Partition Functions Of Twisted Supersymmetric Gauge Theories On Four-Manifolds Via u-Plane Integrals

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Mostly review. Includes new work with Iurii Nidaiev & Jan Manschot

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1. Introduction

2. Cambrian: Witten Reads Donaldson

3. Carboniferous: SW Theory & u-plane Derivation Of Witten's Conjecture.

4. Holocene: Three New Apps

5. Holomorphic Anomaly & Continuous Metric Dep.

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Donaldson Invariants Of 4-folds

X: Smooth, compact, oriented, $\partial X = \emptyset$, $(\pi_1(X) = 0)$

$P \rightarrow X$: Principal SO(3) bundle.

X has metric $g_{\mu\nu}$. Consider moduli space of instantons:

$$\mathcal{M} := \{ A: F + * F = 0\} \, mod \, \mathcal{G}$$

Donaldson defines cohomology classes in $\mathcal{M}$ associated to points and surfaces in X : $\mu(p)$ & $\mu(S)$

$$\wp_D(p^\ell S^r) := \int_{\mathcal{M}} \mu(p)^\ell \, \mu(S)^r$$

Independent of metric! $\Rightarrow$ smooth invariants of X.

Combined with Freedman theorem: Spectacular!

Witten's Interpretation: Topologically Twisted SYM On X

Consider $\mathcal{N} = 2$ SYM theory on X for gauge group G

Witten's ``topological twisting'': Couple to special external gauge fields for certain global symmetries.

Result: Fermion fields and susy operators are differential forms; The twisted theory is defined on non-spin manifolds.

And there is a scalar susy operator with $Q^2 = 0$

Formally: Correlation functions of operators in Q-coho. localize to integrals over the moduli spaces of G-ASD connections (or generalizations thereof).

Witten's proposal: For $G = SU(2)$ correlation functions of the special operators are the Donaldson polynomials.

Local Observables

$$\wp \in Inv(\mathfrak{g}) \Rightarrow \quad U = \wp(\phi) \qquad \phi \in \Omega^0(adP \otimes \mathbb{C})$$

$$\begin{tikzcd}
H_*(X,\mathbb{C}) \arrow[rr, "\text{Descent formalism}"] \arrow[dr, "\mu"'] & & H^*(Fieldspace; Q) \arrow[dl, "\text{Localization identity}"] \\
& H^*(\mathcal{M}) &
\end{tikzcd}$$

$$\mathfrak{g} = \mathfrak{su}(2) \qquad U(p) = Tr_2\left(\frac{\phi^2}{8\pi^2}\right) \qquad U(S) \sim \int_S Tr(\phi F + \psi^2)$$

Donaldson-Witten Partition Function

$$Z_{DW}(p,S) = \langle e^{U(p)+U(S)} \rangle_{\Lambda_0}$$

$$\int_X Tr\, F^2 = 8\,\pi^2 k$$

$$= \sum_{k\geq 0} \Lambda_0^{4k} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}$$

Strategy: Evaluate in LEET $\Rightarrow$ Witten (1994) introduces the Seiberg-Witten invariants.

Major success in Physical Mathematics.

What About Other N=2 Theories?

Natural Question: Given the successful application of $\mathcal{N}=2$ SYM for SU(2) to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N}=2$ field theories?

Topological twisting just depends on $SU(2)_R$ symmetry and makes sense for any $\mathcal{N}=2$ theory.

$$Z^{\mathcal{T}} := \langle e^{U(p)+U(S)} \rangle_{\mathcal{T}}$$

Also an interesting exercise in QFT to compute correlation functions of nontrivial theories in 4d.

SU(2) With Matter On X

$\mathcal{R} = 2^{\oplus N_{fl}}$ Mass parameters $m_f \in \mathbb{C},\ f = 1, \ldots, N_{fl}$

$M \in \Gamma(S^+ \otimes E^{\oplus N_{fl}})$ ⇒ $\mathrm{w}_2(\mathrm{P}) = w_2(X)$

$\mathcal{N} = 2^*\!: \mathcal{R} = (adj_{\mathbb{C}} \oplus adj_{\mathbb{C}}^*)$ $m \in \mathbb{C}$

$M \in \Gamma(S^+ \otimes \mathcal{L}^{\frac{1}{2}} \otimes adP)$ ⇒ $c_1(\mathcal{L}) \equiv w_2(X)$

[Labastida-Marino '98]

UV Interpretation

$$Z(p,S) = \langle e^{U(p)+U(S)} \rangle_{\mathcal{T}}$$

$$= \sum_k \Lambda^k \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)} \mathcal{E}$$

But now $\mathcal{M}_k$: is the moduli space of:

$$F^+ = \mathcal{D}(M, \bar{M}) \qquad \gamma \cdot D\, M = 0$$

``Generalized monopole equations”

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

U(1) case: Seiberg-Witten equations.

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Coulomb Branch Vacua On $\mathbb{R}^4$

$SU(2) \rightarrow U(1)$ by vev of adjoint Higgs field ϕ:

Order parameter: $u = \langle\, U(p) \rangle \in \mathbb{C}$

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/W \cong \mathbb{C}$ $\quad adP \rightarrow L^2 \oplus \mathcal{O} \oplus L^{-2}$

Photon: Connection A on L $\quad U(1)\ VM$: (a, A, χ, ψ, η)

a: complex scalar field on $\mathbb{R}^4$:

Compute couplings in U(1) LEET.

Then compute path integral with this action.

Then integrate over vacua.

Seiberg-Witten Theory: 1/2

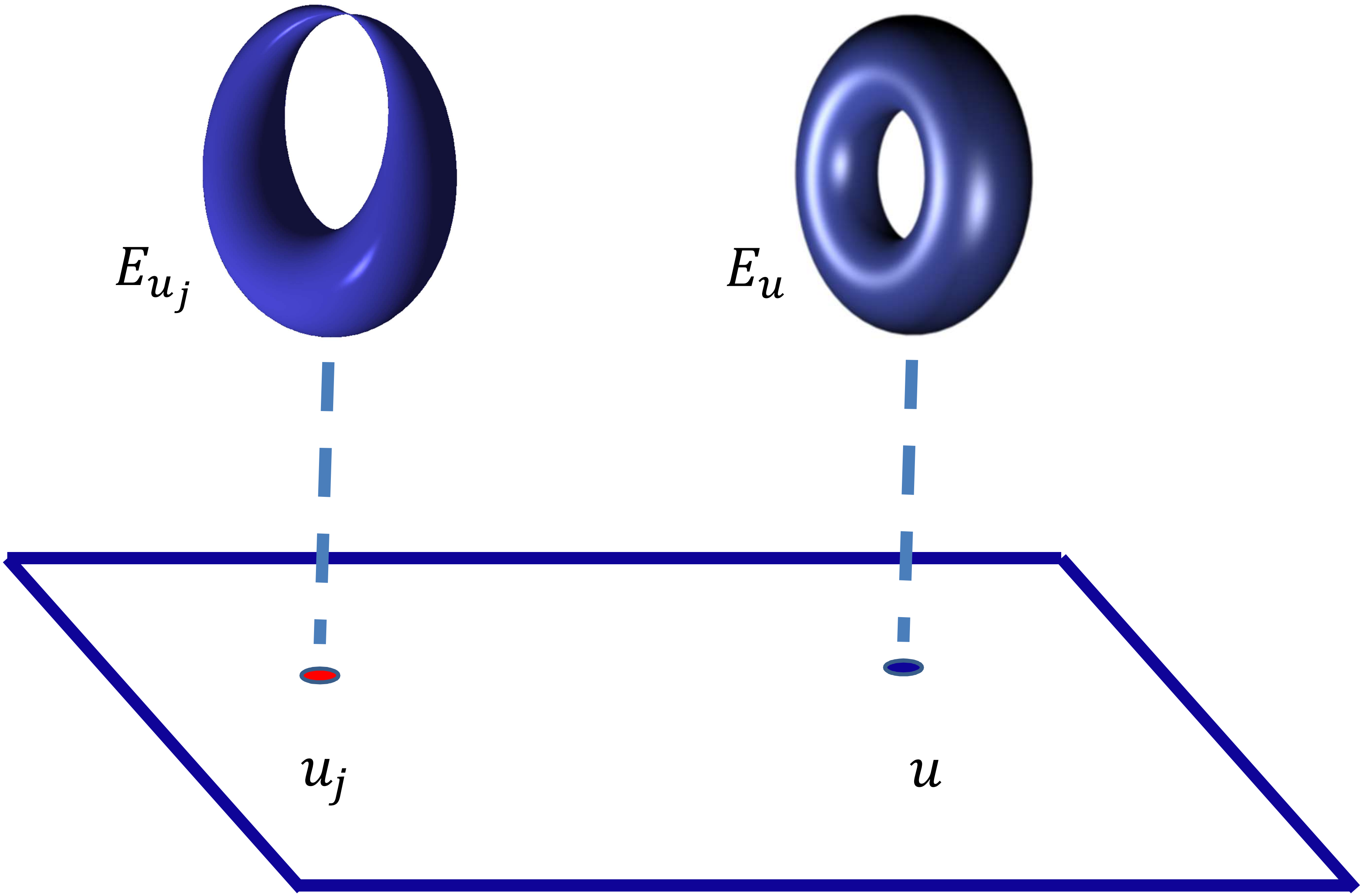
For G=SU(2) SYM coupled to matter the LEET can be deduced from a holomorphic family of elliptic curves with differential:

$$E_u: \quad y^2 = 4x^3 - g_2 x - g_3 \qquad \frac{d\lambda}{du} = \frac{dx}{y} \qquad u \in \mathbb{C}$$

g_2, g_3 are polynomials in u, masses, Λ, modular functions of τ_0

$\Delta := 4(g_2^3 - 27\, g_3^2)$: polynomial in u

$\Delta(u_j) = 0$: Discriminant locus



E_{u_j}
E_u
u_j
u

Examples From SW '94

$$N_{fl} = 0: \quad y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)$$

$$\mathcal{N} = 2^*: y^2 = \prod_{i=1}^{3}(x-\alpha_i) \qquad \alpha_i = u\, e_i(\tau_0) + \frac{m^2}{4} e_i(\tau_0)^2$$

$$N_{fl} = 4: \quad y^2 = W_1 W_2 W_3 + \eta^{12} \sum_{i=1}^{3} R_4^i W_i + \eta^{24} R_6$$

$$W_i = x - u\, e_i(\tau_0) - R_2 e_i(\tau_0)^2$$

Δ: 6th order polynomial in u with $\sim 3 \times 10^3$ terms

$$\mathfrak{m} = \begin{pmatrix} 0 & m_1 & & & & & & \\ -m_1 & 0 & & & & & & \\ & & 0 & m_2 & & & & \\ & & -m_2 & 0 & & & & \\ & & & & 0 & m_3 & & \\ & & & & -m_3 & 0 & & \\ & & & & & & 0 & m_4 \\ & & & & & & -m_4 & \end{pmatrix} \in \mathfrak{spin}(8)$$

$$R_2 \sim Tr_{8_i}\mathfrak{m}^2 \qquad R_4^i \sim Tr_{8_i}\mathfrak{m}^4 \qquad R_6 \sim Tr_{8_i}\mathfrak{m}^6$$

$$SL(2,\mathbb{Z}) \to Out\big(\mathfrak{spin}(8)\big) \cong S_3$$

Local System Of Charges

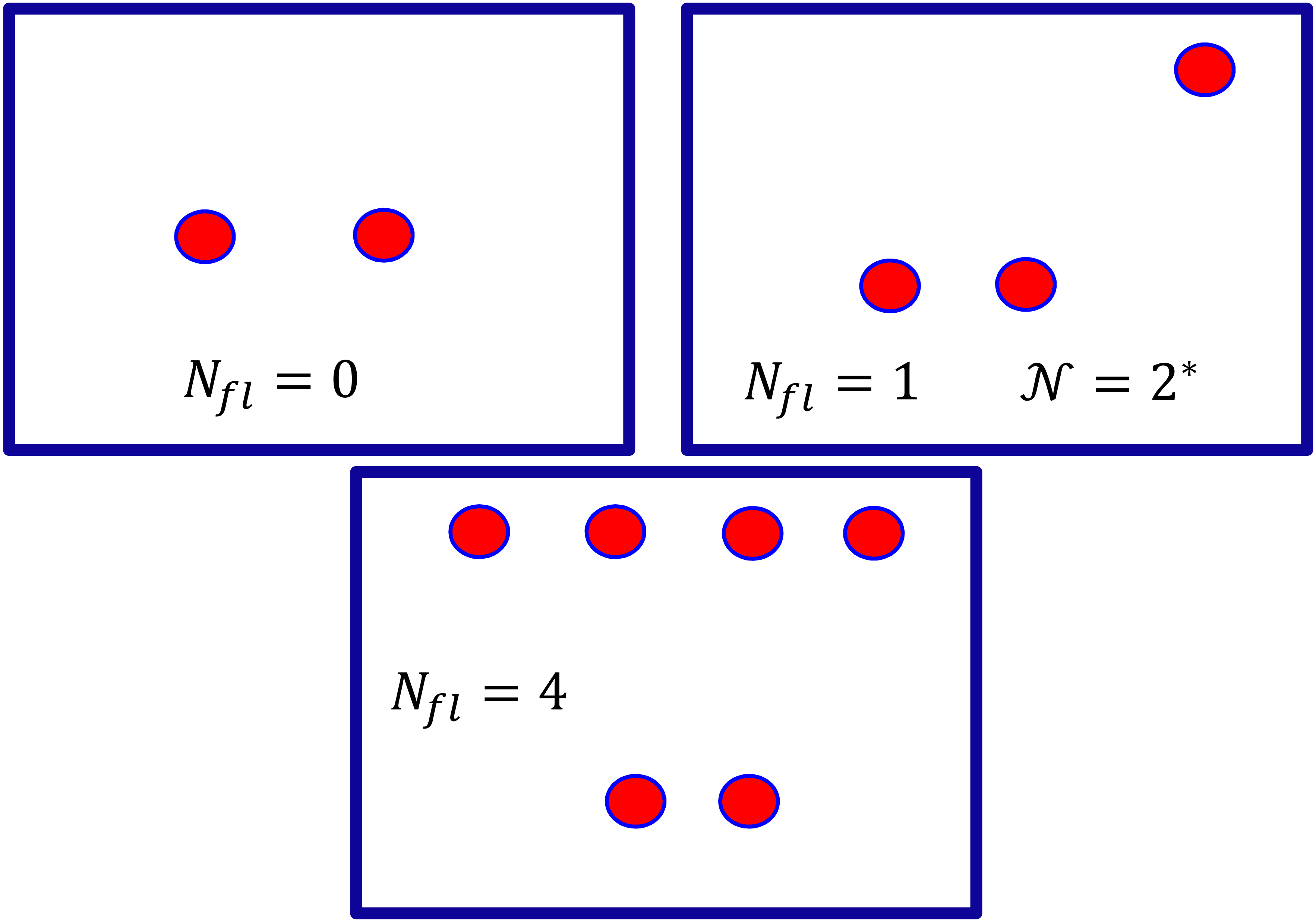
Electro-mag. charge lattice: $\Gamma_u = H_1(E_u;\mathbb{Z})$

has nontrivial monodromy around discriminant locus: $\Delta(u_j) = 0$

LEET: Requires choosing a duality frame:

$$\Gamma_u \cong \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m \Rightarrow \tau(u)$$

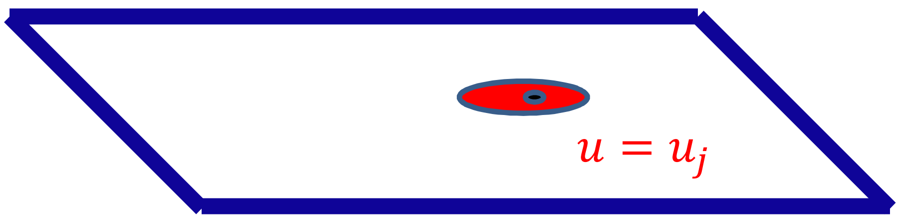
$$S \sim \int_X \bar{\tau} F_+^2 + \tau F_-^2 + \cdots$$



LEET breaks down at $u = u_j$ where $Im(\tau) \to 0$

Seiberg-Witten Theory: 2/2

LEET breaks down because there are new massless fields associated to BPS states



Charge 1 HM: $(M = q \oplus \tilde{q}^*, \cdots)$

LEET in $\mathcal{U}_j$: $+$

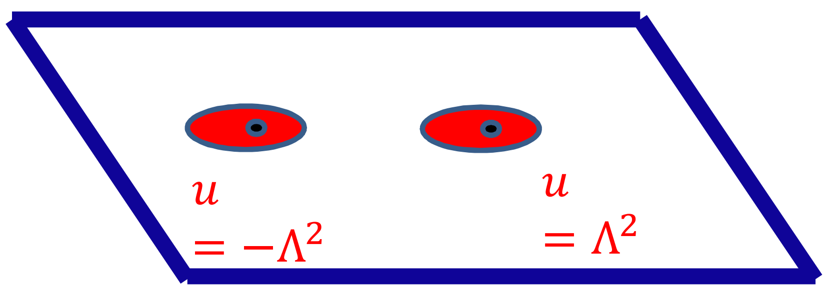
$U(1)_j\ VM$: $(a, A, \chi, \psi, \eta)_j$

Evaluate Z_{DW} Using LEET

$$Z_{DW}(p,S) = \langle e^{U(p)+U(S)} \rangle_{\Lambda}$$

$$= \sum_{k} \Lambda^{k} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}\,\mathcal{E}$$

$$\boxed{Z_{DW} = Z_u + \sum_{u_j} Z_j^{SW}}$$



u-Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$. When $b_2^+ = 1$:

$$Z_u = \int du\, d\bar{u}\ \mathcal{H}\ \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: Sum over line bundles for the U(1) photon.

(Remnant of sum over SU(2) gauge bundles.)

$$\Psi \sim \sum_{\lambda = c_1(L)} e^{-i\,\pi\bar{\tau}(\bar{u})\lambda_+^2 - i\,\pi\tau(u)\lambda_-^2}$$

NOT holomorphic and ***metric- DEPENDENT***

Comments On Z_u

Z_u is a very subtle integral.

It requires careful regularization and definition.

It is also related to integrals from number theory such as `` Θ –lifts'' and mock modular forms.
It is very nearly an integral of a total derivative…

But first let's finish writing down the full answer for the partition function.

Contributions From $\mathcal{U}_j$

Path integral for $U(1)_j$ VM + HM: LEET: Need unknown couplings: $\boldsymbol{C(u)}^{\lambda^2} P(u)^{\sigma} E(u)^{\chi}$

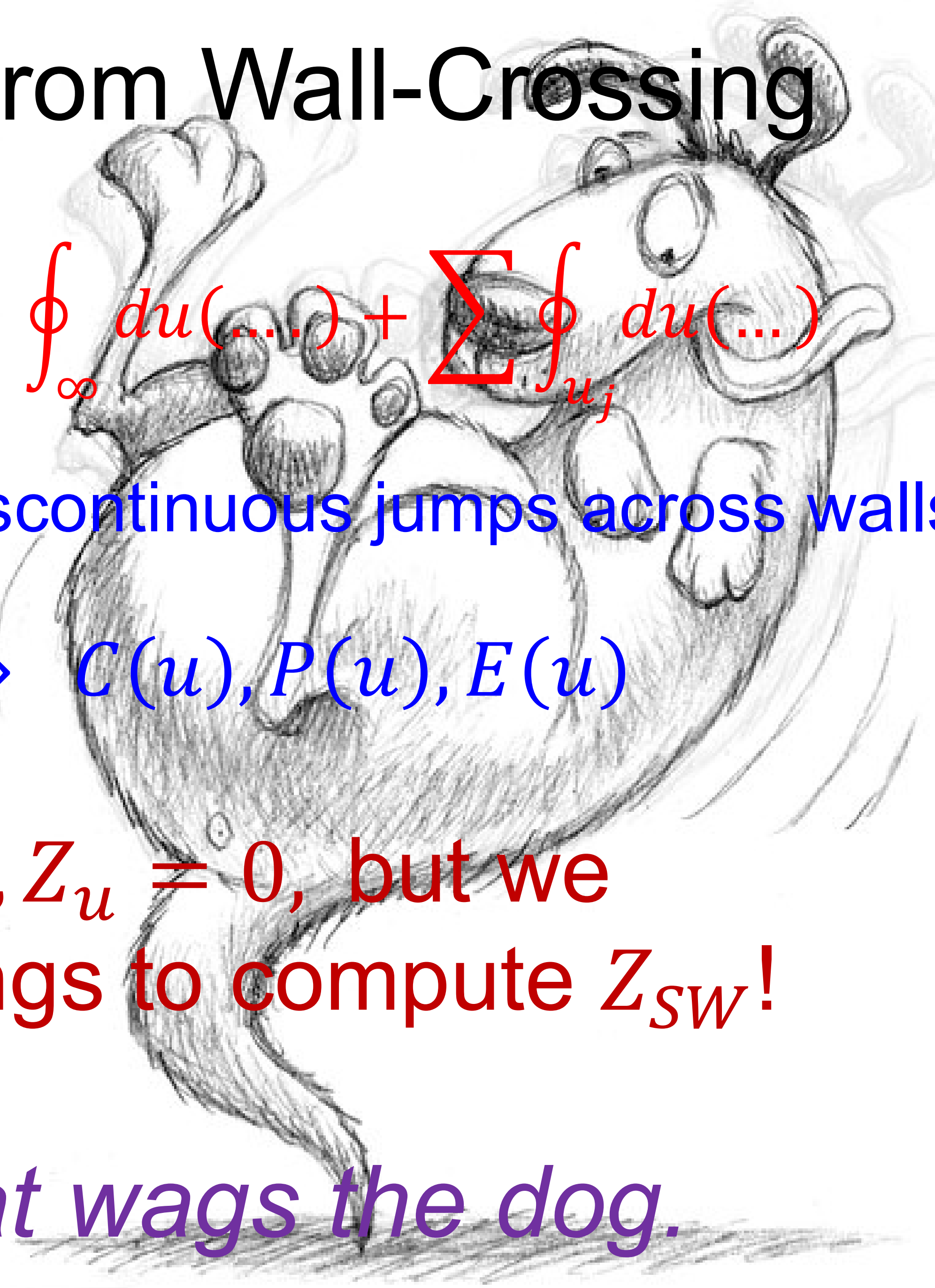
$$\sum_{\lambda \in \frac{1}{2} w_2(X) + H^2(X,\mathbb{Z})} SW(\lambda) e^{2\pi i\, \lambda\cdot\lambda_0}\, R_\lambda(p,S)$$

$$R_\lambda(p,S) = Res\left[\left(\frac{d a_j}{a_j^{1+\frac{d(\lambda)}{2}}} \right) \boldsymbol{e}^{\boldsymbol{2pu+S^2T(u)+i\left(\frac{du}{da_j}\right)S\cdot\lambda}} \boldsymbol{C(u)}^{\lambda^2} P(u)^{\sigma} E(u)^{\chi} \right]$$

Special coordinate: $a_j \quad u = u_j + \kappa_j a_j + \mathcal{O}(a_j^2)$

$$d(\lambda) = \frac{(2\lambda)^2 - c^2}{4} \qquad c^2 = 2\chi + 3\,\sigma$$

Deriving C,P,E From Wall-Crossing



$$\frac{d}{dg_{\mu\nu}} Z_u = \int Tot\ deriv = \oint_{\infty} du(\ldots) + \sum \oint_{u_j} du(\ldots)$$

Z_u piecewise constant: Discontinuous jumps across walls:

$$\Delta_j Z_u + \Delta Z_j^{SW} = 0 \Rightarrow \quad C(u), P(u), E(u)$$

Then for $b_2^+ > 1$, $Z_u = 0$, but we know the couplings to compute Z_{SW}!

Z_u is the tail that wags the dog.

Witten Conjecture

$\Rightarrow$ a formula for all X with $b_2^+ > 0$. For $b_2^+(X) > 1$ we derive ``Witten's conjecture'':

$$Z_{DW}(p,S) = 2^{c^2-\chi_h}\,(\,e^{\frac{1}{2}S^2+2p}\sum_\lambda SW(\lambda)e^{2\pi i\,\lambda\cdot\lambda_0}e^{S\cdot\lambda} +$$

$$+\,i^{\chi_h}e^{-\frac{1}{2}S^2-2p}\sum_\lambda SW(\lambda)e^{2\pi i\,\lambda\cdot\lambda_0}e^{-iS\cdot\lambda})$$

$$\chi_h = \frac{\chi+\sigma}{4}$$

$$c^2 = 2\chi + 3\,\sigma$$

Example: $X = K3$:

$$Z = \Lambda_0^6 \sinh\left(\frac{(\Lambda_0 S)^2}{2} + 2\Lambda_0^2\,p\right)$$

SWST

SWST = <u>S</u>eiberg-<u>W</u>itten <u>S</u>imple <u>T</u>ype

$SW(\lambda) \neq 0$ ONLY when moduli space of solutions to Seiberg-Witten is zero-dimensional.

All known X with $b_2^+ > 1$ are of SWST

Generalization To $N_{fl} > 0$

$b_2^+ > 1$ $\qquad Z(p;S;m_f) = \sum_{j=1}^{2+N_{fl}} Z(p,S;m_f;u_j)$

$$Z(p,S;m_f;u_j) = \sum_{\lambda} SW(\lambda) e^{2\pi i\, \lambda\cdot\lambda_0} R_j(p,S)$$

X is SWST $\Rightarrow$

$R_j(p,S)$ is computable explicitly as a function of $p, S, m_f, \Lambda(\tau_0)$ from first order degeneration of the SW curve.

$$Z(p, S; m_f; \Lambda) =$$

$$\alpha^{\chi} \beta^{\sigma} \sum_{j=1}^{2+N_{fl}} \mu_j^{\chi_h} \nu_j^{\chi_h+\sigma} e^{2\,p\,u_j+S^2 T_j} F_j(S)$$

$$\mu_j = g_2(u_j)^3 / \prod_{t\neq j} (u_j - u_t) \qquad \nu_j \sim \left(g_2(u_j)\right)^{\frac{1}{4}}$$

$$F_j(S) = \sum_{\lambda} SW(\lambda) e^{i\,\pi\lambda\cdot c_1(\mathfrak{s})} \cos(\nu_j \lambda \cdot S)$$

$$y^2 = 4x^3 - g_2(u, m_j)x - g_3(u, m_j)$$

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Application 1: S-Duality Of $N_{fl} = 4$

$Z(p, S; \tau_0)$ is expected to have modular properties:

$$\alpha^{\chi} \beta^{\sigma} \sum_{j=1}^{6} \mu_j^{\chi_h} \, \nu_j^{\chi_h+\sigma} e^{2\, p\, u_j + S^2 T_j} \, F_j(S)$$

$$\mu_j = g_2(u_j)^3 / \prod_{t \neq j} (u_j - u_t) \qquad \nu_j \sim \left(g_2(u_j)\right)^{\frac{1}{4}}$$

Sum over j gives ***symmetric rational*** functions of $u_j \;\Rightarrow\; Z$ will be modular:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

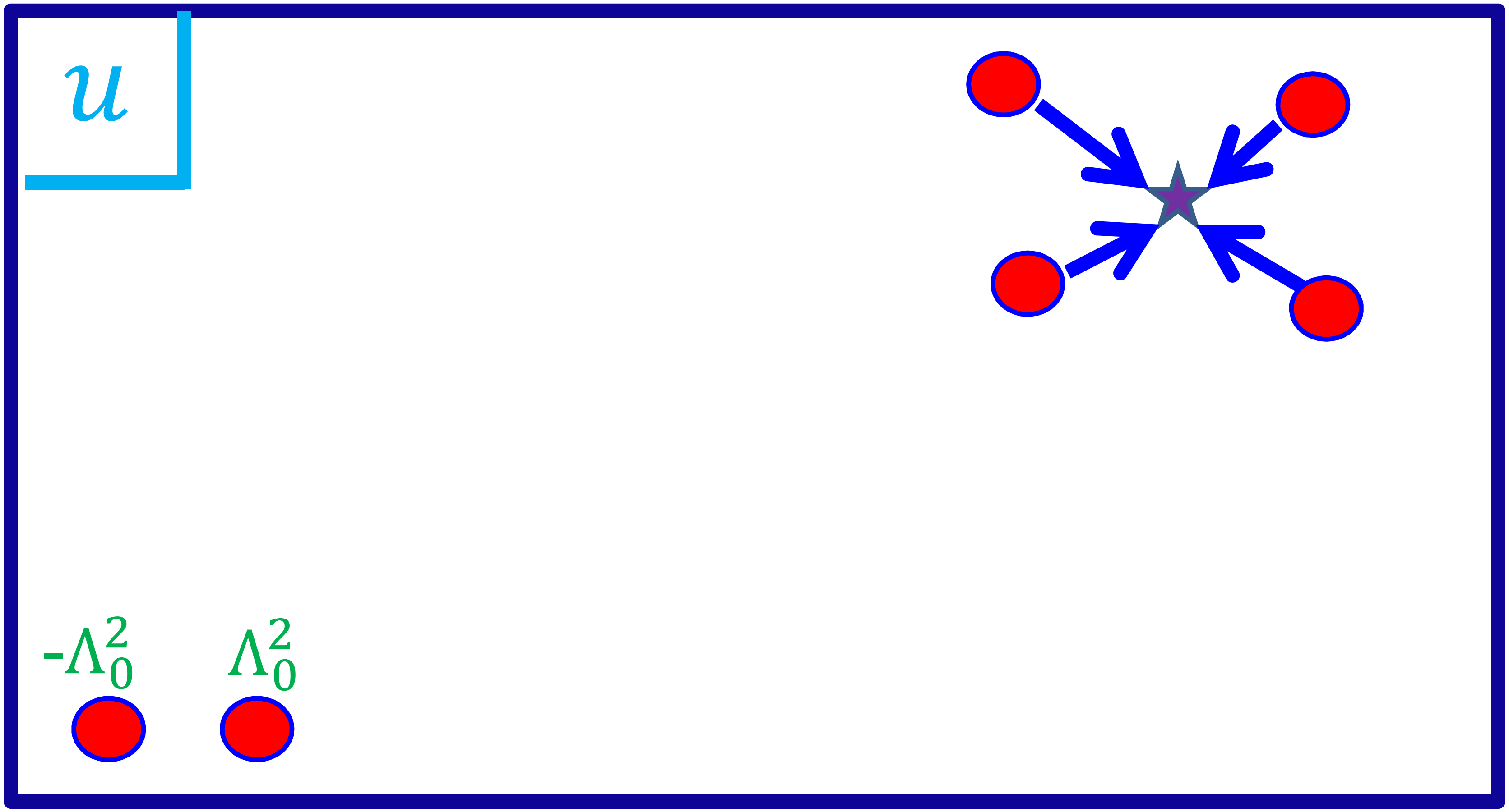
$$Z\left(\frac{p + c(c\tau_0 + d)S^2}{(c\,\tau_0 + d)^2}, \frac{S}{(c\tau_0 + d)^2}; \frac{a\tau_0 + b}{c\tau_0 + d}; \gamma \cdot \mathfrak{m}\right)$$

$$= (c\tau_0 + d)^w \, Z(p, S; \tau_0; \mathfrak{m})$$

$$w = \frac{2\chi + 3\,\sigma}{2}$$

Application 2:
Collision Of Mutually Local Singularities

Interesting things when mass parameters m_j take values so that u_j collide.



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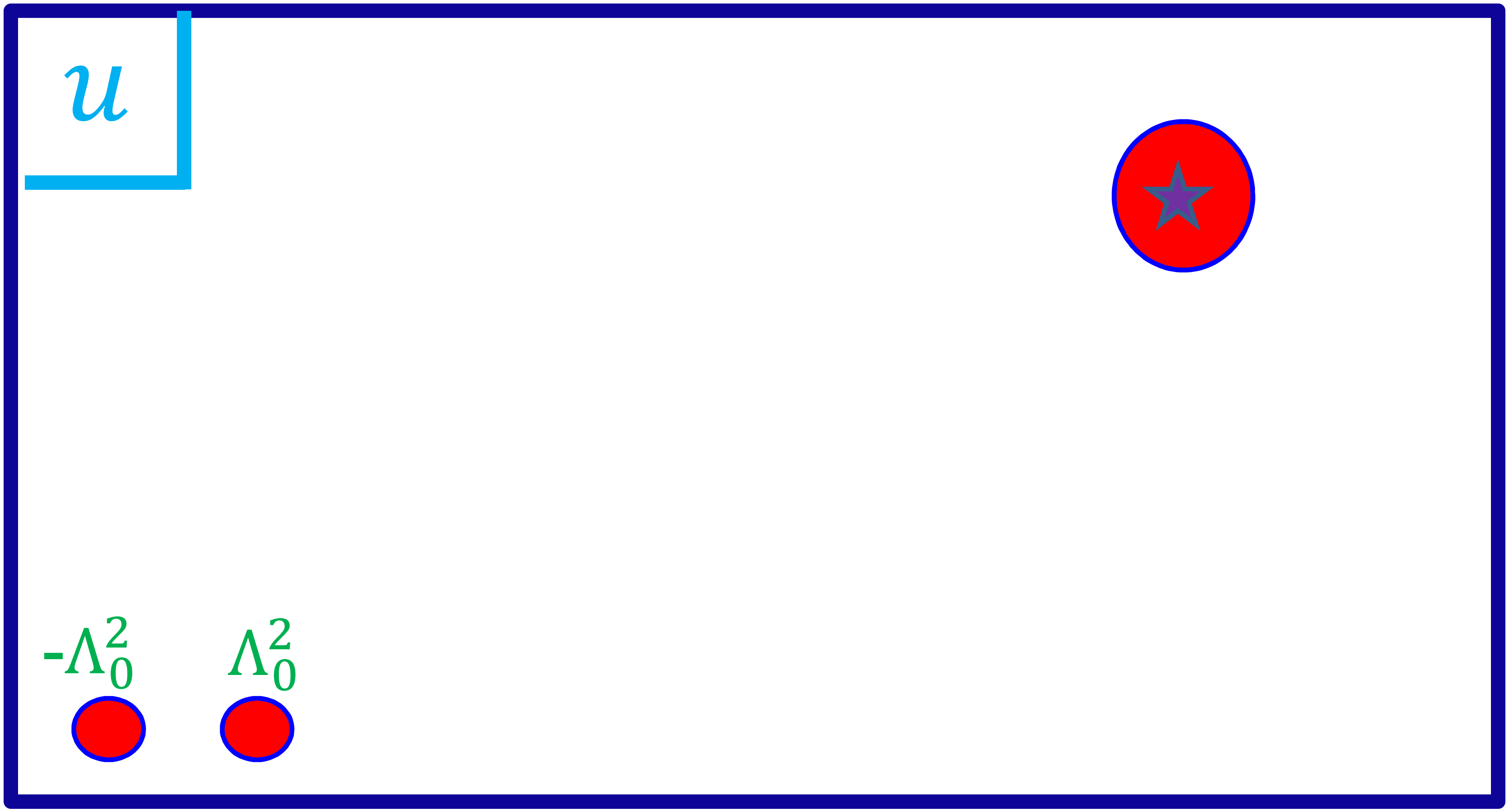


$$e^{i\,\pi\tau_0} \sim \frac{\Lambda_0^4}{m_1 m_2 m_3 m_4} \qquad m_j = M + \delta m_j$$

$$W^{eff} \sim \sum_f \delta m_f \; \tilde{Q}^f Q_f$$

$\Rightarrow Z$ should be $SU(N_{fl})$ −equivariant integral over moduli space of $U(1)$ −multimonopole equations:

$$F^+ = \sum_f \bar{M}^f M_f \qquad \gamma \cdot D M_f = 0$$

In the limit $\delta m_j \to 0$ the dominant term in Z is:

$$J(\tilde{S}) \sum_{f} \prod_{t \neq f} (\delta m_f - \delta m_t)^{-\chi_h} e^{\delta m_f \tilde{p}}$$

$$J(\tilde{S}) = e^{\tilde{S}^2/2} \sum_{\lambda} e^{i\pi\lambda\cdot c_1(\mathfrak{s})} SW(\lambda) \cos(\lambda \cdot \tilde{S})$$

This agrees with computations of Dedushenko, Gukov and Putrov of the equivariant integrals on $U(1)$-multimonopole space.

Application 3: AD3 Partition Function

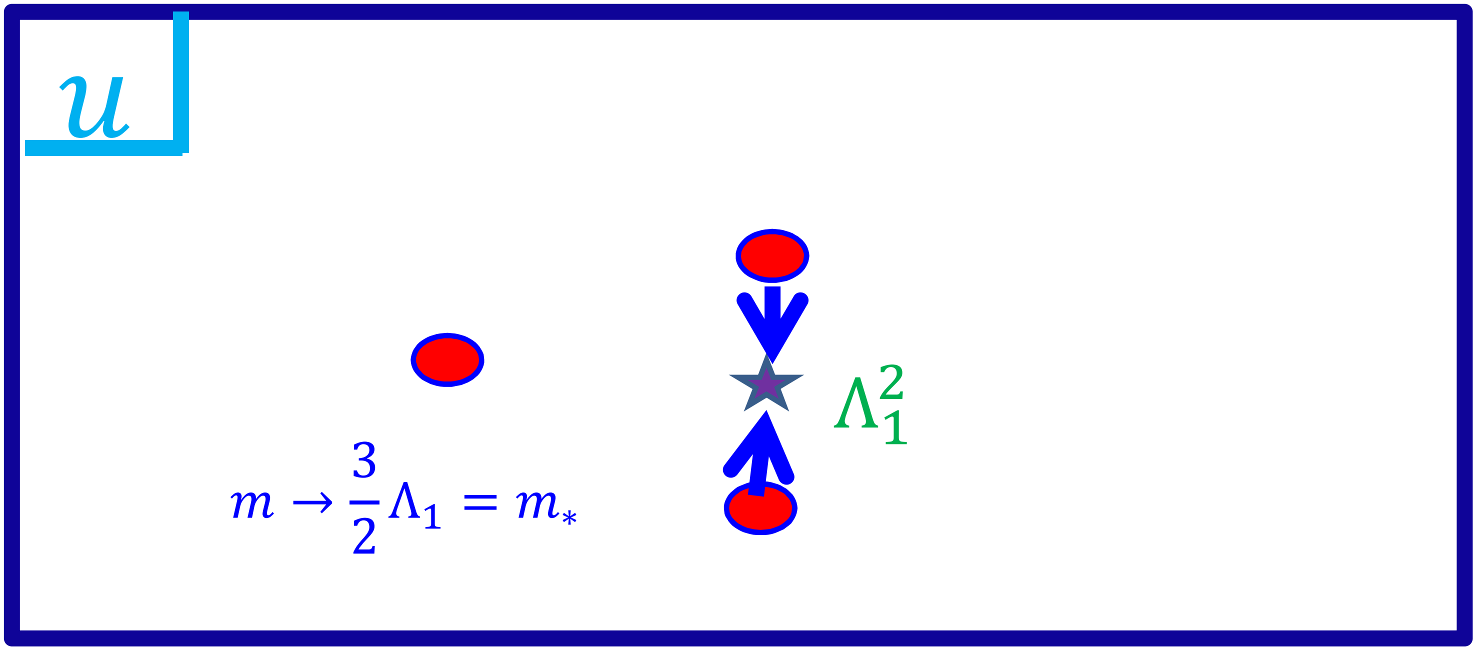
Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities $u_\pm$ collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres,Plesser,Seiberg,Witten]

Two mutually nonlocal BPS states have vanishing mass:

$$\oint_{\gamma_1} \lambda \to 0 \qquad \oint_{\gamma_2} \lambda \to 0 \qquad \gamma_1 \cdot \gamma_2 \neq 0$$

Physically: No local Lagrangian for the LEET : Signals a nontrivial superconformal field theory appears in the IR in the limit $m \to m_*$

AD3 From $SU(2)\ N_{fl} = 1$



SW curve in the scaling region:

$$y^2 = x^3 - 3\,\Lambda_{AD}^2\, x + u_{AD}$$

$$\Lambda_{AD}^2 \sim (m - m_*)$$

Call it the ``AD3-Family over the u_{AD} –plane”

AD3 Partition Function - 1

No obvious UV definition of the invariant.

The $SU(2)$ $N_{fl} = 1$ u-plane integral has a nontrivial contribution from the scaling region $u_{\pm} \to u_{*}$

$$\lim_{m \to m_*} Z_u \; - \; \int du\, d\bar{u} \lim_{m \to m_*} Measure(u, \bar{u}; m) \; \neq 0$$

Limit and integration commute except in an infinitesimal region around u_*

Attribute the discrepancy to the contribution of the AD3 theory

AD3 Partition Function - 2

1. Limit $m \rightarrow m_*$ exists. (No noncompact Higgs branch.)

2. The partition function is a sum over all Q-invariant field configurations.

3. Scaling region near u_* governed by AD3 theory.

$\lim_{m \rightarrow m_*} Z^{SU(2),N_{fl}=1}$ ``contains" the AD3 partition function

Extract it from the scaling region. Our result:

$$Z^{AD3} = \lim_{\Lambda_{AD} \rightarrow 0} \left(Z_u^{AD3-family} + Z_{SW}^{AD3-family} \right)$$

Claim: This is the AD3 TFT on X for $b_2^+ > 0$.

AD3 Partition Function: Evidence 1/2

Existence of limit is highly nontrivial. It follows from ``superconformal simple type sum rules'' :

Theorem [Marino,Moore,Peradze, 1998] If the superconformal simple type sum rules hold:

a.) $\chi_h - c^2 - 3 \leq 0$

b.) $\sum_\lambda SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4$

Then the limit $m \to m_*$ exists

It is now a rigorous theorem that SWST ⇒ SCST

Feehan & Leness

AD3 Partition Function: Evidence 2/2

For $b_2^+ > 0$ recover the expected selection rule:

$$\langle U(p)^{\ell} U(S)^{r} \rangle \neq 0 \text{ only for } \frac{6\ell + r}{5} = \frac{\chi_h - c^2}{5}$$

Consistent with background charge for AD3 computed by Shapere-Tachikawa.

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

$$Z^{AD3} = \frac{1}{\mathfrak{B}!} K_1 K_2^{\chi} K_3^{\sigma} \sum_{\lambda} e^{2\pi i\, \lambda \cdot w_2} SW(\lambda) \cdot$$

$$\cdot \{(\lambda \cdot S)^{\mathfrak{B}-2}(24(\lambda \cdot S)^2 + \mathfrak{B}(\mathfrak{B}-1)S^2)\}$$

$$\mathfrak{B} = \chi_h - c^2 = -\frac{7\,\chi + 11\,\sigma}{4}$$

Surprise! p drops out: U(p) is a ``null vector''

Should We Be Surprised?

Already for $N_{fl} = 0$ and SWST it is also true that $U(p)^2 - \Lambda_0^4$ is a null vector.

Without a good physics reason why these should be null vectors one suspects that there are (standard) 4-manifolds not of SWST.

Z_u Wants To Be A Total Derivative

$$Z_u = \int \mathrm{d}u\, \mathrm{d}\,\bar{\mathrm{u}}\ \mathcal{H}(u)\ \Psi$$

$$\Psi = \sum_{\lambda} \frac{d}{d\bar{u}} \left(\mathcal{E}(r_\lambda^\omega, b_\lambda) \right) e^{-i\,\pi\tau\lambda^2 - i\,S\cdot\frac{du}{da} + 2\,\pi i\,(\lambda-\lambda_0)\cdot w_2}$$

$$\mathcal{E}(r,b) = \int_b^r e^{-2\pi t^2} dt \qquad r_\lambda^\omega = \sqrt{y}\lambda_+ - \frac{i}{4\pi\sqrt{y}} S_+ \frac{du}{da}$$

Lower limit b_λ must be independent of ū
but can depend on λ, ω, u.

[Korpas & Manschot; Moore & Nidaiev]

$$Z_u \overset{?}{=} \int \mathrm{d}u\, \mathrm{d}\,\bar{\mathrm{u}} \frac{d}{d\bar{u}} \left(\mathcal{H}(u) \Theta^\omega \right)$$

Not obvious: No obvious choice for b_λ with both convergence and modular invariance.

BUT: The difference for two metrics CAN be evaluated by residues!

$$Z_u^{\omega} - Z_u^{\omega_0} = \int du\, d\bar{u}\, \frac{d}{d\bar{u}} \left(\mathcal{H}\, \widehat{\Theta}^{\omega,\omega_0}\right)$$

$$\widehat{\Theta}^{\omega,\omega_0} = \sum_{\lambda} \left(\mathcal{E}(r_\lambda^{\omega}) - \mathcal{E}\left(r_\lambda^{\omega_0}\right)\right) e^{-i\,\pi\tau(u)\lambda^2 + \cdots}$$

$$r_\lambda^{\omega} = \sqrt{y}\lambda_+ - \frac{i}{4\pi\sqrt{y}} S_+ \frac{du}{da}$$

Modular completion of indefinite theta function of Vigneras, Zwegers, Zagier

Holomorphic Anomaly & Metric Dependence

For theories with a manifold of superconformal couplings, $\tau(u) \to \tau_0$ when $u \to \infty$

$$\widehat{\Theta}^{\omega,\omega_0}(\tau(u), \dots) \to \widehat{\Theta}^{\omega,\omega_0}(\tau_0, \dots)$$

Contour integral at $\infty \Rightarrow$ $For \quad b_2^+ = 1,$ many ``topological" correlators:

Nonholomorphic in τ_0

Depend ***continuously*** on metric

[Moore & Witten, 1997 -- albeit *sotto voce*]

Special Case Of $\mathbb{CP}^2$

No walls: $\omega \in H^2(X,\mathbb{R}) \cong \mathbb{R}$, so we only see holomorphic anomaly.

$\Rightarrow$ ***Path integral*** derivation of the holomorphic anomaly.

Vafa-Witten Partition Functions

VW twist of N=4 SYM formally computes the ``Euler character'' of instanton moduli space.

(Not really a topological invariant. True mathematical meaning unclear, but see recent work of Tanaka & Thomas; Gholampour, Sheshmani, & Yau.)

Physics suggests the partition function is ***both*** modular (S-duality) ***and*** holomorphic.

Surprise! Computations of Klyachko and Yoshioka for $X = \mathbb{CP}^2$ show that the holomorphic generating function is only mock modular.

But a nonholomorphic modular completion exists.

This has never been properly derived from a path integral argument.

But we just derived completely analogous results for the $N_{fl} = 4$ theory from a path integral, suggesting VW will also have continuous metric dependence.

Indeed, continuous metric dependence in VW theory for $SU(r > 1)$ has been predicted in papers of Jan Manschot using rather different methods.

We hope that a similar path integral derivation can be found for the $\mathcal{N} = 4$ Vafa-Witten twisted theory.

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Future Directions

There is an interesting generalization to invariants for families of four-manifolds.

Couple to ``topologically twisted supergravity"

$$Z \in H^*(BDiff(X))$$

Generalization to all theories in class S: Many aspects are clear – this is under study.

u-plane for class S: General Remarks

UV interpretation is not clear in general.
These theories might give new 4-manifold invariants.

The u-plane is an integral over the base $\mathcal{B}$ of a Hitchin fibration with a theta function associated to the Hitchin torus. It will have the form

$$Z_u = \int_{\mathcal{B}} du\, d\bar{u}\, \mathcal{H}\, \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: ***NOT holomorphic*** and ***metric- DEPENDENT***
``theta function''

Class S: General Remarks

$$\mathcal{H} = \alpha^{\chi}\, \beta^{\sigma}\, \det\left(\frac{da^i}{du_j}\right)^{1-\frac{\chi}{2}} \Delta_{phys}^{\frac{\sigma}{8}}$$

Δ_{phys} a holomorphic function on $\mathcal{B}$ with first-order zeros at the loci of massless BPS hypers

α, β will be automorphic forms on Teichmuller space of the UV curve C

α, β are related to correlation functions for fields in the (0,2) QFT gotten from reducing 6d (0,2)

Class S: General Remarks

$$\Psi \sim \sum_{\lambda} e^{i\pi\lambda\cdot\xi} e^{-i\pi\bar{\tau}(\lambda_+,\lambda_+)-i\pi\tau(\lambda_-\cdot\lambda_-)+\cdots}$$

$$\lambda \in \lambda_0 + \Gamma \otimes H^2(X;\mathbb{Z})$$

$$\xi \in \Gamma \otimes H^2(X;\mathbb{R})$$

$$\Gamma \subset H^1(\Sigma;\mathbb{Z})$$

Lagrangian sublattice

If $\xi = \rho \otimes w_2(X)\ mod\ 2$ ***then*** WC from interior of $\mathcal{B}$ will be cancelled by SW invariants

$\Rightarrow$ No new four-manifold invariants…

Ψ comes from a ``partition function'' of a level 1 SD 3-form on $M_6 = \Sigma \times X$

Quantization: Choose a QRIF Ω on $H^3(M_6;\mathbb{Z})$

Natural choice: [Witten 96,99; Belov-Moore 2004]

$$\Omega(x) = \exp\Big(i\,\pi \;\; WCS(\theta \cup x;\;\; S^1 \times M_6 \,)\Big)$$

Choice of weak-coupling duality frame + natural choice of $spin^c$ structure gives

$$\xi = \rho \otimes w_2(X)$$

Case Of SU(2) $\mathcal{N} = 2^*$

Using the tail-wagging-dog argument, analogous formulae were worked out for $\mathcal{N} = 2^*$, by Moore-Witten and Labastida-Lozano in 1998, but *only* in the case when X is spin.

L&L checked S-duality for the case $b_2^+ > 1$

The generalization to X which is NOT spin is nontrivial: The standard expression from Moore-Witten and Labastida-Lozano is NOT single-valued on the u-plane.

This is not surprising: The presence of external $U(1)_{baryon}$ gauge field $F_{baryon} \sim c_1(\mathfrak{s})$ means there should be new interactions:

$$e^{\kappa_1(u) c_1(\mathfrak{s})^2 + \kappa_2(u) \lambda \cdot c_1(\mathfrak{s})}$$

Shapere & Tachikawa

Holomorphy, 1-loop singularities, single-valuedness forces:

$$(u - u_1)^{-\frac{c_1(\mathfrak{s})^2}{8}} e^{-i \frac{\partial a_D}{\partial m} \lambda \cdot c_1(\mathfrak{s})}$$

Surprise!!

It doesn't work!

Correct version appears to be non-holomorphic.

With Jan Manschot we have an alternative which is currently being checked.

Does the u-plane integral make sense for ***ANY*** family of Seiberg-Witten curves ?

どうもありがとうございます