FOUR-DIMENSIONAL WALL-CROSSING
FROM
THREE-DIMENSIONAL FIELD THEORY

WORK DONE WITH

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OUTLINE

1. INTRODUCTION

THIS TALK IS ABOUT THE BPS SPECTRUM OF $\mathcal{N}=2$, $D=4$ FIELD THEORIES.

THE BPS SPECTRUM OF THE THEORY ON $\mathbb{R}^4$ IS A "PIECEWISE CONSTANT" FUNCTION OF THE BOUNDARY CONDITIONS AT ∞ OF VECTORMULTIPLET SCALARS.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF). THIS TALK WILL GIVE A PHYSICAL INTERPRETATION AND PROOF OF A FAMOUS WCF OF KONTSEVICH + SOIBELMAN.

2. REVIEW $N=2, D=4$ WALL CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$
WITH $\mathcal{N}=2$ SUPERPOINCARÉ SYMMETRY

LET $\mathcal{H}$ BE THE ONE-PARTICLE
HILBERT SPACE.

AS A REPRESENTATION OF THE
$\mathcal{N}=2$ SUPERPOINCARÉ ALGEBRA $\mathcal{S}$, $\mathcal{H}$
DEPENDS ON THE BOUNDARY VALUES
OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED
IN THE MODULI SPACE OF VACUA: $\mathcal{B}$

FOR $u \in \mathcal{B}$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{B}$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE SYMPLECTIC LATTICE Γ OF ELEC. + MAG. CHARGES. (OF RANK $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \mathcal{A}$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$

RECALL THE $\mathcal{N}=2$, $D=4$ SUSY ALGEBRA

$$\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$$

$$\mathcal{S}_0 = \left(\underset{M_{\mu\nu}}{\mathrm{Spin}(1,3)} \ltimes \underset{P_\mu}{\mathbb{R}^4} \right) \oplus \mathfrak{u}(2) \oplus \underset{Z}{\mathbb{R}}$$

$$\mathcal{S}_1 = \left[\mathrm{Spinor} \otimes \mathbb{C}^2 \right]_{\mathbb{R}}$$

$$Q_{\alpha I}, \ \bar{Q}^{\dot{\alpha} I}$$

$$\{ Q_{\alpha I}, \bar{Q}_{\dot{\beta}}{}^{J} \} = 2 P_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta_I{}^J$$

$$\{ Q_{\alpha I}, Q_{\beta J} \} = 2 Z \epsilon_{\alpha\beta} \epsilon_{IJ}$$

UNITARY IRREPS SATISFY BPS BOUND:

$$E \geq |Z|$$

DEF. $\mathcal{H}^{BPS}_{\gamma,u}$ = SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_\gamma(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS

[Cecotti, Fendley, Intriligator, Vafa ; Seiberg & Witten]

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ SO

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|\right) \leq 0$$

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$$MS(\gamma_1,\gamma_2) := \{u \mid Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+\}$$

WCF: BE MORE QUANTITATIVE ABOUT "HOW MANY" STATES DECAY

DEFINE THE BPS INDEX

$$\Omega^{Ph}(\gamma;u) := -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}}(2J_3)^2(-1)^{2J_3}$$

DENEF & MOORE GAVE FORMULAE FOR $\Delta\Omega$ FOR DECAYS $\gamma \to \gamma_1 + \gamma_2$ WHERE AT LEAST ONE OF γ_1, γ_2 ARE PRIMITIVE.

THE DERIVATION IS BASED ON DENEF'S MULTICENTERED SOLUTIONS OF $\mathcal{N}=2$ SUGRA AND QUIVER QUANTUM MECHANICS

THE METHODS ARE DIFFICULT TO USE WHEN <u>BOTH</u> γ_1, γ_2 ARE NON-PRIMITIVE

KONTSEVICH & SOIBELMAN PROPOSED A REMARKABLE W.C.F. FOR AN INDEX $\Omega^{DT}(\gamma;u)$,

"GENERALIZED DONALDSON-THOMAS INVARIANT OF A CALABI-YAU 3-FOLD"

WE EXPECT THAT

$$\Omega(\gamma;u) = \Omega^{Ph}(\gamma;u) = \Omega^{DT}(\gamma;u)$$

SO THE KS WCF APPLIES TO PHYSICAL BPS DEGENERACIES.

THEIR FORMULA APPLIES TO ALL DECAYS $\gamma \to \gamma_1 + \gamma_2$.

THIS TALK PROVES THE KS WCF FOR $\Omega^{Ph}(\gamma;u)$ IN $\mathcal{N}=2$ FIELD THEORIES

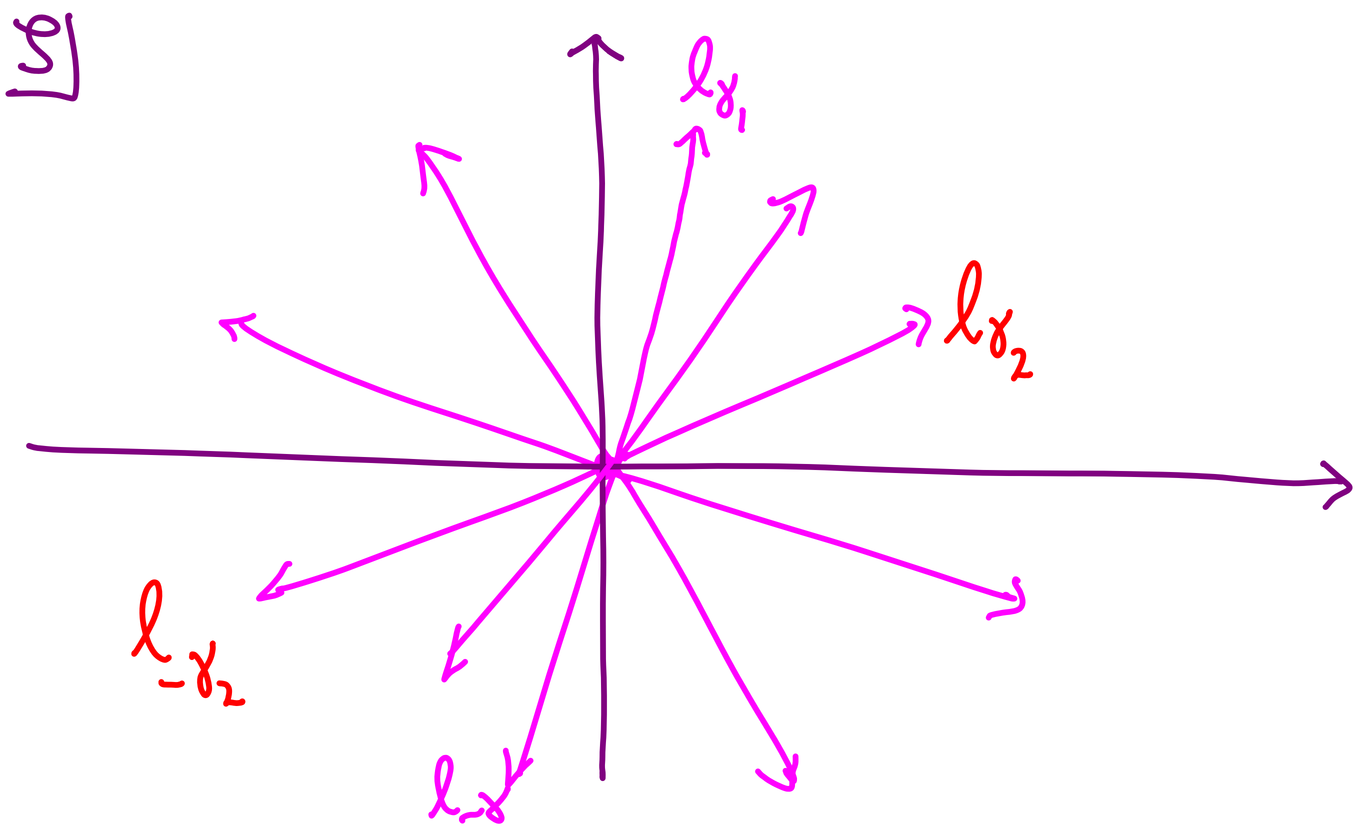
3. THE KONTSEVICH–SOIBELMAN FORMULA

DATA:

1. SYMPLECTIC LATTICE Γ
2. CENTRAL CHARGES $Z_\gamma(u)$, $u \in \mathcal{B}$
3. PIECEWISE CONSTANT $\Omega(\gamma; u) \in \mathbb{Z}$

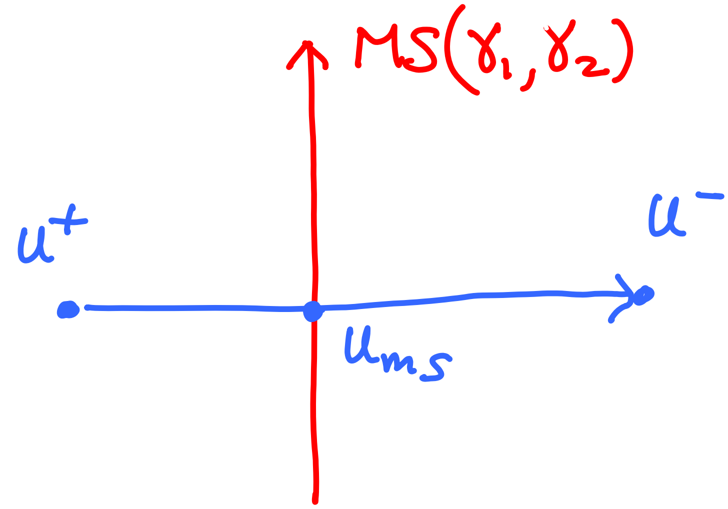
BPS RAYS: FOR $u \in \mathcal{B}$, $\gamma \in \Gamma$

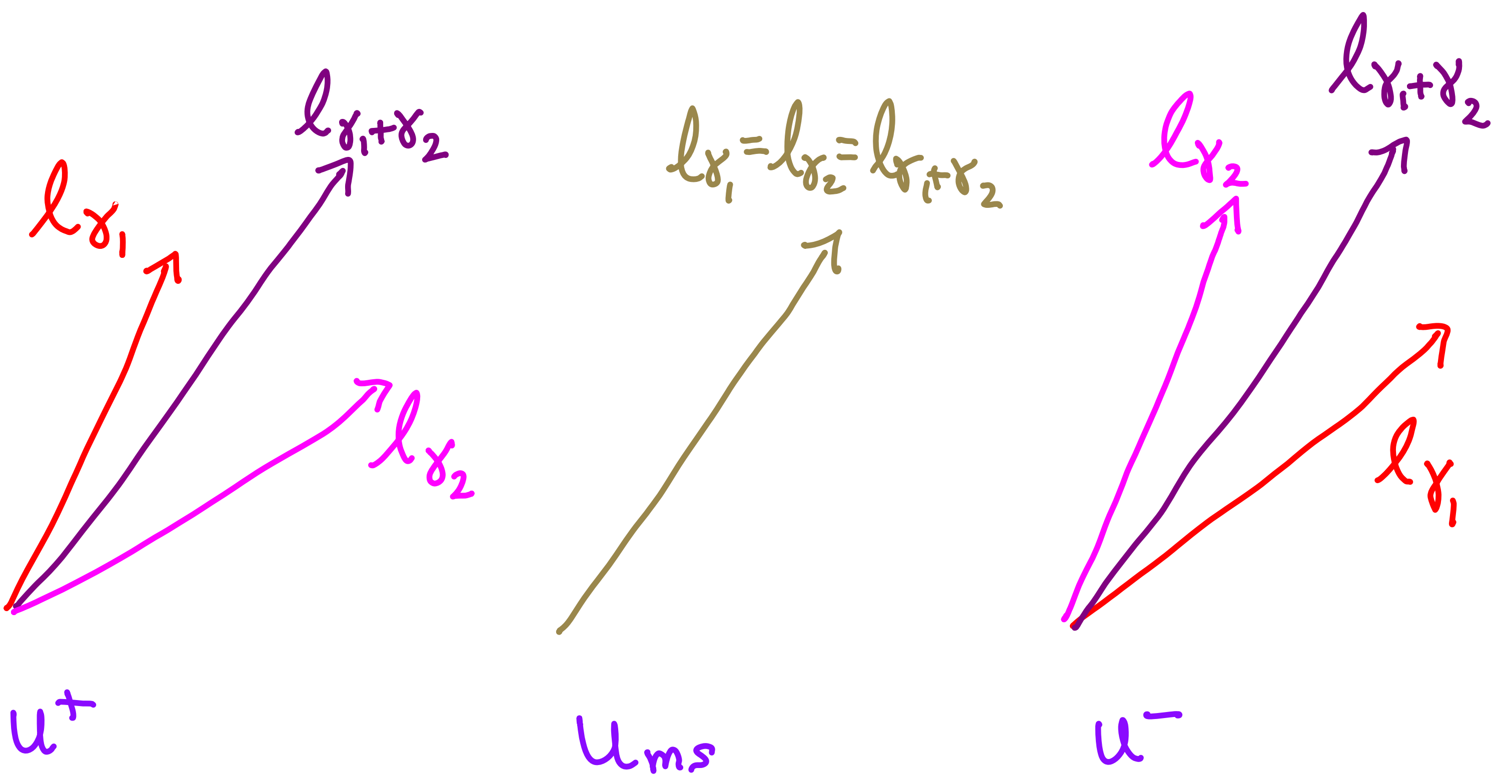
$$\ell_\gamma := Z_\gamma(u)\mathbb{R}_- = \left\{ \zeta \,\middle|\, \zeta / Z_\gamma(u) \in \mathbb{R}_- \right\}$$



AS u VARIES THE SLOPES OF THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1, \gamma_2)$ BPS RAYS WILL COALESCE





SECOND INGREDIENT: A SYMPLECTIC TORUS:

- INTRODUCE THE COMPLEX TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots\cdots \times \mathbb{C}^*}_{2r}$$

$$\gamma \in \Gamma \implies \text{FUNCTION } X_\gamma : T \to \mathbb{C}^*$$

"HOLOMORPHIC FOURIER MODES"

CHOOSING A BASIS γ_i FOR $\Gamma \implies$

$$(e^{\theta_1}, \cdots\cdots, e^{\theta_{2r}}) \in T, \quad \theta_i \in \mathbb{C}$$

$$X_\gamma = \exp[\gamma \cdot \theta]$$

- HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^T := \frac{1}{2}\epsilon^{ij}\frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}} \qquad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A SYMPLECTOMORPHISM:

$$K_\gamma : \; X_{\gamma'} \longrightarrow X_{\gamma'}\left(1 - \sigma(\gamma) X_\gamma\right)^{\langle \gamma', \gamma \rangle}$$

$$X_{\gamma'} \rightarrow X_{\gamma'} \exp\left[\langle \gamma, \gamma' \rangle \log\left(1 - \sigma(\gamma) X_\gamma\right)\right]$$

$$\sigma(\gamma) = \pm 1$$

ABOUT THE SIGN:

K & S INTRODUCE A LIE ALGEBRA

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle\, e_{\gamma_1+\gamma_2}$$

DEFINE A GROUP ELEMENT

$$U_\gamma := \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

AND WORK WITH U_γ INSTEAD OF K_γ

CHOOSE A <u>QUADRATIC REFINEMENT</u>

$$\frac{\sigma(\gamma_1+\gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle\gamma_1,\gamma_2\rangle}$$

$\sigma(\gamma)\, e_\gamma$ GENERATE THE LIE ALGEBRA OF SYMPLECTIC VECTOR FIELDS.

EXAMPLE

$$r=1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b),(a',b') \rangle = ab' - a'b$$

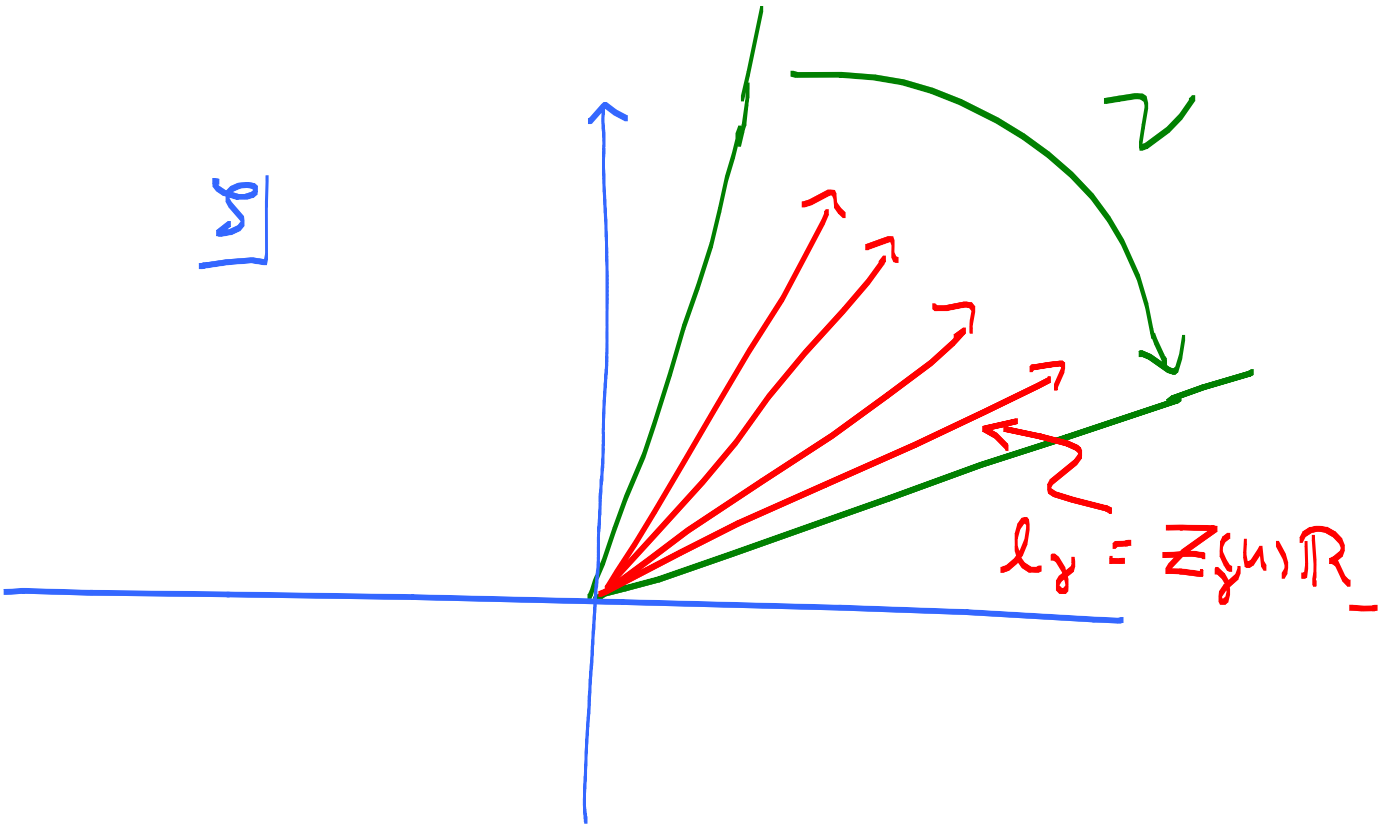
$$T \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \qquad y = X_{0,1}$$

$$\omega^T = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$K_{a,b} : \begin{cases} x \longrightarrow x\left(1-(-1)^{ab}x^a y^b\right)^b \\ y \longrightarrow y\left(1-(-1)^{ab}x^a y^b\right)^{-a} \end{cases}$$

NOW CHOOSE A CONVEX CONE $\mathcal{V}$



FOR EACH BPS RAY DEFINE

$$S_\gamma := \prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma';u)}$$

AND THEN DEFINE:

$$A_{\mathcal{V}} := \overrightarrow{\prod_{\ell_\gamma \subset \mathcal{V}}} S_\gamma$$

$$\mathcal{A}_\nu := \overrightarrow{\prod_{\ell_\gamma \subset \nu}} S_\gamma = \overrightarrow{\prod_{-Z_\gamma \in \nu}} K_\gamma^{\Omega(\gamma;u)}$$

THE PRODUCT IS TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

$\mathcal{A}_\nu$ DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma;u)$ DEPEND ON u ...

THE KS FORMULA STATES THAT

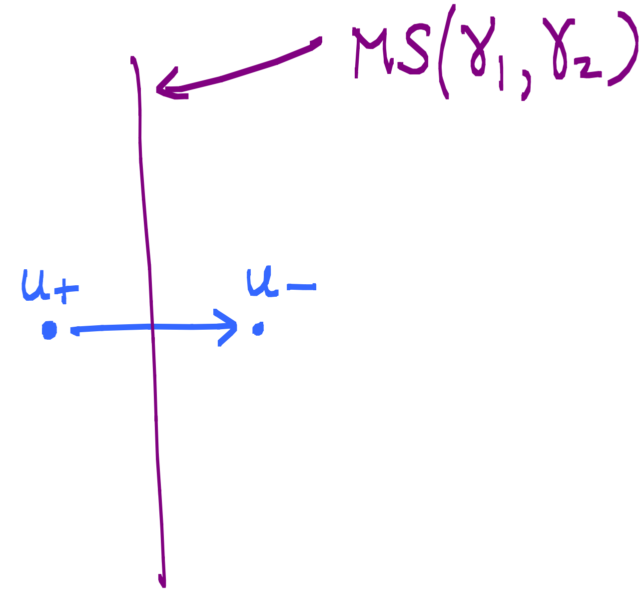
NEVERTHELES,

$$A_{\nu} = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} K_{\gamma}^{\Omega(\gamma;u)}$$

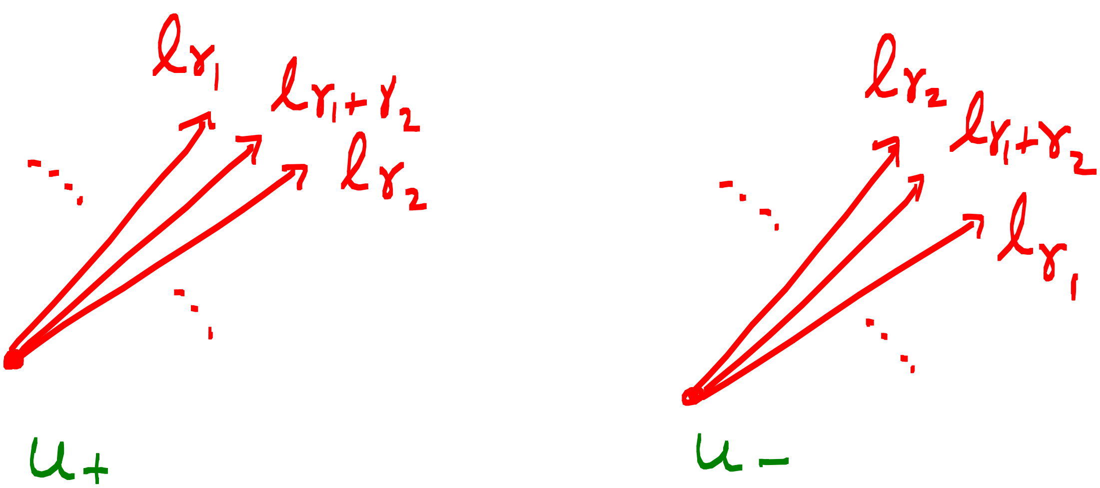
IS <u>CONSTANT IN u</u> AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR ν.

THIS IS A WALL-CROSSING FORMULA...

AS u CROSSES A WALL



$$\ell_\gamma = Z_\gamma(u) \cdot \mathbb{R}_- \qquad \text{ROTATES}$$



$\Longrightarrow$ EXCHANGE ORDER IN

$$A_\nu = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} K_\gamma^{\Omega(\gamma;u)}$$

$\Omega(\gamma;u)$ MAKES A COMPENSATING CHANGE

ONE CAN RECOVER THE PRIMITIVE & SEMI-PRIMITIVE W.C.F. FROM THIS FORMULATION...

[with Wu-yen Chuang]

4. COMPACTIFICATION OF $N=2, D=4$ FIELD THEORIES

A. SEIBERG-WITTEN SOLUTION

G - COMPACT S.S. GAUGE GROUP, RANK $= r$

$\Longrightarrow$ $D=4, N=2$ FIELD THEORY

(CAN ALSO INCLUDE HM's)

$$\mathcal{B} = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r : \quad \begin{array}{l} u_2 = \langle \mathrm{Tr}\, \Phi^2 \rangle \\ u_3 = \langle \mathrm{Tr}\, \Phi^3 \rangle \\ \quad \vdots \end{array}$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

IN TERMS OF

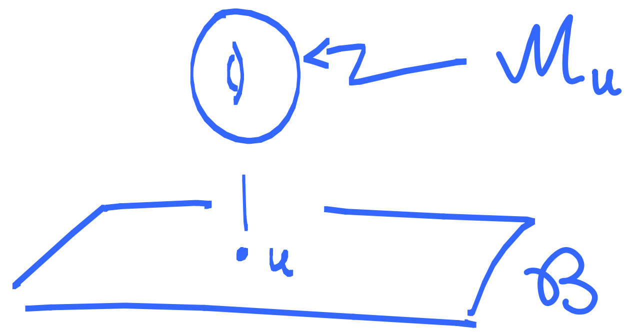
SPECIAL KÄHLER GEOMETRY

REVIEW SPECIAL KÄHLER GEOM:

c.f. D. FREED, hep-th/9712042

VIEW Γ AS A LOCAL SYSTEM OVER $\mathcal{B}$

$$\begin{tikzcd} \mathcal{M}_u \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ u \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$



$$\mathcal{M}_u = \Gamma_u^* \underset{\mathbb{Z}}{\otimes} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

FIBERS = ABELIAN VARIETIES

IN REGIONS OF $\mathcal{B}$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag}, \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \mathrm{Span}\{\alpha_I\} \oplus \mathrm{Span}\{\beta^I\}$$

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0 \qquad \langle \alpha_I, \beta^J \rangle = \delta_I^{\ J}$$

CHOOSING A DUALITY FRAME,

$\mathcal{M}_u$ HAS PERIOD MATRIX τ_{IJ}

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1. LOW ENERGY LAGRANGIAN:

$$\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im}\tau_{IJ}\left(da^I * d\bar{a}^J + F^I * F^J\right)$$

$$+ \frac{1}{4\pi} \operatorname{Re}\tau_{IJ} \; F^I \wedge F^J$$

$$a^I = Z_{\alpha_I}(u) \qquad I = 1, \dots, r$$

LOCAL COORD'S ON $\mathcal{B}$

2. CENTRAL CHARGE FUNCTION

$$\boxed{\begin{gathered} Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg} \\ a^I = Z_{\alpha_I}(u) \qquad a_{D,I} = Z_{\beta^I}(u) \end{gathered}}$$

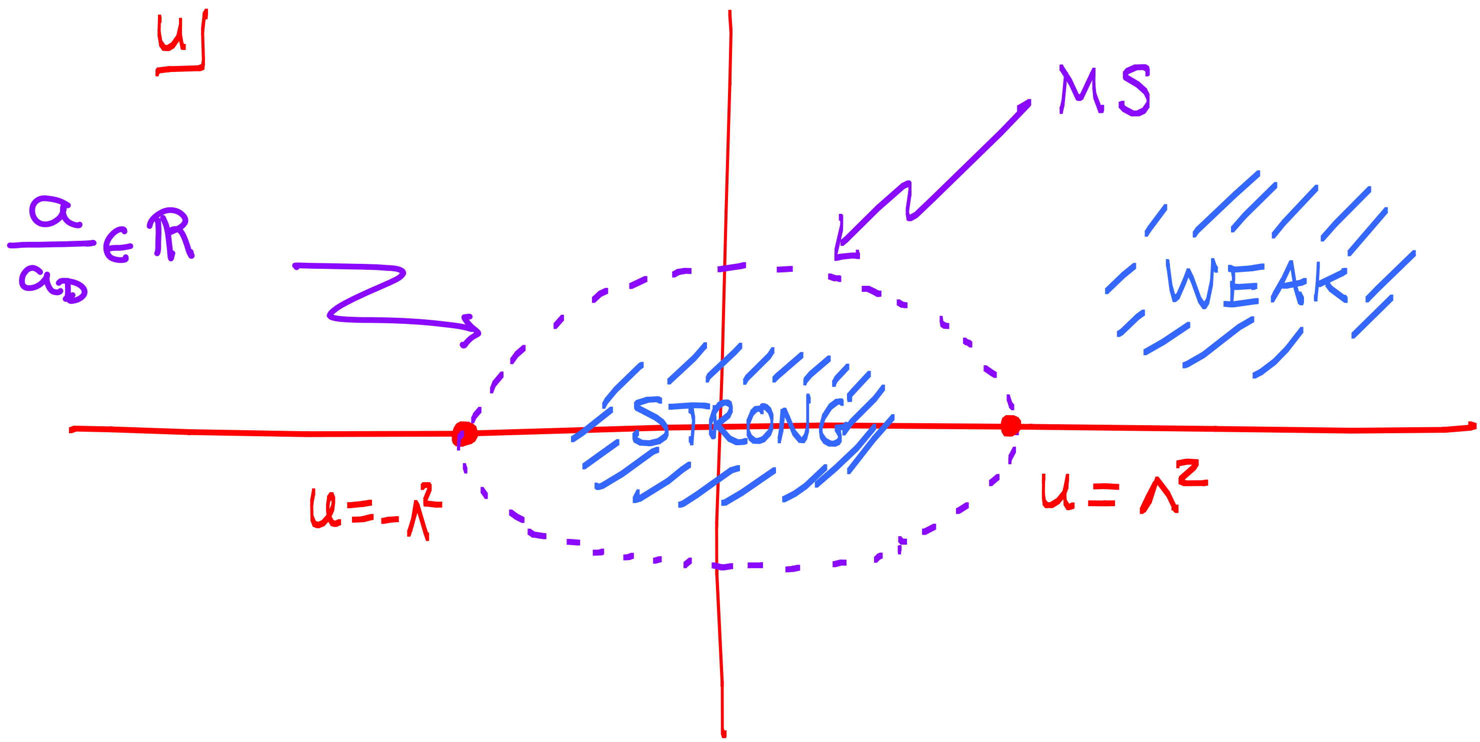
S & W IDENTIFY $\mathcal{M}_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES, NOW CALLED THE SEIBERG-WITTEN CURVE.

MOREOVER, THE CENTRAL CHARGE FUNCTION IS THE PERIOD OF A MEROMORPHIC 1-FORM, THE SW DIFFERENTIAL:

$$Z_\gamma(u) = \oint_\gamma \lambda_{SW}$$

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u : \quad y + \frac{\Lambda^4}{y} = x^2 - 2u$$



$$\lambda_{SW} = x \frac{dy}{y}$$

$$a = \oint_\alpha x \frac{dy}{y} \qquad a_D = \oint_\beta x \frac{dy}{y}$$

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0) \oplus \text{CONJUGATE}$$

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1) \oplus \text{CONJUGATE}$$

[Bilal & Ferrari]

KS IDENTITY:

$$K_{2,-1}K_{0,1} = K_{0,1}K_{2,1}K_{4,1}\cdots K_{2,0}^{-2}\cdots K_{6,-1}K_{4,-1}K_{2,-1}$$

IT IS TRUE !!!

REMARK: ADDING MASSIVE FLAVORS GENERALIZES THE W.C.F.

$$\Gamma \longrightarrow \Gamma \oplus \Gamma^f$$
$$\gamma \longrightarrow \gamma + \gamma^f$$

Introduce constants $\log\mu \in (\Gamma^f)^* \otimes \mathbb{C}^*$

$$X_{\gamma,\gamma_f} = X_\gamma \cdot \prod_a (\mu^a)^{\gamma_a^f}$$

$$K_{\gamma,\gamma_f} : X_{\gamma'} \rightarrow X_{\gamma'} \left(1 - \sigma(\gamma) X_{\gamma,\gamma_f}\right)^{\langle\gamma',\gamma\rangle}$$

WCF AGAIN HOLDS WITH

$$Z_{\gamma,\gamma f}(u) = Z_\gamma(u) + m^a \gamma_a^f$$

B. COMPACTIFY ON A CIRCLE.

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.
- LOW ENERGY THEORY IS A 3D σ-MODEL: $\mathbb{R}^3 \longrightarrow \mathcal{M}$

$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_4^I \, dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,4})_I \, dx^4 \end{aligned}\right\} \text{PERIODIC!}$$

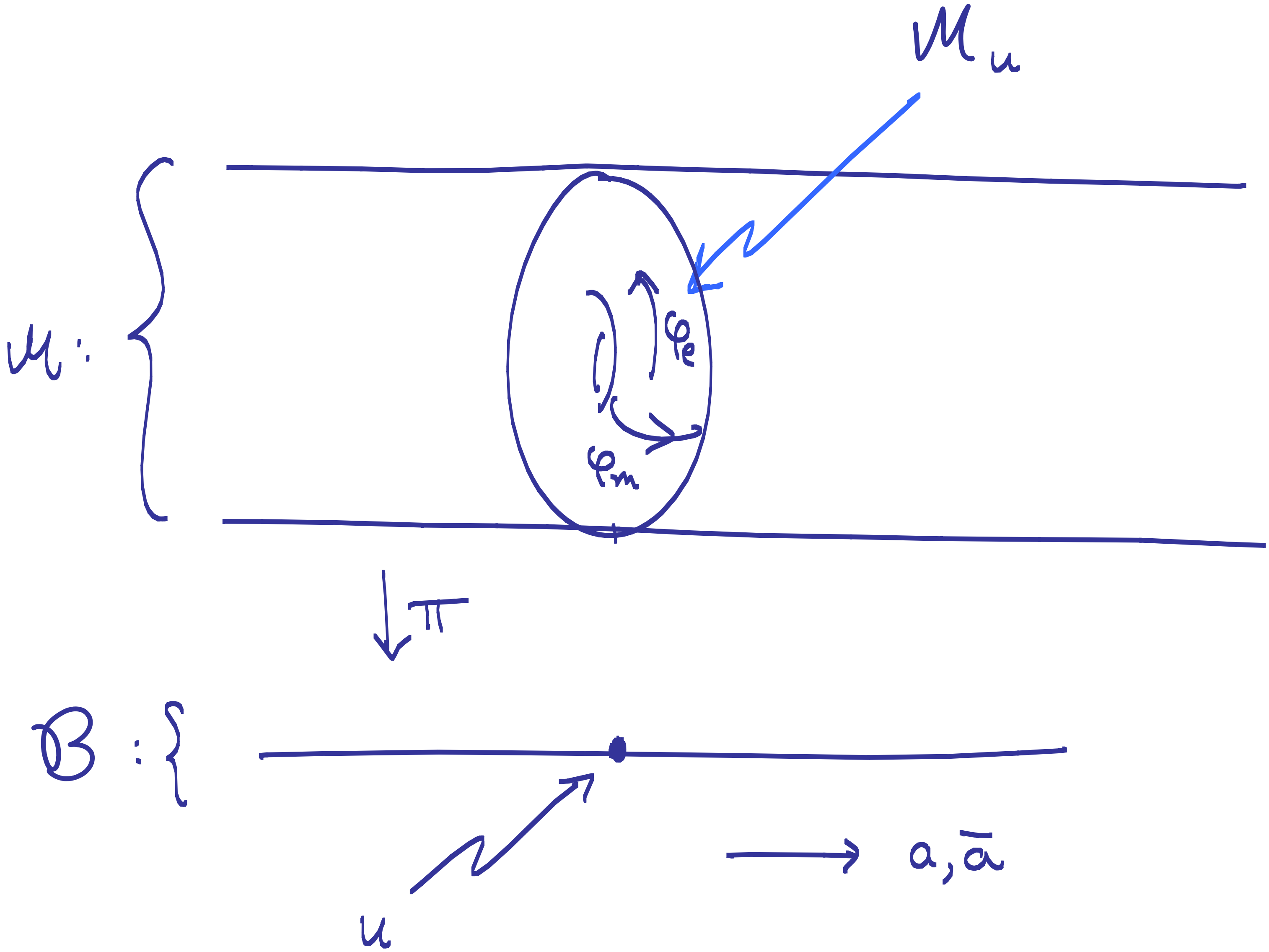
- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

TOPOLOGICALLY $\mathcal{M}$ IS A TORUS FIBRATION OVER $\mathcal{B}$:

IT IS EXACTLY $\mathcal{M} = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$ THAT APPEARED ABOVE:



THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J$$

$$- \frac{1}{8\pi^2 R} (\operatorname{Im}\tau)^{-1,\, IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R(\operatorname{Im}\tau)|da|^2 + \frac{1}{4\pi^2 R}(\operatorname{Im}\tau)^{-1}|dz|^2$$

C. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS <u>DEPEND</u> ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{B}$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.

5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN'S THEOREM: KNOWING $(\mathcal{M}, g)$ IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := \mathcal{M} \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

<u>THEOREM</u>: IF $(\mathcal{M}, g)$ IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p: Z \longrightarrow \mathbb{CP}^1$$

$$M^\zeta = p^{-1}(\zeta) = M \text{ IN COMPLEX STRUCTURE } \zeta$$

2. $\exists$ HOLOMORPHIC SECTION

ϖ OF $\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$

$\varpi_\zeta := \varpi\big|_{M^\zeta}$ = HOLOMORPHIC SYMPLECTIC FORM ON M^ζ

3. $\forall x \in M$, $\exists$ HOLOMORPHIC SECTION $s_x: \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

4. $\exists$ ANTI-HOLOMORPHIC $\sigma: Z \to Z$ COVERING $\zeta \to -1/\bar{\zeta}$

CONVERSELY,

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

FOR $\zeta \in \mathbb{C}^*$:

$$\varpi_\zeta = -\frac{i}{2\zeta}\,\omega_+ + \omega_3 - \frac{i}{2}\,\zeta\,\omega_-$$

$\uparrow$

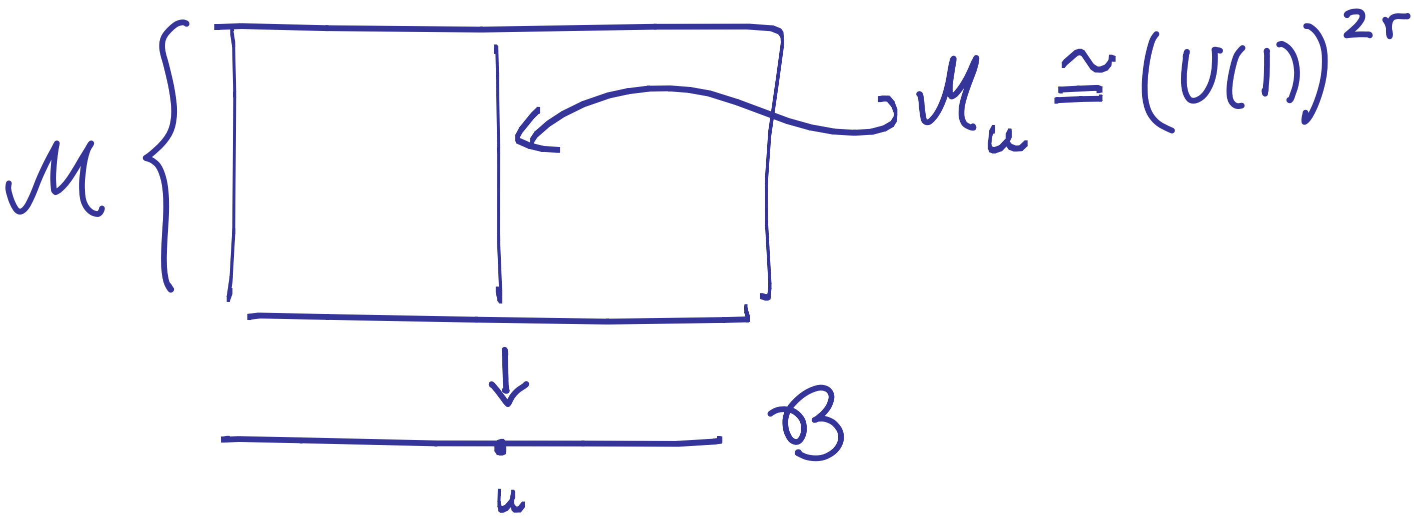
KÄHLER FORM

$$\omega_+ = \omega_1 + i\,\omega_2$$

OUR STRATEGY IS TO CONSTRUCT ϖ_ζ EXPLICITLY FOR $\mathcal{M}^\zeta$ USING A "NICE" SET OF HOLOMORPHIC FUNCTIONS ON TWISTOR SPACE:

$$\mathcal{X}_\gamma \ , \quad \gamma \in \Gamma$$

USE THE TORUS FIBRATION OF $\mathcal{M}$:



FOR $\zeta \neq 0, \infty$ $\mathcal{M}_u$ IS NOT HOLOMORPHIC

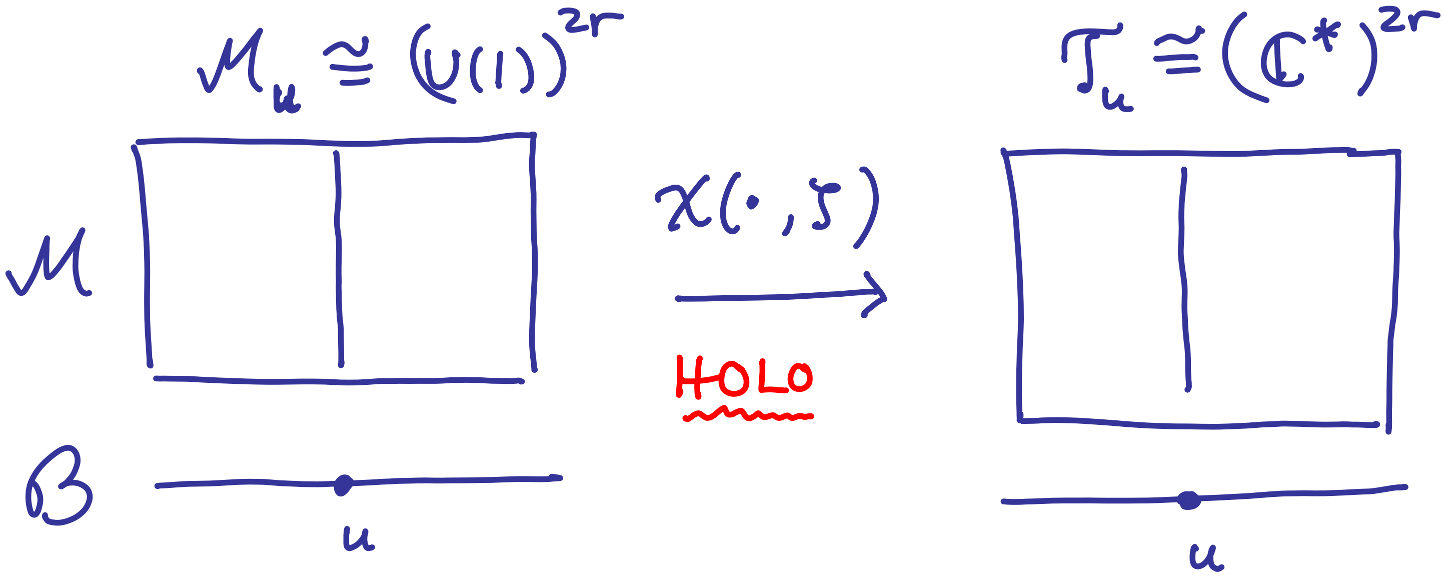
BUT CONSIDER $\mathcal{T} := \Gamma^* \otimes \mathbb{C}^*$

- $\mathcal{T}$ HAS A FIXED COMPLEX STRUCTURE WITH HOLOMORPHIC FIBERS $\cong (\mathbb{C}^*)^{2r}$
- $\mathcal{T}$ HAS HOLOMORPHIC FUNCTIONS X_γ
- $\mathcal{T}$ HAS A FIBERWISE HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^{\mathcal{T}} := \frac{1}{2} \epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}}$$

WE SEARCH FOR A HOLOMORPHIC MAP

$$\mathcal{X}: \mathcal{Z} \longrightarrow \mathcal{T} = \Gamma^* \otimes \mathbb{C}^*$$



SO THAT: $$\omega_\zeta = \mathcal{X}^*(\varpi^{\mathcal{T}})$$

IF WE DEFINE $$\mathcal{X}_\gamma := \mathcal{X}^*(X_\gamma)$$

$$\Rightarrow \quad \omega_\zeta = \frac{1}{2} \epsilon^{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

WE CAN VIEW

$$\varpi_\zeta = \frac{1}{2}\epsilon^{ij}\,\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

IN TWO WAYS:

- KNOW THE METRIC $\Rightarrow$ CONSTRUCT $\mathcal{X}_\gamma$

  – DO THIS FOR THE SEMIFLAT METRIC AND FIRST QUANTUM CORRECTION

- ULTIMATELY, WE DEFINE THE $\mathcal{X}_\gamma$ AND USE THEM TO DEFINE ϖ_ζ (AND HENCE THE HK METRIC)

EXAMPLE: SEMI-FLAT LIMIT

WE KNOW $g^{sf} \Rightarrow$ COMPUTE

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

$$= \frac{i}{8\pi\zeta} da^I \wedge dz_I + \cdots + \frac{i\zeta}{8\pi}\overline{da^I \wedge dz_I}$$

FIND $\mathcal{X}_{\gamma_i}$ SO THAT:

$$\varpi_\zeta = \frac{1}{8\pi^2 R}\epsilon^{ij}\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

SOLUTION: DEFINE $\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$

$$\boxed{\mathcal{X}_\gamma^{sf} = \exp\left[\pi R\zeta^{-1} Z_\gamma + i\,\theta_\gamma + \pi R\zeta\,\overline{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

- LEADING APPXT. TO $\mathcal{X}_\gamma$ FOR $R \to \infty$
- NO Q.C.'s FROM BPS STATES.

6. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.

- CONSIDER A POINT $u_* \in \mathcal{B}$

WHERE A SINGLE HM HAS $M \to 0$

$\Longrightarrow$ DOMINANT CONTRIBUTION NEAR u_*.

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$

KK REDUCTION $\Longrightarrow$ TARGET SPACE METRIC IS A GIBBONS-HAWKING ANSATZ: [Seiberg Witten; Ooguri Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left(\frac{d\varphi_m}{2\pi} + A \right)^2 + V(\vec{x})\, d\vec{x}^2$$

$$F = * dV \qquad \vec{x} \in \mathbb{R}^3$$

INTEGRATE OUT KK TOWER:

$$V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{g^2R^2|a|^2 + \left(g\frac{\varphi_e}{2\pi} + n\right)^2}}$$

$$a = x^1 + i x^2$$

$$\varphi_e = 2\pi R\, x^3 \qquad \text{PERIODIC}$$

$$V(\vec{x}) = V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi} \sum_{n\neq 0} e^{i n g \varphi_e} K_0\left(2\pi R |n g a|\right)$$

$$\sim e^{-2\pi R |nga|} \quad : \quad \text{INSTANTON CONTRIBUTION}$$

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\mathcal{X}_\gamma\}$ ON TWISTOR SPACE IS GENERATED BY:

$$\mathcal{X}_e := \mathcal{X}_{(1,0)} = \exp\{ i\varphi_e + \cdots \}$$

$$\mathcal{X}_m := \mathcal{X}_{(0,1)} = \exp\{ i\varphi_m + \cdots \}$$

$$\mathcal{X}_{(k_1,k_2)} = (\mathcal{X}_e)^{k_1}(\mathcal{X}_m)^{k_2}$$

$$(k_1, k_2) \in \mathbb{Z}^2 \cong \Gamma_u$$

DETERMINE χ_e AND χ_m

FROM A DIFFERENTIAL EQUATION

$$\boxed{\varpi_5 = -\frac{1}{4\pi^2 R}\,\frac{d\chi_e}{\chi_e}\wedge\frac{d\chi_m}{\chi_m}}$$

HK STRUCTURE: $\alpha = 1,2,3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A\right) + \frac{1}{2} V \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Rightarrow$ COMPUTE $\varpi_5 = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$

WE FIND:

$$\mathcal{X}_e = \mathcal{X}_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i\varphi_e + \pi R \zeta \bar{a} \right]$$

BUT

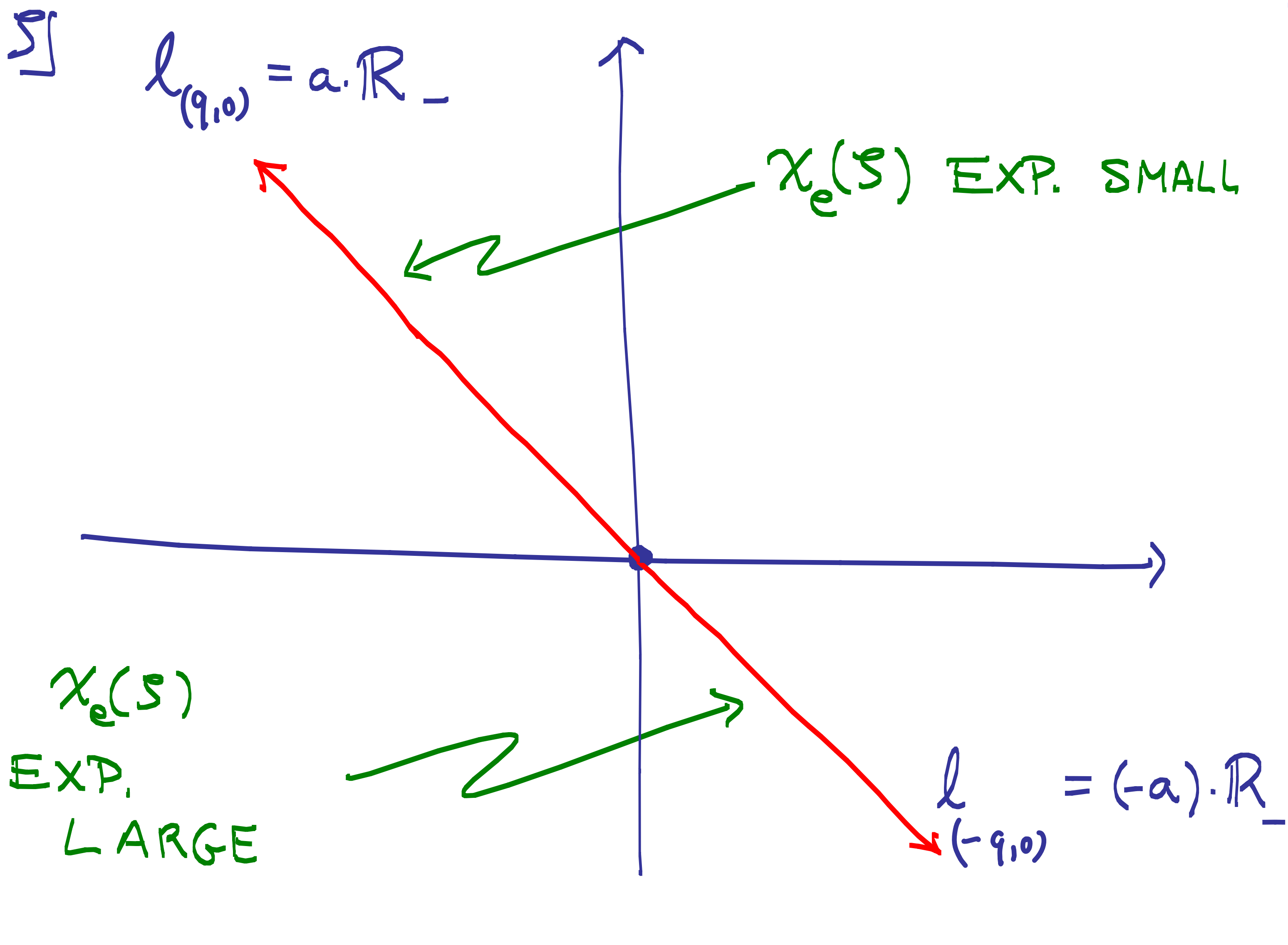
$$\mathcal{X}_m = \mathcal{X}_m^{s.f.} \cdot \mathcal{X}_m^{inst.}$$

$$\mathcal{X}_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i\varphi_m + \pi R \zeta \overline{a_D} \right]$$

$$a_D = \frac{q^2}{2\pi i}\left(a \log \frac{a}{e\Lambda} \right)$$

$\mathcal{X}_m^{inst}$ = INSTANTON CONTRIBUTION

$$\boxed{\chi_m^{inst}(\zeta) = \exp\left\{ \frac{iq}{4\pi} \int_{\ell_{(q,0)}} \frac{d\zeta'}{\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \log\left(1-\chi_e(\zeta')^{q}\right) - \frac{iq}{4\pi} \int_{\ell_{(-q,0)}} \frac{d\zeta'}{\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \log\left(1-\chi_e(\zeta')^{-q}\right) \right\}}$$



EMERGENCE OF THE KS TRANSFORMATION

AS A FUNCTION OF ζ, $\mathcal{X}_m$

IS DISCONTINUOUS ACROSS THE

BPS RAYS OF THE HYPERMULTIPLET

OF CHARGE $(\pm q, 0)$

$$\ell_\gamma := \left\{ \zeta \;\middle|\; \frac{Z_\gamma}{\zeta} \in \mathbb{R}_- \right\}$$

ACROSS THESE RAYS:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = K_{(\pm q, 0)} (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= \left(\mathcal{X}_e, \mathcal{X}_m (1 - \mathcal{X}_e^{\pm q})^{\mp q} \right)^{ccw}$$

KEY FEATURES OF $\mathcal{X}_\gamma$:

1. $\mathcal{X}_\gamma$ ARE HOLOMORPHIC ON $\mathcal{Z}$

2. $\mathcal{X}_\gamma \cdot \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$

3. $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$

4. $\mathcal{X}_\gamma \sim \mathcal{X}_\gamma^{s.f.}$ FOR $R \to \infty$

5. $$\left.\begin{array}{l} \lim\limits_{\zeta \to 0} \mathcal{X}_\gamma \exp\left(-\frac{\pi R}{\zeta} Z_\gamma(u)\right) \\ \lim\limits_{\zeta \to \infty} \mathcal{X}_\gamma \exp\left(-\pi R \zeta \overline{Z_\gamma(u)}\right) \end{array}\right\} \text{FINITE}$$

6. $\mathcal{X}_{\gamma'}(\zeta)$ TRANSFORMS BY $K_\gamma^{\Omega(\gamma;u)}$ ACROSS THE BPS RAY ℓ_γ.

7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: PROPERTIES 1–6

HOLD FOR THE <u>EXACT</u> FUNCTIONS $\mathcal{X}_{\gamma_g}$ USING ALL THE BPS RAYS ℓ_γ WITH DISCONTINUITY $K_\gamma^{\Omega(\gamma;u)}$

THIS WILL DETERMINE THEM UNIQUELY

OBSERVATION: $\mathcal{X}_\gamma$ ARE THE SOLUTION OF A RIEMANN-HILBERT PROBLEM.

(R-H PROBLEM:

FIND A PIECEWISE HOLO. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.)

SUMMARIZING THE $\mathcal{X}_\gamma$ BY A SINGLE MAP $\mathcal{X}$ (RECALL $\mathcal{X}_\gamma = \mathcal{X}^*(X_\gamma)$)

$\Rightarrow$ A RIEMANN-HILBERT PROBLEM IN THE ζ- PLANE FOR THE MAP

$$\mathcal{X}(\zeta) : \mathcal{M}^\zeta \longrightarrow \mathcal{T} = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ :

$$\mathcal{X}^{cw} = S_\gamma\left(\mathcal{X}^{ccw}\right)$$

$$\left[\text{RECALL: } S_\gamma = \prod_{\ell_{\gamma'}=\ell_\gamma} K_{\gamma'}^{\Omega(\gamma';u)}\right]$$

2.) $\mathcal{X}(\zeta)$ HAS ASYMPTOTICS FOR $\zeta \longrightarrow 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $\mathcal{O}(1)$ CORRECTIONS

$$\Upsilon := (\mathcal{X}^{sf})^{-1}\mathcal{X} : \mathcal{M} \to \mathcal{M}$$

i.e.

$$\Upsilon_0 = \lim_{\zeta\to 0} \Upsilon(\zeta) \;\&\; \Upsilon_\infty = \lim_{\zeta\to\infty} \Upsilon(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf}(\zeta) \cdot$$

$$\exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \right.$$

$$\left. \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_{\gamma'}(\zeta') \right] \right\}$$

ITERATING THIS EQUATION

(AS A SUM OVER TREES...)

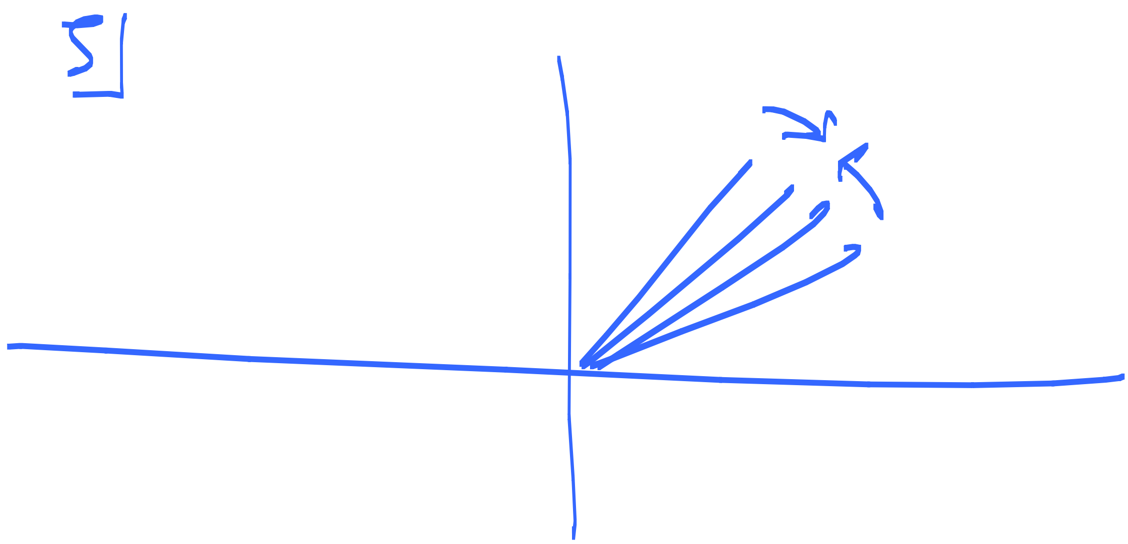
GIVES THE FULL INSTANTON EXPANSION!

$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S

- WE RECONSTRUCT THE METRIC FROM

$$\varpi = \frac{1}{4\pi^2 R} \mathcal{X}^*(\varpi^T)$$

- AS u CROSSES A WALL OF MS BPS RAYS PILE UP



$u \to u^+$: DISCONTINUITY IN RH PROBLEM ALONG $\ell_{\gamma_1} = \ell_{\gamma_2}$

$$= \overset{\curvearrowright}{\prod_{\substack{n \geq 0 \\ m \geq 0}}} K_{n\gamma_1+m\gamma_2}^{\Omega^+(n\gamma_1+m\gamma_2)}$$

$u \to u^-$: DISCONTINUITY is $\overset{\curvearrowleft}{\prod}$

THUS, THE RH PROBLEM REMAINS UNCHANGED AS u CROSSES THE WALL IF THE $\Omega(\gamma;u)$ OBEY THE KSWCF.

CONCLUSION: $\mathcal{X}_\gamma(u,\zeta)$ IS CONTINUOUS IN $\mathcal{B}\times\mathbb{C}^*$ EXCEPT ALONG BPS RAYS $\ell_\gamma\subset\mathbb{C}^*$.

ALONG ℓ_γ $\mathcal{X}_\gamma$ JUMPS BY A SYMPLECTOMORPHISM SO $\widetilde{\omega}_\zeta$ IS <u>CONTINUOUS</u>.

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE HK. METRIC ACROSS WALLS OF M.S. !

THE RESULTING METRIC PASSES A NUMBER OF CONSISTENCY TESTS.

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM?

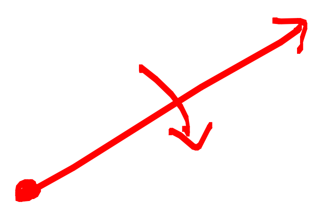
8. PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$\mathcal{A}_\zeta = \mathcal{X}^{-1}\,\zeta\partial_\zeta \mathcal{X}$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$\mathcal{X}^{-1}\,\zeta\partial_\zeta\mathcal{X} \to (S\mathcal{X})^{-1}\,\zeta\partial_\zeta(S\mathcal{X})$$

$$= \mathcal{X}^{-1}\,\zeta\partial_\zeta\,\mathcal{X}$$

$\Rightarrow \mathcal{A}_\zeta$ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Longrightarrow \quad \zeta \partial_\zeta X = X \mathcal{A}_\zeta$$

STRUCTURE GROUP: SYMPL(Γ)

ASYMPTOTICS $\Longrightarrow$

$$\mathcal{A}_\zeta = \zeta^{-1} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(+1)}$$

(NOTE: $\mathcal{A}_\zeta^{(-1)}$ CONJUGATE TO $i\pi Z_{\gamma_i} \overleftarrow{\frac{\partial}{\partial \theta^i}}$)

SINCE S_γ IS INDPT. OF R, u, Λ ....

SAME ARGUMENT $\Rightarrow$ X SATISFIES A

SET OF DIFFERENTIAL EQATIONS:

$$\frac{\partial}{\partial u}\chi = \chi \mathcal{A}_u$$

$$\frac{\partial}{\partial \bar{u}}\chi = \chi \bar{\mathcal{A}}_u$$

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \chi \mathcal{A}_\Lambda$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \chi \mathcal{A}_{\bar{\Lambda}}$$

$$R \frac{\partial}{\partial R}\chi = \chi \mathcal{A}_R$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \chi \mathcal{A}_\zeta$$

$$\mathcal{A}_i = \zeta^{-1}\mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta\,\mathcal{A}_i^{(+1)}$$

KEY POINT: THESE EQUATIONS ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!

$$\frac{\partial}{\partial u}\chi = \chi \mathcal{A}_u$$

$$\frac{\partial}{\partial \bar{u}}\chi = \chi \mathcal{A}_{\bar{u}}$$

HOLOMORPHY ON $\mathcal{M}^5$

ALSO HOLOMORPHY...

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \chi \mathcal{A}_\Lambda$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \chi \mathcal{A}_{\bar{\Lambda}}$$

VIEW Λ AS BACKGROUND VEV OF A VM.

$$R \frac{\partial}{\partial R}\chi = \chi \mathcal{A}_R$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \chi \mathcal{A}_\zeta$$

ANOMALOUS SCALE AND R-SYMMETRY

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.

$\mathcal{A}_{\gamma}^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS ℓ_{γ}

DENOTE STOKES FACTORS BY $\mathcal{S}_{\gamma}$

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\mathcal{S}_{\gamma}$ ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1-INSTANTON APPROXIMATION:

$$\mathcal{S}_{\gamma} = S_{\gamma}^{K.S.}$$

9. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE HYPERKÄHLER METRIC FOLLOWS FROM THE KS WCF.

4. IT IS USEFUL TO WORK WITH THE TWISTOR TRANSFORM AND HOLOMORPHIC FUNCTIONS ON TWISTOR SPACE.

10. CONCLUSION

— OTHER THINGS WE HAVE DONE —

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAFA.

$\mathcal{B}$ = FAMILY OF MASSIVE $d=2$ $\mathcal{N}=(2,2)$ Q.F.T.

$V \longrightarrow \mathcal{B}$ BUNDLE OF (c,c) OPERATORS ($\Longleftrightarrow$ R GROUNDSTATES)

ANALOGY $V = \mathcal{C}^\infty(\mathcal{M}_u)$

CECOTTI & VAFA DEFINE tt^* CONNECTION

$$\nabla_i \psi = \left(\frac{\partial}{\partial t^i} + \zeta^{-1} C_i + g^{-1} \partial_i g \right) \psi = 0$$
$$\bar{\nabla}_i \psi = \left(\frac{\partial}{\partial \bar{t}^i} + \zeta \bar{C}_i \right) \psi = 0$$

FLATNESS $\Rightarrow$ tt^* EQUATIONS

ANALOGY: HOLOMORPHY OF $\mathcal{X}$ ON $\mathcal{M}^\zeta$

MOREOVER, ANOMALOUS SCALE + $U(1)_R$ SYMMETRY $\Longrightarrow$

$$\zeta \frac{\partial}{\partial \zeta} \psi = \left(R \zeta C + Q - R \zeta^{-1} \bar{C} \right) \psi$$
$$R \frac{\partial}{\partial R} \psi = \left(R \zeta C + Q + R \zeta^{-1} \bar{C} \right) \psi$$

BPS WALL-CROSSING EXPRESSED THROUGH STOKES FACTORS

STOKES FACTORS $S_{ij} = \mathbb{1} + e_{ij} \quad i \neq j$

BPS CHARGES LABELED BY PAIRS $i \neq j$

EXAMPLE $\mathrm{RK}(V) = 3$. CROSSING A WALL FOR BPS STATE (13):

$$S_{12}^{\Omega_{12}} S_{13}^{\Omega_{13}^{+}} S_{23}^{\Omega_{23}} = S_{23}^{\Omega_{23}} S_{13}^{\Omega_{13}^{-}} S_{12}^{\Omega_{12}}$$

$$\Longleftrightarrow \qquad \Omega_{13}^{+} = \Omega_{13}^{-} - \Omega_{12}\Omega_{23}$$

NOTE $\Omega_{ij} = \mathrm{Tr}_{\mathcal{H}_{ij}}\left(F(-1)^F e^{-R\cdot H}\right)$

4D ANALOG ?

- THE FUNCTIONS $\mathcal{X}_\gamma$ ARE 'tHOOFT–WILSON–MALDACENA LOOP OPERATOR VEV'S; MOREOVER THERE IS A NICE INTERPRETATION IN TERMS OF A 3D TFT [TO APPEAR]

- ANALOG FOR SUPERGRAVITY:

$\mathcal{M} \longrightarrow$ Q.K. MANIFOLD FIBERED BY HEISENBERG GROUPS

$\mathcal{T} \longrightarrow$ CANONICAL H.G. EXTENSION OF $\Gamma^* \otimes \mathbb{C}^*$ GIVEN BY Q.R.

HITCHIN THM. $\longrightarrow$ LE BRUN THM.

$\varpi_\zeta \longrightarrow$ CONTACT STRUCTURE

BUT.... CONVERGENCE???

• RELATION TO HITCHIN SYSTEMS

FOR A LARGE CLASS OF $d=4, N=2$ THEORIES THE MODULI SPACE $(\mathcal{M}, g)$ IS ALSO THE MODULI SPACE OF A HITCHIN SYSTEM (WITH SINGULARITIES) ON $\mathbb{P}^1$.

THIS FOLLOWS FROM WITTEN'S CONSTRUCTION OF $d=4, N=2$ THEORIES USING M5-BRANES.

FOR THESE THEORIES WE HAVE CONSTRUCTED THE $\mathcal{X}_\gamma$ AND VERIFIED THE PROPERTIES (1–6) ABOVE

THE KS TRANSFORMATIONS AND $\zeta \to 0, \infty$ ASYMPTOTICS EMERGE VERY NATURALLY...

SEE ANDY NEITZKE'S TALK.

— TO DO —

- SINGULARITIES AT SUPERCONFORMAL POINTS REMAIN TO BE UNDERSTOOD

- RELATIONS TO INTEGRABLE SYSTEMS AND THE "MOTIVIC W.C.F."

- RELATION TO THE WORK OF JOYCE & BRIDGELAND / TOLEDANO LAREDO

- SOME OF THE DISCUSSION GENERALIZES NICELY TO SUGRA, BUT PUZZLES REMAIN

- MIGHT GIVE EXPLICIT FORMULATION OF Q.C.'S TO HYPERMULTIPLET MODULI SPACES.