SURFACE DEFECTS IN D=4, N=2 THEORIES: BPS STATES, WALL-CROSSING, & HYPERKÄHLER GEOMETRY

SCGP, MARCH 25, 2011

WORK DONE WITH
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"4D WC + 3D FT" 0807.4723

"WC, HITCHIN, WKB" 0907.3987

"FRAMED BPS STATES" 1006.0146

* "WALL-CROSSING IN COUPLED 2D-4D" 1103.2598

I. SUMMARY OF MAIN RESULTS

1. WE COMBINE AND GENERALIZE THE CECOTTI-VAFA AND K-S WALL-CROSSING FORMULA, AND EXPLAIN ITS ROLE IN THE PHYSICS OF SURFACE DEFECTS

2. WE CONSTRUCT HYPERHOLOMORPHIC CONNECTIONS (BBB BRANES) OVER SEIBERG-WITTEN MODULI SPACES USING A GENERALIZATION OF THE INVERSE SCATTERING METHOD
($\Rightarrow$ EXPLICIT SOLUTIONS TO HITCHIN'S EQUATIONS.)

II. BACKGROUND REVIEW

A. $\mathcal{N}=2$ FIELD THEORY

B. BPS STATES & INDICES

C. LINE DEFECTS

D. H.K. GEOMETRY

E. THEORIES OF CLASS S

IIA: $\mathcal{N}=2$ INGREDIENTS

1. $\mathcal{B}$ = COULOMB BRANCH OF A $d=4$, $\mathcal{N}=2$ FIELD THRY

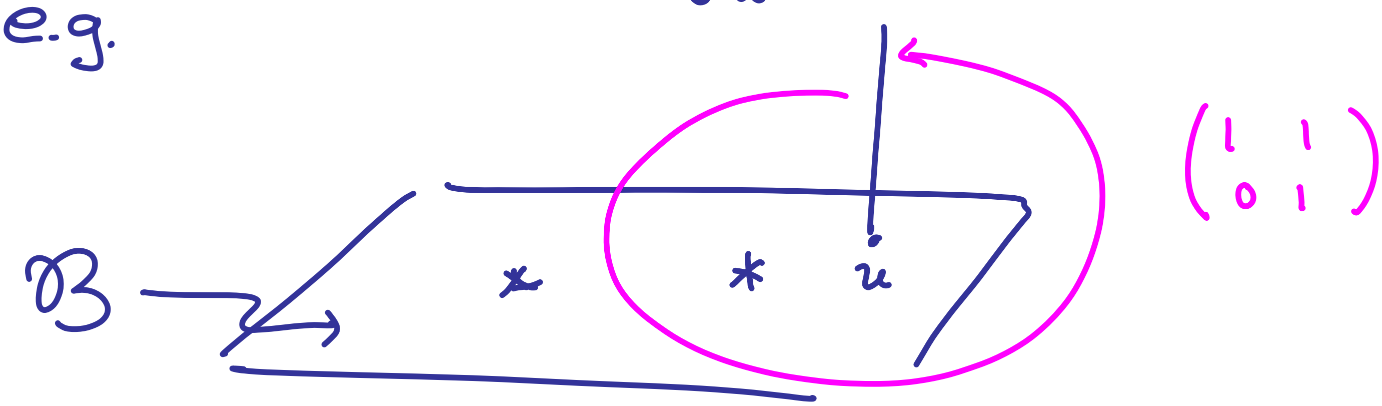
2. LOW ENERGY THEORY HAS UNBROKEN RANK r

$$\boxed{\text{SELF-DUAL ABELIAN GAUGE THEORY}}$$

3. LOCAL SYSTEM $\Gamma_g \to \mathcal{B}$ OF RANK $2r$ SYMPLECTIC LATTICES

$(\Gamma_g)_u \cong \mathbb{Z} \oplus \mathbb{Z}$

e.g.



4. $V = \Gamma_g \otimes \mathbb{R}$ HAS COMPATIBLE POSITIVE COMPLEX STR. $\mathcal{J}$

- $\mathbb{F} \in \Omega^2(M^4) \otimes V$
- $\mathbb{F} = (* \otimes \mathcal{J})\mathbb{F}$
- $d\mathbb{F} = 0.$

5. CHOOSING A DUALITY FRAME

$\{e_I, e^I\}$ FOR Γ_g

- $\mathbb{F} = e_I F^I + e^I G_I$
- $\mathcal{J} \Longrightarrow \tau_{IJ}$
- ACTION

$$S = \frac{1}{4\pi} \int \mathrm{Im}\,\tau_{IJ} F^I * F^J + \mathrm{Re}\,\tau_{IJ} F^I F^J$$

6. INCLUDING FLAVOR CHARGES:

$$0 \to \Gamma_f \longrightarrow \Gamma \longrightarrow \Gamma_g \longrightarrow 0$$

- $\Gamma_f = \mathrm{Ann}(\langle \cdot, \cdot \rangle)$
- NOT GLOBALLY SPLIT

7. CENTRAL CHARGE $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$

8. LAGRANGIAN SUBVARIETY:

$$a^I = Z(e^I) \qquad a_{D,I} = Z(e_I) = \frac{\partial \mathcal{F}}{\partial a^I}$$

9. $$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma ,$$

$E \geq |Z_\gamma|$ ON $\mathcal{H}_\gamma$

$$\mathcal{H}_\gamma^{BPS} := \{ \psi \mid E\psi = |Z_\gamma| \psi \}$$

II B. PROTECTED SPIN CHARACTER

$$\mathcal{H}^{BPS}_{\gamma} = \rho_{hh} \otimes h_{\gamma}$$

$$\Omega(\gamma; u; y) = \mathrm{Tr}_{h_\gamma} (-1)^{2J_3} (-y)^{2\mathcal{J}_3}$$

$$\mathcal{J}_3 = J_3 + I_3$$

BPS INDEX $\quad \Omega(\gamma; u) = \Omega(\gamma; u; y=-1)$

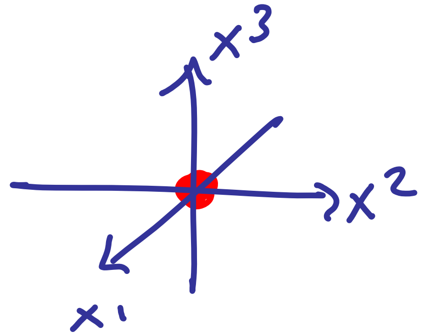
THESE INDICES JUMP ACROSS

$$W(\gamma_1, \gamma_2) = \{ u \in \mathcal{B} \mid Z_{\gamma_1} \parallel Z_{\gamma_2} \}$$

IIC: LINE DEFECTS

1. CONSIDER A LINE DEFECT

$$\{\vec{x}=0\}\times\mathbb{R}$$



DEF: WE SAY L IS OF TYPE ζ IF IT PRESERVES THE SUBALGEBRA

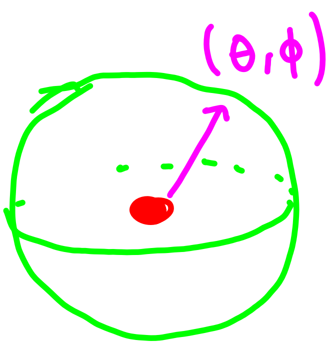
$$R_\alpha{}^A \sim Q_\alpha{}^A + \zeta\,\sigma^0_{\alpha\dot\beta}\,\bar{Q}^{\dot\beta A}$$

EXAMPLES

1.) WILSON

$$L_g \sim P\exp \int_{\mathbb{R}\times\vec{0}} \left(\frac{\varphi}{2g} - iA - \frac{g}{2}\bar{\varphi}\right)$$

2.) 't HOOFT



$$F \sim p \otimes \sin\theta \, d\theta \, d\phi \qquad p \in \mathfrak{t}$$

$$\varphi/g \sim \frac{p}{r} + \varphi_\infty/g$$

(B.) FRAMED BPS STATES

HILBERT SPACE IS MODIFIED

$$\mathcal{H}_L = \bigoplus_{\gamma \in \Gamma_L} \mathcal{H}_{L,\gamma}$$

(IN GENERAL $\Gamma_L = \Gamma$- TORSOR)

$\Longrightarrow$ MODIFIED BPS BOUND

$$E \geq -\mathrm{Re}\left(Z_\gamma / \zeta\right)$$

THINK OF $\mathcal{H}_{L,\gamma}$ AS SECTOR CREATED BY ∞-LY HEAVY DYON OF CHARGE γ, AND NOTE THAT:

$$\lim_{M \to \infty} \left(| \zeta M - Z_\gamma | - M \right) = -\mathrm{Re}\left(Z_\gamma / \zeta\right)$$

DEF: FRAMED BPS STATES SATURATE THIS BOUND

DEF: THE FRAMED PROTECTED SPIN CHARACTER

$$\underline{\overline{\Omega}} := \mathrm{Tr}_{\mathcal{H}^{BPS}_{L_\zeta,\gamma,u}} (-1)^{2J_3} (-y)^{2\mathcal{J}_3}$$

$$\mathcal{J}_3 = J_3 + I_3$$

$$\underline{\overline{\Omega}}(L,\gamma;y;\zeta;u)$$

PIECEWISE CONSTANT IN ζ, u:

$\exists$ WALL-CROSSING FOR $\underline{\overline{\Omega}}$

(C.) FRAMED BPS WALL-CROSSING

THE "HALO METHOD" LEADS TO A W.C.F. FOR $\overline{\underline{\Omega}}$:

$\overline{\underline{\Omega}}$ WILL JUMP ACROSS "BPS WALLS"

$$W_{\gamma_h} := \left\{ (u,\zeta) \;\middle|\; \frac{Z_{\gamma_h}(u)}{\zeta} \in \mathbb{R}_- \right\}$$

$$\subset \mathcal{B} \times \mathbb{C}^*$$

(THESE PLAY THE ROLE OF STOKES WALLS.)

TO WRITE THE WCF:

$$F(L) = \sum_{\gamma \in \Gamma_L} \overline{\underline{\Omega}}(L, \gamma, y) X_\gamma$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

ACROSS W_{γ_h}

$$F^+(L) = \Phi_{\gamma_h}(X_{\gamma_h}) F^-(L) \, \Phi_{\gamma_h}(X_{\gamma_h})^{-1}$$

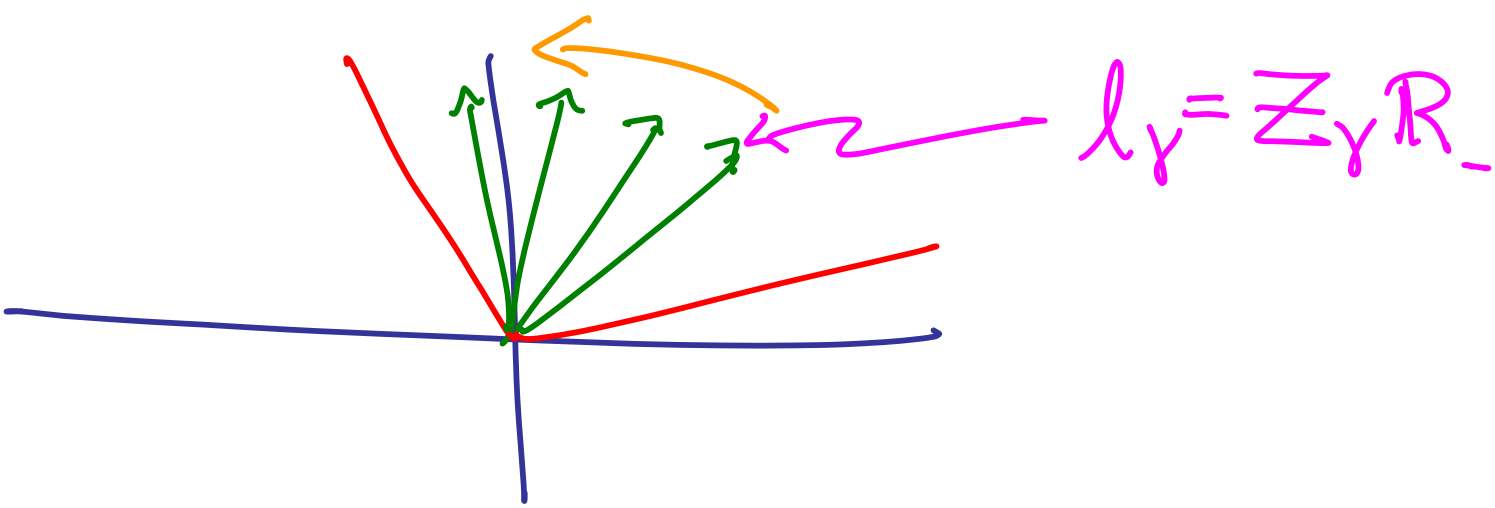
- $\Phi_{\gamma_h}(X_{\gamma_h})$ DEPENDS ON $\Omega(\gamma_h; y; u)$
- CONSISTENCY OF WCF FOR $\overline{\underline{\Omega}}$

$\Longrightarrow$ "motivic" KSWCF

SIMPLIFICATION FOR $y \to -1$

$$K^{\Omega}_{\gamma_h} : X_\gamma \to (1 - X_{\gamma_h})^{\langle \gamma, \gamma_h \rangle \Omega(\gamma_h)} \cdot X_\gamma$$

$$A(\sphericalangle) = : \prod_{l_\gamma \subset \sphericalangle} K^{\Omega}_{\gamma} :$$



$A(\sphericalangle)$ IS <u>CONSTANT</u> AS FUNCTION OF u

II D: HK GEOMETRY & COMPACTIFICATION TO 3DIM'S

(A.) PUT THEORY ON $\mathbb{R}^3 \times S^1_R$

LOW ENERGY EFFECTIVE THEORY:

σ - MODEL : $\mathbb{R}^3 \longrightarrow \mathcal{M}$

FIELDS : $u(\vec{x}) \in \mathcal{B}$ [SEIBERG & WITTEN]

<u>AND</u>

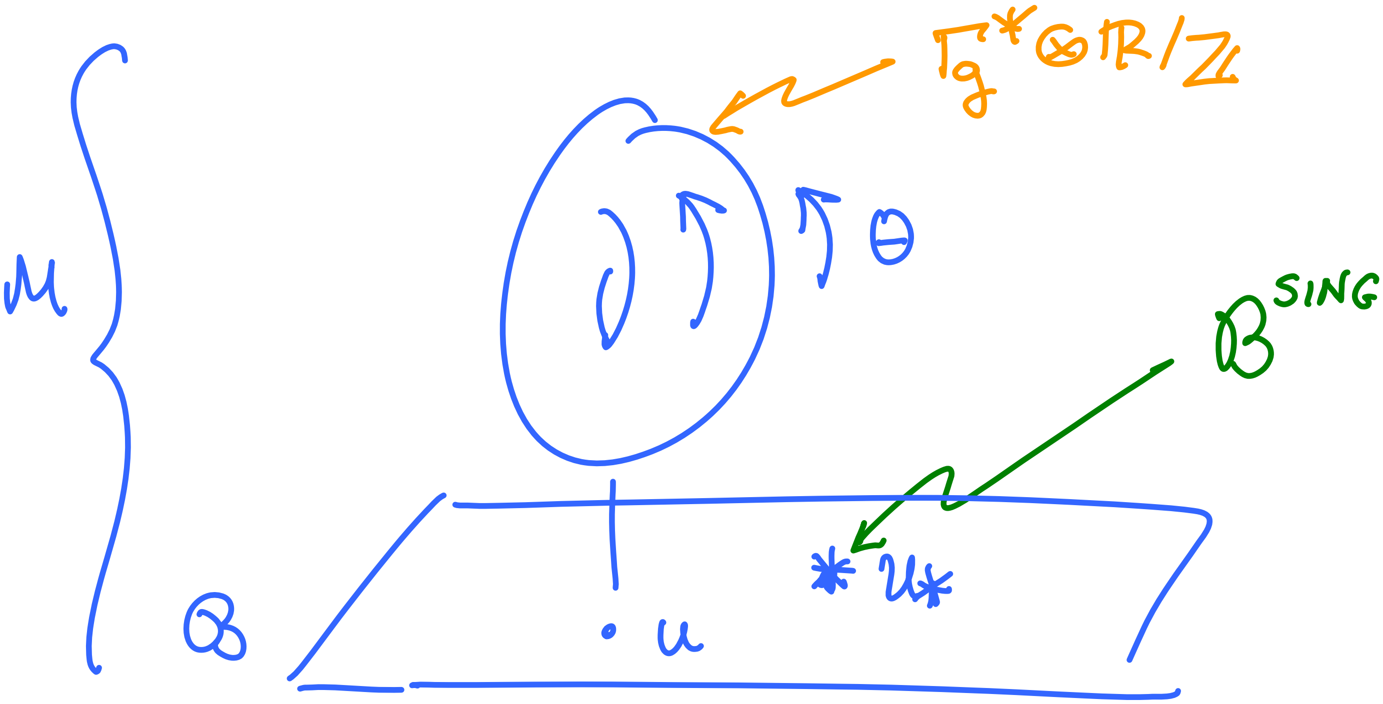
WILSON LINE SCALARS

$$\theta_\gamma = \oint_{S^1} \langle \gamma, A \rangle$$

ELEC. + MAG. WILSON LINES:

PERIODIC SCALARS: $\theta \in \Gamma_g^* \otimes \mathbb{R}/\mathbb{Z}$

$\mathcal{M}$ IS A TORUS FIBRATION



- SUSY $\Rightarrow$ $\mathcal{M}$ IS HYPERKÄHLER
- THERE CAN BE SINGULARITIES ABOVE THE DISCRIMINANT LOCUS WHERE POPULATED CHARGES γ ARE MASSLESS:

$$\mathcal{B}^{SING} = \{ u \mid Z(\gamma; u) = 0 \ \& \ \Omega(\gamma; u) \neq 0 \}$$

(B.) "DARBOUX COORD'S"

IT IS INSTRUCTIVE TO CONSTRUCT THE HK METRIC ON $\mathcal{M}$ EXPLICITLY

HK $\Longleftrightarrow$ FAMILY OF HOLOMORPHIC-SYMPLECTIC $\mathcal{M}^{\zeta}$, $\zeta \in \mathbb{P}^1$

GMN '08 : CONSTRUCTED A FAMILY OF FUNCTIONS $\mathcal{Y}_\gamma$ ON $\mathcal{M}^{\zeta} \times \mathbb{C}^*$

1. PIECEWISE HOLOMORPHIC ON $\mathcal{M}^{\zeta} \times \mathbb{C}^*$

2. $\mathcal{Y}_\gamma \mathcal{Y}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{Y}_{\gamma+\gamma'}$

3. $\{\mathcal{Y}_\gamma, \mathcal{Y}_{\gamma'}\}_{\omega^{\zeta}} = \langle \gamma, \gamma' \rangle \mathcal{Y}_\gamma \mathcal{Y}_{\gamma'}$

4. $\mathcal{Y}_\gamma \xrightarrow[\zeta \to 0]{} \mathcal{Y}_\gamma^{sf} := \exp\left(\frac{\pi R Z_\gamma}{\zeta} + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right)$

5. ACROSS W_{γ_h}: $\mathcal{Y}_\gamma^{+} = \left(1 - \mathcal{Y}_{\gamma_h}\right)^{\langle \gamma, \gamma_h \rangle \Omega(\gamma_h)} \mathcal{Y}_\gamma^{-}$

$\Rightarrow$ CONSTRUCT THE $\mathcal{Y}_\gamma$

VIA INTEGRAL EQUATION:

$$\boxed{\log \mathcal{Y}_\gamma = \log \mathcal{Y}_\gamma^{sf} + \sum_{\gamma'} \langle \gamma, \gamma' \rangle \Omega(\gamma') \mathbb{K}_{\gamma'} * \log(1 - \mathcal{Y}_{\gamma'})}$$

$$\mathbb{K}_{\gamma'} * f = \int_{\ell_{\gamma'}} d\zeta' K(\zeta, \zeta') f(\zeta')$$

$$\ell_{\gamma'} = \{ \zeta \mid Z_{\gamma'}/\zeta < 0 \}$$

$$d\zeta' K(\zeta, \zeta') = \frac{-1}{4\pi i} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta}$$

- FORMALLY ZAMOLODCHIKOV'S TBA
- $\sum_{\gamma'}$: INSTANTON CONTRIBUTIONS

(C.) RELATION TO LINE DEFECTS

- WRAP L_ζ ON S^1 $\Rightarrow$

LOCAL OPERATOR IN 3D THEORY

THE "DARBOUX EXPANSION"

$$\langle L_\zeta \rangle_m = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma) \, \mathcal{Y}_\gamma(m, \zeta)$$

HAS NO WALL-CROSSING

- NONCOMMUTATIVE DEFORMATION

$$F(L_\zeta) = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma; y) X_\gamma$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

DEFORMS ALGEBRA OF FUNCTIONS ON $\mathcal{M}^\zeta$

II E. M5-BRANES & HITCHIN SYSTEMS

(A.) THEORIES OF CLASS $\mathcal{S}$

GENERALIZING A CONSTRUCTION OF WITTEN, GMN INTRODUCED THE CLASS $\mathcal{S}$ OF $D=4, \mathcal{N}=2$ THEORIES:

CONSIDER NONABELIAN $(2,0)$ $\mathfrak{g}$ - THEORY ON RIEMANN SURFACE C WITH PUNCTURES

C: "UV CURVE"

BREAK $so(5)_R \longrightarrow so(2) \oplus so(3)$

AND PARTIALLY TWIST TO PRODUCE

$d=4, \mathcal{N}=2$ THEORY ON $\mathbb{R}^{1,3}$

BOUNDARY COND'S AT PUNCTURES

2 SPECIFIED BY BEHAVIOR OF BPS OPERATORS $\Longrightarrow$ $T[C, \mathfrak{g}, m]$

THIS IS THE CLASS $\mathcal{S}$.

(B.) WHEN COMPACTIFYING THEORIES IN CLASS $\mathcal{S}$ ON S^1 WE CAN IDENTIFY $\mathcal{M}$ WITH MODULI SPACE OF HITCHIN SYSTEMS ON C :

$$
\begin{tikzcd}
 & 6D\ \mathfrak{g}\text{-THEORY}/C\times S^1\times\mathbb{R}^{1,2} \arrow[dl, "C"'] \arrow[dr, "S^1"] & \\
T[C,\mathfrak{g},m]/S^1\times\mathbb{R}^{1,2} \arrow[dr, "S^1"'] & & 5D\ SYM/C\times\mathbb{R}^{1,2} \arrow[dl, "C"] \\
 & \sigma\text{-MODEL } \mathbb{R}^{1,2}\to\mathcal{M} &
\end{tikzcd}
$$

(C.) SW MODULI = HITCHIN MODULI

- REDUCTION OF 5D $\mathcal{Y}$-SYM ON C $\Rightarrow$ BPS EQS = HITCHIN EQS.:

$$F + R^2 [\varphi, \bar{\varphi}] = 0$$

$$\bar{\partial}_A \varphi = 0$$

- SPECTRAL CURVE $\Sigma \subset T^*C$

Σ: $$\det(\lambda - \varphi(z,\bar{z})) = 0$$

$$\Sigma \longrightarrow C$$

Σ = SEIBERG-WITTEN CURVE

- $\lambda = p\,dq$ on T^*C

$$\lambda|_{\Sigma} = \text{SW DIFFERENTIAL}$$

- $\Gamma = H_1(\Sigma, \mathbb{Z})$

2

(D.) SW MODULI = HITCHIN MODULI

= MODULI OF FLAT G_c CONN'S.

(A, φ) SOLVE HITCHIN EQS. $\Longrightarrow$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

IS FLAT: $d\mathcal{A} + \mathcal{A}^2 = 0 \quad \forall \zeta \in \mathbb{C}^*$

IN FACT:

$$\mathcal{M}^{\zeta}_{\mathrm{HITCHIN}} \cong \mathcal{M}_{\mathrm{FLAT}}(\mathfrak{g}_{\mathbb{C}}\text{-CONN})$$

TYPICAL BOUNDARY CONDITION

$$\varphi \sim \frac{1}{z - z_a}\begin{pmatrix} m_a & \\ & -m_a \end{pmatrix} + \cdots$$

z_a = PUNCTURES OF C

(E.) BPS STATES FOR $T[C, A_1, m]$

DEF: A <u>WKB PATH</u> ON C

$$\langle \lambda , \partial_t \rangle = e^{i\vartheta}$$

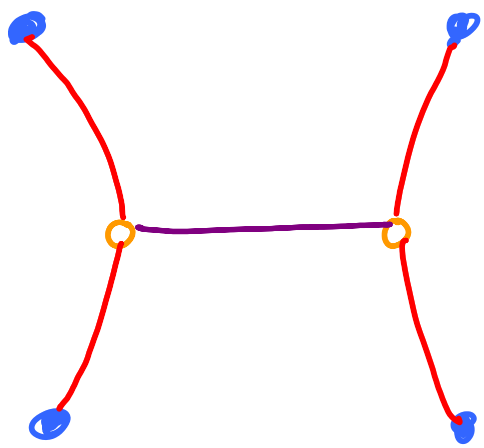
GENERIC WKB PATHS HAVE BOTH ENDS ON SINGULAR POINTS z_a:

AT CRITICAL VALUES OF $\vartheta = \vartheta_*$ <u>FINITE</u> WKB PATHS APPEAR. THEY LIFT TO CLOSED CYCLES $\gamma \subset \Sigma$.

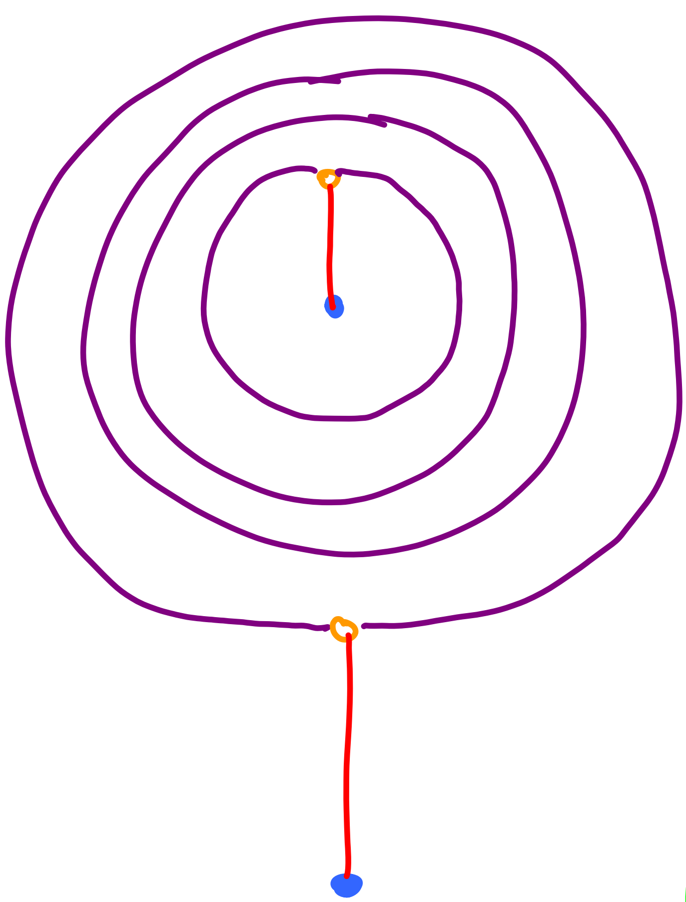
THESE "ARE" BPS STATES WITH

$$Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda = e^{i\vartheta_*} |Z_\gamma|$$

C

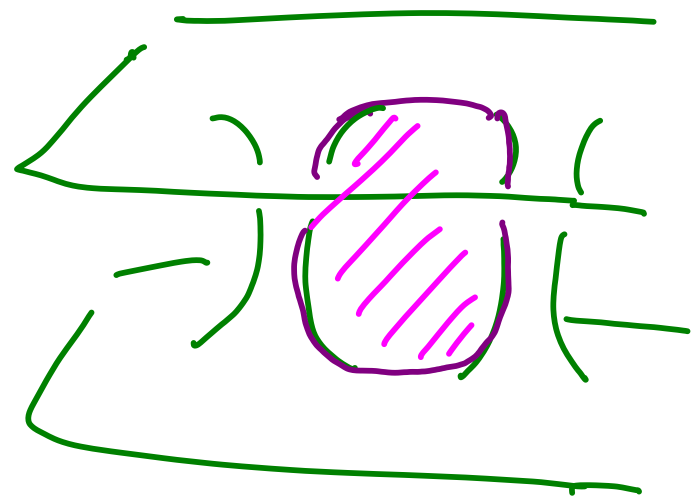


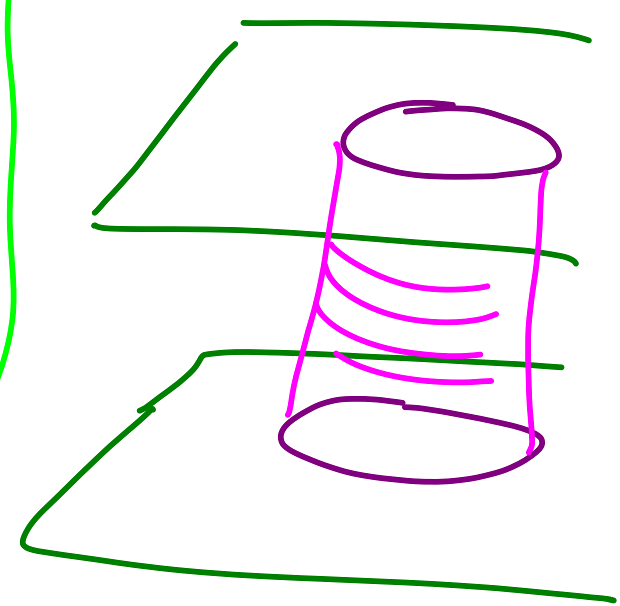
HYPER



VECTOR

Σ





(F.) A NATURAL SET OF LINE DEFECTS

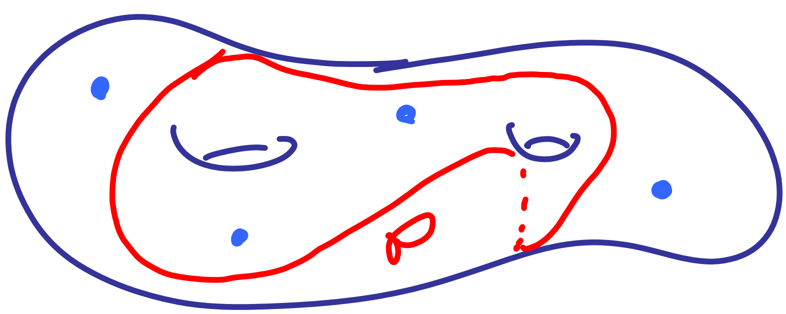
6D $\mathfrak{g}$-THEORY HAS SUSY

<u>SURFACE OP'S</u> $\mathbb{S}(R, \sigma)$

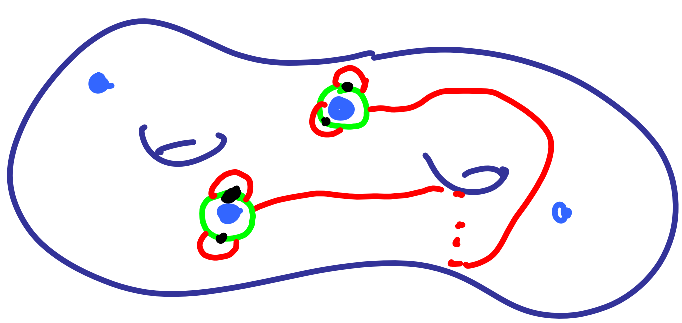
$R \sim \mathfrak{g}$-REP, σ = SURFACE

CHOOSE CLOSED PATH $\wp \subset C$

C:



[MORE GENERALLY $\wp$ = LAMINATION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- CHOOSE $\sigma = \mathbb{R} \times \wp$:

6D SURFACE DEFECT $\mathbb{S}(\mathcal{R}, \sigma)$

$$\Longrightarrow$$

4D LINE DEFECT $L_\varsigma(\mathcal{R}, \wp)$

IN THEORY $T(C, \mathfrak{g}, m)$

- IF WE TAKE $\sigma = S^1_R \times \wp$:

$$\langle \mathbb{S}(\mathcal{R}, \sigma) \rangle = \langle L_\varsigma(\mathcal{R}, \wp) \rangle$$

$$= \mathrm{Tr}_{\mathcal{R}} \left(P\exp \int_{\wp} \frac{R\varphi}{\varsigma} + A + R\varsigma\bar{\varphi} \right)$$

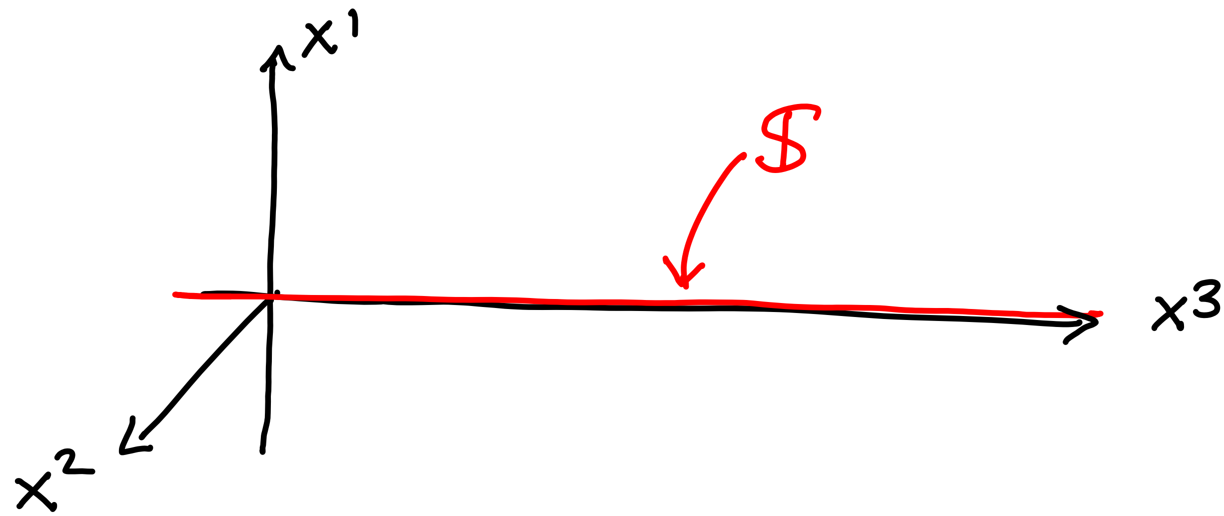
$$= \mathrm{Tr}_{\mathcal{R}} \left(\mathrm{Hol}_{\wp}(\mathcal{A}) \right)$$

$\Rightarrow$ <u>COMPUTABLE</u>

III. SURFACE DEFECTS & 2D/4D WCF

(A.) UV DEFINITION

SUPPORT: $x^1 = x^2 = 0$ IN $\mathbb{R}^{1,3}$



- POINCARE' SUBALGEBRA

$$\{ P_0, P_3, M_{12}, F_A, F_V \}$$

$$\oplus$$

$$\{ Q_1^{\ 1}, Q_2^{\ 2}, \bar{Q}_{\dot{1}1}, \bar{Q}_{\dot{2}2} \}$$

ISOMORPHIC TO $D=2$ $(2,2)$ SUSY

(WITHOUT SETTING $P_1 = P_2 = 0$!!)

$$\begin{pmatrix} \mathcal{N}=2 \\ VM \end{pmatrix} \supset \begin{matrix} D=2 \ (2,2) \\ \text{TWISTED CHIRAL MULTIPLET} \end{matrix}$$

$$Y = \varphi + \theta\cdot\lambda + \theta^2\left(F_{03} - i\left(F_{12} - D_{12}\right)\right)$$

$\Rightarrow$ TWO UV DEF'S

1

- CHOOSE $D=2$ $(2,2)$ THEORY T_{2d} ON S WITH G- GLOBAL SYMMETRY
- CHOOSE $D=4$ $\mathcal{N}=2$ THEORY T_{4d} WITH G-GAUGE SYMMETRY
- USING RESTRICTION OF TWISTED CHIRAL MULTIPLET, GAUGE THE 2D G-GLOBAL SYMMETRY.

2

(GUKOV-WITTEN):

USING D=4 THEORY T_{4d} WITH G-GAUGE SYMMETRY, REDUCE THE STRUCTURE GROUP ALONG $S \subset M_4$ TO $T \subset G$ AND INTRODUCE:

$$\int_{M_4} d^4x\, d^4\theta\, \mathcal{F} + \int_S dx^0 dx^3 d^2\tilde{\theta}\; W(\gamma) + c.c.$$

$$\gamma \in \mathfrak{t} = \mathrm{Lie}(T)$$

(C.) CHIRAL RINGS (GAIOTTO 0911.1316)

EXAMPLE: 2D $\mathbb{CP}^1$ MODEL COUPLED TO 4D PURE SU(2) MODEL

FOR 2D $\mathbb{CP}^1$ MODEL:

CHIRAL RING: $x^2 = \Lambda_{2d}^2 e^t$

ADD A TWISTED MASS:

$$x^2 = \Lambda_{2d}^2 e^t + 2u$$

GAIOTTO : 4D INSTANTONS $\Longrightarrow$

$$x^2 = \Lambda_{2d}^2 e^t + 2u + \frac{\Lambda_{4d}^4}{\Lambda_{2d}^2 e^t}$$

$$z = e^t \qquad \lambda = x\, dt$$

$$\lambda^2 = \left(\frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z} \right) (dz)^2$$

SEIBERG-WITTEN CURVE !!

GENERALIZE TO $T[C, A_n, m]$:

$z \in C \Rightarrow \mathbb{S}_z$: CANONICAL SURFACE DEFECT

$\mathbb{S}_z \sim$ OPEN M2-BRANE ENDING AT

$$\{x^1 = x^2 = 0\} \times \{z \in C\} \subset \mathbb{R}^4 \times C$$

- DECOUPLE GRAVITY $\Rightarrow$ (2,0) THEORY WITH SURFACE DEFECT

CLAIM: CHIRAL RING =

EQUATION FOR SW CURVE:

$$\lambda^n + \lambda^{n-2}\phi_2(z) + \cdots + \phi_n(z) = 0$$

$$\Downarrow$$

$$x^n + x^{n-2} r_2(e^t) + \cdots + r_n(e^t) = 0$$

(D.) IR DESCRIPTION

- ASSUME $\mathfrak{S}$ HAS A FINITE SET OF MASSIVE VACUA $i \in \mathcal{V}$

- Ex1: T_{2d} = L.G. MODEL WITH MORSE CRITICAL POINTS

- Ex2: $T[C, A_n, m]$ WITH $\mathfrak{S}_z$:
VACUA = BRANCHES $\lambda_1, \dots \lambda_n$ OVER z

- VACUA FOR THE 2D-4D SYSTEM ARE DESCRIBED BY (u, i) AND TOGETHER FORM A FINITE COVER:

$$
\begin{array}{ccc}
\mathcal{V} & \longrightarrow & \mathcal{B}_{\mathfrak{S}} \\
 & & \downarrow \\
 & & \mathcal{B}
\end{array}
$$

IR LAGRANGIAN

- 4D : UV $\longrightarrow$ IR

$$\mathcal{F} \longrightarrow \mathcal{F}^{eff}\left(V^I = a^I + \cdots \right)$$

4D ABELIAN VM'S
IN SOME DUALITY FRAME

- 2D+4D : UV $\longrightarrow$ IR

$$S^{eff} = \int d^4x\, d^4\theta\, \mathcal{F}^{eff}(V^I) + \int dx^0 dx^3 d^2\tilde{\vartheta}\, \mathcal{W}^{eff}(\Upsilon^I)$$

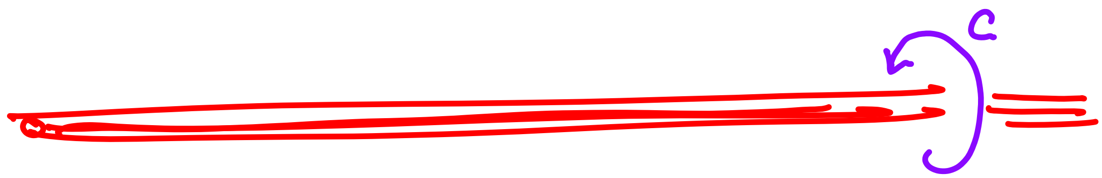
- $\mathcal{F}^{eff}$ DEPENDS ON $u \in \mathcal{B}$
- $\mathcal{F}^{eff}, \mathcal{W}^{eff}$ ONLY LOCALLY DEFINED
- NAIVELY, $\mathcal{W}^{eff}$ DEPENDS ON $(u, i) \in \mathcal{B}_{\mathbb{S}}$

(E) THE EFFECTIVE SOLENOID

PHYSICAL INTERPRETATION:

SOLENOID

FLUX $= \gamma_i$ FOR VAC $= i$



$$\mathbb{F} = dA = 0$$

BUT $\oint_C A \in \overline{V} = \Gamma_g \otimes \mathbb{R}$

CHOOSE DUALITY FRAME

$$\gamma_i = \oint_C A = \eta_I e^I + \alpha^I e_I$$

FROM LAGRANGIAN COMPUTE

$$\eta_I + \tau_{IJ}\alpha^J = \frac{\partial W^{eff}}{\partial a^I} := t_I$$

"GUKOV-WITTEN PARAMETERS"

(F.) TORSORS OF SUPERPOTENTIALS

N.B. AHARONOV-BOHM PHASE OF A PROBE PARTICLE $\gamma_t \in \Gamma_g$

$$\Phi_{AB} = \exp\left[2\pi i \langle \gamma_t, \gamma_i \rangle\right]$$

THIS ONLY DETERMINES $\gamma_i \bmod \Gamma_g$

BUT IN FACT,

THE FLUX IN THE SOLENOID IS PHYSICALLY RELEVANT:

THE SUPERPOTENTIAL DEPENDS ON γ_i, NOT JUST ON i

WHY?

SUPPOSE A 4D PARTICLE OF CHARGE γ^e BECOMES MASSLESS AT $u = u_*$: $Z(\gamma^e; u) \longrightarrow 0$ @ u^*

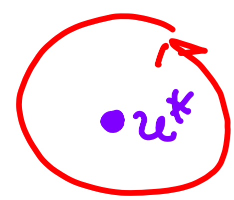
- THIS PARTICLE ALSO BINDS TO THE SOLENOID $\Rightarrow$ 2D CHIRAL MULTIPLET BECOMES MASSLESS

- WITTEN (1993) $\Rightarrow$

$$\Delta W^{eff} = \frac{1}{2\pi i} Z_{\gamma^e} \log Z_{\gamma^e}$$

$\Rightarrow$ MONODROMY IN $u \in \mathcal{B}$:

$$W^{eff} \longrightarrow W^{eff} + n Z_{\gamma^e}$$



$$\Rightarrow \quad t_I \longrightarrow t_I + n \frac{\partial}{\partial a^I} Z_{\gamma^e}$$

TAKING INTO ACCOUNT GENERAL PATHS IN $\mathcal{B}$:

$$\boxed{\Rightarrow \text{ SUPERPOTENTIALS LIVE IN A } \Gamma\text{-TORSOR: } \Gamma_i}$$

- A CHOICE OF SUPERPOTENTIAL $\iff$ A CHOICE OF GAUGE γ_i

- INTRODUCE THE NOTATION

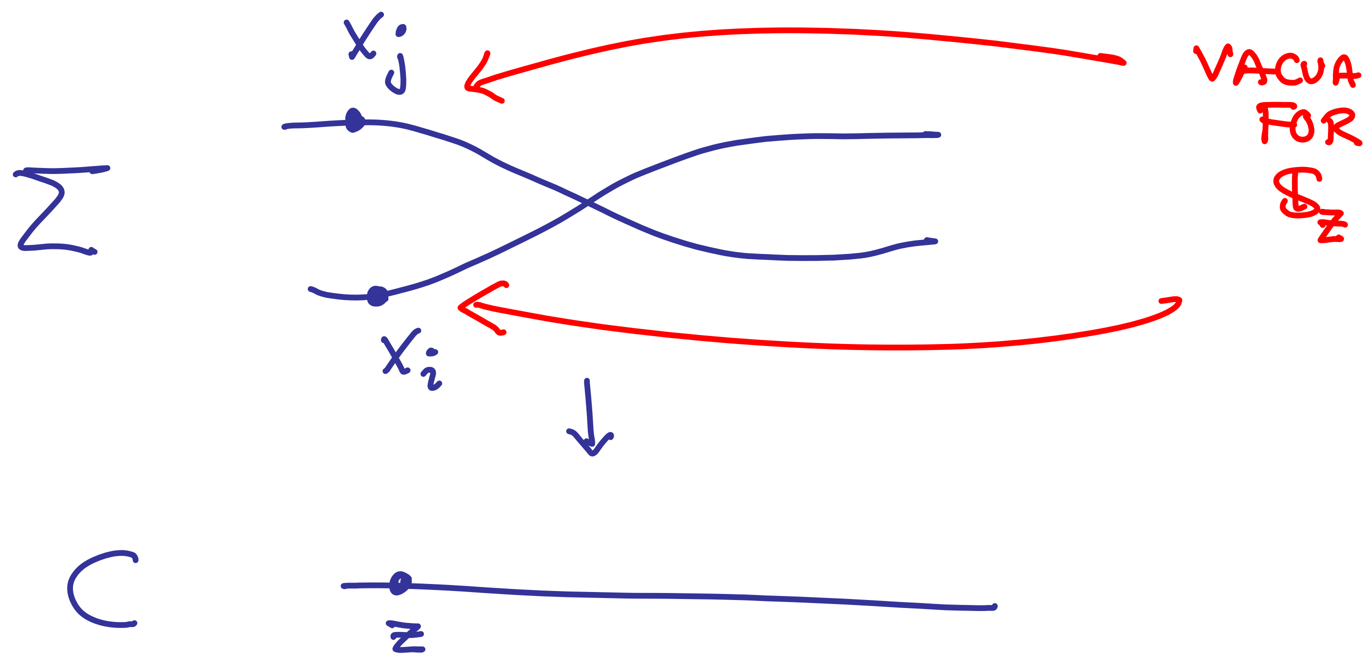
$$W^{eff} = Z_{\gamma_i}$$

FOR SUPERPOTENTIAL IN FLUX γ_i

$$Z_{\gamma_i+\gamma} = Z_{\gamma_i} + Z_\gamma$$

SO THIS EXTENDS THE CENTRAL CHARGE Z TO Γ_i

EXAMPLE: $T[C, A_n, m]$:



DIFF. OF SUPERPOT. ON $\mathbb{S}_z$:

$$Z_{\gamma_i} - Z_{\gamma_j} = \frac{1}{\pi} \int_{\gamma_{ij}} \lambda$$

γ_{ij} = <u>OPEN</u> <u>CURVE</u> JOINING x_i TO x_j

$$\in H_1\left(\Sigma, \{x_i, x_j\}; \mathbb{Z}\right)$$

$\in \Gamma_{ij}$: A Γ-TORSOR

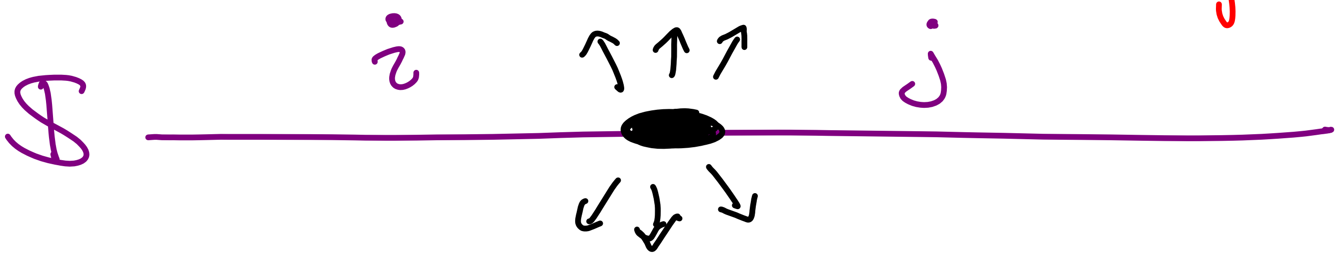
UP TO SOME SUBTLETIES:

$$\Gamma_{ij} = \Gamma_i - \Gamma_j$$

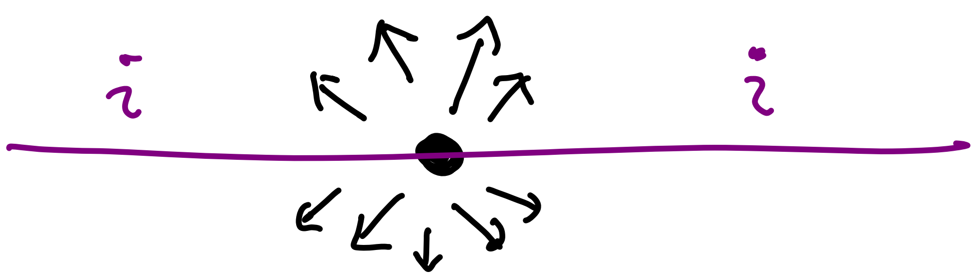
(G.) <u>NEW</u> BPS DEGENERACIES

- $\Omega(\gamma; u)$ AS BEFORE

- $\mu(\gamma_{ij})$: # 2D SOLITONS BETWEEN VAC i & j OF 4D CHARGE $\gamma_{ij} \in \Gamma_{ij}$



- THERE ARE ALSO 2D PARTICLES IN VACUUM i:



WITH 4D CHARGE $\gamma \in \Gamma$

<u>SURPRISE:</u> THE # OF SUCH PARTICLES DEPENDS ON THE "CHOICE OF GAUGE" Z_{γ_i}

WE COUNT THEM WITH DEGENERACY $\omega(\gamma, \gamma_i)$ SUCH THAT

$$\omega(\gamma, \gamma_i + \gamma') = \omega(\gamma, \gamma_i) + \underbrace{\Omega(\gamma)\langle\gamma, \gamma'\rangle}_{\text{LANDAU LEVEL DEGENERACY}}$$

ESSENTIAL PHYSICS: THERE CAN BE BOUNDSTATES OF 4D PARTICLES WITH THE SURFACE DEFECT

(DEFINE $\omega(\gamma, \gamma') := \Omega(\gamma)\langle\gamma, \gamma'\rangle$; ALL BPS DEGEN'S ARE $\mu + \omega$)

EXAMPLE: BPS STATES IN $T[C, A_1, m]$

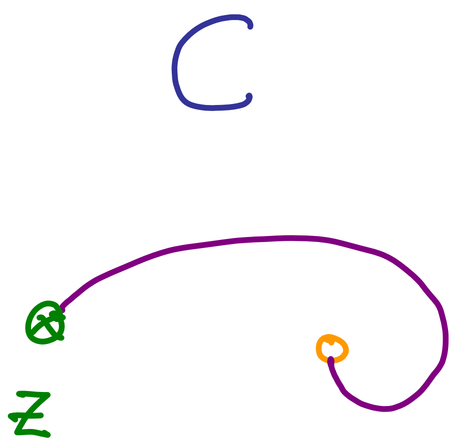
$$\Sigma \xrightarrow{2:1} C$$

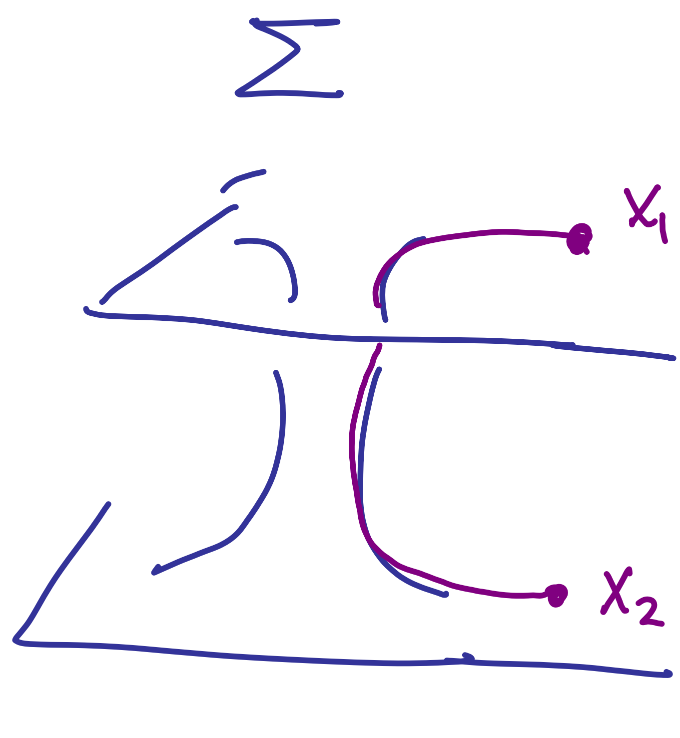
$$x_1, x_2 \longrightarrow z$$

- VACUA OF $\mathcal{S}_z$: x_1, x_2

- BPS SOLITON OF CHARGE $\gamma_{12} \in \Gamma_{12}$:

WKB CURVE JOINING x_1 TO x_2





BPS DEGENERACIES FOR $T[C, A_1, m]$

$\mu(\gamma_{ij})$ = SIGNED SUM OF WKB CURVES FROM x_i TO x_j PASSING THROUGH A RAMIFICATION POINT

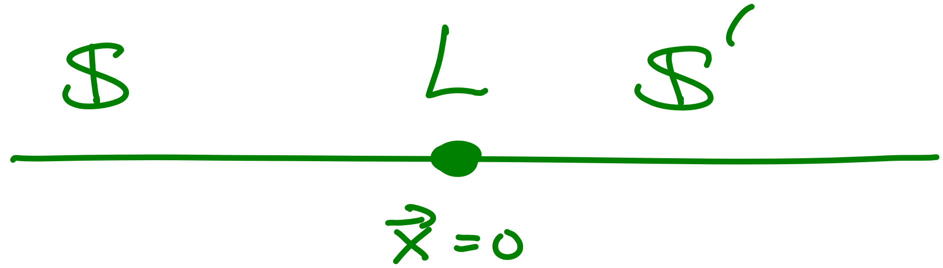
$$\omega(\gamma, \gamma_{ij}) = \omega(\gamma, \gamma_i) - \omega(\gamma, \gamma_j)$$

$$= \Omega(\gamma; u) \langle \gamma, \gamma_{ij} \rangle$$

γ_{ij}: AN OPEN PATH

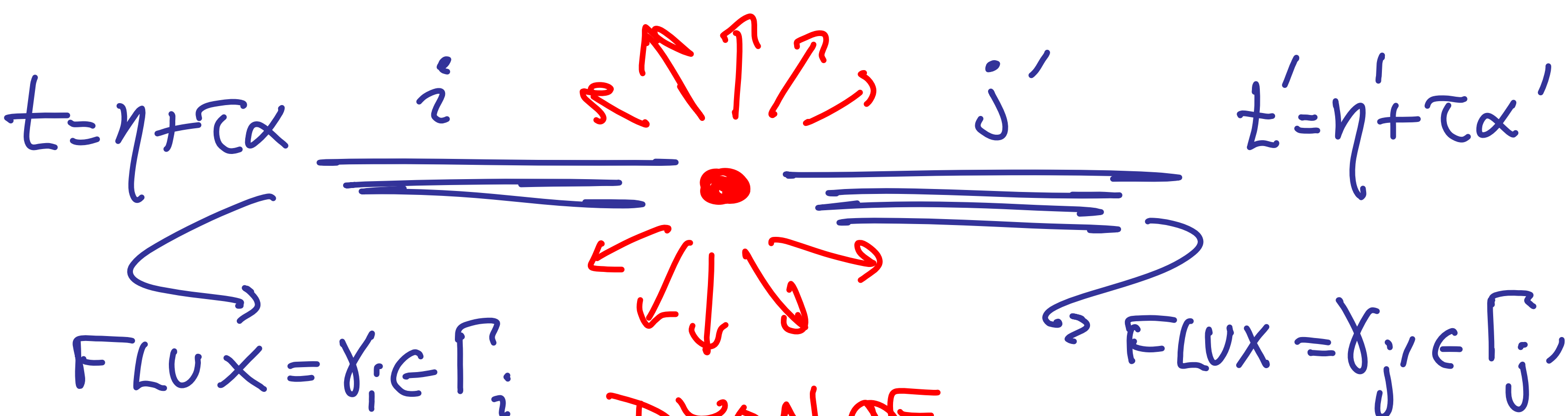
γ : WKB Rep.

(H.) LINE DEFECTS = DOMAIN WALLS



$$\mathcal{H}_{\mathfrak{S}L\mathfrak{S}'} = \bigoplus_{\gamma_{ij'}\in\Gamma_{ij'}} \mathcal{H}_{\mathfrak{S}L\mathfrak{S}',\gamma_{ij'}}$$

IR: DYON EMBEDDED IN A SOLENOID:



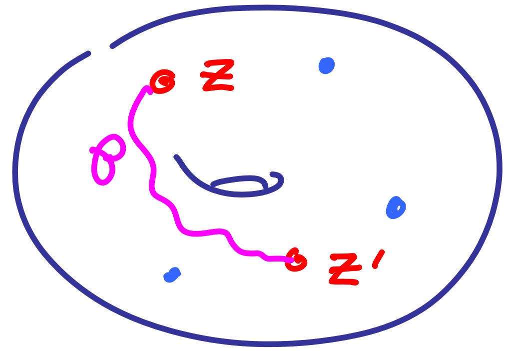
DYON OF CHARGE $\gamma_i-\gamma_{j'}+\gamma\in\Gamma_{ij'}$

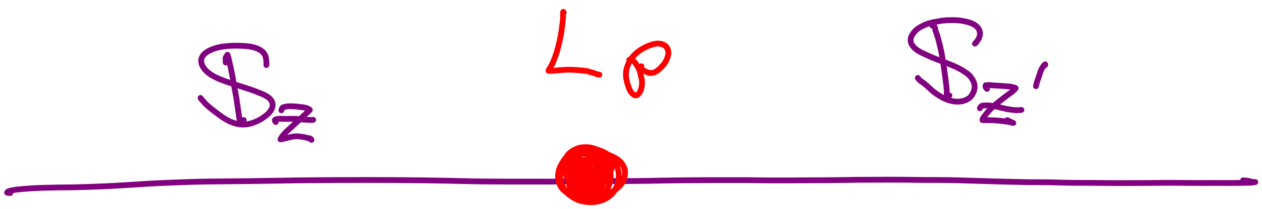
DYON CHARGE LIVES IN Γ-TORSOR $\Gamma_{ij'}$

EXAMPLE OF $T[C, A_1, m]$:

CHOOSE A PATH ρ IN C (UP TO HOMOTOPY) FROM z TO z':

C:





$$\mathcal{H}_{\mathfrak{S} L \mathfrak{S}'} = \bigoplus_{\gamma_{ij'} \in \Gamma_{ij'}} \mathcal{H}_{\mathfrak{S} L \mathfrak{S}', \gamma_{ij'}}$$

$\Gamma_{x_i, x_j'}$ = HOMOLOGY CLASS OF OPEN PATHS x_i TO x_j' IN Σ

(I.) FRAMED BPS STATES

SLS' PRESERVES

$$Q_1^1 - \zeta^{-1}\bar{Q}^1_{\dot{2}} \;\&\; Q_2^2 + \zeta^{-1}\bar{Q}^2_{\dot{1}}$$

$\Rightarrow$ CAN DEFINE FRAMED BPS STATES

$$F(L_\wp) = \sum_{\Gamma_{ij'}} \overline{\underline{\Omega}}(L_\wp, \gamma_{ij'})\, X_{\gamma_{ij'}}$$

$$X_{\gamma_{ij'}} X_{\gamma_{k''\ell'''}} = \begin{cases} \pm X_{\gamma_{ij'}+\gamma_{j'\ell'''}} & j' = k'' \\ 0 & j' \neq k'' \end{cases}$$

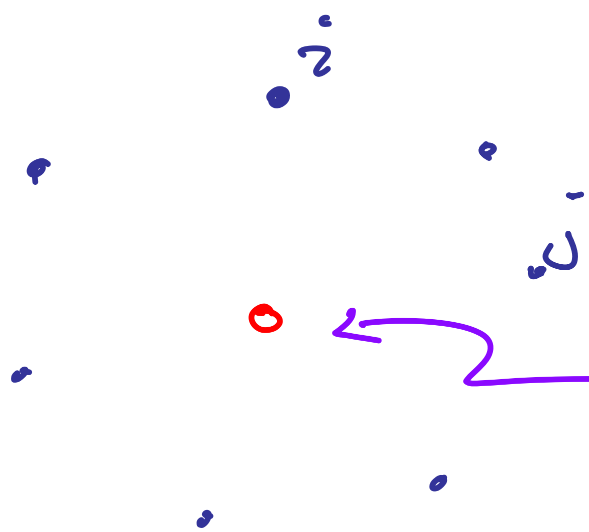
(NONCOMMUTATIVE EVEN FOR $y=-1$)

- CONSISTENCY OF WC OF $\overline{\underline{\Omega}} \Rightarrow$ WCF FOR μ & ω

(J.) FORMAL STATEMENT OF 2D4D WCF

4 PIECES OF DATA

① GROUPOID OF VACUA $\mathbb{V}$:



DISTINGUISHED OBJECT: "VACUUM OF THE ZERO SURFACE DEFECT"

MORPHISMS: $i \xrightarrow{\gamma} j$

DRAW AN ARROW FOR EACH $\gamma \in \Gamma$ TO GET $\mathrm{Hom}(i,j) = \Gamma$-TORSOR.

$\mathrm{Hom}(0,0) = \Gamma$, $\mathrm{Hom}(i,0) = \Gamma_i$, ...

COMPOSITION OF MORPHISMS

$$\begin{tikzcd} & j \arrow[dr, "\gamma'"] & \\ i \arrow[ur, "\gamma"] \arrow[rr, "\gamma+\gamma'"'] & & k \end{tikzcd}$$

etc.

LET $a \in \Gamma, \Gamma_i, \Gamma_{ij}$

DENOTE COMPOSITION $a+b$
<u>WHEN DEFINED</u>

(2) CENTRAL CHARGE $Z \in \mathrm{Hom}(\mathbb{V}, \mathbb{C})$

$$Z(a) + Z(b) = Z(a+b)$$

WHEN $a+b$ IS DEFINED

3. BPS DATA:

$$\Omega(\gamma) \in \mathbb{Z}$$

$$\mu(\gamma_{ij}) \in \mathbb{Z}$$

$$\omega(\gamma, \gamma_a) \in \mathbb{Z}$$

$$\omega(\gamma, \gamma_a + \gamma') = \omega(\gamma, \gamma_a) + \Omega(\gamma) \langle \gamma, \gamma' \rangle$$

PIECEWISE CONSTANT IN STABILITY DATA Z

4. TWISTING FUNCTION

$\sigma(a,b) \in \mathbb{Z}_2$ WHEN $a+b$ DEFINED

$$\sigma(a,b)\,\sigma(a+b,c) = \sigma(a,b+c)\,\sigma(b,c)$$

3 DEFINITIONS

1. DEF: A BPS RAY IS A RAY IN CPLX PLANE:

- $\mathbb{R}_- \cdot Z(\gamma)$ IF $\omega(\gamma, \cdot) \neq 0$
- $\mathbb{R}_- Z(\gamma_{ij})$ IF $\mu(\gamma_{ij}) \neq 0$

2. DEFINE THE (TWISTED) GROUPOID ALGEBRA $\mathbb{C}[\mathbb{V}]$:

$$X_a X_b = \begin{cases} \sigma(a,b)\, X_{a+b} & \text{IF } a+b \text{ COMPOSABLE} \\ 0 & \text{ELSE} \end{cases}$$

3. DEFINE TWO AUTOMORPHISMS OF $\mathbb{C}[\mathbb{V}]$:

- CV-LIKE: $S^{\mu}_{\gamma_{ij}}$

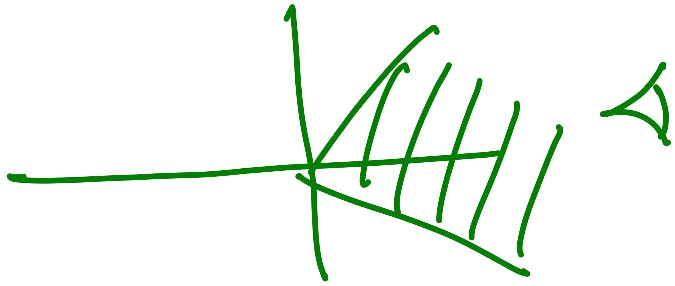
$$X_a \to \left(1-\mu(\gamma_{ij})X_{\gamma_{ij}}\right)\cdot X_a \cdot \left(1+\mu(\gamma_{ij})X_{\gamma_{ij}}\right)$$

- KS-LIKE K^{ω}_{γ}:

$$X_a \longrightarrow \left(1-X_{\gamma}\right)^{-\omega(\gamma,a)} X_a$$

2D/4D WALL-CROSSING FORMULA

FOR CONVEX SECTOR



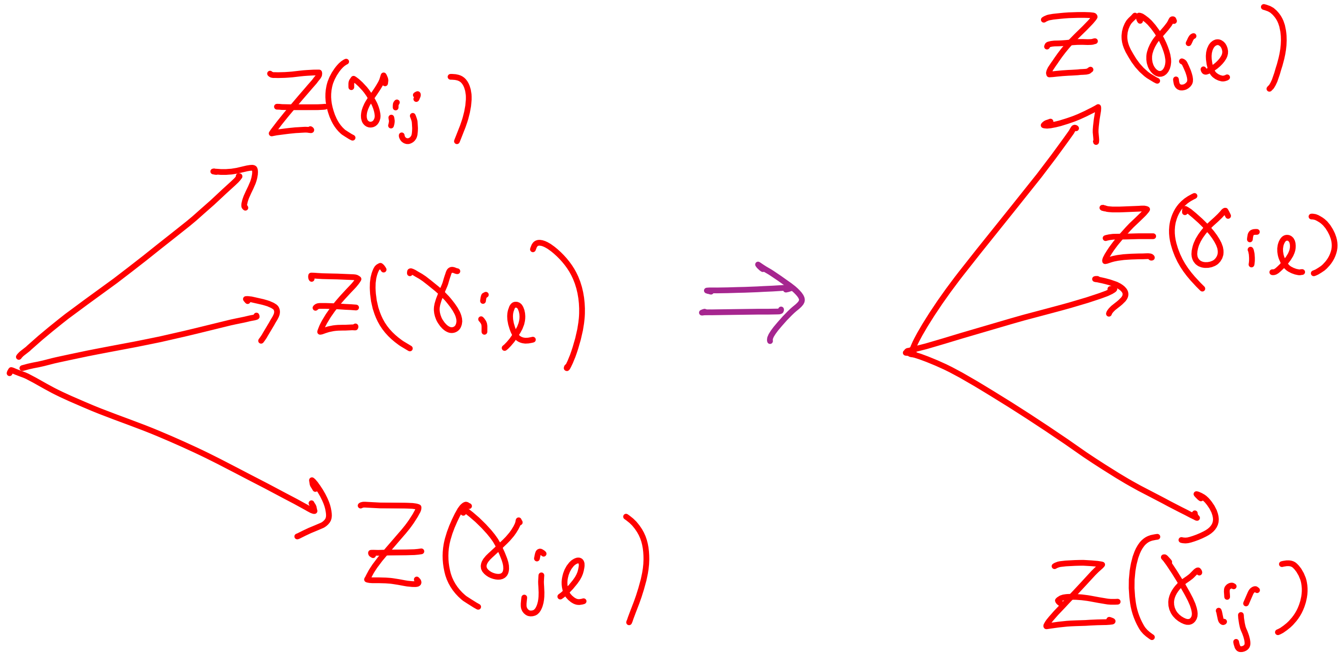
$$\mathcal{A}(\measuredangle) = :\prod_{Z(\gamma_{ij})\in\measuredangle} S^{\mu}_{\gamma_{ij}} \prod_{Z(\gamma)\in\measuredangle} K^{\omega}_{\gamma}:$$

ORDERED BY PHASE OF Z

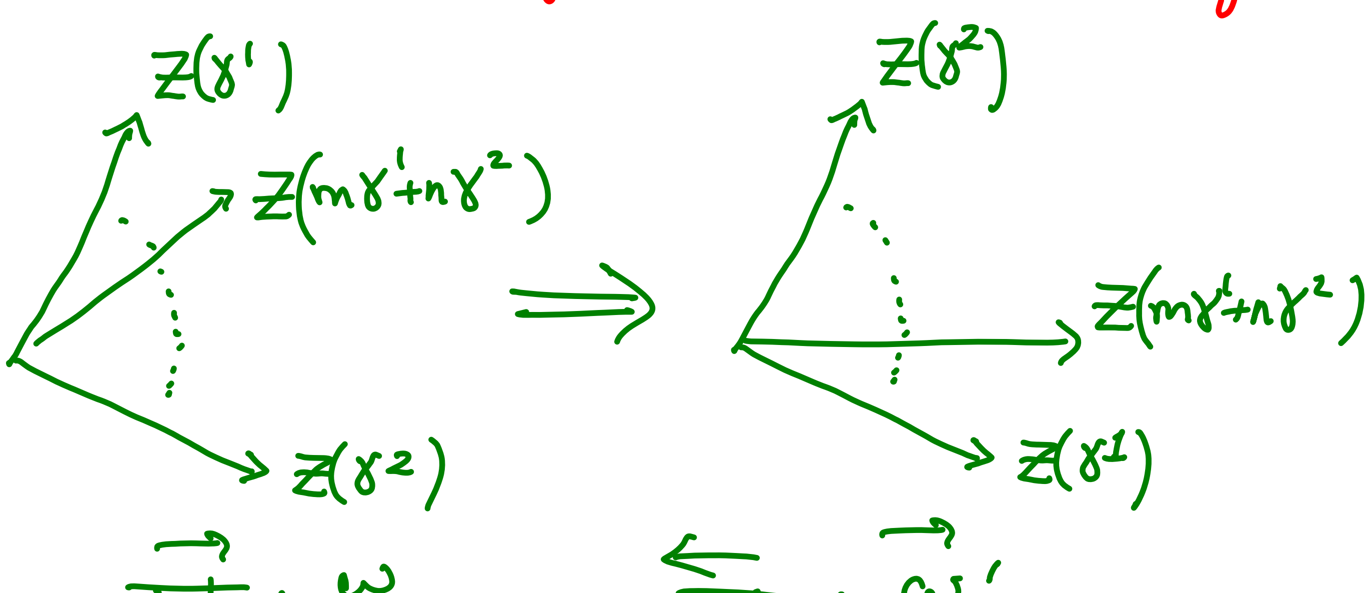
<u>W.C.F.</u>: $\mathcal{A}(\measuredangle)$ IS CONSTANT SO LONG AS NO BPS LINE ENTERS/LEAVES THE SECTOR $\measuredangle$.

CV + KS WCF ARE SPECIAL CASES

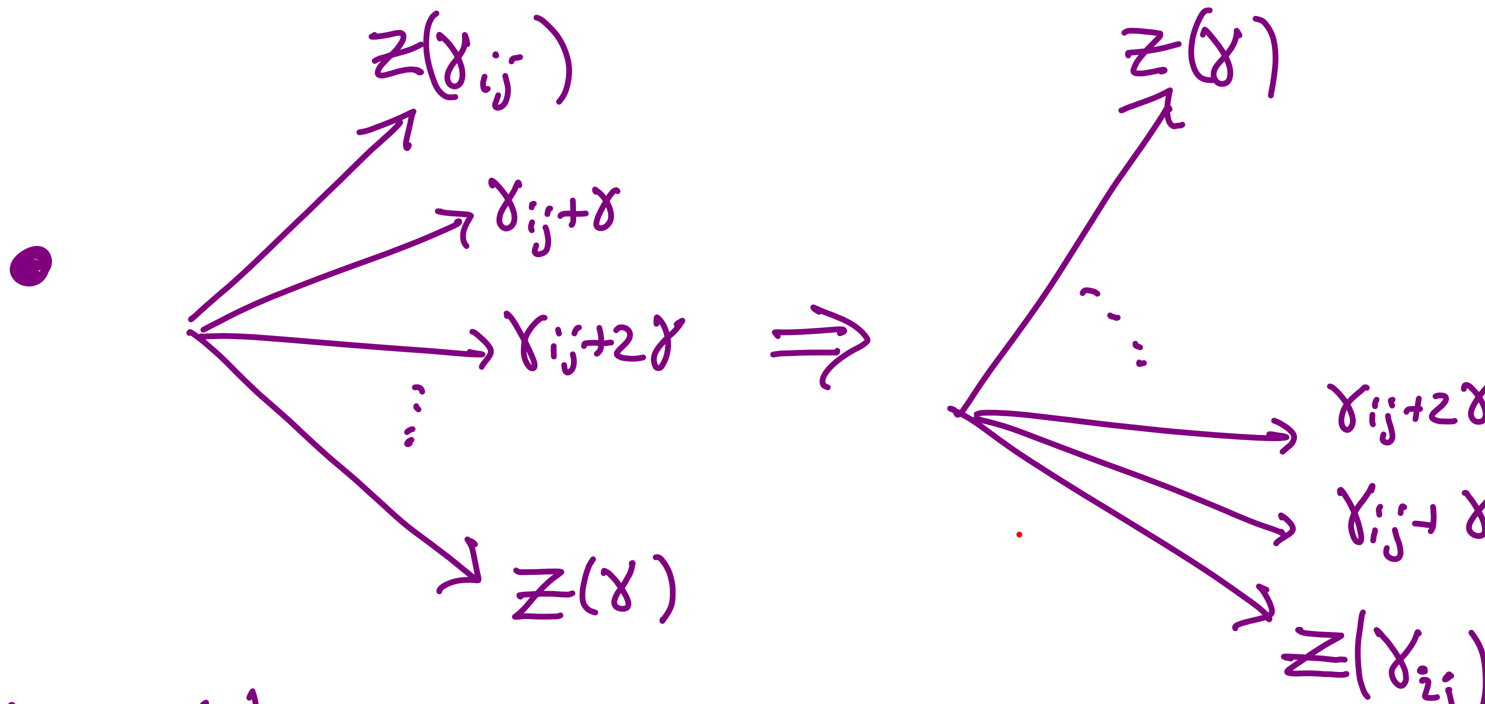
(K.) TYPES OF WALL-CROSSING

- 

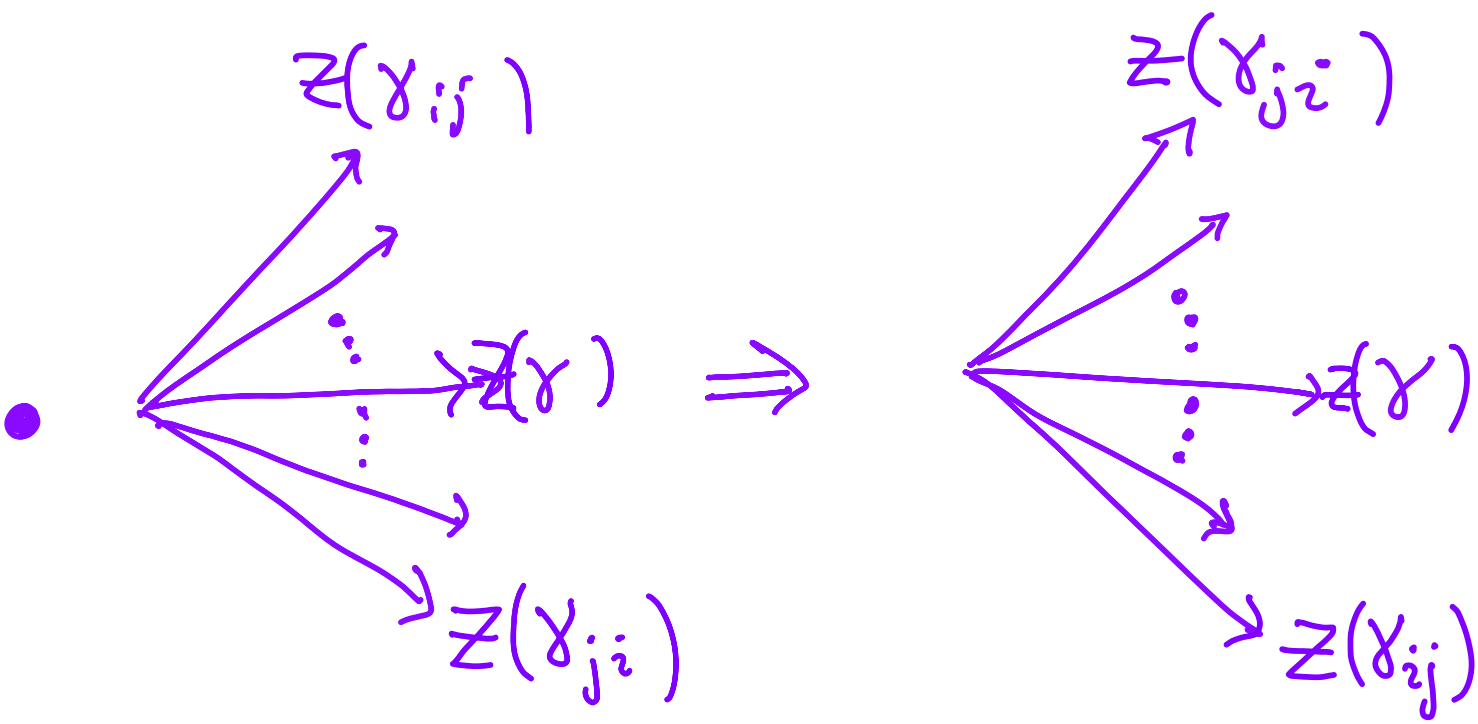
$$S^{\mu}_{\gamma_{ij}} S^{\mu}_{\gamma_{il}} S^{\mu}_{\gamma_{jl}} = S^{\mu'}_{\gamma_{jl}} S^{\mu'}_{\gamma_{il}} S^{\mu'}_{\gamma_{ij}}$$

- 

$$\overrightarrow{\prod} K^{\omega} = \overleftarrow{\prod} K^{\vec{\omega}'}$$



$$(\alpha)\quad K_\gamma^{\omega} \prod S^{\mu}_{\gamma_{ij}+n\gamma} = \prod S^{\mu'}_{\gamma_{ij}+n\gamma} K_\gamma^{\omega'}$$



$$(\beta)\quad \prod S^{\mu}_{\gamma_{ij}+n\gamma} K_\gamma^{\omega} \prod S^{\mu}_{\gamma_{ji}+n\gamma} = \prod S^{\mu'}_{\gamma_{ji}+n\gamma} K_\gamma^{\omega'} \prod S^{\mu'}_{\gamma_{ij}+n\gamma}$$

$\boxed{\text{THM: MIXED WCF CAN BE SOLVED EXPLICITLY}}$

$$\Sigma_{ij} := \sum_{n=0}^{\infty} \mu(\gamma_{ij} + n\gamma)\, X_\gamma^n$$

$$\Pi_{ij} := \prod_{n=1}^{\infty} \left(1 - X_\gamma^n\right)^{\omega(n\gamma, \gamma_{ij})}$$

(α):

$$\Sigma_{ij}' = \left(1 - X_\gamma\right)^{-\omega(\gamma, \gamma_{ij})} \Sigma_{ij}$$

$$\Pi_{ij}' = \Pi_{ij}$$

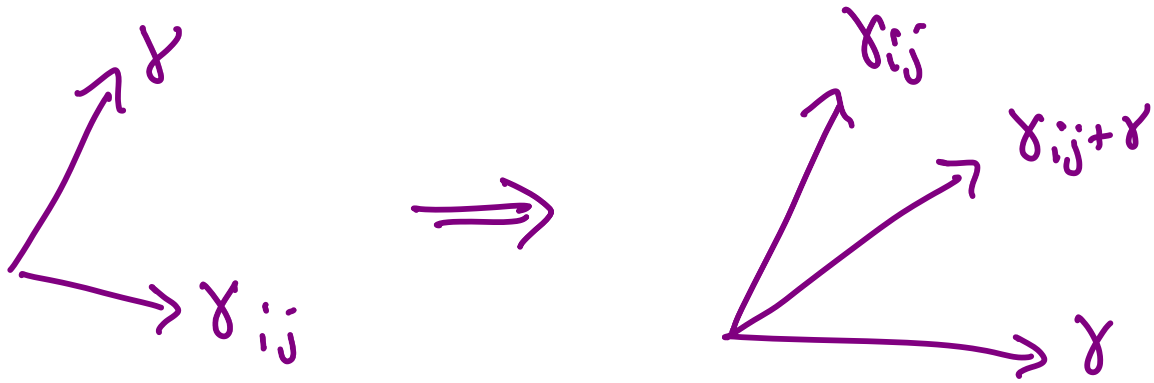
(β)

$$\Pi_{ji}' = \Delta^{-2} \Pi_{ij}$$

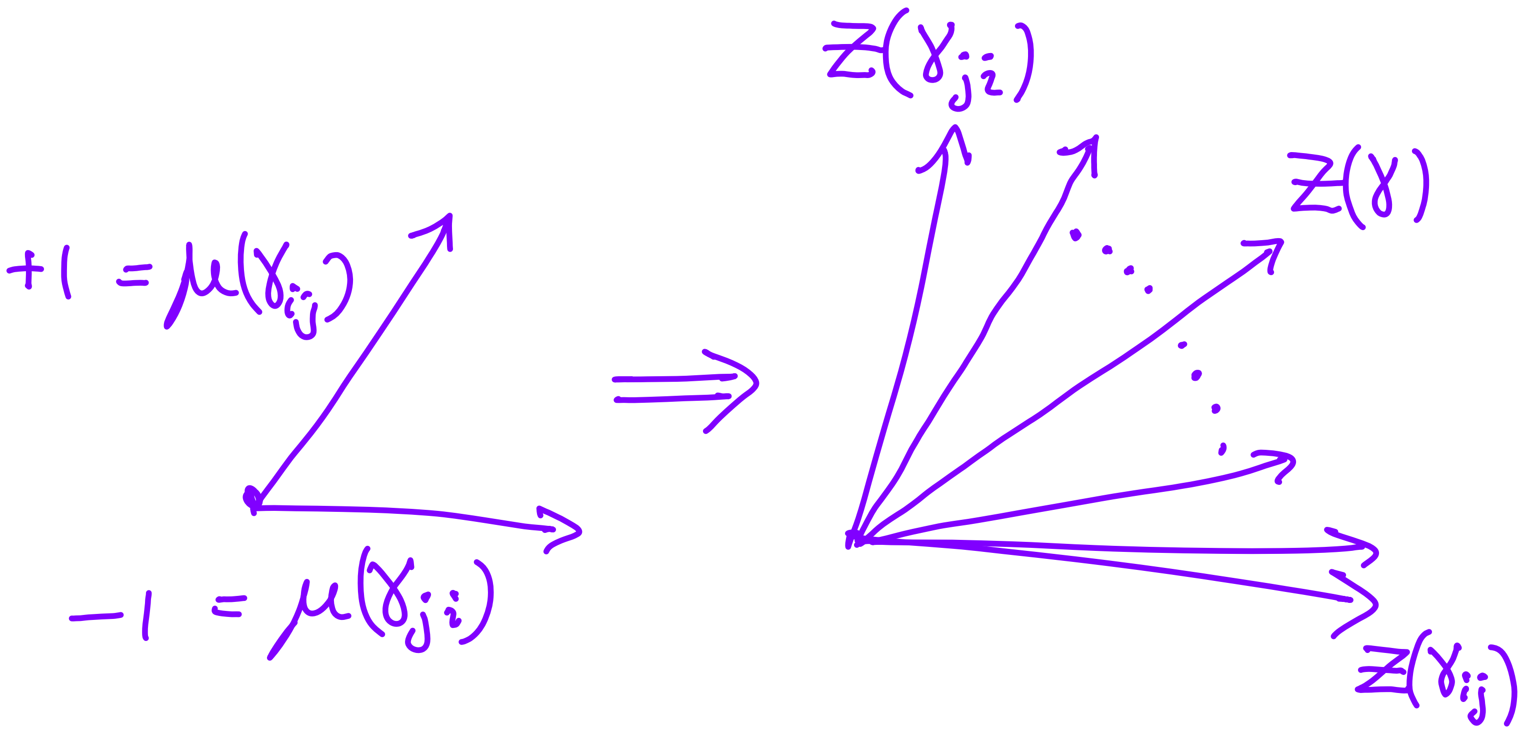
$$\Sigma_{ij}' = \Delta^{-1} \Sigma_{ij}$$

$$\Delta = \Pi_{ji} + \Sigma_{ij} \Sigma_{ji} X_\gamma$$

SPECIAL CASES

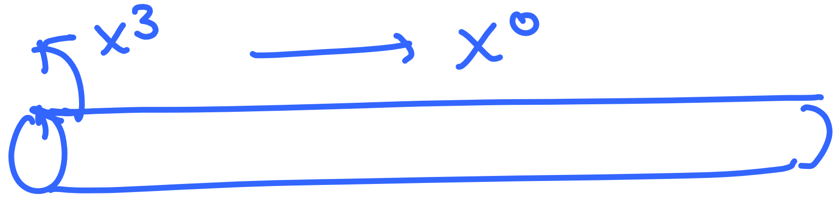


$$K_\gamma^{\omega} S_{\gamma_{ij}}^{\mu} = S_{\gamma_{ij}}^{\mu'} S_{\gamma_{ij}+\gamma}^{\mu'} K_\gamma^{\omega'}$$



$$S_{\gamma_{ij}}^{\mu} S_{\gamma_{ji}}^{\mu} = \prod_{\nearrow} S_{\gamma_{ji}+n\gamma}^{\mu'} K_\gamma^{\omega'} \prod_{\searrow} S_{\gamma_{ij}+n\gamma}^{\mu'}$$

(L.) 3D/1D SYSTEM & HK GEOMETRY



SURFACE DEFECT IN 4D THEORY

$\Downarrow$

LINE DEFECT IN 3D THEORY

$$\exp i \int \left(\eta_I F_{03}^{\ I} - \alpha^I G_{03\,I} \right) dx^0 dx^3$$

$$\Downarrow$$

$$\exp\ i \int_{\mathbb{R}_{x_0}} \left(\eta_I d\theta_{el}^{\ I} + \alpha^I d\theta_{mg\,I} \right)$$

$$d\theta_{el}^{\ I} = \oint_{S'_R} F^I \qquad d\theta_{mg,I} = \oint_{S'_R} G_I$$

$\eta_I \, d\theta_{el}^I + \alpha^I d\theta_{mg,I}$ IS A

LOCALLY DEFINED 1-FORM ON $\mathcal{M}$

CROSSING PATCHES WE FIND

A CONNECTION A^{sf} ON

$$(\mathcal{L}_S)_i \to \mathcal{M}$$

WHEN $\mathcal{M}$ HAS THE

SEMIFLAT HK METRIC

$$\boxed{A_i^{sf} \text{ IS HYPER-HOLOMORPHIC}}$$

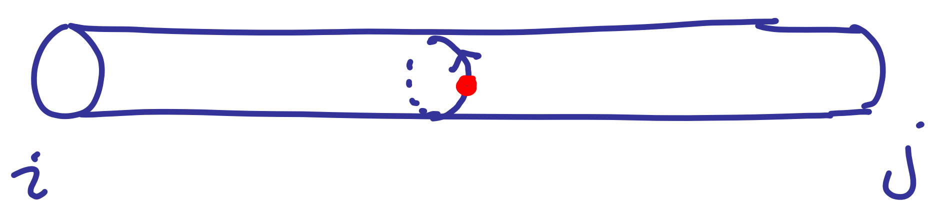
(F_i^{sf} IS TYPE (1,1) IN ALL

COMPLEX STRUCTURES)

(M.) QUANTUM CORRECTIONS

JUST AS WORLDLINES OF 4D BPS PARTICLES $\Rightarrow$ INSTANTON CORRECTIONS TO 3D SEMIFLAT HK METRIC...

WORLDLINES OF 2D BPS PARTICLES $\Rightarrow$ INSTANTON CORRECTONS TO THE 3D SEMIFLAT CONNECTION:



3D LINE DEFECT

$$\mathrm{Tr}\, P \exp \int dx^0 \left(\varphi^* \oplus + \mathrm{Fermi}^2 \right)$$

SUSY $\Rightarrow \oplus$ IS HYPER-HOLOMORPHIC

THESE INSTANTONS MIX THE LINE BUNDLES OF VACUA:

SO THE QUANTUM-CORRECTED CONNECTION $\oplus$ ONLY MAKES SENSE ON THE VECTOR BUNDLE:

$$V_{\mathcal{S}} = \bigoplus_{i \in V} (\mathcal{L}_{\mathcal{S}})_i$$

ANALOGOUS TO THE METRIC...

CONSTRUCT $\oplus$ VIA ITS HOLOMORPHIC SECTIONS

$$y_{\gamma_i} \in H^0(V_{\mathcal{S}})$$

ANALOGOUS TO TBA: CONSTRUCT $\mathcal{Y}_{\gamma_i}$ FROM AN INTEGRAL EQUATION:

Split: $\mathcal{Y}_{\gamma_i} = g_i\, y_{\gamma_i}$

$$y_{\gamma_i} \in H^0\left((\mathcal{L}_S)_i \right)$$

$$g_i \in \mathrm{Hom}\left((\mathcal{L}_S)_i ,\ \bigoplus_j (\mathcal{L}_S)_j \right)$$

$$y_{\gamma_i} := \mathcal{Y}^{sf}_{\gamma_i} \exp\left[-\sum_{\gamma} \omega(\gamma,\gamma_i)\, K_\gamma * \log\left(1 - \mathcal{Y}_\gamma\right) \right]$$

$$\boxed{g_i = g_i^0 - \sum_{\substack{j \neq i \\ \gamma_{ji}}} \mu(\gamma_{ji})\, K_{\gamma_{ji}} * \left(g_j\, y_{\gamma_{ji}} \right)}$$

REMARKS:

1. SMOOTHNESS OF CONNECTION (#) IS GUARANTEED BY 2d-4d WCF.

2. CONSTRUCTION CAN BE CARRIED OUT EXPLICITLY IN SOME SIMPLE EXAMPLES, SUCH AS INSTANTONS ON PERIODIC TAUB-NUT SPACE.

3. INTEGRAL EQUATION $\Rightarrow$ EXPLICIT CONSTRUCTION OF FLAT SECTIONS IN $T[C, A_1, m]$ EXPLES

$\Rightarrow$ EXPLICIT CONSTRUCTION OF SOLUTIONS TO HITCHIN EQS.

4. EQUATION FOR g_i IS A VERSION OF THE INVERSE SCATTERING METHOD; IN A SPECIAL CASE COINCIDES WITH RECENT RESULT OF LUKYANOV-ZAMOLODCHIKOV

5. FOR $y \neq \pm 1$ $F(L_p)$

GENERATE A NONCOMMUTATIVE ALGEBRA DEFORMING ALGEBRA OF HOLOMORPHIC FUNCTIONS ON $\mathcal{M}^S$

FOR INTERFACES, AT $y=-1$

$$\langle L_p \rangle \in H^0\left(\mathrm{Hom}(V_S, V_S)\right)$$

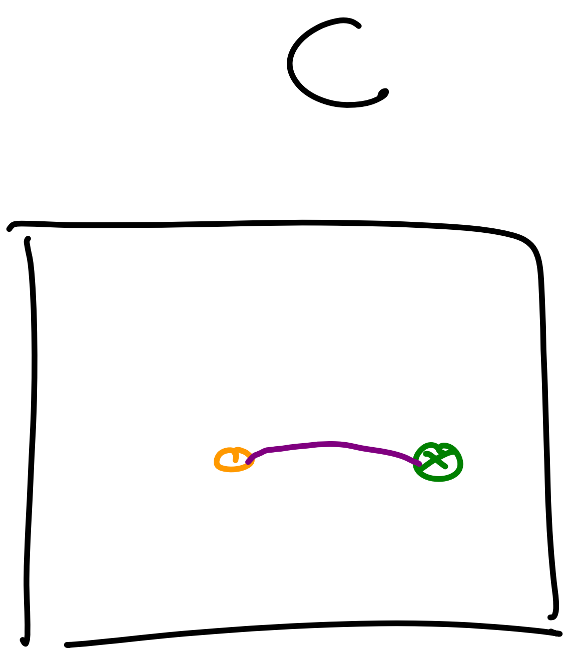
SO $y \neq -1$ ALGEBRA OF $F(L_p)$ DEFORMS IT: GOOD FOR ANYTHING?

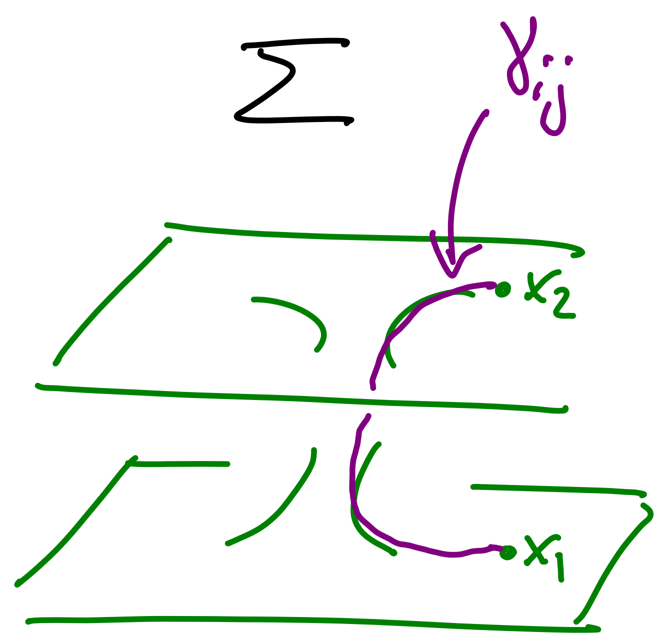
IV. SIMPLE EXAMPLES

RECALL THAT FOR $T[C, A_1, m]$

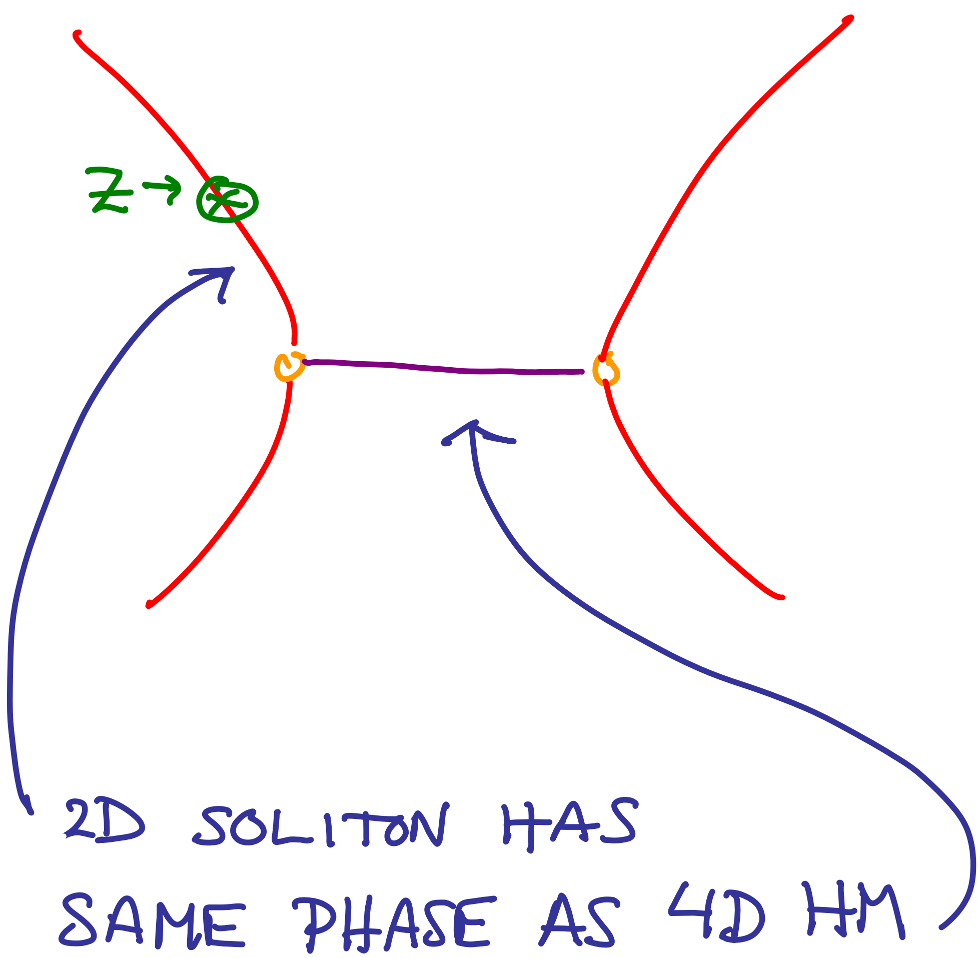
- 4D BPS PARTICLES ARE FINITE OR CLOSED WKB PATHS

- GENERALIZE TO SOLITONS: OPEN WKB PATHS CONNECTING x_1 TO x_2 THROUGH RAMP POINT





N.B. AS A FUNCTION OF $z \in C$ THE WALLS OF MARGINAL STABILITY <u>ARE</u> THE CRITICAL WKB CURVES OF OCCUPIED 4D STATES:



ALSO RECALL THE CHIRAL RING FOR $\mathbb{CP}^1$ COUPLED TO $\mathcal{N}=2, d=4$:

$$x^2 = \Lambda_{2d}^2 e^t + 2u + \frac{\Lambda_{4d}^4}{\Lambda_{2d}^2 e^t}$$

$$z = e^t \qquad \lambda = x\, dt$$

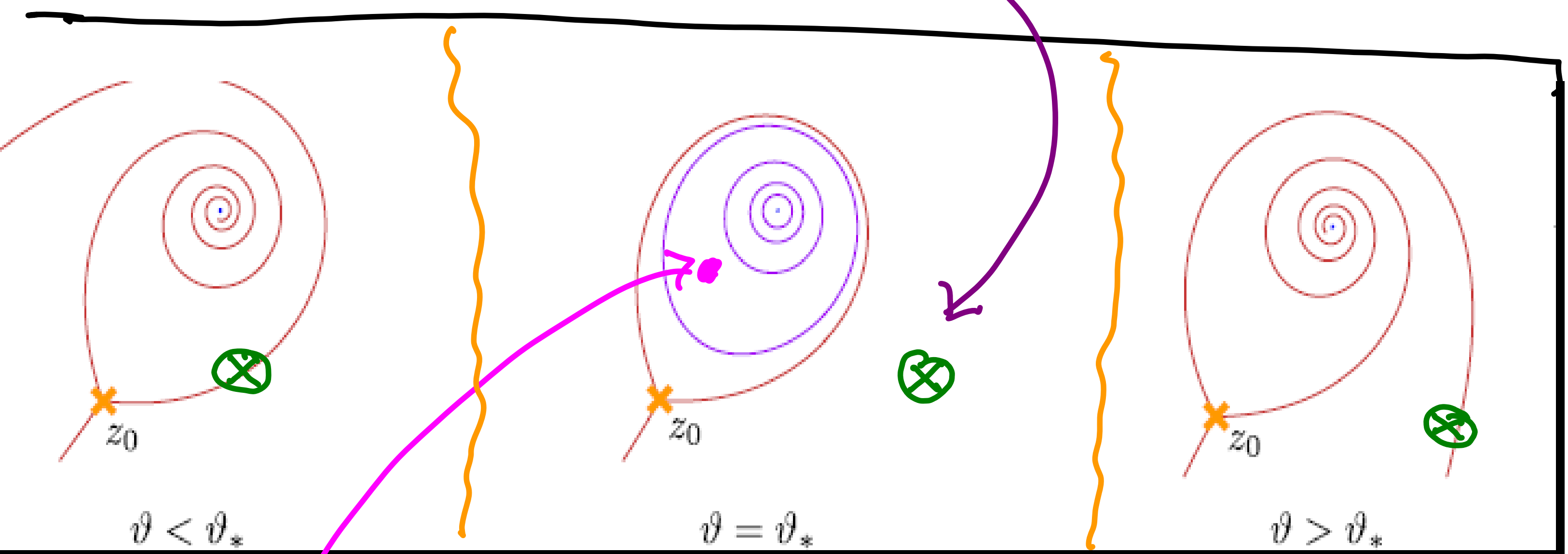
$$\lambda^2 = \left(\frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z} \right) (dz)^2$$

FIRST CONSIDER $\Lambda_{4d} \to 0$:

PURE $\mathbb{CP}^1$ MODEL

$$\lambda^2 = \left(\frac{\Lambda_{2d}^2}{z} + \frac{m^2}{z^2} \right) (dz)^2$$

FOR Z IN STRONG COUPLING REGION THERE ARE TWO SOLITONS



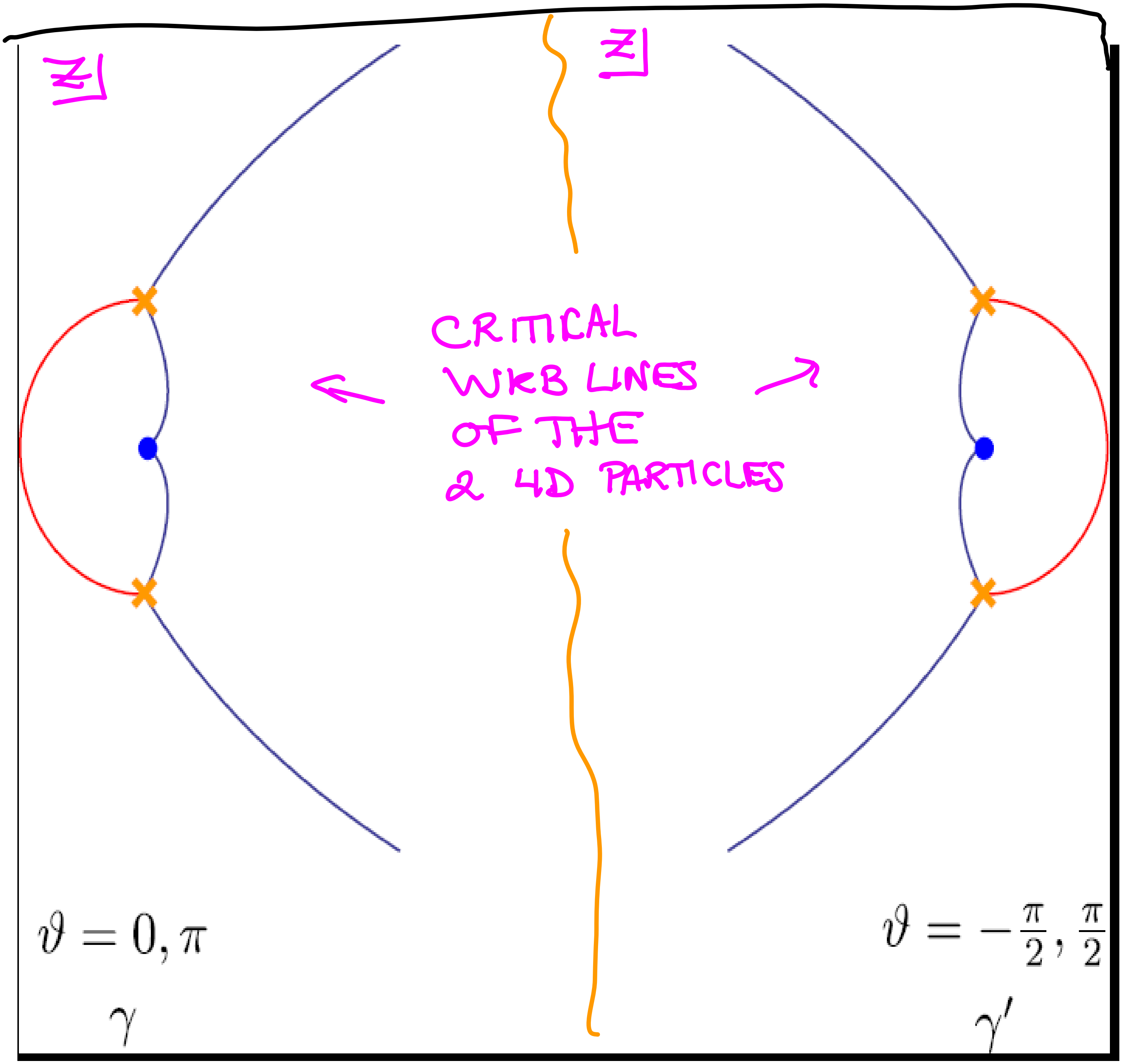
$z \in$ WEAK COUPLING: ∞-ly MANY BPS STATES

$$S^{\mu}_{\gamma_{ij}} S^{\mu}_{\gamma_{ji}} = \prod S^{\mu'}_{\gamma_{ji}+n\gamma} K^{\omega'}_{\gamma} \prod S^{\mu'}_{\gamma_{ij}+n\gamma}$$

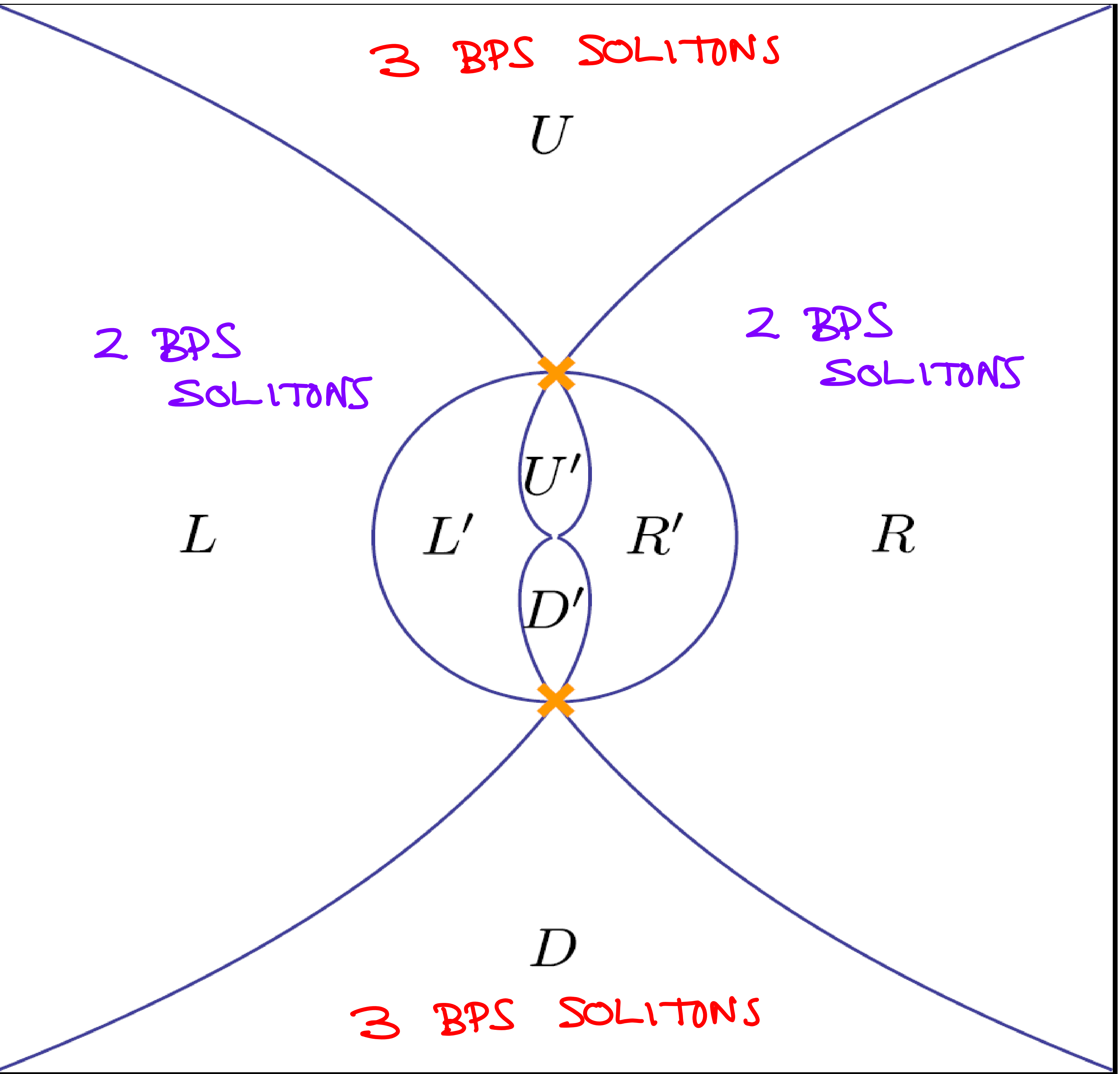
VERIFIES RESULT OF DOREY

$$\lambda^2 = \left(\frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z} \right) (dz)^2$$

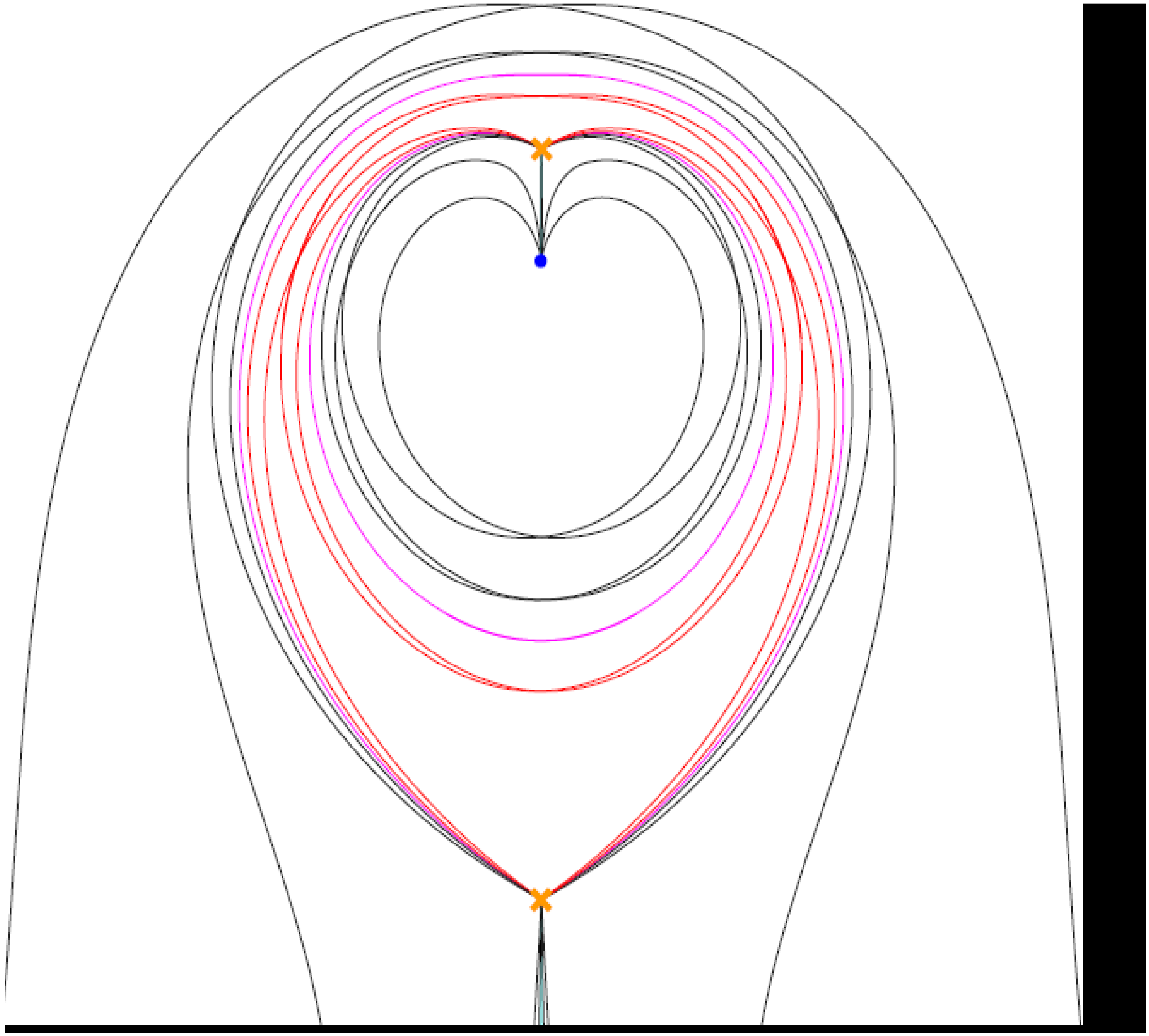
$u \in$ STRONG COUPLING



$\boxed{u = 0}$


3 BPS SOLITONS
U
2 BPS SOLITONS
2 BPS SOLITONS
U′
L
L′
R′
R
D′
D
3 BPS SOLITONS

$u \in$ WEAK COUPLING



- FINITE SOLITON SPECTRUM: $z \in$ STRONG CPLNG
- INFINITE SOLITON SPECTRUM: $z \in$ WEAK CPLNG
- UNCOUNTABLY MANY "CHAMBERS" FOR $z \in$ WEAK COUPLING.

V. CURRENT WORK

1. GENERALIZE A_1-THEORIES TO HIGHER RANK: QUALITATIVELY NEW PHENOMENA

2. GENERALIZATION TO SURFACE DEFECTS $\mathbb{S}_{z_1, \ldots, z_n}$ FOR SEVERAL M2 BRANES

$\vdots$