out}) := \sum_{Y: X_{in} \to X_{out}} \frac{\mathcal{Z}(Y)}{|Aut(Y)|}$$

$Aut(Y)$: Automorphism group of homeomorphism type restricting to the *identity* on the boundary.

Some Questions:

$$\mathcal{A}(X_{in}, X_{out}) \coloneqq \sum_{Y: X_{in} \to X_{out}} \frac{\mathcal{Z}(Y)}{|Aut(Y)|} \in Hom(\mathcal{Z}(X_{in}), \mathcal{Z}(X_{out}))$$

1. Does it exist?

2. Is it computable?

3. What properties does it have ?

4. Extension to the fully local TQFT ?

Some Answers:

1. It exists for d=1,2 and does not exist for $d \geq 3$, at least not in the most naïve sense…

2. Yes, when it exists.

3. From explicit computations: Splitting Property

4. For d=2, this is the extension to open-closed TQFT.

The Total Amplitude

Consider all summed amplitudes simultaneously as a linear transformation on the tensor algebra:

$$\mathcal{A} \in End\left(T^*\left(\oplus_X \mathcal{Z}(X)\right)\right)$$

$\oplus_X$: Direct sum over all smooth connected (d-1)-manifolds (up to diffomorphism - a countable sum)

The summed amplitudes descend to

$$\bar{\mathcal{A}} \in End\left(S^*\left(\oplus_X \mathcal{Z}(X)\right)\right) := End(Fock(\mathcal{Z}))$$

The Splitting Property

For $\kappa = \mathbb{C}$ we can put an inner product structure on $Fock(\mathcal{Z})$ and there exists an inner product space $\mathcal{W}$ such that

$$\Phi : Fock(\mathcal{Z}) \to \mathcal{W}$$

$$\bar{\mathcal{A}} = \Phi\Phi^*$$

Our Convention:

$$Hom(V_1, V_2) \cong V_1^\vee \otimes V_2$$

$$T_{12} \in Hom(V_1, V_2) \qquad T_{23} \in Hom(V_2, V_3)$$

$$T_{12}T_{23} \in Hom(V_1, V_3)$$

$$T_{12} \otimes T_{23} \in V_1^\vee \otimes V_2 \otimes V_2^\vee \otimes V_3 \mapsto T_{12}T_{23} \in V_1^\vee \otimes V_3$$

$$\bar{\mathcal{A}} = \Phi\Phi^*$$

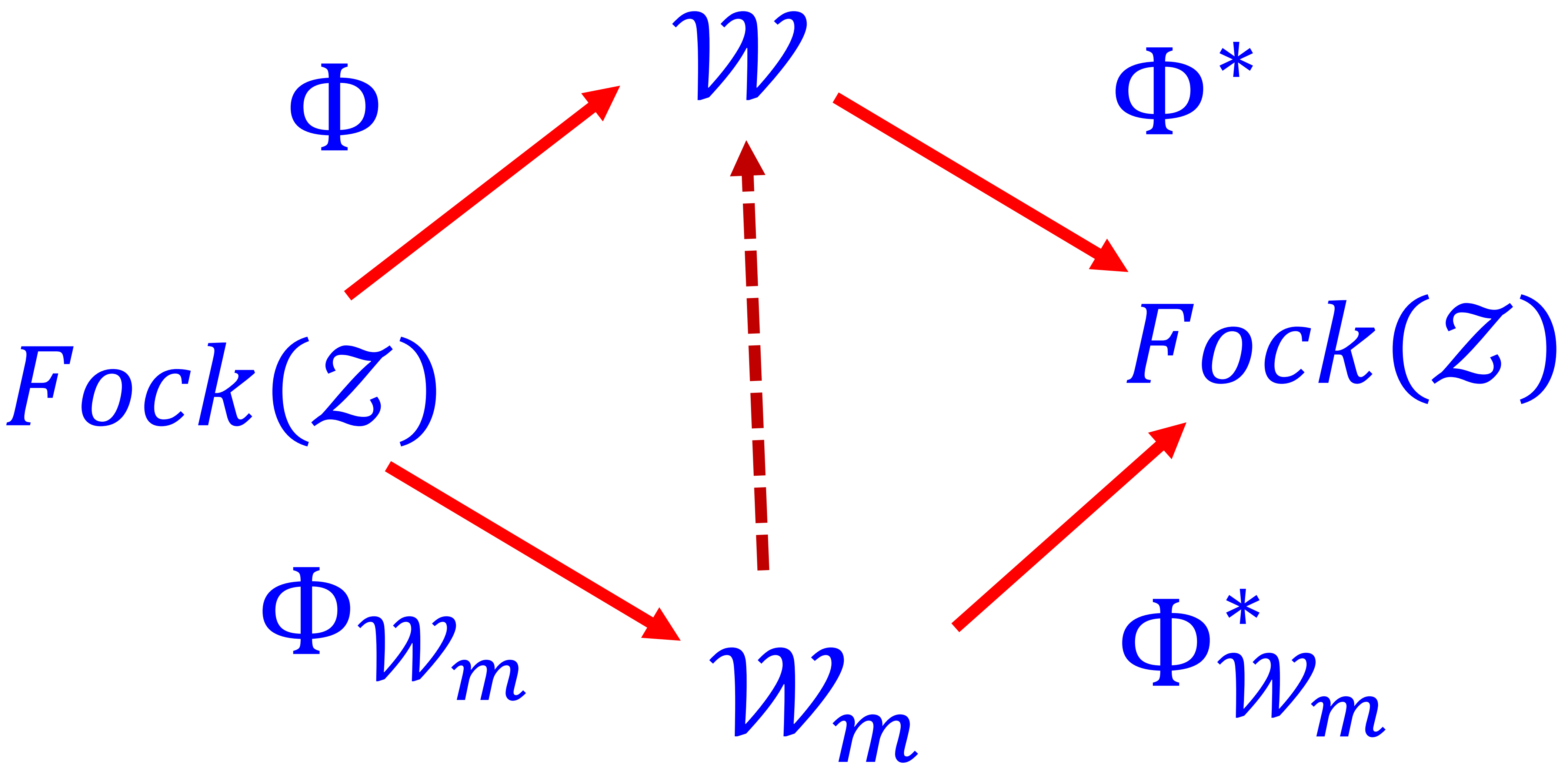
1. $\bar{\mathcal{A}}$ need not be positive definite.

2. Even if existence is trivial , explicitly finding $\mathcal{W}$ and Φ in examples seems to be slightly nontrivial.

3. $\mathcal{W}$ is not unique: $\mathcal{W} \to \oplus_\alpha \mathcal{W}_\alpha$

$$\Phi \to \oplus_\alpha \sqrt{p_\alpha}\Phi_\alpha \qquad \sum p_\alpha = 1$$

4. There might be a ``minimal'' $\mathcal{W}_m$



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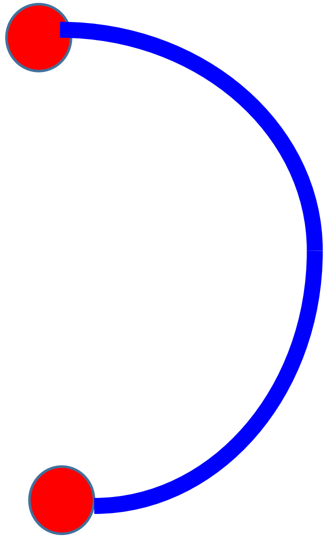
IV. Example: d=1, unoriented

$\mathcal{Z}$ is determined by a f.d. vector space $V = \mathcal{Z}(pt)$ and a symmetric nondegenerate bilinear form $b: V \otimes V \to \kappa$



$$Fock(\mathcal{Z}) = Fock(V) = S^*V = \kappa \oplus V \oplus S^2V \oplus \cdots$$

Start with $X_{in} = X_{out} = \emptyset$

$$\mathcal{Z}(S^1) = \dim_\kappa V$$

$$\mathcal{A}(\emptyset, \emptyset) = \exp \dim_\kappa V$$

Hartle-Hawking Vector & Covector

HH vector: The sum of nothing to something:

$$\kappa \hookrightarrow Fock(\mathcal{Z}) \overset{\bar{\mathcal{A}}}{\to} Fock(\mathcal{Z}) : 1 \mapsto \Psi_{HH} \in Fock(\mathcal{Z})$$

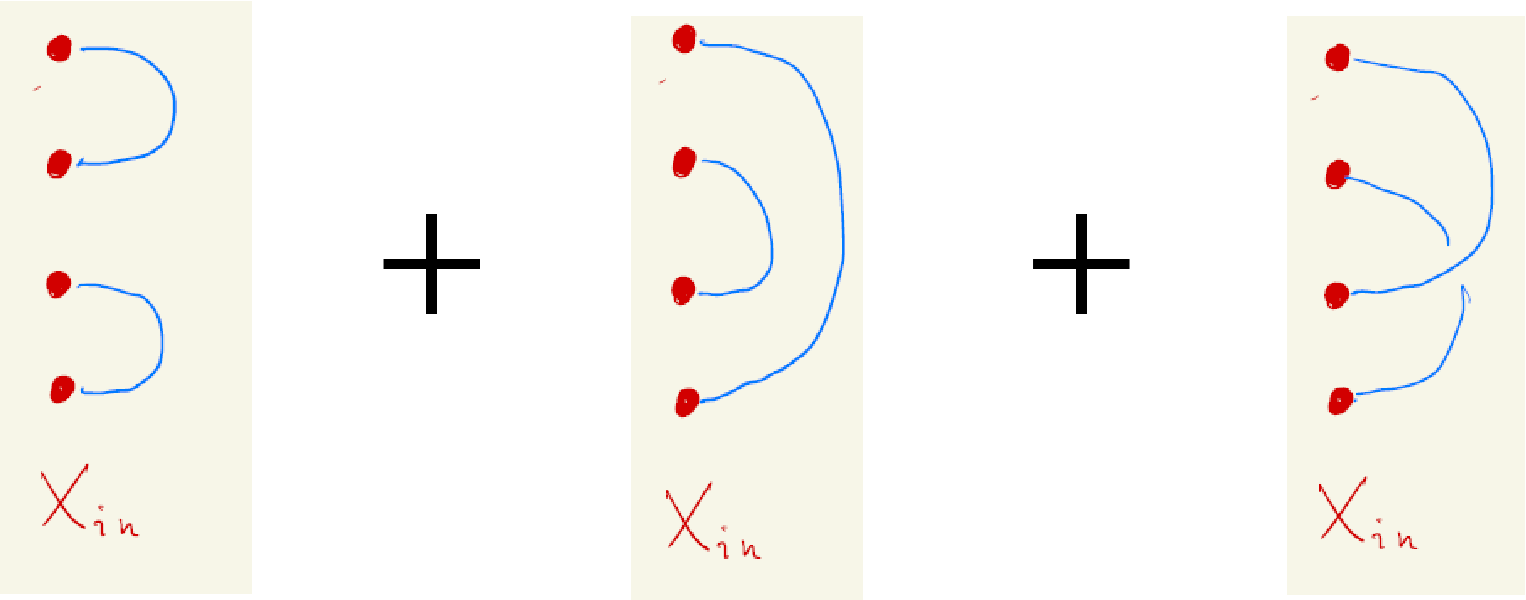
HH covector: The sum of anything to nothing:

$$Fock(\mathcal{Z}) \overset{\bar{\mathcal{A}}}{\to} Fock(\mathcal{Z}) \to \kappa : \quad \Psi_{HH}^{\vee} \in Hom(Fock(\mathcal{Z}), \kappa)$$

Simplest example: Suppose $\dim V = 1$

Take $\kappa = \mathbb{C}$ and choose v with $b(v, v) = 1$

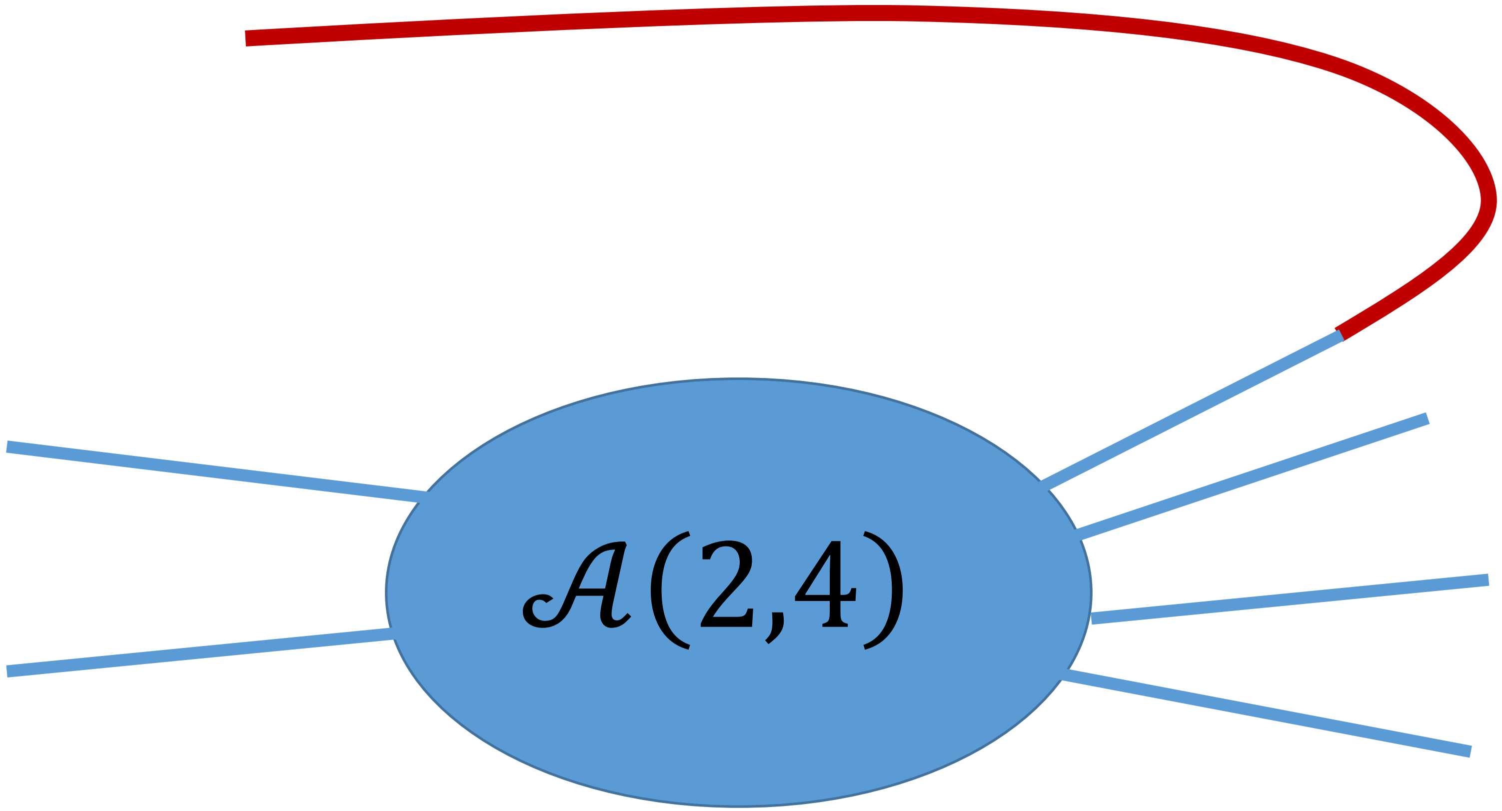
$$\Psi^{\vee}_{HH} = \exp(1) \sum_{n=0}^{\infty} \frac{(2n)!}{n!\, 2^n} (v^{\vee})^{2n} \in S^* V^{\vee}$$



Nondegenerate $b \Rightarrow$ canonical isomorphisms

$$b^{\vee}: V \to V^{\vee} \qquad b_{\vee}: V^{\vee} \to V$$

$$b^{\vee} \circ \mathcal{A}(n_i, n_o) = \mathcal{A}(n_i + 1, n_0 - 1)$$



$$\bar{\mathcal{A}} = \exp(1) \sum_{n_{in}+n_{out}=2n}^{\infty} \frac{(2n)!}{n!\,2^n} (v^\vee)^{n_{in}} \otimes v^{n_{out}}$$

$$\in S^*V^\vee \otimes S^*V \cong End\big(Fock(V)\big)$$

Splitting

Wick's theorem: $$\frac{(2n)!}{n!\,2^n} = \int \frac{dh}{\sqrt{2\pi}}\, h^{2n}\, e^{-\frac{1}{2}h^2}$$

$$\psi_n = (2\pi)^{-\frac{1}{4}} h^n e^{-\frac{1}{4}h^2} \in L^2(\mathbb{R}) = \mathcal{W}$$

$$\langle \psi_n, \psi_m \rangle = \delta_{n+m=0(2)} \frac{(2n+2m)!}{(n+m)!2^{n+m}}$$

$$\Phi = \exp\left(\frac{1}{2}\right) \sum_n (v^\vee)^n \otimes \psi_n \in Hom(Fock(V), \mathcal{W})$$

Generalizes to $\dim V > 1$

$$\Psi_{HH}^{\vee}(e^{v}) = \exp[\dim V + \frac{1}{2} b(v,v)] \qquad v \in V$$

$\Psi_{HH}^{\vee}$ is multilinear & totally symmetric $\Rightarrow$
determined by values on the diagonal.

Results can be extended to oriented d=1 theory
$\mathcal{Z}$ determined solely by a single vector space V

$$\mathcal{Z}(pt_{+}) = V \qquad \mathcal{Z}(pt_{-}) = V^{\vee} \qquad \mathcal{W} = L^{2}(T^{2})$$

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V. Example: d=2 & Oriented

$\mathcal{Z}(S^1) = \mathcal{C}$: f.d. commutative Frobenius algebra

[Friedan, Dijkgraaf, Segal,…]

$\mathcal{Z}(Disk)$: $\theta_{\mathcal{C}}: \mathcal{C} \to \kappa$

$b(\phi_1, \phi_2) = \theta_{\mathcal{C}}(\phi_1 \phi_2)$: Symmetric nondegenerate form

Open-closed case discussed later.

Complete proof of the sewing theorem (including equivariant case): Moore & Segal 2002

	Semisimple	Non-semisimple
Closed	Yes	Examples
Open-closed	Yes	????

$\mathcal{Z}(S^1)$ Semisimple

$$\mathcal{C} = \bigoplus_{x \in \mathcal{X}} \mathcal{C}_x = \bigoplus_{x \in \mathcal{X}} \mathbb{C}\varepsilon_x$$

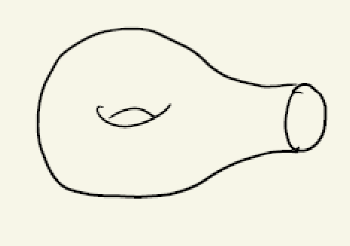
$$\varepsilon_x \varepsilon_y = \delta_{x,y} \varepsilon_x \qquad \theta(\varepsilon_x) = \theta_x \in \kappa^*$$

2d Topological String Theory with target space $\mathcal{X} = Spec(\mathcal{C})$ and dilaton $\theta_x = g_{string,x}^{-2}$

$$\bar{\mathcal{A}}(\emptyset,\emptyset) = \exp(\,\mathcal{Z}(Y_0) + \mathcal{Z}(Y_1) + \cdots)$$

$$= \exp(\,\theta_{\mathcal{C}}\left(\frac{1}{1-h}\right)) \quad = \exp \sum_{x \in \mathcal{X}} \lambda_x$$

$h \in \mathcal{C}$: Handle-adding element defined by the one-hole torus with one outgoing S^1



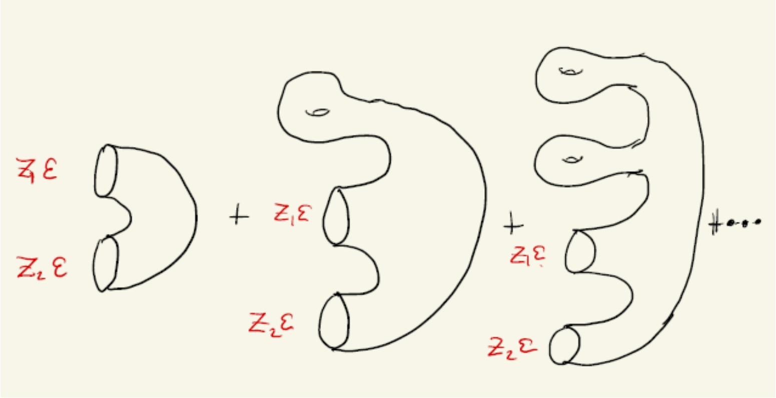
$$\lambda_x = \frac{\theta_x}{1-\theta_x^{-1}} = g_x^{-2} + 1 + g_x^2 + \cdots$$

$$\bar{\mathcal{A}}(S^1 \coprod S^1, \emptyset)(\phi_1, \phi_2) = ?$$

For simplicity: Take $\dim \mathcal{C} = 1$ $\phi_1 = z_1 \varepsilon$, $\phi_2 = z_2 \varepsilon \in \mathcal{C}$

Bordisms with *one connected component*:

$$\phi_1 \otimes \phi_2 \mapsto z_1 z_2 \lambda$$

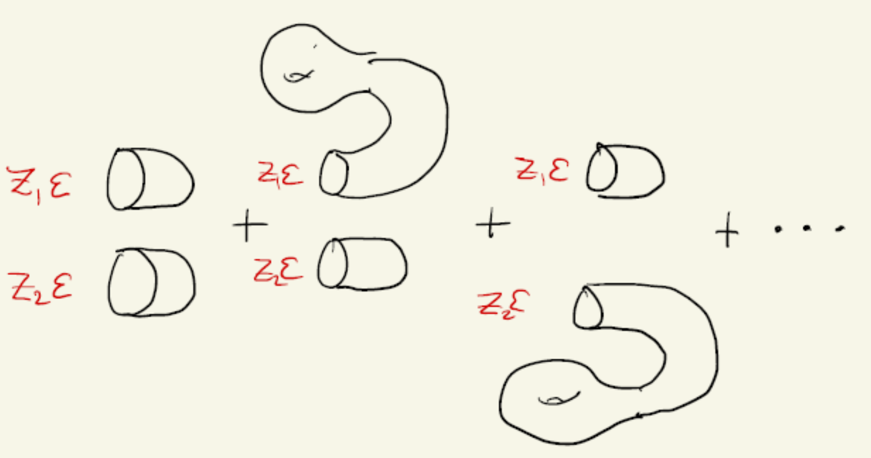


Bordisms with *one connected component* and n ingoing circles:

$$\phi_1 \otimes \cdots \otimes \phi_n \mapsto (z_1 \cdots z_n)\lambda$$

Returning to 2 ingoing circles: We can also have bordisms with two connected components:

$$\phi_1 \otimes \phi_2 \mapsto z_1 z_2 \lambda^2$$



Altogether: $\bar{\mathcal{A}}(2,0)(\phi_1, \phi_2) = z_1 z_2\, e^{\lambda}(\lambda + \lambda^2)$

Marolf-Maxfield recognize $B_2(\lambda)$ as a Bell polynomial $= z_1 z_2\, e^{\lambda} B_2(\lambda)$

Bell Polynomials

$B_n(x_1, \ldots, x_n)$: A polynomial that counts the ways a set of n elements can be partitioned

Coefficient of $x_1^{k_1} x_2^{k_2} \cdots$: counts disjoint decompositions with

k_1 subsets of cardinality 1

k_2 subsets of cardinality 2

Etc.

$$B_n(\lambda) := B_n(\lambda, \lambda, \ldots, \lambda)$$ (Touchard polynomials)

Dividing a bordism $\coprod_1^n S^1 \to \emptyset$ into connected components will have k_j connected components with j ingoing circles. Each such component, when summed over handles gives a factor of λ

Upshot is:

$$\Psi_{HH}^{\vee} = e^{\lambda} \sum_{n=0}^{\infty} B_n(\lambda)(\varepsilon^{\vee})^n$$

Applying $b_\vee$ →

$$\bar{\mathcal{A}} = e^{\lambda} \sum_{n_i, n_o} B_{n_i+n_o}(\lambda)(\varepsilon^\vee)^{n_i} \otimes \left(\frac{\varepsilon}{\theta}\right)^{n_o}$$

$$e^{\lambda} B_n(\lambda) = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^n$$

Used extensively in the Marolf-Maxfield paper.

$$\bar{\mathcal{A}} = e^{\lambda} \sum_{n_i, n_o} B_{n_i+n_o}(\lambda)(\varepsilon^{\vee})^{n_i} \otimes \left(\frac{\varepsilon}{\theta}\right)^{n_o}$$

$$e^{\lambda} B_n(\lambda) = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^n$$

$$\bar{\mathcal{A}} = \sum_{n_i, n_o \geq 0} (\varepsilon^{\vee})^{\otimes n_i} \left(\sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^{n_i+n_o}\right) \otimes \left(\frac{\varepsilon}{\theta}\right)^{\otimes n_o}$$

Frobenius structure gives canonical sesquilinear form

$$\varepsilon^* = \theta\varepsilon^\vee \qquad (\varepsilon^\vee)^* = (\theta^*)^{-1}\,\varepsilon$$

$$\mathcal{W} = Fock(\mathcal{C})$$

For θ real, but not necessarily positive,

$$\bar{\mathcal{A}} = \Phi\Phi^* \qquad \Phi = \sum_{\ell,d\in\mathbb{Z}_+} \sqrt{\frac{\lambda^d}{d!}}\,(d\,\varepsilon^\vee)^\ell \otimes \left(\frac{\varepsilon}{\sqrt{\theta}}\right)^d$$

$$\Phi \otimes \Phi^* = \sum_{\ell_i, d_i, \ell_o, d_o} \sqrt{\frac{\lambda^{d_i}}{d_i!}} \sqrt{\frac{\bar{\lambda}^{d_o}}{d_o!}}$$

$$(d_i\, \varepsilon^\vee)^{\ell_i} \otimes \left(\frac{\varepsilon}{\sqrt{\theta}}\right)^{d_i} \otimes \left(\frac{\theta}{\sqrt{\bar{\theta}}} \varepsilon^\vee\right)^{d_o} \otimes \left(\frac{d_o}{\bar{\theta}} \varepsilon\right)^{\ell_o}$$

Contracting inner two factors puts $d_i = d_o$
and recovers $\bar{\mathcal{A}}$ if θ is real.

Relation To Coherent States - 1/2

$$[a, a^*] = 1 \qquad \frac{1}{\sqrt{d!}}(a^*)^d|0\rangle := |d\rangle \leftrightarrow \left(\frac{\varepsilon}{\sqrt{\theta}}\right)^d \in S^d\mathcal{C}$$

$$\Psi_\lambda := \exp\left(\sqrt{\lambda}\, a^*\right)|0\rangle$$

$$N := a^* a \qquad e^\lambda B_n(\lambda) = \langle \Psi_\lambda, N^n\, \Psi_\lambda \rangle$$

Relation To Coherent States - 2/2

$$Z_{ann}: \mathcal{W} \to \mathcal{C}^{\vee} \otimes \mathcal{W} \qquad |d\rangle \mapsto (d\,\varepsilon^{\vee}) \otimes |d\rangle$$

$$Z_{cr}: \mathcal{W}^{\vee} \to \mathcal{W}^{\vee} \otimes \mathcal{C} \qquad \langle d| \mapsto \langle d| \otimes \left(\frac{d}{\theta}\,\varepsilon\right)$$

$$\bar{\mathcal{A}} = \langle\, \Psi_{\lambda}, \frac{1}{1 - Z_{cr}} \frac{1}{1 - Z_{ann}} \Psi_{\lambda} \,\rangle$$

$$\in\ S^{*}\mathcal{C}^{\vee} \otimes S^{*}\mathcal{C}\ \cong\ End\big(Fock(\mathcal{C})\big)$$

In some sense $\mathcal{W}$ is the Hilbert space of a ``dual quantum mechanical system'' to the ``quantum gravity theory.''

$$\bar{\mathcal{A}} = \langle \frac{1}{1 - Z_{cr}} \Psi_\lambda , \frac{1}{1 - Z_{ann}} \Psi_\lambda \rangle \in End(S^* \mathcal{C})$$

Digression: Formulae are reminiscent of ``quantum mechanics with noncommutative amplitudes''

Standard QM: $\langle \psi_1, \psi_2 \rangle \in \mathbb{C}$

QMNA: $\langle \Psi_1, \Psi_2 \rangle \in$ some C^* —algebra

i.e. Ψ in a Hilbert C* module $\mathcal{E}$

Born Rule ?

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^*\Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators $\mathfrak{B}$ are another C* algebra.

Definition: *QMNA observables* are self-adjoint elements of $\mathfrak{B}$

Definition: A ***QMNA state*** is a completely positive unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of φ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

Family of quantum systems over a noncommutative space.

Reinterpret various constructions in quantum information theory in terms of noncommutative geometry

END OF DIGRESSION

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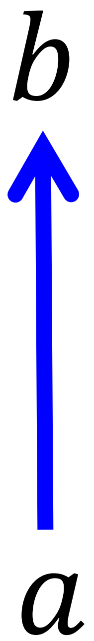
VI. d=2 Open-Closed: Oriented, Semi-simple

In/out manifolds are disjoint unions of circles and oriented intervals

The intervals are 1-morphisms in a category (of manifolds with corners)

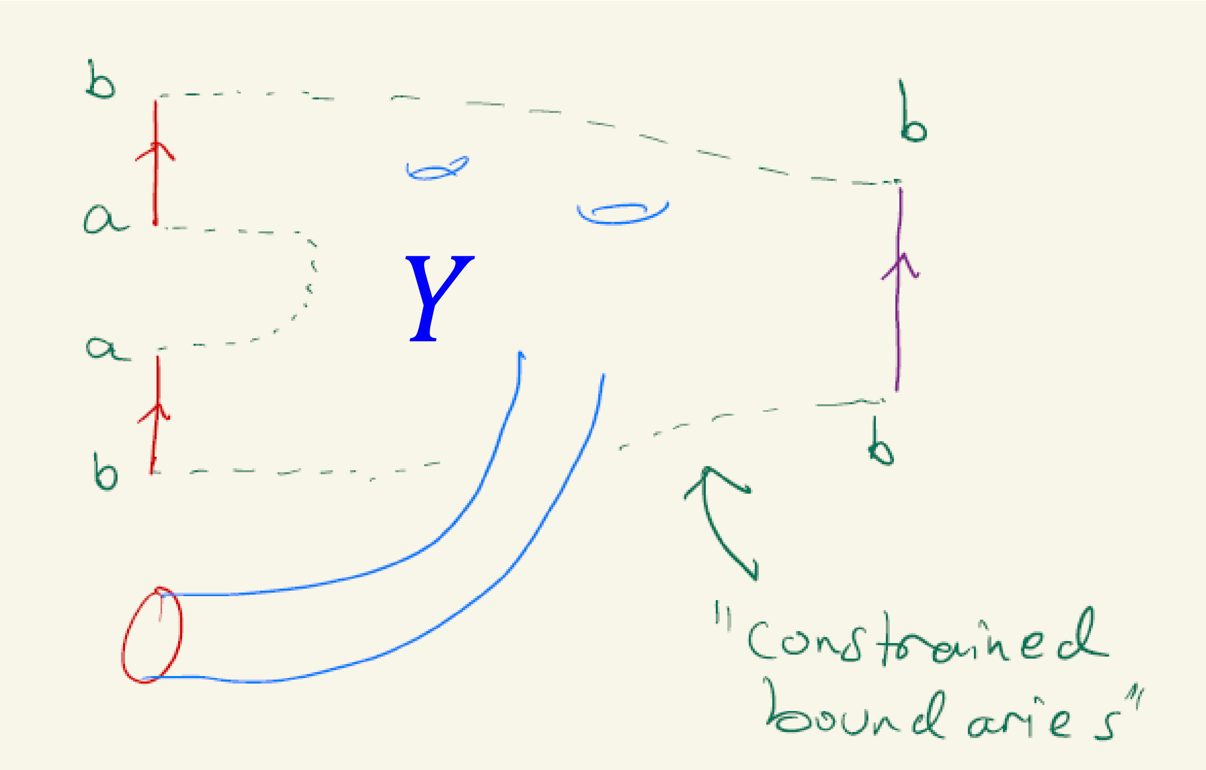


a, b are objects in a category of boundary conditions.

[Moore & Segal]

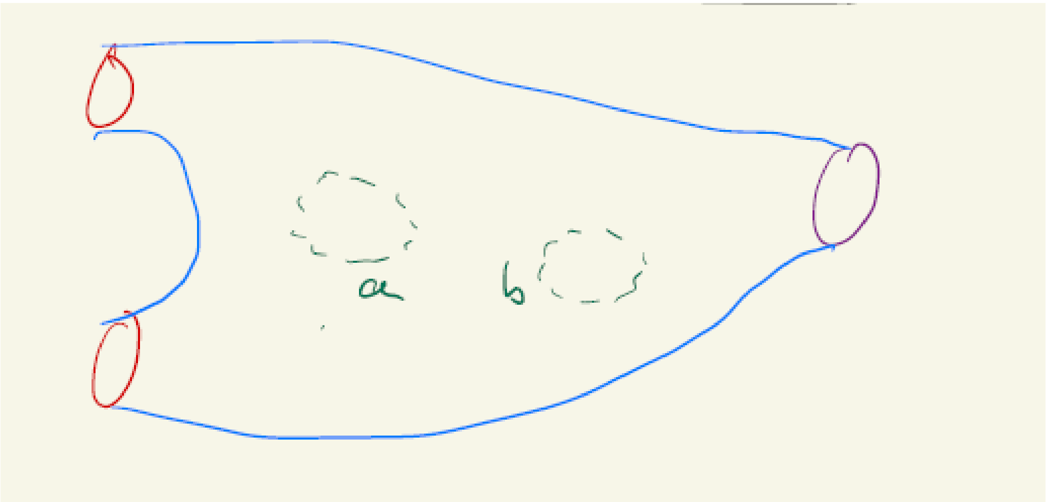
$$\mathcal{Z}(I_{ab}) = Hom(a,b) \coloneqq \mathcal{O}_{ab}$$

The surfaces are now 2-morphisms in a 2-category



$$\partial Y = (\partial Y)_{in} \coprod (\partial Y)_{out} \coprod (\partial Y)_{constrained}$$

We can also have closed constrained boundaries



Three conceptually distinct kinds of boundaries

1. Ingoing/outgoing circles & intervals

2. Constrained boundaries connecting in/out endpoints to in and/or out endpoints of intervals

3. Closed constrained boundaries

Splitting Formula - Simplest Case

For simplicity (we can relax all these conditions):

1. $\dim \mathcal{C} = 1$

2. All constrained boundaries are labeled with single b.c. a with $Hom(a,a) = \mathcal{O}_{aa}$

3. No closed constrained boundaries

4. All in/out manifolds are intervals I_{aa}

μ^{-1} = open string coupling: $\mu^2 = \theta$

For μ real, $\theta > 0$ $\bar{\mathcal{A}} = \Phi\Phi^*$

$$\Phi: \ S^*\mathcal{O}_{aa} \to L^2(\mathcal{E}_{N_a})$$

``Cardy condition'' implies $\mathcal{O}_{aa} \cong Mat_{N_a \times N_a}(\mathbb{C})$ [Moore&Segal]

$\mathcal{E}_{N_a} =$ vector space of $N_a \times N_a$ Hermitian matrices

$$\Phi = \sum_{n} \sum_{S=\{i_1 j_1, i_2 j_2, \ldots i_n j_n\}} \prod_{a=1}^{n} e^{\vee}_{i_a j_a} \int_{\mathcal{E}_{N_a}} [dH]\, e^{-\frac{1}{2}U(H)} \prod_{a=1} H_{j_a i_a} \langle H |$$

e_{ij} : Basis of matrix units for $\mathcal{O}_{aa}$; $e^{\vee}_{ij}$ is the dual basis

$$e^{-U(H)} = \int_{\sqrt{-1}\mathcal{E}_{N_a}} [dS] \exp\left(\left(\frac{\lambda}{\det(1-S)^{\frac{1}{\mu}}}\right) - Tr(SH)\right)$$

Corollary:

$$\Psi^{\vee}_{HH}(e^{T}) = \exp[\, \lambda/(\det(1-T)^{\frac{1}{\mu}}\,)]$$

Related formula in Gardiner-Megas

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G- Equivariant Generalization: Finite Groups

1. Bordism category: Principal G-bundles

2. Replace Frobenius algebra by Turaev algebra $\mathcal{C}$

3. Sum over bordisms splits into two parts: Sum over bundles and sum over topologies of surfaces

4. Sum over bundles is gauging. It simply replaces $\mathcal{C}$ by ss Frobenius algebra $\mathcal{C}^{orb}$ and we are back in the previous case.

Topological Gravity Coupled To 2D YM

G: Compact connected simple Lie group

Morphisms are surfaces with area A , which is additive under gluing.

$$\mathcal{Z}(S^1) = L^2(G)^G \otimes \mathcal{C}$$

ON basis: $\chi_R \otimes \varepsilon/\sqrt{\theta} \quad R \in G^\vee$:

$G^\vee$: Unitary dual: Irreps of G

$$\log \mathcal{A}(\emptyset,\emptyset) = \int_0^\infty \frac{dA}{A}\; A^p \sum_{g=0}^{\infty} \sum_{R\in\mathcal{G}^\vee} \left(\theta \dim(R)^2\right)^{1-g} e^{-A\left(g_{ym}^{-2} C_2(R)+\Lambda_0\right)}$$

$\Lambda_0 > 0$ Cosmological constant

p: Determines the measure for summing over bordisms

Depends on the nature of the quantum gravity we couple to.

$$\log \mathcal{A}(\emptyset,\emptyset) = \int_0^\infty \frac{dA}{A}\; A^p \sum_{g=0}^{\infty} \sum_{R\in \mathcal{G}^\vee} (\theta \dim(R)^2)^{1-g} e^{-A(\mu C_2(R)+\mu_0)}$$

$$= \sum_{R\in G^\vee} \lambda_R \qquad \lambda_R = \frac{\theta(\dim(R)^2)}{1-(\theta(\dim(R))^2)^{-1}} \; \frac{\Gamma(p)}{\left(g_{ym}^{-2} C_2(R) + \Lambda_0\right)^p}$$

Converges for $Re(p) > |\Delta_+(\mathfrak{g})| + \frac{1}{2}$

Expected to admit analytic continuation in p

$$\Psi^{\vee}_{HH}\left(e^{z\,\varepsilon \otimes U}\right) = \prod_{R \in G^{\vee}} \exp[\ \lambda_R \exp\left(\ z \frac{\chi_R(U)}{\dim(R)}\right)]$$

Compare with result for semisimple 2d closed case:

$$\Psi^{\vee}_{HH}(e^{z_x \varepsilon_x}) = \prod_{x \in \mathcal{X}} \exp[\ \lambda_x \exp(\ z_x)\ ]$$

strongly suggests there will be factorization with

$$\mathcal{W} = Fock\left(\mathcal{Z}(S^1)\right) = S^*(\ L^2(G)^G \otimes \mathcal{C})$$

1. Motivation
2. Reminders On TQFT
3. Summed & Total Amplitudes: Splitting Property
4. Example: $d = 1$
5. Example: $d = 2$, closed
6. Example: $d = 2$, open-closed
7. Coupling To 2d YM
8. $d \geq 3$: Comments
9. Summary & Some Open Problems

VIII. Comments On $d \geq 3$

Can we extend these ideas to d=3 TQFT ?

Classification of manifolds is ***MUCH*** more difficult !!

$$\mathcal{A}(\emptyset, \emptyset) = \exp(\sum_{Y} \mathcal{Z}(Y))$$

Sum over closed connected 3-folds Y

That includes the sum over $Y = S^1 \times \Sigma_g$

$$\mathcal{Z}(Y) = \dim \mathcal{Z}(\Sigma_g)$$

For standard fully local TQFT, $\dim \mathcal{Z}(\Sigma_g)$ grows with g

The sum is irretrievably divergent.

Can we have $\dim \mathcal{Z}(\Sigma_g) = 0$ for sufficiently large g ?

Sergei Gukov: No!

Cut along the boundary of a handlebody for any g

If $\dim \mathcal{Z}(\Sigma_g) = 0$ for <u>any</u> g then all amplitudes vanish!!

Is there some way to modify the domain and/or codomain categories to produce interesting examples for $d>2$?

IX. Summary And Open Problems

For suitable parameters of our TQFT, the total amplitude

$$\bar{\mathcal{A}} \in End(\otimes_X S^*(\mathcal{Z}(X))) := End\big(Fock(\mathcal{Z})\big)$$

Has a splitting: $\bar{\mathcal{A}} = \Phi\Phi^*$

$$\Phi: Fock(\mathcal{Z}) \to \mathcal{W}$$

We also worked out some examples for non-semi-simple d=2 TQFT. The splitting persists

Potential conceptual interpretations: Holography & QMNA

Extensions of the d=2 results

1.The general non-semisimple case, closed, and open

2. Other tangential structures: Unorientable, (s)pin, …

3. Topological string theory: $\bar{\mathcal{A}} \in End\left(S^* H_q^*(X)\right)$

4. A splitting formula for JT gravity might have interesting implications for the ongoing discussion about the role of ensemb averages in AdS/CFT

Is the existence of a splitting formula deep or a trivial consequence of linear algebra ?

Rough idea: The total amplitude is symmetric under exchange of all in-going boundaries for all out-going boundaries.

But any symmetric (f.d. complex) matrix S can be written as $S = \Phi\Phi^{tr}$

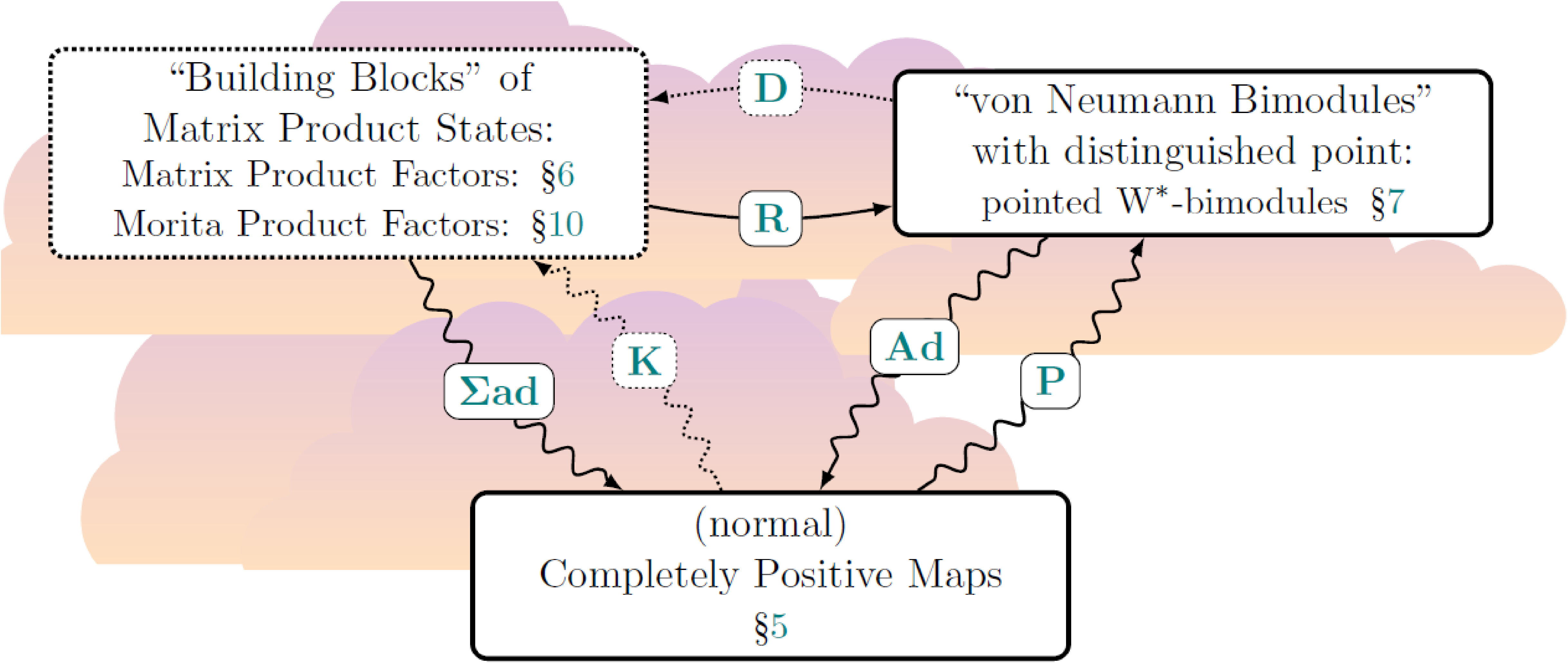
If it doesn't just follow by linear algebra, is there an a priori reason why it should hold?

And what to do about $d \geq 3$???

… and before we finish …

… an advertisement for an upcoming paper with Roman Geiko and Tom Mainiero

Equivalence of 2-categories of



Thanks for your attention!

SUPPLEMENT 1

MM construction of ``baby universe Hilbert space''

A sesquilinear form on $S^*\mathcal{C}$ is defined by

$$\langle \phi_1, \phi_2 \rangle = \Psi_{HH}^{\vee}(K(\phi_1)\phi_2)$$

$$\phi = \sum_{n=0}^{\infty} c_n \varepsilon^n \;\to\; f_\phi(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\langle \phi_1, \phi_2 \rangle = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} \left(f_1(d)\right)^* f_2(d)$$

$Ann(\langle\cdot,\cdot\rangle) \cong$ A vector space of order ≤ 1 entire functions that vanish on $\mathbb{Z}_+$

$S^*\mathcal{C}$ is viewed as a $*$-algebra.

MM then imitate the GNS construction and define a ``baby universe Hilbert space''

$$\mathcal{H}_{BU} := S^*\mathcal{C}/Ann(\langle\cdot,\cdot\rangle)$$

$$\cong \{ (\xi_0, \xi_1, \ldots) \in \mathbb{C}^\infty \mid \sum \frac{\lambda^d}{d!} |\xi_d|^2 < \infty \}$$

$(\lambda > 0)$ $\cong$ H.O. representation of Heisenberg algebra

Expectation values in a coherent state are then interpreted as stochastic expectations of a ``universe creation operator Z ''

$\Psi_{HH}^{\vee}$ is viewed as defining an expectation value on polynomials in a stochastic variable Z on $S^*\mathcal{C}$ where $Z(\varepsilon)$ has the interpretation of the partition function of a 1d TQFT chosen from an ensemble with Poisson probability distribution

$$p(d) = e^{-\lambda} \frac{\lambda^d}{d!}$$

for an ensemble of 1d TQFTs with $\dim V = d$.

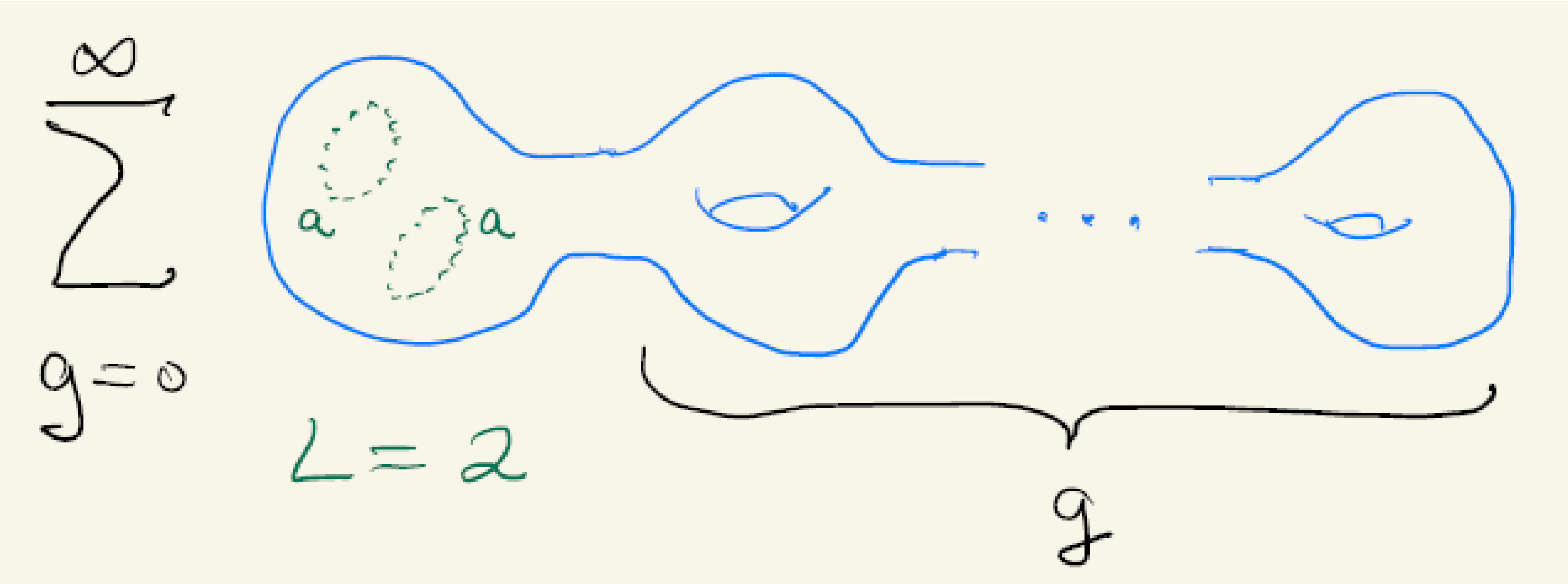
END OF SUPPLEMENT 1

BEGIN SUPPLEMENT 2

VII. Constrained Boundaries & An Ensemble Interpretation

MM paper aimed to give an interpretation of the 2d model in terms of an ensemble average of 1d models.

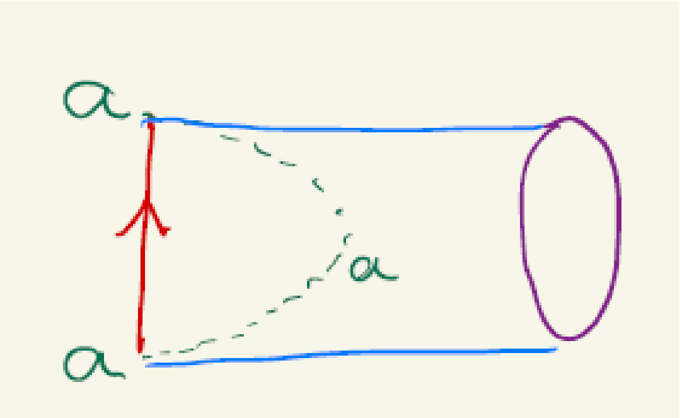
Sum over bordisms $\emptyset \to \emptyset$ with L constrained boundaries of type a



+ Disconnected surfaces

$$\frac{\bar{\mathcal{A}}_{\{L\}}(\emptyset,\emptyset)}{\bar{\mathcal{A}}(\emptyset,\emptyset)} = B_L(x_1,\ldots,x_L) \qquad x_j = \theta_{\mathcal{C}}\left(\frac{1}{1-h}B_a^j\right)$$

$$B_a := \iota_a\left(1_{\mathcal{O}_{aa}}\right) \in \mathcal{C}$$



$$\dim\mathcal{C} = 1: \qquad B_a = \frac{N_a}{\mu}\,\varepsilon$$

$$\frac{\bar{\mathcal{A}}_{\{L\}}(\emptyset,\emptyset)}{\bar{\mathcal{A}}(\emptyset,\emptyset)} = \left(\frac{1}{\mu}\right)^L \sum_{d=0}^{\infty} e^{-\lambda} \frac{\lambda^d}{d!} \, (dN_a)^L$$

$$= \left(\frac{1}{\mu}\right)^L \langle \mathcal{Z}(S^1)^L \rangle_{\mathcal{E}}$$

$\mathcal{Z}$ is a stochastic variable on an ensemble $\mathcal{E}$ of 1d oriented TQFT's $\mathcal{Z}_d$ labeled by $d \in \mathbb{Z}_+$ with

$$p(\mathcal{Z}_d) = e^{-\lambda} \frac{\lambda^d}{d!} \qquad \mathcal{Z}_d(S^1) = \dim V_d = d\, N_a$$

It would be interesting to give an ensemble interpretation to the full set of open/closed amplitudes.

This suggests it could be interesting to consider TQFT's where the target category is the category of f.d. vector bundles over measure spaces as a way to model ensemble averages of field theories.

END OF SUPPLEMENT 2

BEGIN SUPPLEMENT 3

Quantum Systems

Set of physical ``states'' $\mathcal{S}$

Set of physical ``observables'' $\mathcal{O}$

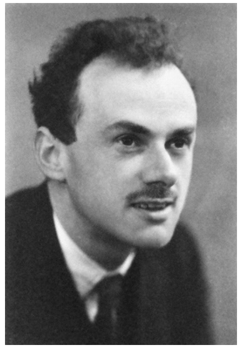
Born Rule: $BR : \mathcal{S} \times \mathcal{O} \to \mathcal{P}$

$\mathcal{P}$ Probability measures on $\mathbb{R}$.

$m \in \mathfrak{M}(\mathbb{R})$ ⟹ $0 \leq \wp(m) \leq 1$

$m = [r_1, r_2] \subset \mathbb{R}$ $BR(\mathrm{s}, \mathrm{O})([r_1, r_2])$

is the probability that a measurement of the observable O in the state s has value between r_1 and r_2 .



Dirac-von Neumann Axioms

$\mathcal{S}$ Density matrices ρ: Positive trace class operators on Hilbert space of trace =1

$\mathcal{O}$ Self-adjoint operators T on Hilbert space

Spectral Theorem: There is a one-one correspondence of self-adjoint operators T and projection valued measures:

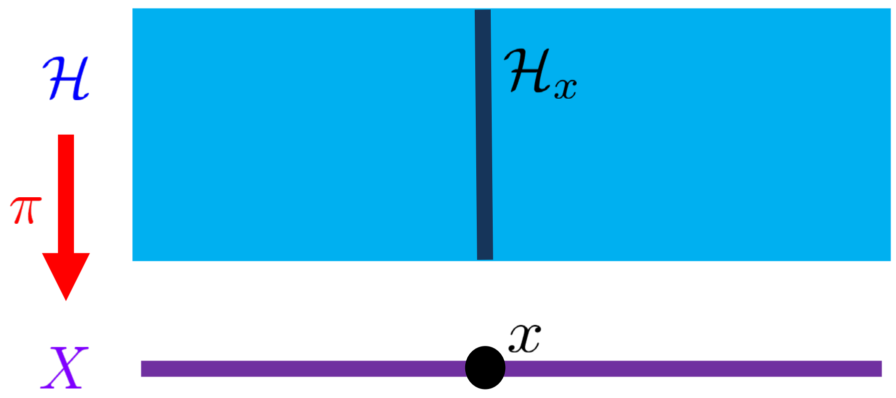
$$m \in \mathfrak{M}(\mathbb{R}) \quad \rightarrow \quad P_T(m)$$

Example: $T = \sum_\lambda \lambda P_\lambda \qquad P_T([r_1, r_2]) = \sum_{r_1 \leq \lambda \leq r_2} P_\lambda$

$$m \in \mathfrak{M}(\mathbb{R}) \qquad BR(\rho, T)(m) = \mathrm{Tr}_{\mathcal{H}} \left(\rho P_T(m) \right)$$

Continuous Families Of Quantum Systems

Hilbert bundle over space X of control parameters.



For each x get a probability measure $\wp_x$:

$$m \in \mathfrak{M}(\mathbb{R}) \mapsto \wp_x(m) := \mathrm{Tr}_{\mathcal{H}_x}(\rho_x P_{T_x}(m))$$

$$BR : \mathcal{S} \times \mathcal{O} \times X \to \mathcal{P}$$

$$BR(\rho, T, x) = \wp_x$$

Noncommutative Control Parameters

We would like to define a family of quantum systems parametrized by a NC manifold whose ``algebra of functions'' is a general C* algebra $\mathfrak{A}$

What are observables?

What are states?

What is the Born rule?

What replaces the Hilbert bundle?

Noncommutative Hilbert Bundles

Definition: Hilbert C* module $\mathcal{E}$ over C*-algebra $\mathfrak{A}$.

Complex vector space $\mathcal{E}$ with a right-action of $\mathfrak{A}$
and an ``inner product'' valued in $\mathfrak{A}$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

$$(\Psi_1, \Psi_2)^*_{\mathfrak{A}} = (\Psi_2, \Psi_1)_{\mathfrak{A}}$$

$(\Psi, \Psi)_{\mathfrak{A}} \geq 0$ (Positive element of the C* algebra.)

$$(\Psi_1, \Psi_2 a) = (\Psi_1, \Psi_2) a \quad .....$$

Like a Hilbert space, but ``overlaps'' are valued in a (possibly) noncommutative algebra.

Quantum Mechanics With Noncommutative Amplitudes

Basic idea: Replace the Hilbert space by a Hilbert C* module

$$\mathcal{H} \to \mathcal{E}$$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

Overlaps are valued in a possibly noncommutative algebra.

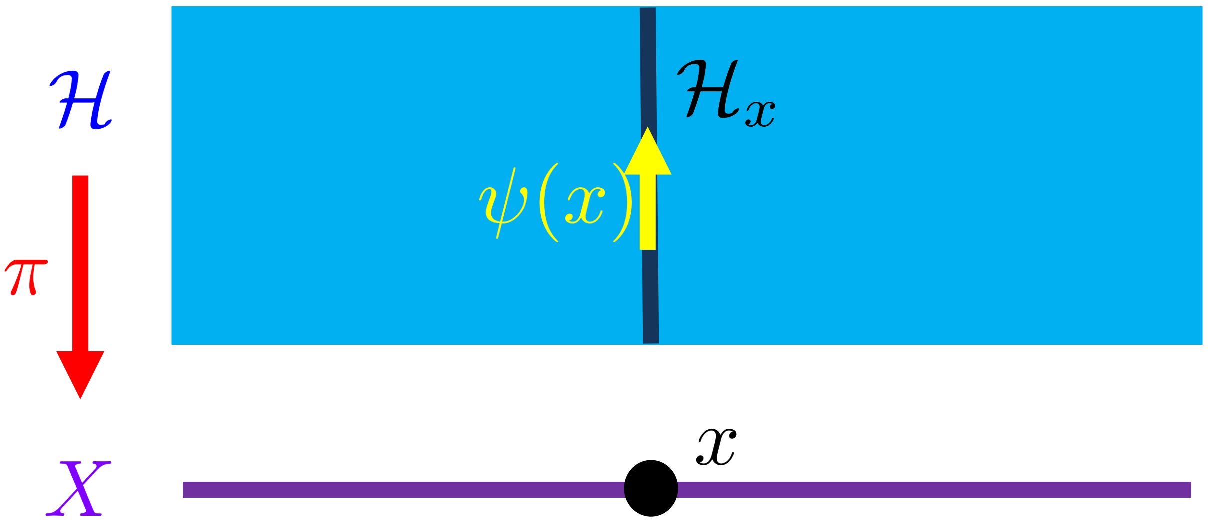
QM: $0 \leq \wp(\lambda) = (\psi_\lambda, \psi)(\psi_\lambda, \psi)^* \leq 1$

QMNA: $(\Psi_\lambda, \Psi)(\Psi_\lambda, \Psi)^* \in \mathfrak{A}$

Example 1: Hilbert Bundle Over A Commutative Manifold

$$\mathcal{E} = \Gamma[\mathcal{H} \to X] \qquad \mathfrak{A} = C(X)$$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$



$$(\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A} := C(X)$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}}(x) := (\psi_1(x), \psi_2(x))_{\mathcal{H}_x} \in \mathbb{C}$$

Example 2: Hilbert Bundle Over A Fuzzy Point

Def: ``fuzzy point'' has $\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C})$

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C})$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}} = \Psi_1^{\dagger}\Psi_2$$

Observables In QMNA

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^*\Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators
$\mathfrak{B}$ are another C* algebra.

Definition: *QMNA observables*
are self-adjoint elements of $\mathfrak{B}$

(Technical problem: There is no spectral theorem for self-adjoint elements of an abstract C* algebra.)

C* Algebra States

Definition: A *C*-algebra state* $\omega \in \mathcal{S}(\mathfrak{A})$ is a positive linear functional

$$\omega : \mathfrak{A} \to \mathbb{C} \qquad \omega(\mathbf{1}) = 1$$

$$\mathfrak{A} = C(X) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$\omega(f) = \int_X f d\mu$ $\quad d\mu$ = a positive measure on X:

$$\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C}) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$\omega(T) = \mathrm{Tr}_{\mathcal{H}}(\rho T)$ $\quad \rho$ = a density matrix

QMNA States

Definition: A ***QMNA state*** is a completely positive unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

``Completely positive'' comes up naturally both in math and in quantum information theory.

Positive: $\varphi : \mathfrak{B}_{\geq 0} \to \mathfrak{A}_{\geq 0}$

Unital: $\varphi(1_{\mathfrak{B}}) = 1_{\mathfrak{A}}$

Completely positive

$$\varphi \otimes 1 : (\mathfrak{B} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0} \to (\mathfrak{A} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0}$$

QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of φ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

Family Of Quantum Systems Over A Fuzzy Point

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C}) = \mathbb{C}^b \otimes \mathbb{C}^a = \mathcal{H}_{\mathrm{Bob}} \otimes \mathcal{H}_{\mathrm{Alice}}$$

$$\mathfrak{A} = Mat_a(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Alice}})$$

$$\mathfrak{B} = Mat_b(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Bob}})$$

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

``A NC measure $\omega \in \mathcal{S}(\mathfrak{A})$'' is equivalent to a density matrix ρ_A on $\mathcal{H}_A$

QMNA state:

$$\varphi(T) = \sum_\alpha E_\alpha^\dagger T E_\alpha \quad \sum_\alpha E_\alpha^\dagger E_\alpha = 1$$

Quantum Information Theory
& Noncommutative Geometry

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

$$= \sum_\alpha \mathrm{Tr}_{\mathcal{H}_A} \rho_A E_\alpha^\dagger (P_T(m)) E_\alpha$$

$$= \sum_\alpha \mathrm{Tr}_{\mathcal{H}_B} E_\alpha \rho_A E_\alpha^\dagger P_T(m)$$

$$= \mathrm{Tr}_{\mathcal{H}_B} \mathcal{E}(\rho_A) P_T(m)$$

Last expression is the measurement by Bob of T in the state ρ_A prepared by Alice and sent to Bob through quantum channel $\mathcal{E}$.