Web Formalism and the IR limit of 1+1 N=(2,2) QFT

- or -

A short ride with a big machine

String-Math, Edmonton, June 12, 2014

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collaboration with
Davide Gaiotto & Edward Witten

draft is ``nearly finished"…

Three Motivations

1. IR sector of ***massive*** 1+1 QFT with N =(2,2) SUSY

2. Knot homology.

3. Categorification of 2d/4d wall-crossing formula.

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

Witten (2010) reformulated knot homology in terms of Morse complexes.

This formulation can be further refined to a problem in the categorification of Witten indices in certain LG models (Haydys 2010, Gaiotto-Witten 2011)

Gaiotto-Moore-Neitzke studied wall-crossing of BPS degeneracies in 4d gauge theories. This leads naturally to a study of Hitchin systems and Higgs bundles.

When adding surface defects one is naturally led to a "nonabelianization map" inverse to the usual abelianization map of Higgs bundle theory. A "categorification" of that map should lead to a categorification of the 2d/4d wall-crossing formula.

Goals & Results - 1

Goal: Say everything we can about massive (2,2) theories in the far IR.

Since the theory is massive this would appear to be trivial.

Result: When we take into account the BPS states there is an extremely rich mathematical structure.

We develop a formalism – which we call the ``web-based formalism'' – (that's the ``big machine'') - which shows that:

Goals & Results - 2

BPS states have ``interaction amplitudes'' governed by an L_∞ algebra

(Using just IR data we can define an L_∞ - algebra and there are ``interaction amplitudes'' of BPS states that define a solution to the Maurer-Cartan equation of that algebra.)

There is an A_∞ category of branes/boundary conditions, with amplitudes for emission of BPS particles from the boundary governed by solutions to the MC equation.

(A_∞ and L_∞ are mathematical structures which play an important role in open and closed string field theory, respectively.)

Goals & Results - 3

If we have a pair of theories then we can construct supersymmetric interfaces between the theories.

Such interfaces define A_∞ functors between Brane categories.

Theories and their interfaces form an A_∞ 2-category.

Given a continuous family of theories (e.g. a continuous family of LG superpotentials) we show how to construct a ``flat parallel transport'' of Brane categories.

The parallel transport of Brane categories is constructed using interfaces.

The flatness of this connection implies, and is a categorification of, the 2d wall-crossing formula.

Outline

- Introduction: Motivations & Results
- Web-based formalism
- Web representations & L_∞
- Half-plane webs & A_∞
- Interfaces
- Flat parallel transport
- Summary & Outlook

Definition of a Plane Web

We now give a purely mathematical construction.

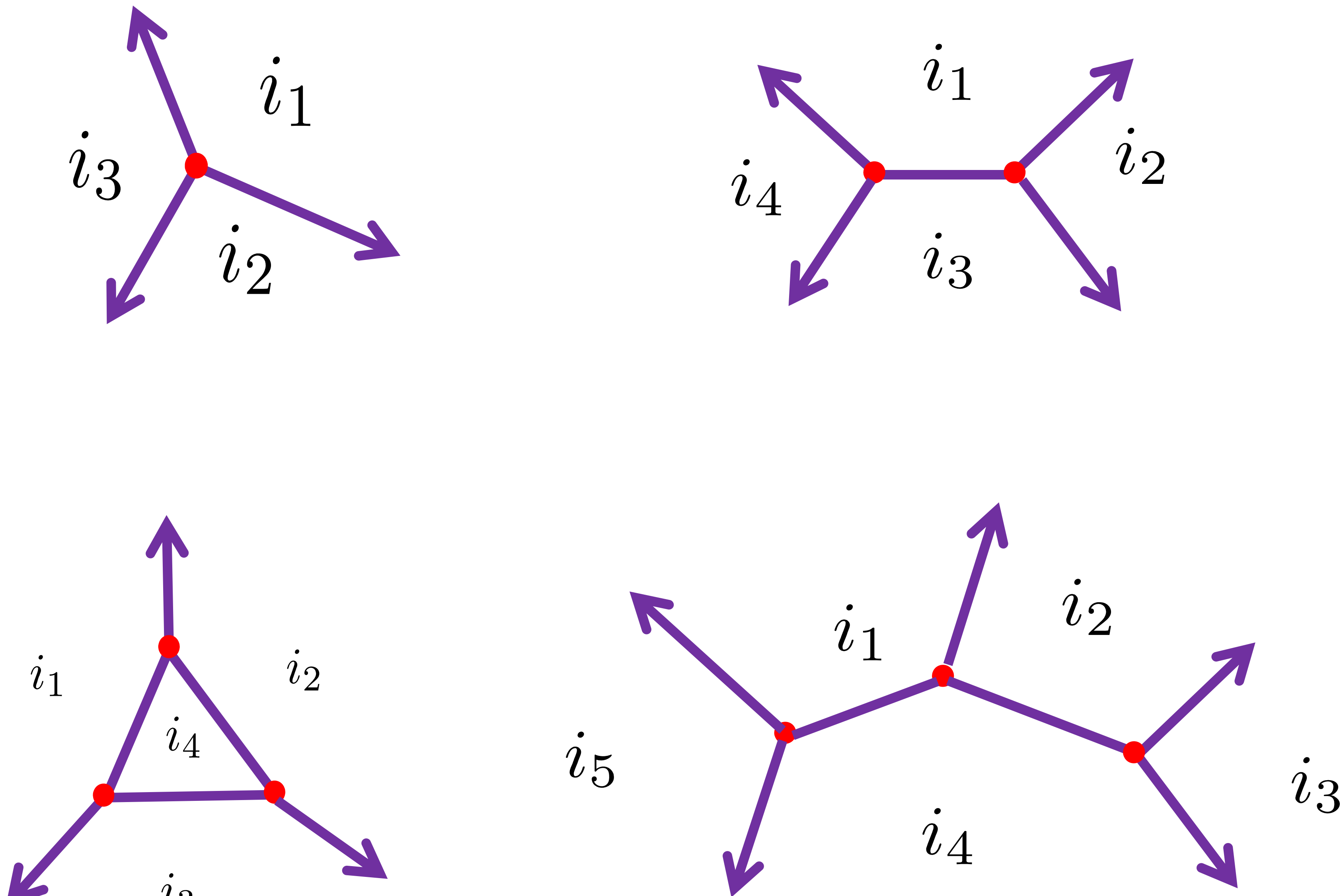
It is motivated from LG field theory.

Vacuum data:

1. A finite set of ``vacua'': $i, j, k, \cdots \in \mathbb{V}$

2. A set of weights $z : \mathbb{V} \to \mathbb{C}$

Definition: A *plane web* is a graph in $\mathbb{R}^2$, together with a coloring of faces by vacua (so that across edges labels differ) and if an edge is oriented so that *i* is on the left and *j* on the right then the edge is parallel to $z_{ij} = z_i - z_j$. (Option: Require vertices at least 3-valent.)


i_1
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i_3
i_1
i_2
i_5
i_3
i_4

Physically, the edges will be worldlines of BPS solitons in the (x,t) plane, connecting two vacua:

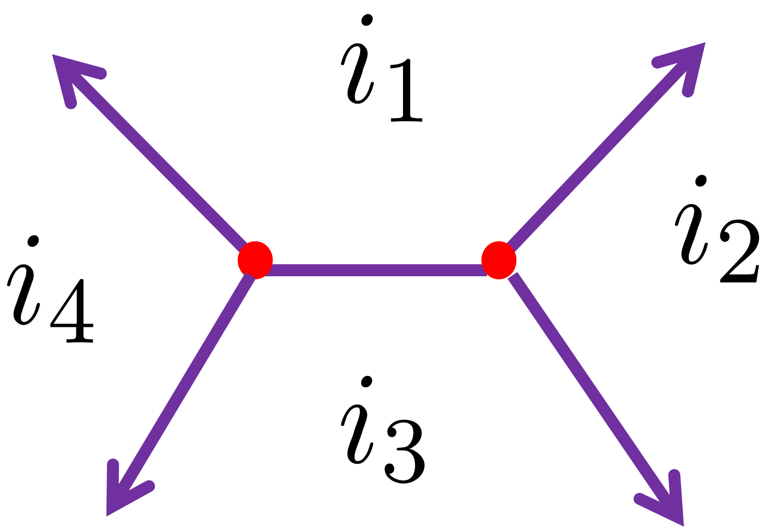
See Davide Gaiotto's talk.

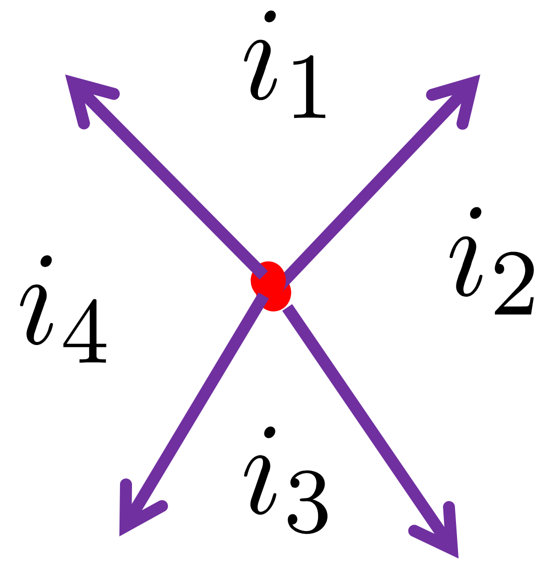
Useful intuition: We are joining together straight strings under a tension z_{ij}. At each vertex there is a no-force condition:

$$z_{i_1,i_2} + z_{i_2,i_3} + \cdots z_{i_n,i_1} = 0$$

Deformation Type

Equivalence under translation and stretching (but not rotating) of strings subject to edge constraints defines ***deformation type***.





Moduli of webs with fixed deformation type

$$\mathcal{D}(\mathfrak{w}) \subset (\mathbb{R}^2)^{V(\mathfrak{w})}$$

$$\dim \mathcal{D}(\mathfrak{w}) = 2V(\mathfrak{w}) - E(\mathfrak{w})$$

(z_i in generic position)

$$\mathcal{D}^{\mathrm{red}}(\mathfrak{w}) = \mathcal{D}(\mathfrak{w})/\mathbb{R}^2_{\mathrm{transl}}$$

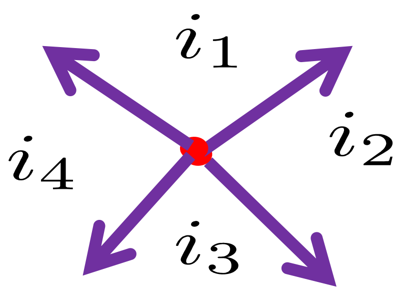
Cyclic Fans of Vacua

Definition: A cyclic fan of vacua is a cyclically-ordered set

$$I = \{i_1, \dots, i_n\}$$

so that the rays $z_{i_k, i_{k+1}} \mathbb{R}_+$ are ordered clockwise

$$I = \{i_1, i_2, i_3, i_4\}$$

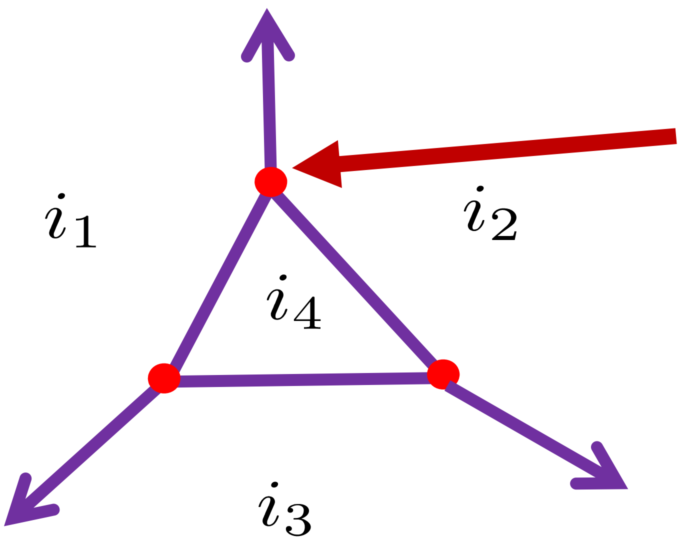


Fans at vertices and at ∞

For a web $\mathfrak{w}$ there are two kinds of cyclic fans we should consider:

Local fan of vacua at a vertex *v*: $I_v(\mathfrak{w})$

Fan of vacua ∞ : $I_\infty(\mathfrak{w})$



$$I_v(\mathfrak{w}) = \{i_1, i_2, i_4\}$$

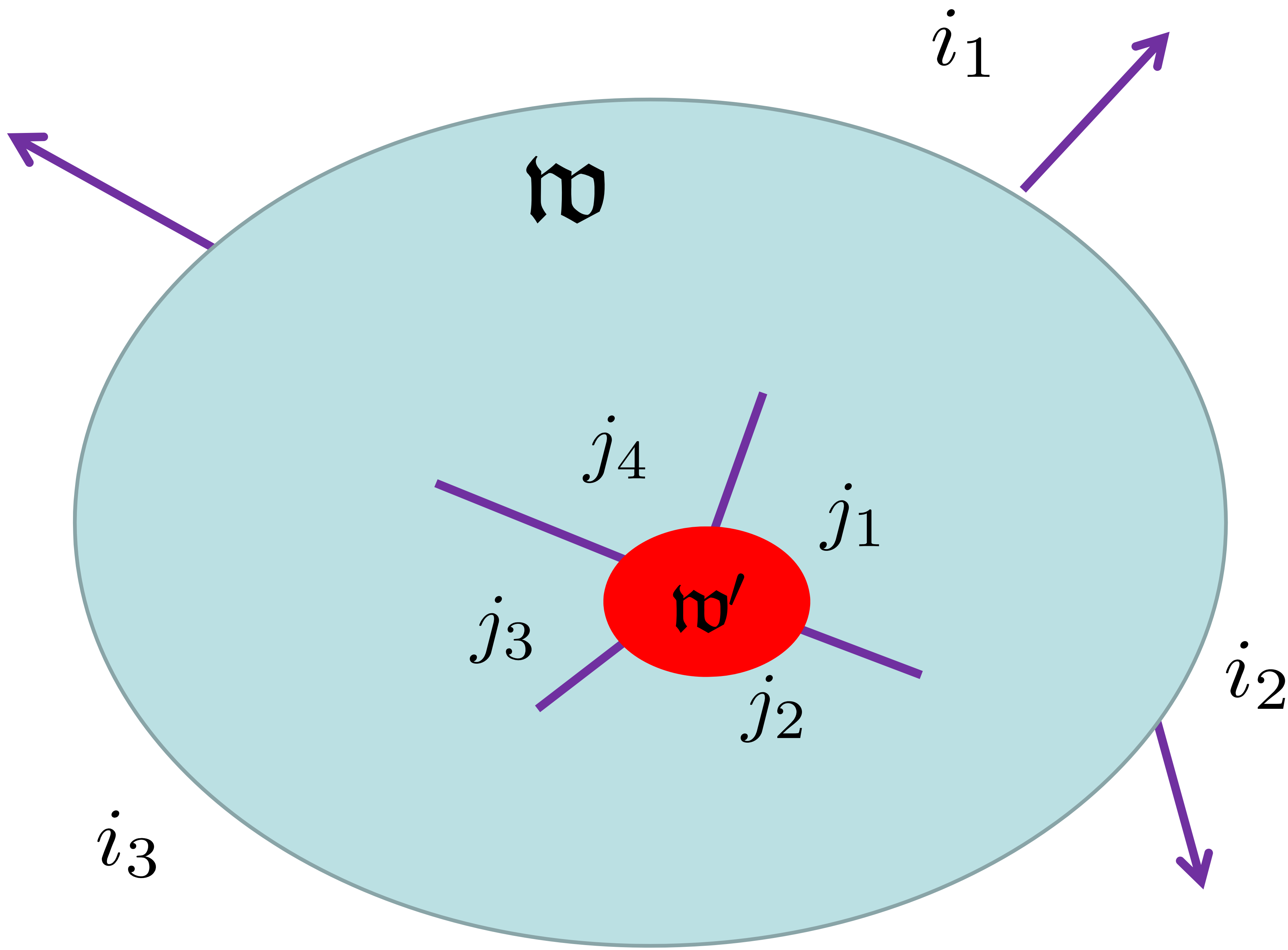
$$I_\infty(\mathfrak{w}) = \{i_1, i_2, i_3\}$$

Convolution of Webs

Definition: Suppose $\mathfrak{w}$ and $\mathfrak{w}'$ are two plane webs and $v \in \mathcal{V}(\mathfrak{w})$ such that

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}')$$

The *convolution of $\mathfrak{w}$ and $\mathfrak{w}'$*, denoted $\mathfrak{w} *_v \mathfrak{w}'$ is the deformation type where we glue in a copy of $\mathfrak{w}'$ into a small disk cut out around *v*.


i_1
$\mathfrak{w}$
j_4
j_1
$\mathfrak{w}'$
j_3
i_2
j_2
i_3

The Web Ring

$\mathcal{W}$ Free abelian group generated by oriented deformation types of plane webs.

``oriented": Choose an orientation o($\mathfrak{w}$) of $\mathcal{D}^{\text{red}}(\mathfrak{w})$

$$* : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$$

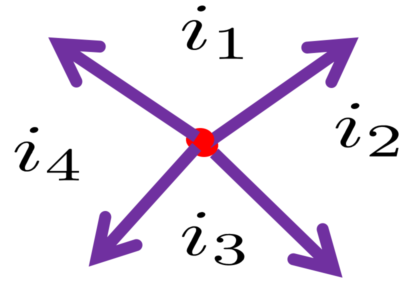
$$I_v(\mathfrak{w}_1) \neq I_\infty(\mathfrak{w}_2) \Rightarrow \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

$$\mathfrak{w}_1 * \mathfrak{w}_2 := \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2$$

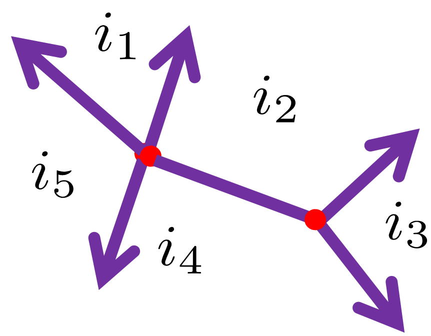
$$o(\mathfrak{w} *_v \mathfrak{w}') = o(\mathfrak{w}) \wedge o(\mathfrak{w}')$$

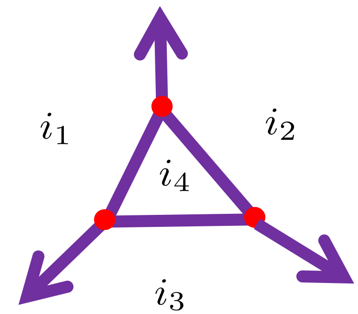
Rigid, Taut, and Sliding

A ***rigid web*** has $d(\mathfrak{w}) = 0$.
It has one vertex:

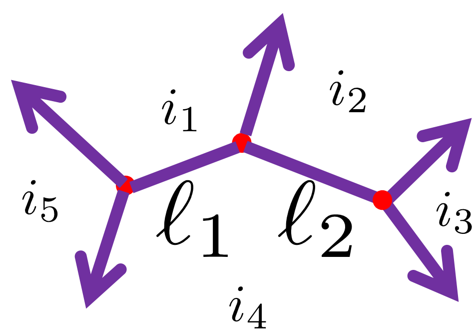


A ***taut web*** has
$d(\mathfrak{w}) = 1$:





A ***sliding web*** has
$d(\mathfrak{w}) = 2$



The taut element

Definition: The taut element $\mathfrak{t}$ is the sum of all taut webs with standard orientation

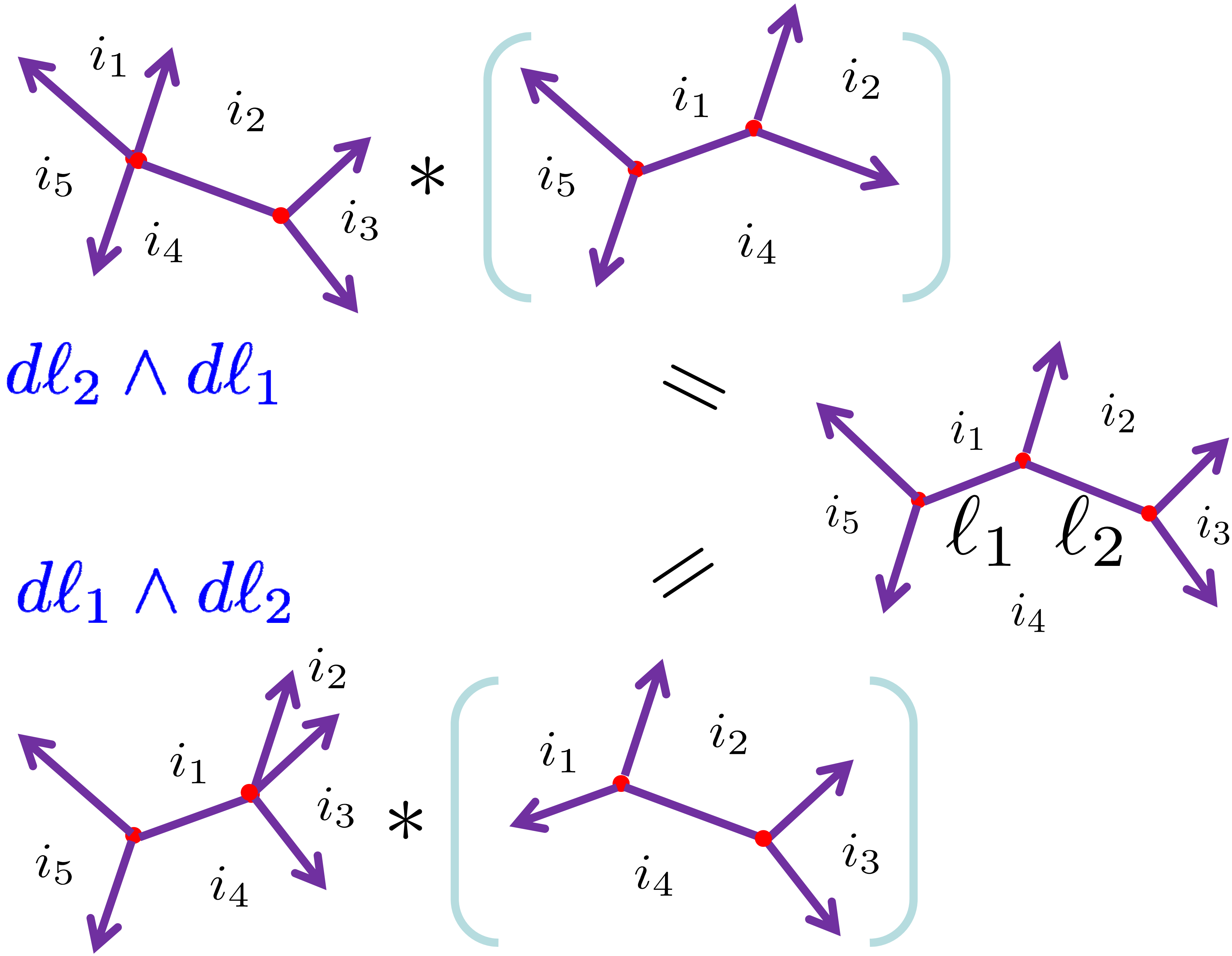
$$\mathfrak{t} := \sum_{d(\mathfrak{w})=1} \mathfrak{w}$$

Theorem: $\mathfrak{t} * \mathfrak{t} = 0$

Proof: The terms can be arranged so that there is a cancellation of pairs:

$$\mathfrak{w}_1 * \mathfrak{w}_2 \qquad \mathfrak{w}_3 * \mathfrak{w}_4$$

Representing two ends of a moduli space of sliding webs



i_1
i_2
i_3
i_4
i_5
$*$
i_1
i_2
i_4
i_5
$d\ell_2 \wedge d\ell_1$
$=$
i_1
i_2
i_3
i_4
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ℓ_1
ℓ_2
$d\ell_1 \wedge d\ell_2$
$=$
i_1
i_2
i_3
i_4
i_5
$*$
i_1
i_2
i_3
i_4

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Web Representations

Definition: A *representation of webs* is

a.) A choice of $\mathbb{Z}$-graded $\mathbb{Z}$-module R_{ij} for every ordered pair ij of distinct vacua.

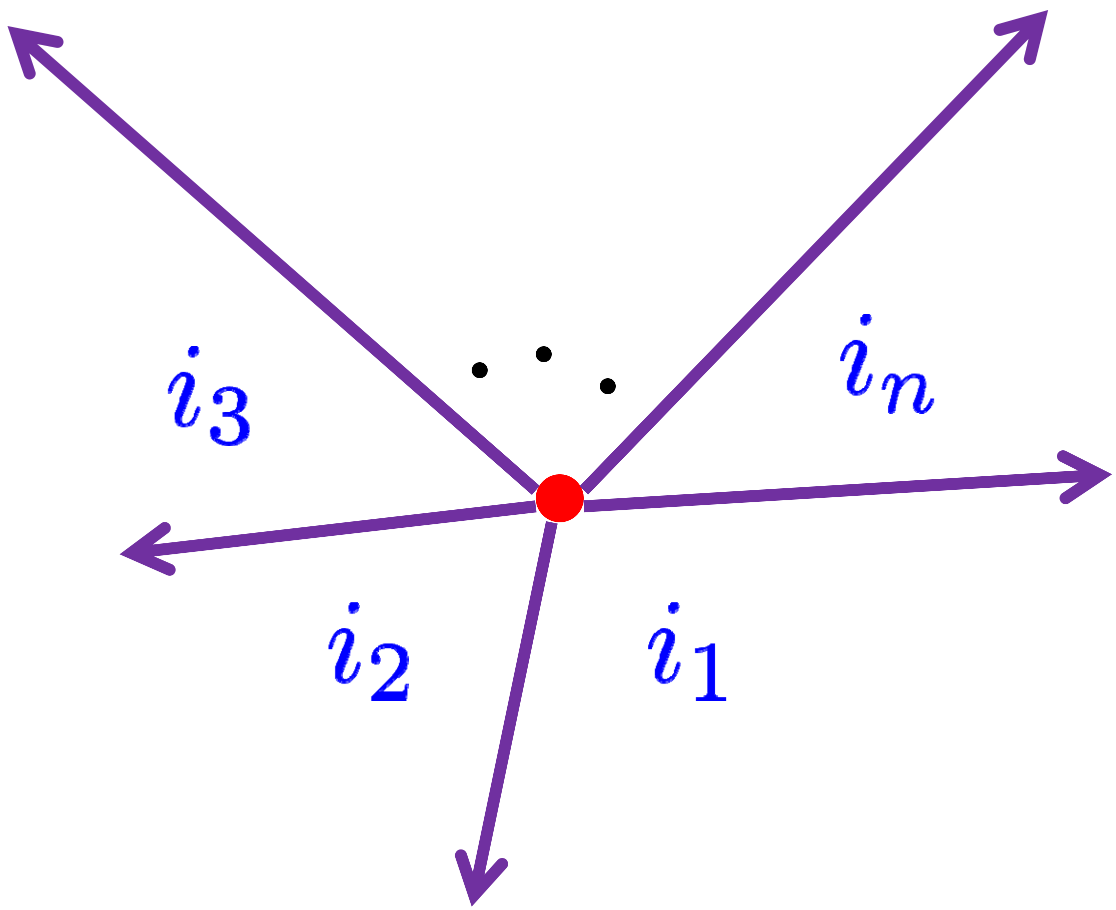
b.) A degree = -1 perfect pairing

$$K : R_{ij} \otimes R_{ji} \to \mathbb{Z}$$

For every cyclic fan of vacua introduce a *fan representation*:

$$I = \{i_1, \dots, i_n\} \Rightarrow$$

$$R_I := R_{i_1, i_2} \otimes \cdots \otimes R_{i_n, i_1}$$



$$R_I := R_{i_1,i_2} \otimes \cdots \otimes R_{i_n,i_1}$$

Web Rep & Contraction

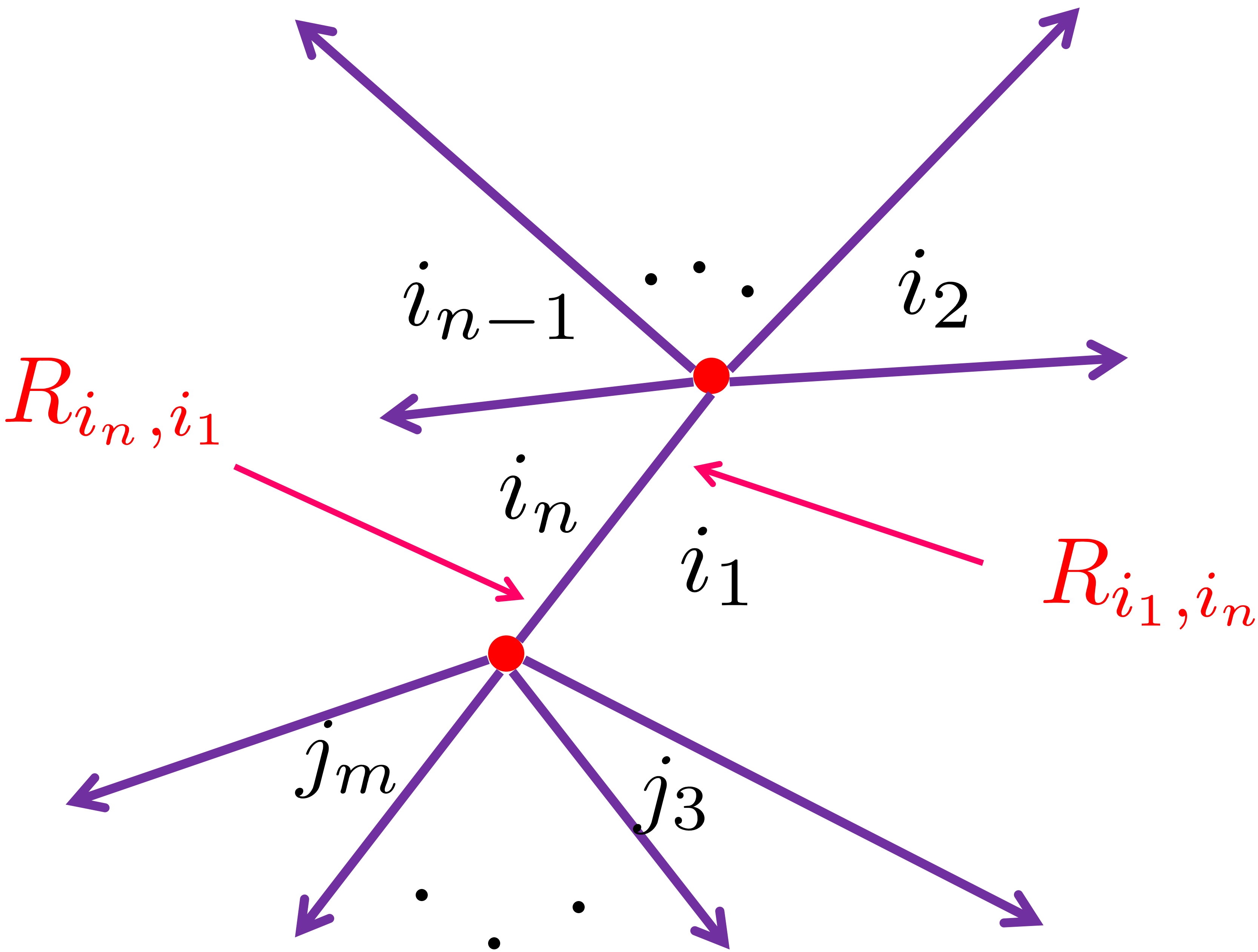
Given a rep of webs and a deformation type $\mathfrak{w}$ we define the *representation of $\mathfrak{w}$*:

$$R(\mathfrak{w}) := \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})}$$

There is a natural contraction operator:

$$\rho(\mathfrak{w}) : R(\mathfrak{w}) \to R_{I_\infty(\mathfrak{w})}$$

by applying the contraction K to the pairs R_{ij} and R_{ji} on each internal edge:



i_{n-1}
i_2
R_{i_n, i_1}
i_n
i_1
R_{i_1, i_n}
j_m
j_3

Extension to Tensor Algebra

$$R^{\mathrm{int}} := \oplus_I R_I$$ Rep of all vertices.

$$\rho(\mathfrak{w}) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

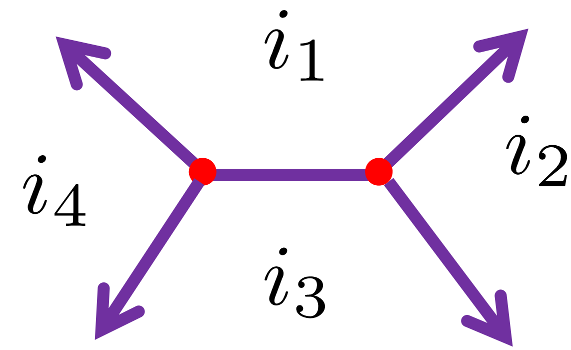
$$r^{(1)} \otimes \cdots \otimes r^{(n)} \in R_{I_1} \otimes \cdots \otimes R_{I_n}$$

$$\rho(\mathfrak{w})[r^{(1)}, \ldots, r^{(n)}]$$

vanishes, unless

$$\{R_{I_1}, \ldots, R_{I_n}\} \Longleftrightarrow \{R_{I_v(\mathfrak{w})}\}$$

Example



$$R(\mathfrak{w}) = R_{i_1 i_3 i_4} \otimes R_{i_1 i_2 i_3}$$

$$\rho(\mathfrak{w})[\underline{r_{i_1 i_3}} r_{i_3 i_4} r_{i_4 i_1} \otimes r_{i_1 i_2} r_{i_2 i_3} \underline{r_{i_3 i_1}}]$$

$$\pm K(r_{i_1 i_3}, r_{i_3 i_1}) r_{i_1 i_2} r_{i_2 i_3} r_{i_3 i_4} r_{i_4 i_1}$$

$$\in R_{i_1 i_2 i_3 i_4}$$

L_∞ -algebras

$$\rho(\mathfrak{t}) : TR^{\text{int}} \to R^{\text{int}}$$

$$\mathfrak{t} * \mathfrak{t} = 0 \quad \Longrightarrow$$

$$\sum_{\mathrm{Sh}_2(S)} \epsilon \; \rho(\mathfrak{t})[\rho(\mathfrak{t})[S_1], S_2] = 0.$$

$$S = \{r_1, \ldots, r_n\} \quad r_i \in R^{\text{int}}$$

$$S = S_1 \amalg S_2 \qquad \epsilon \in \{\pm 1\}$$

L∞ and A∞ Algebras

If A is a vector space (or Z-module) then an ∞-algebra structure is a series of multiplications:

$$m_n(a_1, \ldots, a_n) \in A$$

Which satisfy quadratic relations:

$$S = \{a_1, \ldots, a_n\}$$

$$L_\infty : \quad \sum_{\mathrm{Sh}_2(S)} \epsilon m_{s_1+1}(m_{s_2}(S_2), S_1) = 0$$

$$A_\infty : \sum_{\mathrm{Pa}_3(S)} \epsilon m_{s_1+1+s_3}(S_1, m_{s_2}(S_2), S_3)) = 0$$

The Interior Amplitude

Sum over cyclic fans: $R^{\mathrm{int}} := \oplus_I R_I$

$$\rho(\mathfrak{t}) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

Interior amplitude: $\beta \in R^{\mathrm{int}}$

Satisfies the L_∞ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t})(e^{\beta}) = 0$$

$$e^{\beta} = 1 + \beta + \frac{1}{2!}\beta \otimes \beta + \cdots$$

``Interaction amplitudes for solitons''

Definition of a Theory

By a ***Theory*** we mean a collection of data

$$(\mathbb{V}, z, R_{ij}, K, \beta)$$

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Half-Plane Webs

Same as plane webs, but they sit in a half-plane $\mathcal{H}$.

Some vertices (but no edges) are allowed on the boundary.

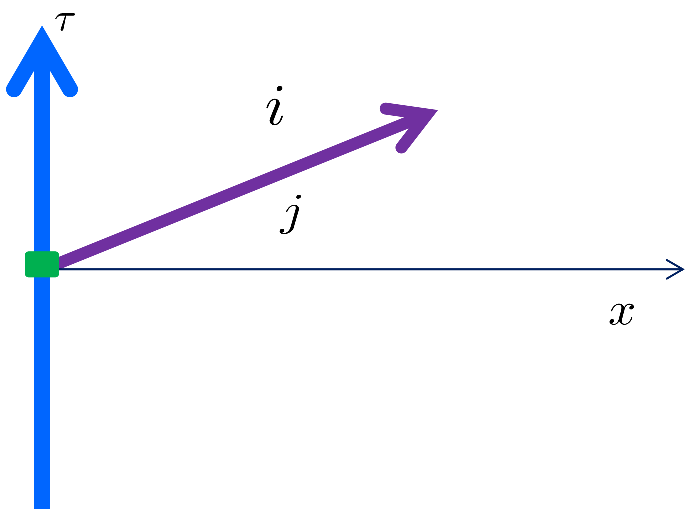
$\mathcal{V}_i(\mathfrak{u})$ Interior vertices

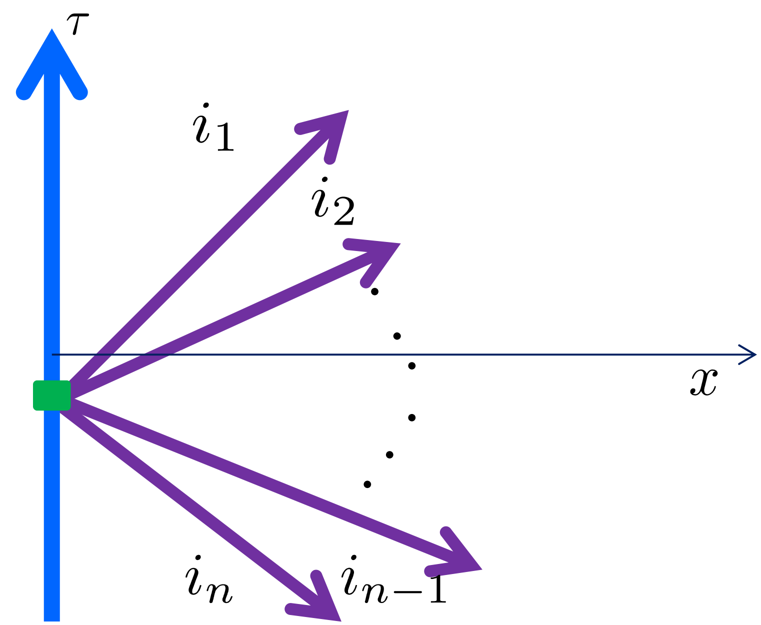
$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1, \dots, v_n\}$ ***time-ordered*** boundary vertices.

deformation type, reduced moduli space, etc. ….

$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}) - 1$$

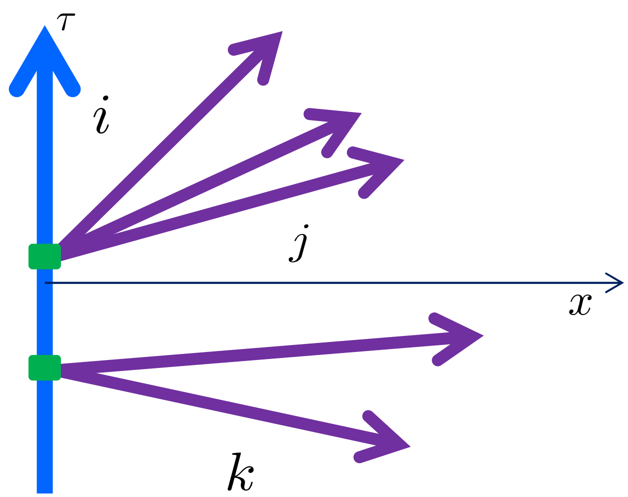
Rigid Half-Plane Webs

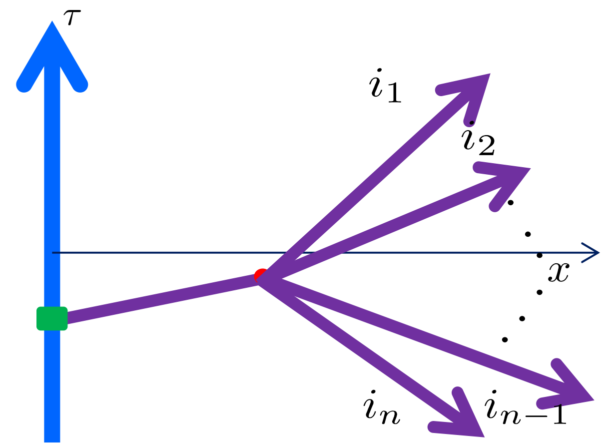




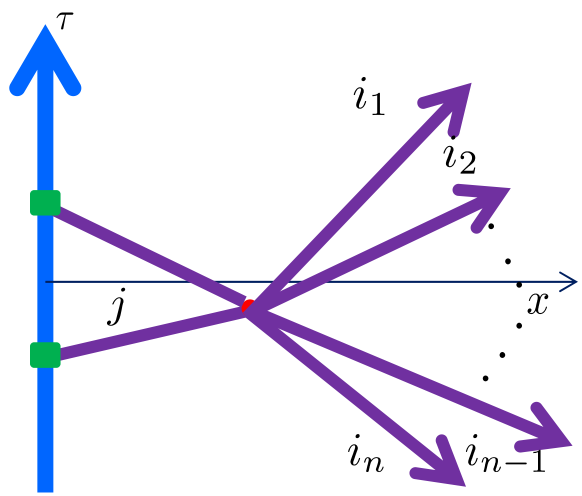
$$d(\mathfrak{u}) = 0$$

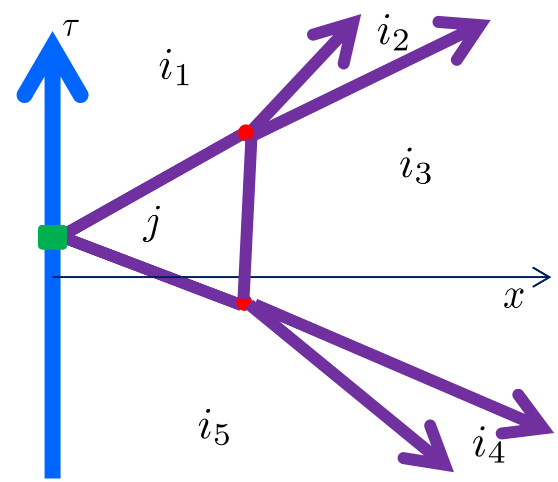
Taut Half-Plane Webs



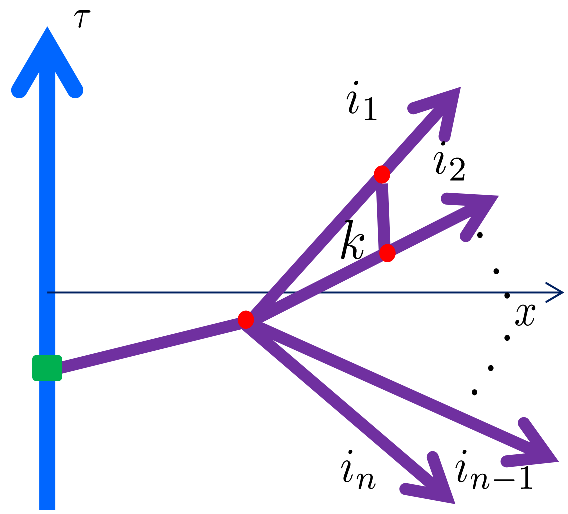


$$d(\mathfrak{u}) = 1$$

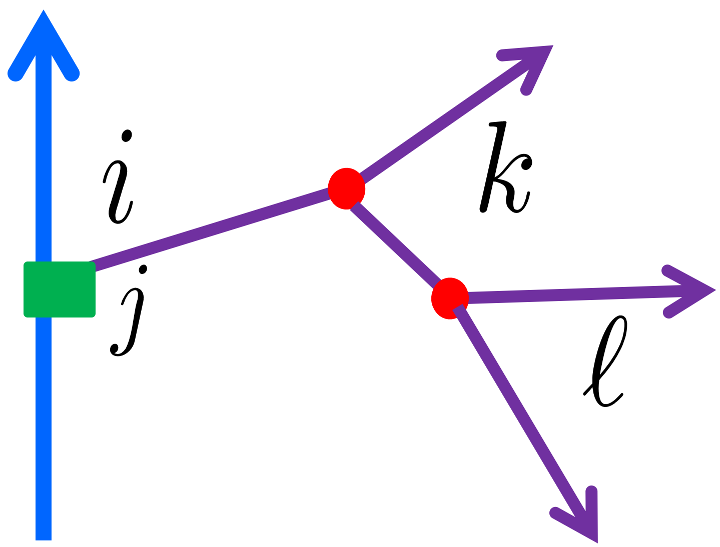


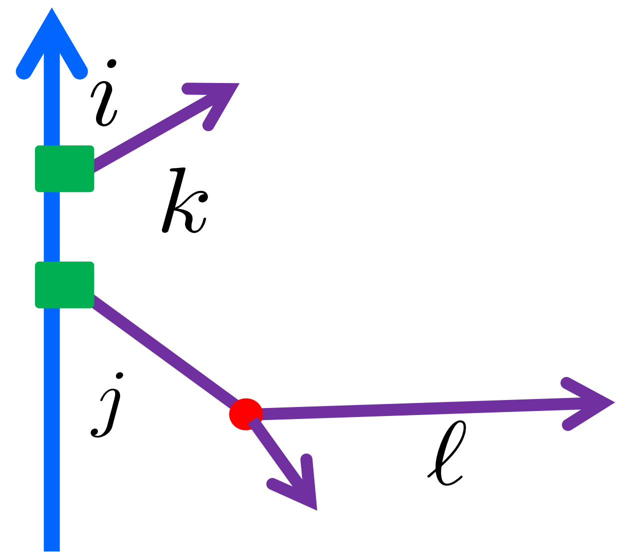


Sliding Half-Plane webs



$$d(\mathbf{u}) = 2$$





Half-Plane fans

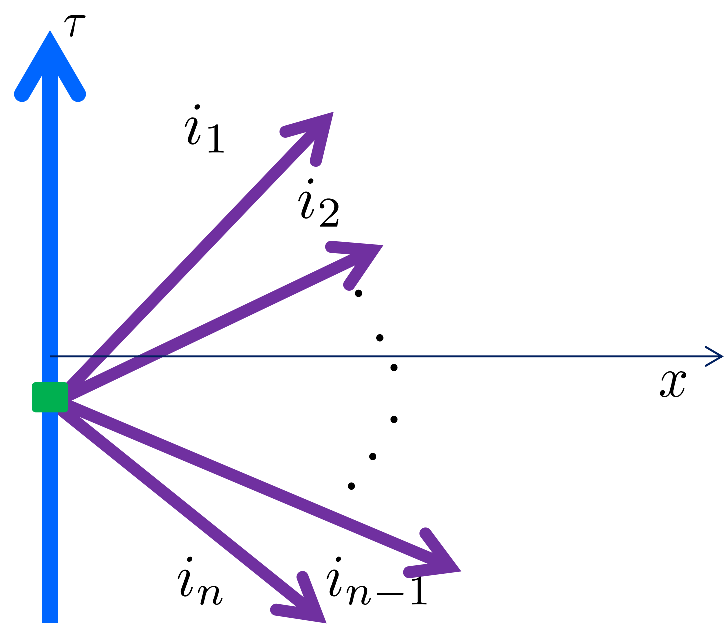
A half-plane fan is an ordered set of vaccua,

$$J = \{i_1, \dots, i_n\}$$

such that successive vacuum weights:

$$z_{i_s, i_{s+1}}$$

are ordered clockwise and in the half-plane:



Convolutions for Half-Plane Webs

We can now introduce a convolution at boundary vertices:

Local half-plane fan at a boundary vertex *v*: $J_v(\mathfrak{u})$

Half-plane fan at infinity: $J_\infty(\mathfrak{u})$

$\mathcal{W}_{\mathcal{H}}$ Free abelian group generated by oriented def. types of half-plane webs

There are now two convolutions:

$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \to \mathcal{W}_{\mathcal{H}}$$

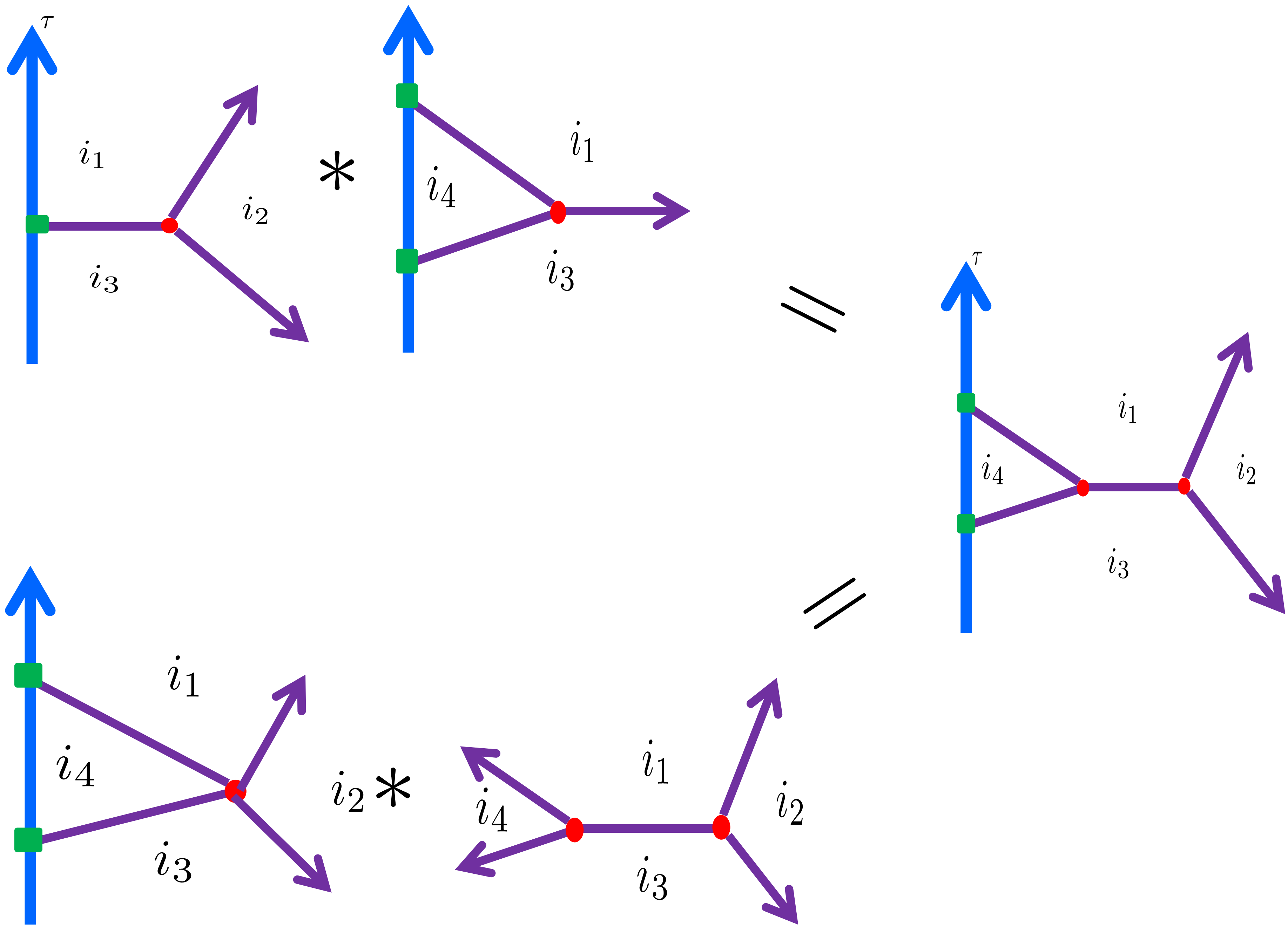
$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

Convolution Theorem

Define the half-plane taut element: $$\mathfrak{t}_{\mathcal{H}} := \sum_{d(\mathfrak{u})=1} \mathfrak{u}$$

Theorem: $$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathrm{p}} = 0$$

Proof: A sliding half-plane web can degenerate (in real codimension one) in two ways: Interior edges can collapse onto an interior vertex, or boundary edges can collapse onto a boundary vertex.



τ
i1
i2
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*
i4
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i3
=
τ
i1
i4
i2
i3
=
i1
i4
i3
i2 *
i4
i1
i3
i2

Half-Plane Contractions

A rep of a half-plane fan: $J = \{j_1, \ldots, j_n\}$

$$R_J := R_{j_1,j_2} \otimes \cdots \otimes R_{j_{n-1},j_n}$$

$\rho(\mathfrak{u})$ now contracts R($\mathfrak{u}$):

$$\bigotimes_{v\in\mathcal{V}_\partial(\mathfrak{u})} R_{J_v(\mathfrak{u})} \bigotimes_{v\in\mathcal{V}_i(\mathfrak{u})} R_{I_v(\mathfrak{u})}$$

$$\to R_{J_\infty(\mathfrak{u})}$$

time ordered!

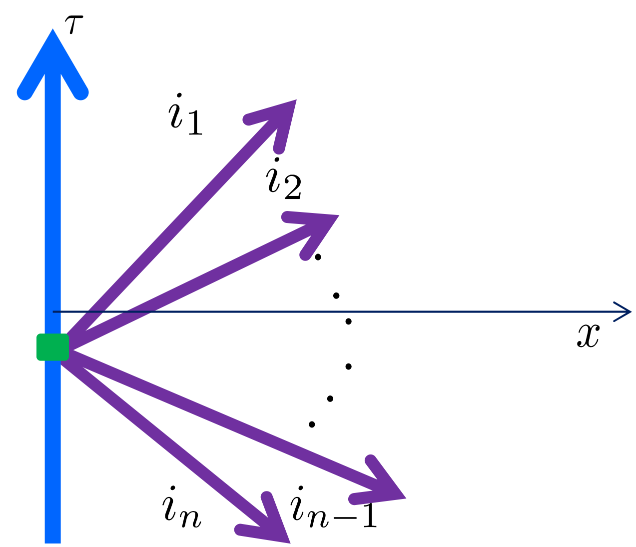
The Vacuum A_∞ Category

(For $\mathcal{H}$ = the positive half-plane)

Objects: i $\in \mathbb{V}$.

Morphisms: $$\mathrm{Hom}(j,i) = \begin{cases} \widehat{R}_{ij} & \mathrm{Re}(z_{ij}) > 0 \\ \mathbb{Z} & i = j \\ 0 & \mathrm{Re}(z_{ij}) < 0 \end{cases}$$

$$\widehat{R}_{i_1,i_n} := \oplus_J R_J$$



$$J = \{i_1, \dots, i_n\}$$

$$\widehat{R}_{i_1,i_n} = R_{i_1,i_n} \oplus \cdots$$

Hint of a Relation to Wall-Crossing

The morphism spaces can be defined by a Cecotti-Vafa/Kontsevich-Soibelman-like product:

Suppose $\mathbb{V} = \{ 1, \ldots, K\}$.
Introduce the elementary K x K matrices e_{ij}

$$\underbrace{\otimes_{\mathrm{Re}(z_{ij})>0}}_{\text{phase ordered!}} (\mathbb{Z}\mathbf{1} \;\oplus\; R_{ij} e_{ij})$$

phase ordered!

$$= \mathbb{Z}\mathbf{1} \;\oplus_{i,j} \widehat{R}_{ij} e_{ij}$$

A_∞ Multiplication

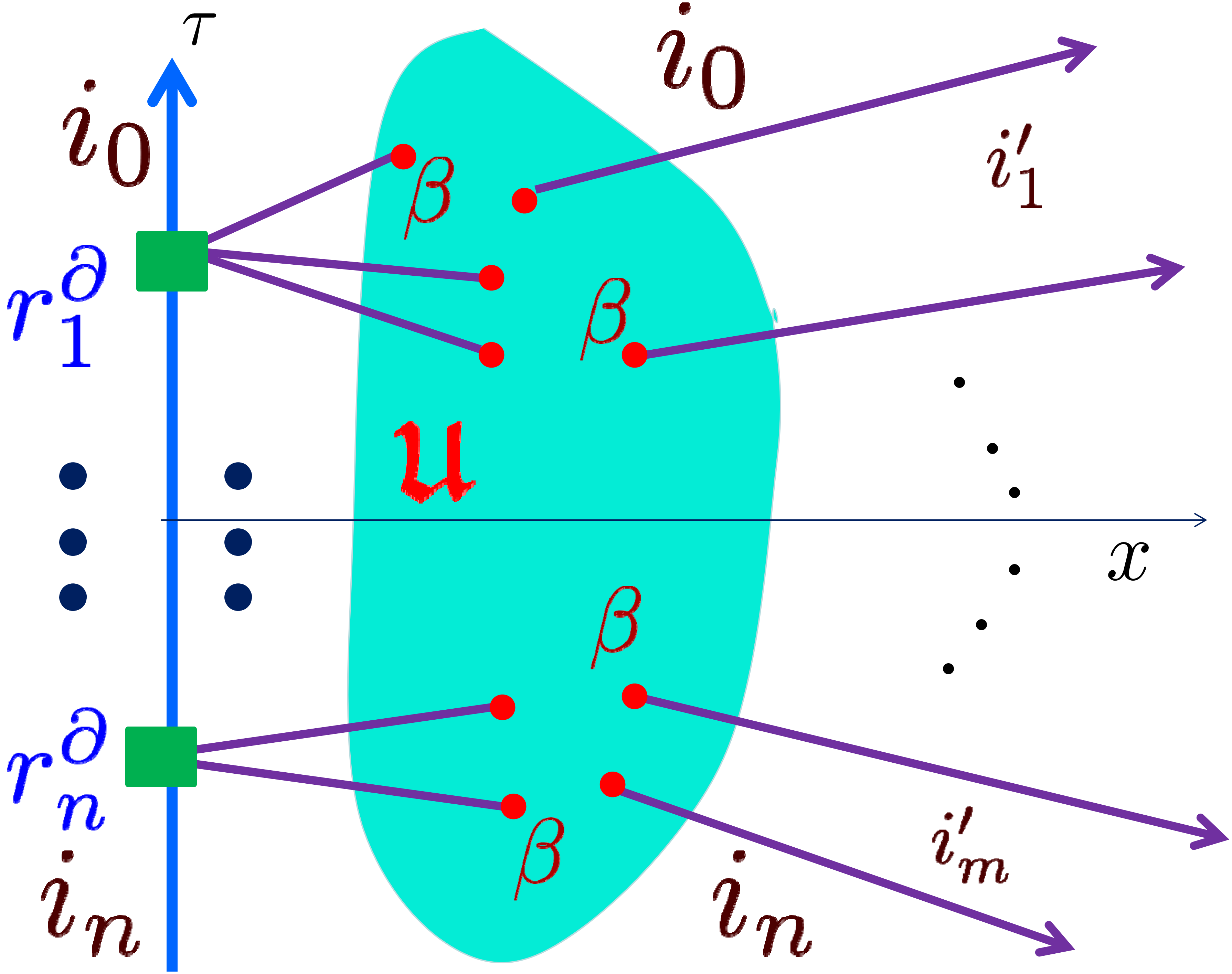
Interior amplitude: $\beta \in R^{\text{int}}$

Satisfies the L_∞ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t}_p)(e^\beta) = 0$$

$$m_n^\beta[r_1^\partial, \dots, r_n^\partial] := \rho(\mathfrak{t}_{\mathcal{H}})[r_1^\partial, \dots, r_n^\partial; e^\beta]$$

$$r_s^\partial \in \mathrm{Hom}(i_{s-1}, i_s)$$



τ
i_0
r_1^∂
r_n^∂
i_n
β
β
β
β
𝔘
i_0
i'_1
x
i'_m
i_n

Enhancing with CP-Factors

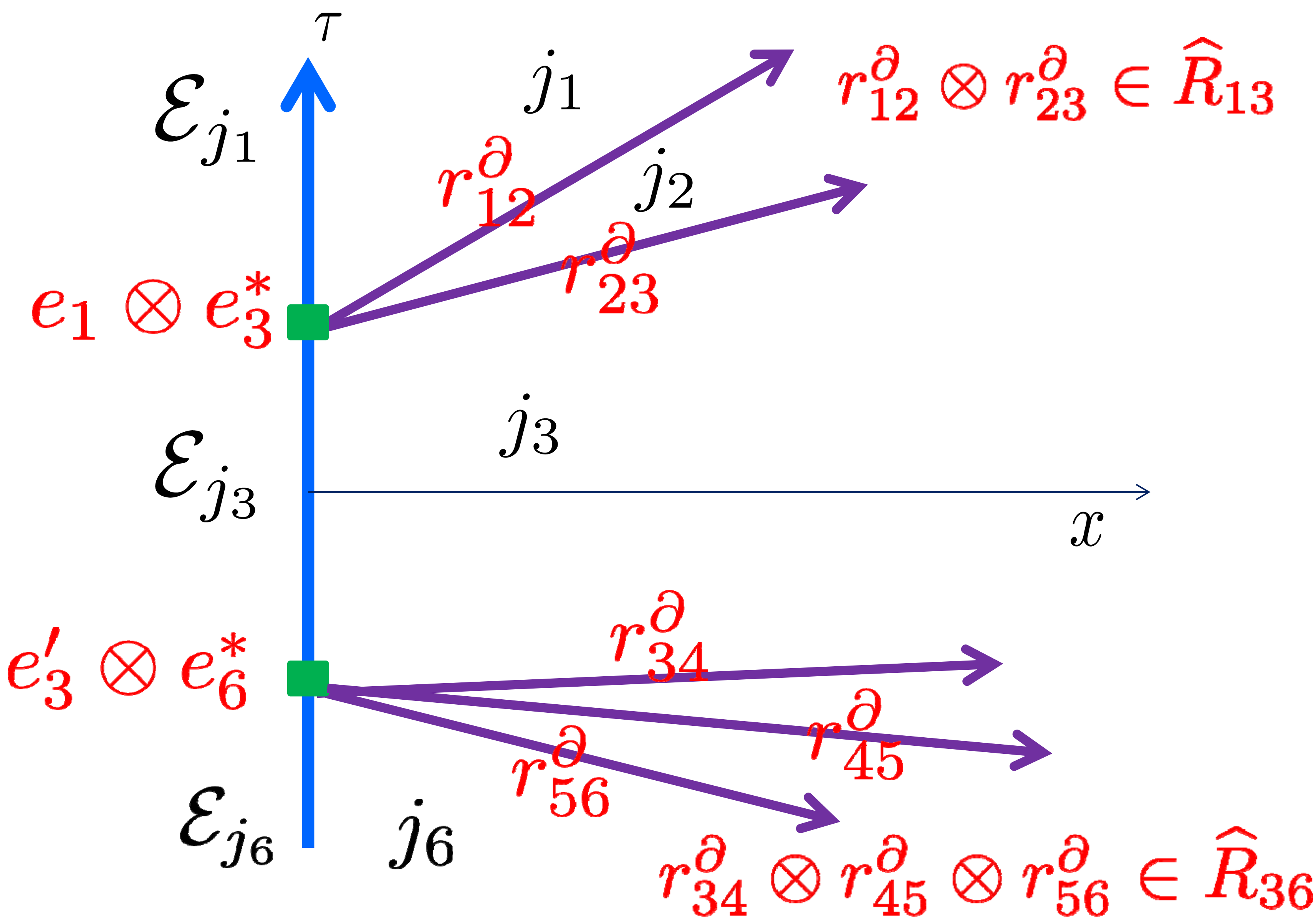
CP-Factors: $i \in \mathbb{V}$ ⟶ $\mathcal{E}_i$ $\mathbb{Z}$-graded module

$$\mathrm{Hop}(i,j) \Longrightarrow \mathcal{E}_i \otimes \mathrm{Hop}(i,j) \otimes \mathcal{E}_j^*$$

$$m_n^\beta \Longrightarrow m_n^\beta \otimes m_2^{\mathrm{CP}}$$

Enhanced A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta; \mathcal{E})$

Example: Composition of two morphisms



Proof of A_∞ Relations

$$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{p} = 0 \Rightarrow$$

$$\sum \epsilon \, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon \, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_{p})[S_1], S_2] = 0.$$

$$S = \{r_1, \dots, r_m\} \quad S = S_1 \amalg S_2$$

$$P = \{r_1^{\partial}, \dots, r_n^{\partial}\} \quad P = P_1 \amalg P_2 \amalg P_3$$

$$r_a \in R^{\mathrm{int}} \qquad r_s^{\partial} \in \widehat{R}_{i_{s-1}, i_s}$$

$$\sum \epsilon \, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon \, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{\beta, \ldots, \beta\}$$

and the second line vanishes.

Hence we obtain the A∞ relations for :

$$m^\beta[P] := \rho(\mathfrak{t}_{\mathcal{H}})[P; e^\beta]$$

Defining an A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta, \mathcal{E})$

Boundary Amplitudes

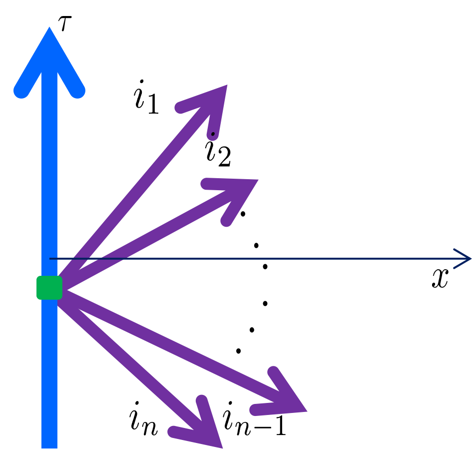
A Boundary Amplitude $\mathcal{B}$ (defining a Brane) is a solution of the A_∞ MC:

$$\mathcal{B} \in \oplus_{i,j} \mathrm{Hop}^{\mathcal{E}}(i,j)$$

$$\mathcal{B} \in \oplus_{\mathrm{Re}(z_{ij})>0} \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*$$

$$\sum_{n=1}^{\infty} m_n^{\beta}[\mathcal{B}^{\otimes n}] = 0$$

$$\rho(\mathfrak{t}_{\mathcal{H}})[\frac{1}{1-\mathcal{B}}; e^{\beta}] = 0$$



``Emission amplitude" from the boundary:

Category of Branes

The Branes themselves are objects in an A_∞ category

$$\mathfrak{Br}(\mathbb{V}, z, R, K, \beta)$$

$$\mathrm{Hop}(\mathcal{B}_1, \mathcal{B}_2)$$

$$= \oplus_{i,j\in\mathbb{V}} \mathcal{E}_i^1 \otimes \mathrm{Hop}(i,j) \otimes (\mathcal{E}_j^2)^*$$

$$M_n(\delta_1, \dots, \delta_n) = \dots.$$

("Twisted complexes": Analog of the derived category.)

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Families of Data

Now suppose the data of a Theory varies ***continuously*** with space:

$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

We have an interface or Janus between the theories at x_{in} and x_{out}.

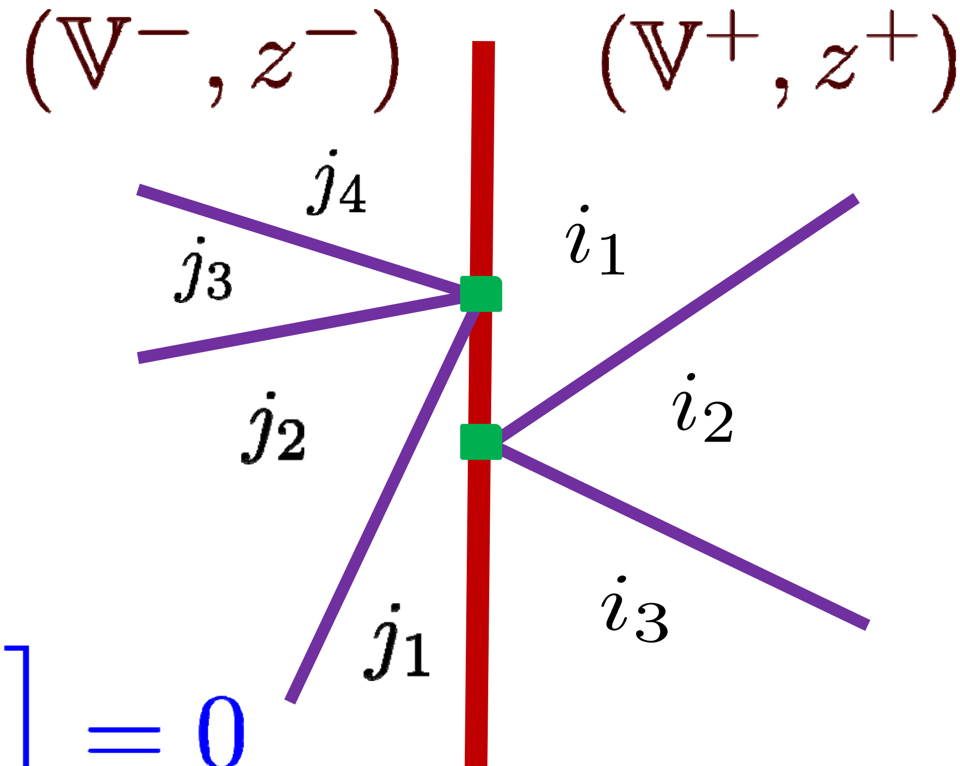
?? How does the Brane category change??

We wish to define a ``flat parallel transport'' of Brane categories. The key will be to develop a theory of supersymmetric interfaces.

Interface webs & amplitudes

Given data $\mathcal{T}^{\pm} = (\mathbb{V}, z, R, K, \beta)^{\pm}$

Introduce a notion of ``interface webs''



These behave like half-plane webs and we can define an *Interface Amplitude* to be a solution of the MC equation:

$$\rho(\mathbf{t}^{-,+})\left[\frac{1}{1-\mathcal{B}^{-,+}}; e^{\beta}\right] = 0$$

Category of Interfaces

Interfaces are very much like Branes,

Chan-Paton: $\mathcal{E}(\mathfrak{I})_{i^-,j^+}$

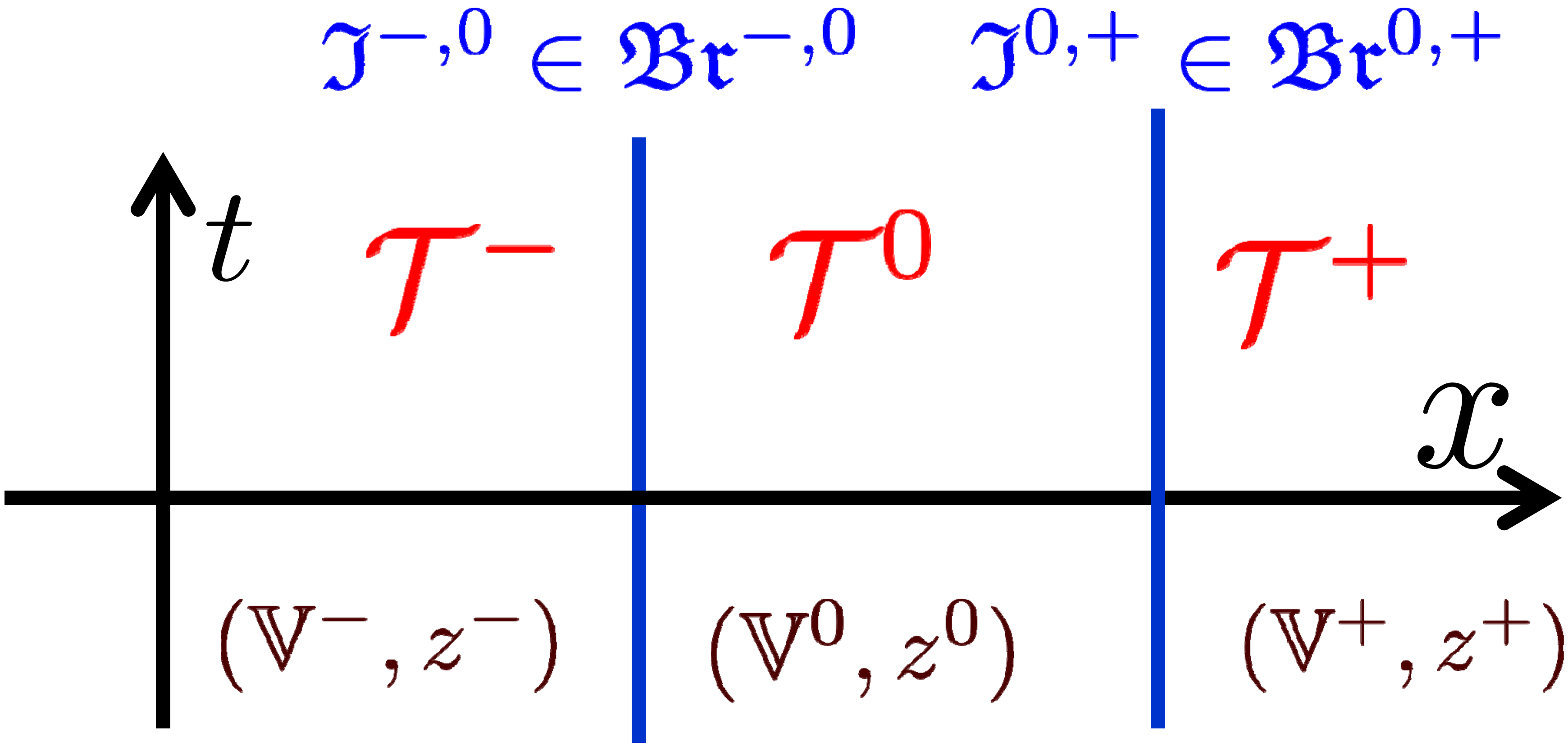
$$(i^-, j^+) \in \mathbb{V}^- \times \mathbb{V}^+$$

In fact we can define an A_∞ category of Interfaces between the two theories:

$$\mathfrak{I}^{-,+} \in \mathfrak{Br}^{-,+}$$

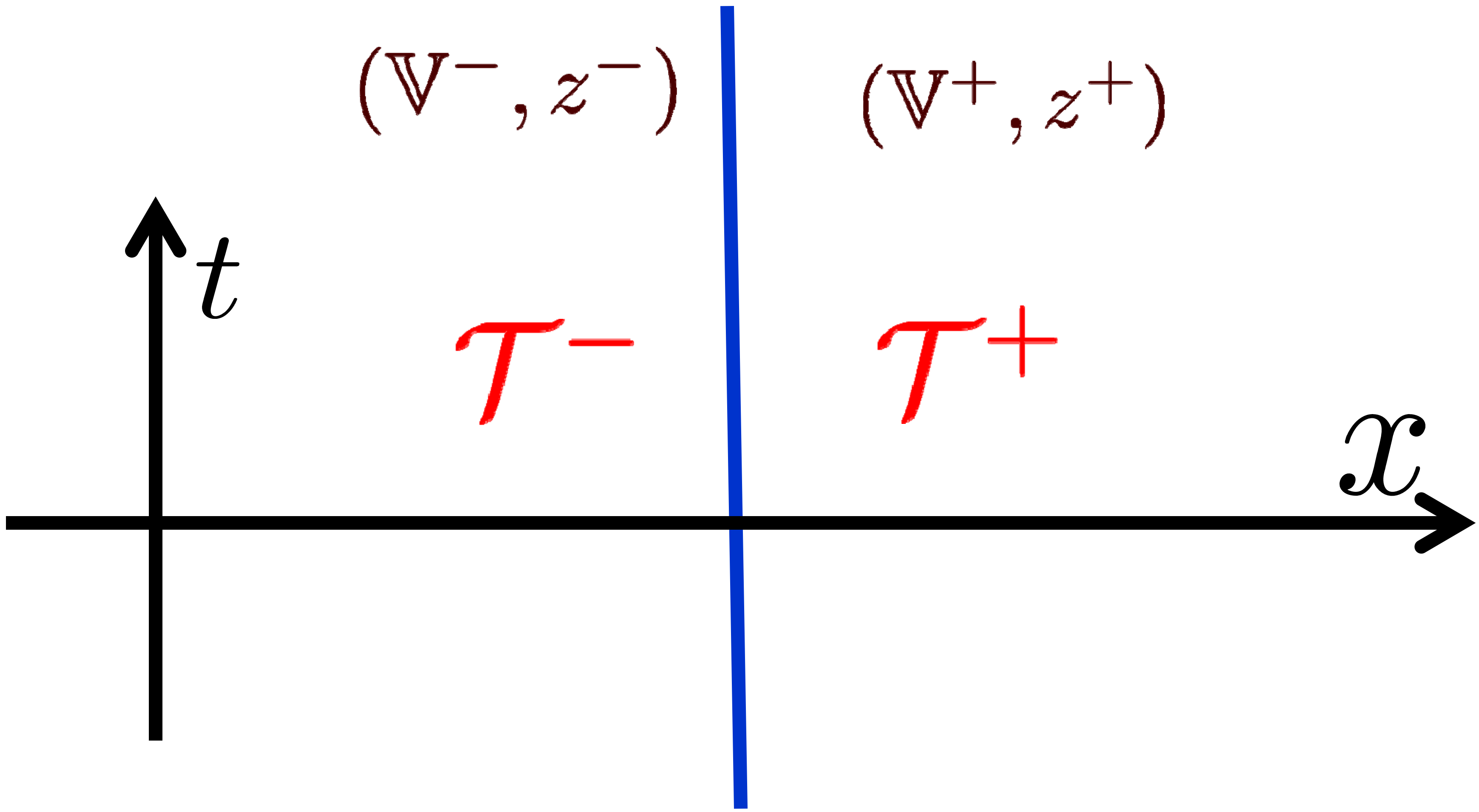
Note: If one of the Theories is trivial we simply recover the category of Branes.

Composition of Interfaces -1



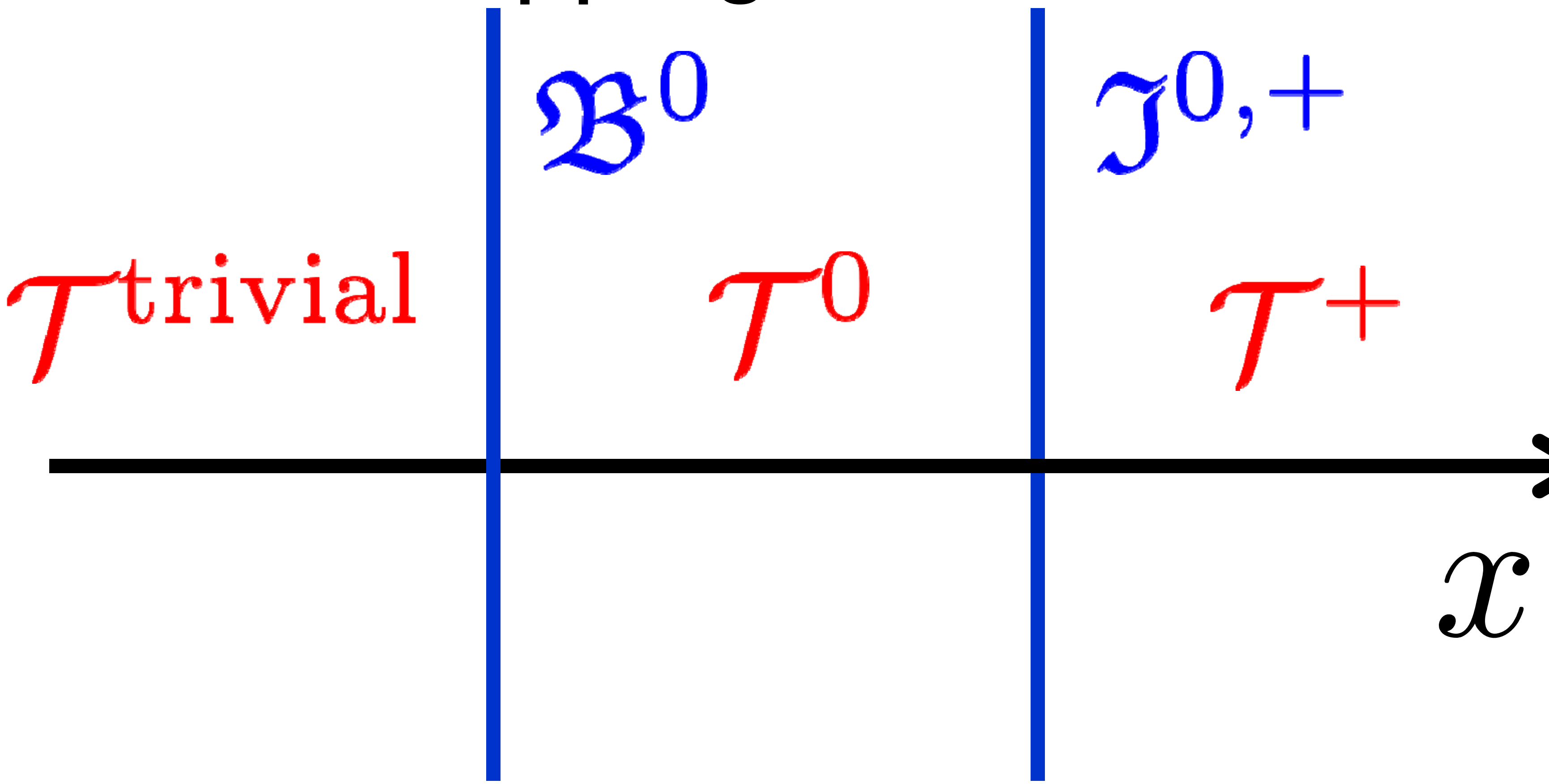
Want to define a ``multiplication'' of the Interfaces…

Composition of Interfaces - 2



$$\mathfrak{I}^{-,+} = \mathfrak{I}^{-,0} \star \mathfrak{I}^{0,+} \in \mathfrak{Br}^{-,+}$$

Mapping of Branes



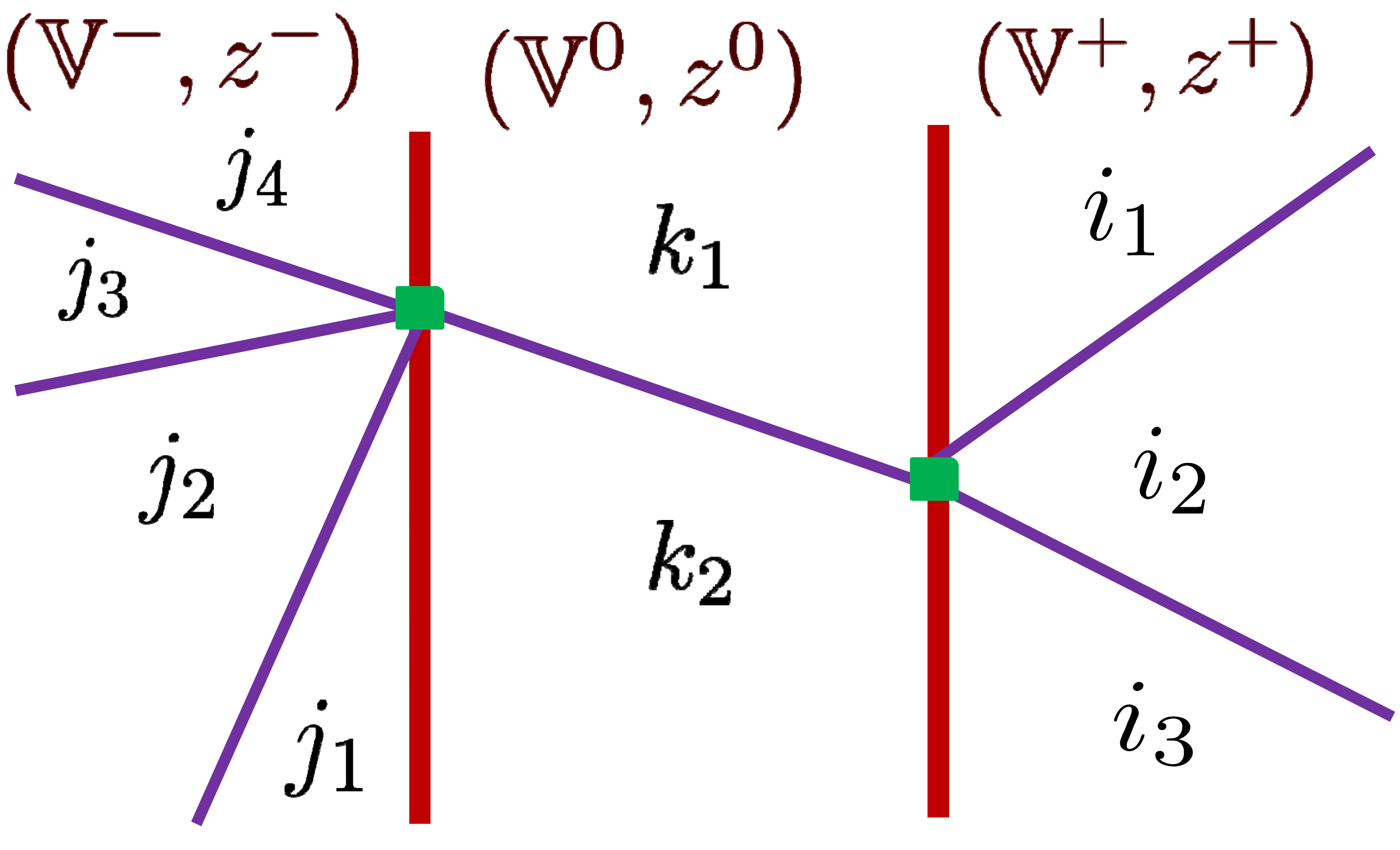
Special case: ``maps'' branes in theory $\mathcal{T}^0$ to branes in theory $\mathcal{T}^+$:

$$\mathfrak{B}^0 \to \mathfrak{B}^+ := \mathfrak{B}^0 \bigstar \mathfrak{I}^{0,+}$$

Technique: Composite webs

Given data $(\mathbb{V}, z, R, K, \beta)^{-,0,+}$

Introduce a notion of ``composite webs''



Def: Composition of Interfaces

A convolution identity implies:

$$\rho(\mathfrak{t}^{-,0,+})\left[\frac{1}{1-\mathcal{B}^{-,0}}, \frac{1}{1-\mathcal{B}^{0,+}}; e^{\beta}\right]$$

Interface amplitude

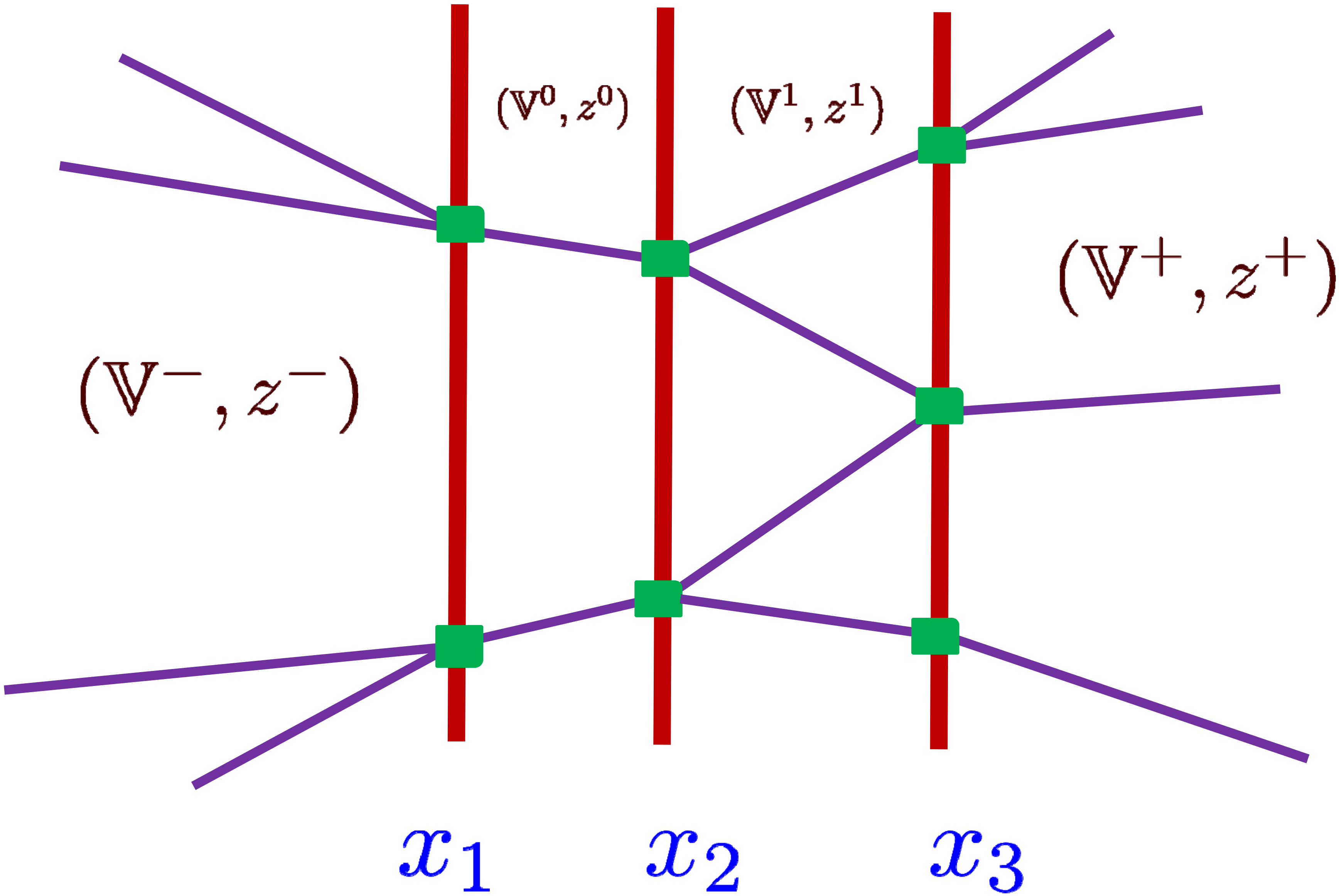
$$\mathcal{E}(\mathfrak{I}^{-,0} \bigstar \mathfrak{I}^{0,+}) = \oplus_{j^0}\mathcal{E}(\mathfrak{I}^{-,0})_{i^- j^0} \otimes \mathcal{E}(\mathfrak{I}^{0,+})_{j^0 k^+}$$

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,+} \to \mathfrak{Br}^{-,+}$$

Physically: An OPE of susy Interfaces

Theorem: The product is an A_∞ bifunctor

Associativity?



Homotopy Equivalence

(Standard homological algebra)

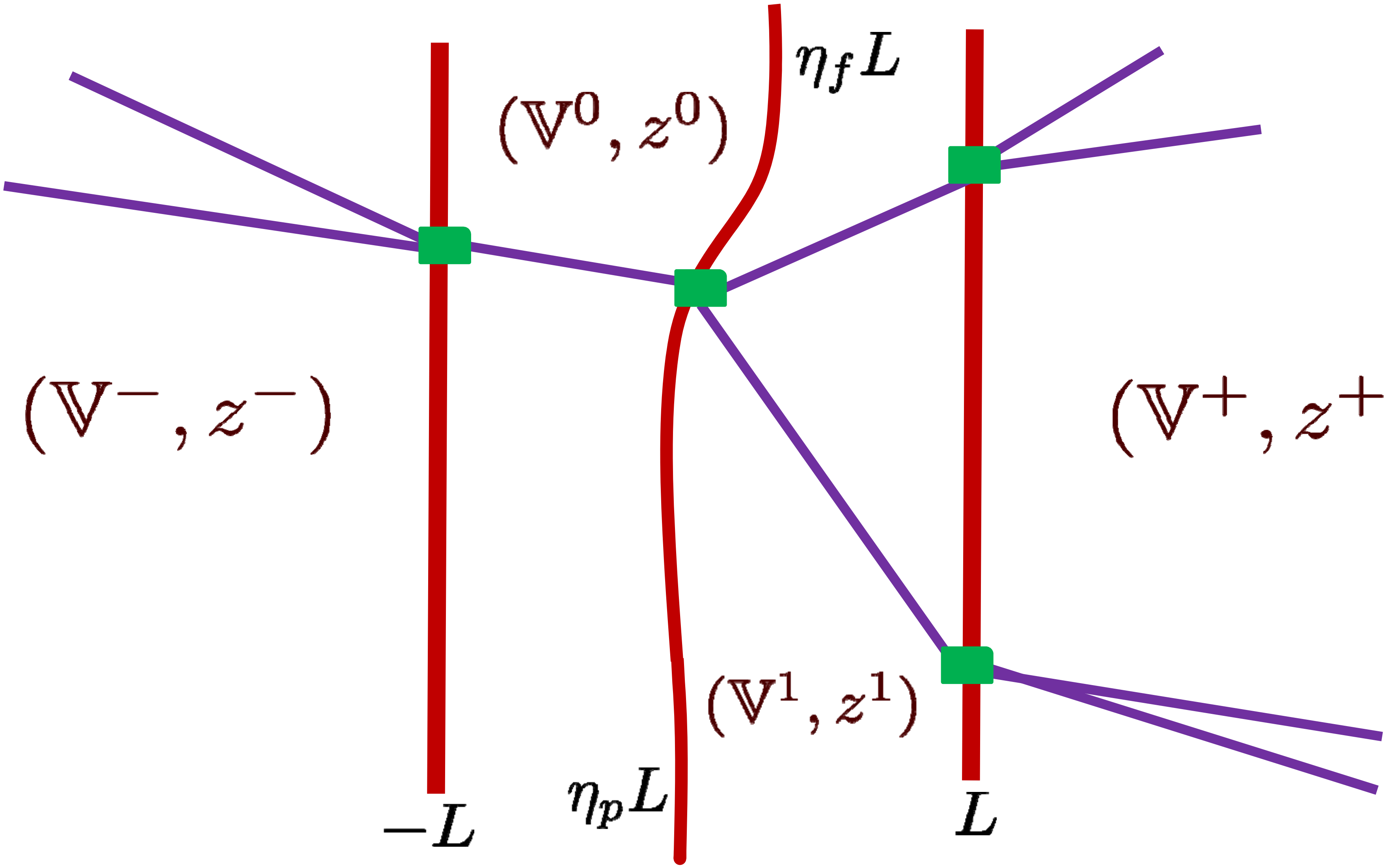
$$\delta_1, \delta_2 \in \mathrm{Hom}(\mathfrak{I}, \mathfrak{I}')$$

$$\delta_1 \sim \delta_2 \quad \Longleftrightarrow \quad \delta_1 - \delta_2 = M_1(\delta_3)$$

$$\mathfrak{I} \sim \mathfrak{I}' \quad \Longleftrightarrow \quad \begin{array}{l} M_2(\delta', \delta) \sim \mathrm{Id} \\ M_2(\delta, \delta') \sim \mathrm{Id} \end{array}$$

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,1} \times \mathfrak{Br}^{1,+} \to \mathfrak{Br}^{-,+}$$

Product is associative up to homotopy equivalence



Webology: Deformation type, taut element, convolution identity, …

An A_∞ 2-category

Objects, or 0-cells are Theories:

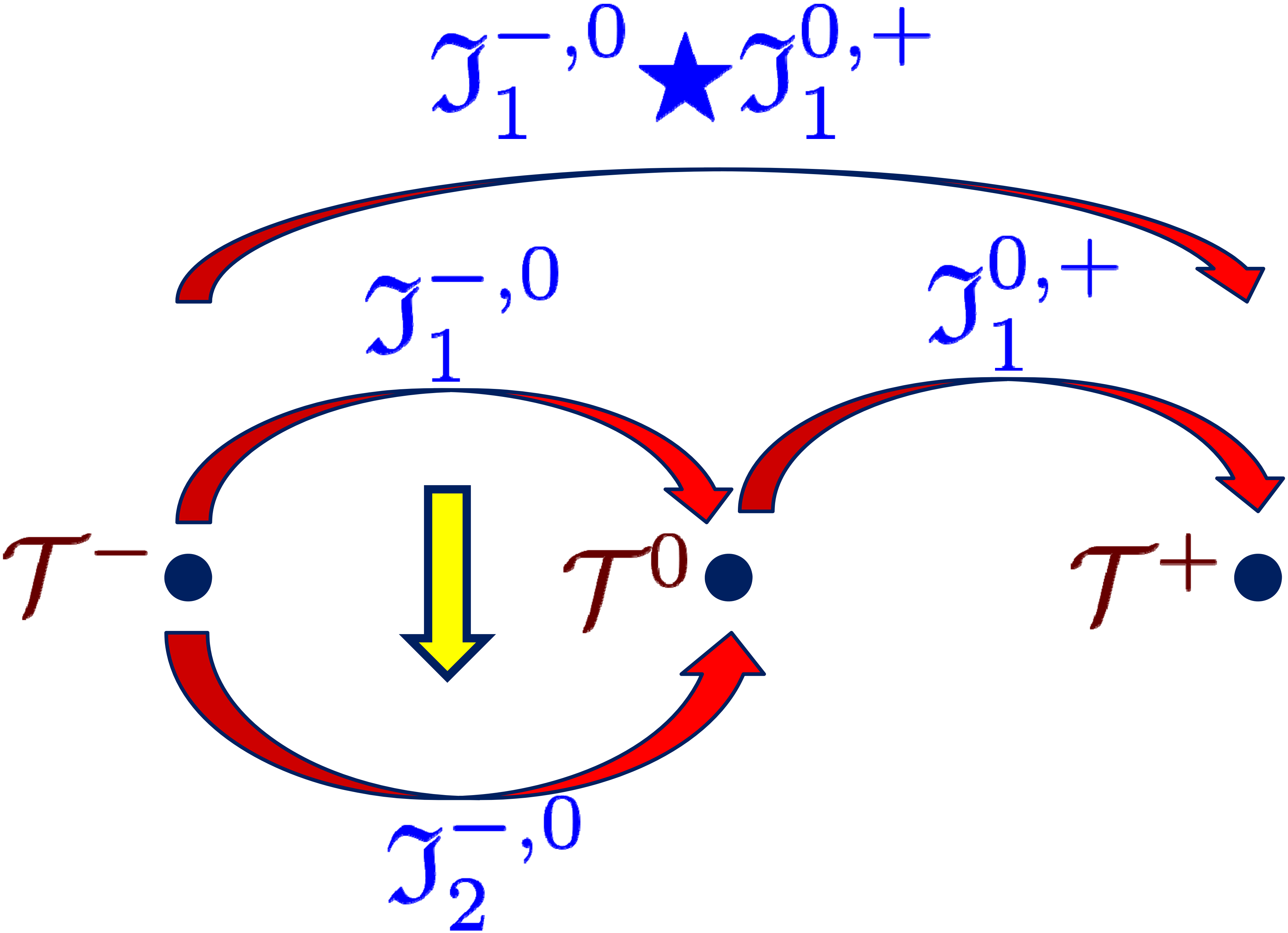
$$\mathcal{T} = (\mathbb{V}, z, R, K, \beta)$$

1-Morphisms, or 1-cells are objects in the category of Interfaces:

$$\mathfrak{I} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$$

2-Morphisms, or 2-cells are morphisms in the category of Interfaces:

$$\delta \in \mathrm{Hop}(\mathfrak{I}_1^{-,+}, \mathfrak{I}_2^{-,+})$$



$\mathfrak{I}_1^{-,0} \bigstar \mathfrak{I}_1^{0,+}$
$\mathfrak{I}_1^{-,0}$
$\mathfrak{I}_1^{0,+}$
$\mathcal{T}^-$
$\mathcal{T}^0$
$\mathcal{T}^+$
$\mathfrak{I}_2^{-,0}$

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Parallel Transport of Categories

For any continuous path:

$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

we ***want*** to associate an A∞ functor:

$$\mathbb{F}[\wp] : \mathfrak{Br}(\mathcal{T}^{\mathrm{in}}) \to \mathfrak{Br}(\mathcal{T}^{\mathrm{out}})$$

$$\mathbb{F}[\wp_1 \circ \wp_2] = \mathbb{F}[\wp_1] \circ \mathbb{F}[\wp_2]$$

$$\wp \sim \wp' \quad \Rightarrow \quad \tau : \mathbb{F}[\wp] \cong \mathbb{F}[\wp']$$

Interface-Induced Transport

Idea is to induce it via a suitable Interface:

$$\mathbb{F}[\wp] : \mathfrak{B}^{\text{in}} \to \mathfrak{B}^{\text{in}} \bigstar \mathfrak{I}^{\text{in,out}}$$

But how do we construct the Interface?

Example: Spinning Weights

$$z_i(x) = e^{\mathrm{i}\vartheta(x)} z_i$$

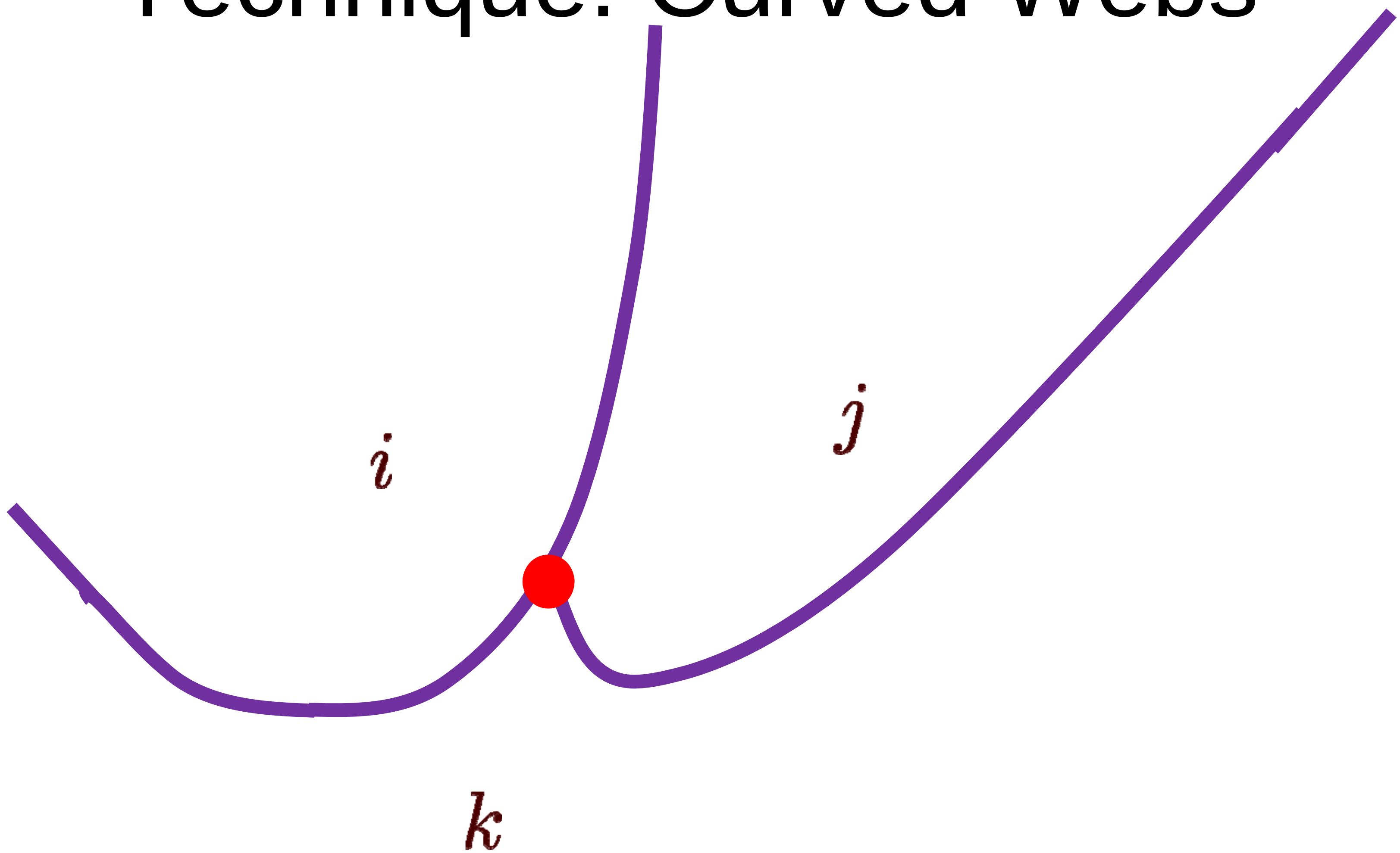
$$(\mathbb{V}, R, K, \beta) \quad \text{constant}$$

We can construct explicitly: $\mathfrak{I}[\vartheta(x)]$

$$\vartheta_1(x) \sim \vartheta_2(x) \implies \mathfrak{I}[\vartheta_1(x)] \sim \mathfrak{I}[\vartheta_2(x)]$$

$$\mathfrak{I}[\vartheta_1 \circ \vartheta_2] \sim \mathfrak{I}[\vartheta_1] \bigstar \mathfrak{I}[\vartheta_2]$$

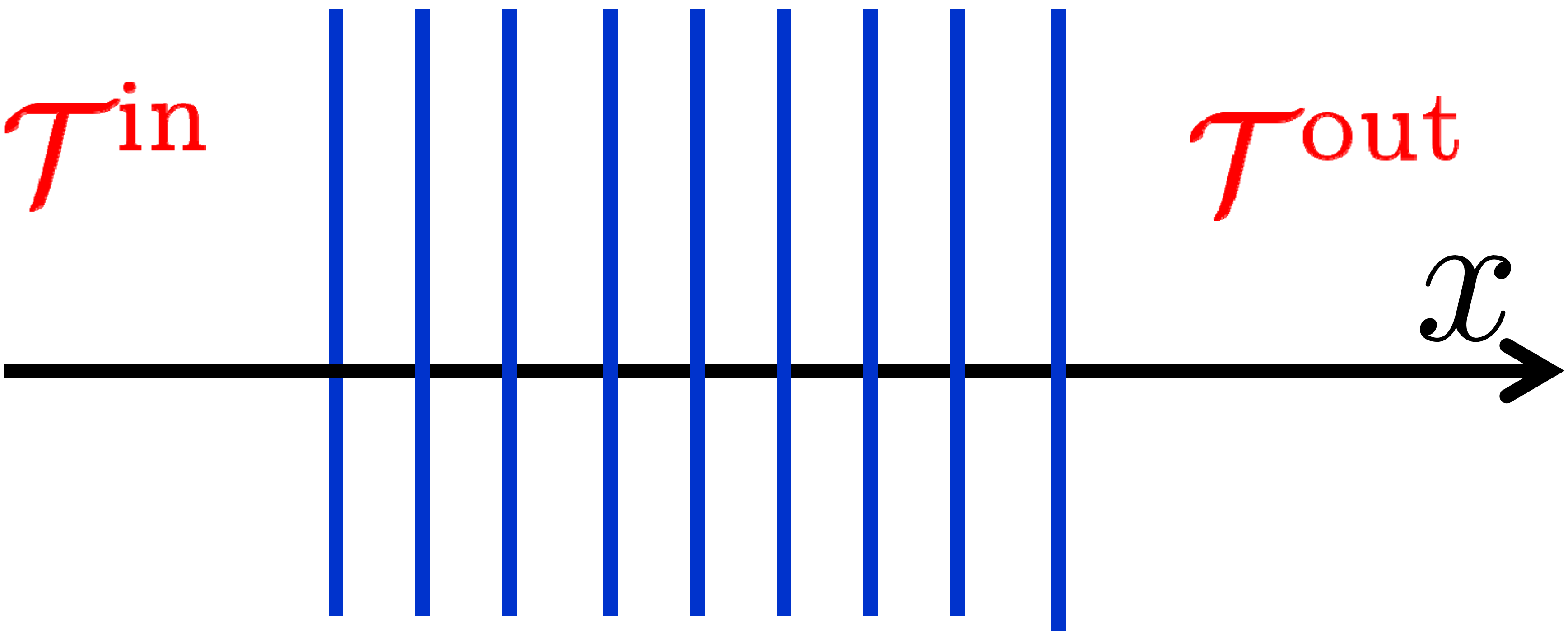
Technique: Curved Webs



Webology: Deformation type, taut element, convolution identity, …

Reduction to Elementary Paths:

$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$



$$\mathfrak{J}^{\mathrm{in,out}} := \mathfrak{J}_1 \bigstar \cdots \bigstar \mathfrak{J}_n$$

Categorified ``S-wall crossing''

For spinning weights this works very well.

$$\mathfrak{I}[\vartheta(x)]$$

decomposes as a product of ``trivial parallel transport Interfaces'' and ``S-wall Interfaces,'' which categorify the wall-crossing of framed BPS indices.

In this way we categorify the ``detour rules'' of the nonabelianization map of spectral network theory.

General Case?

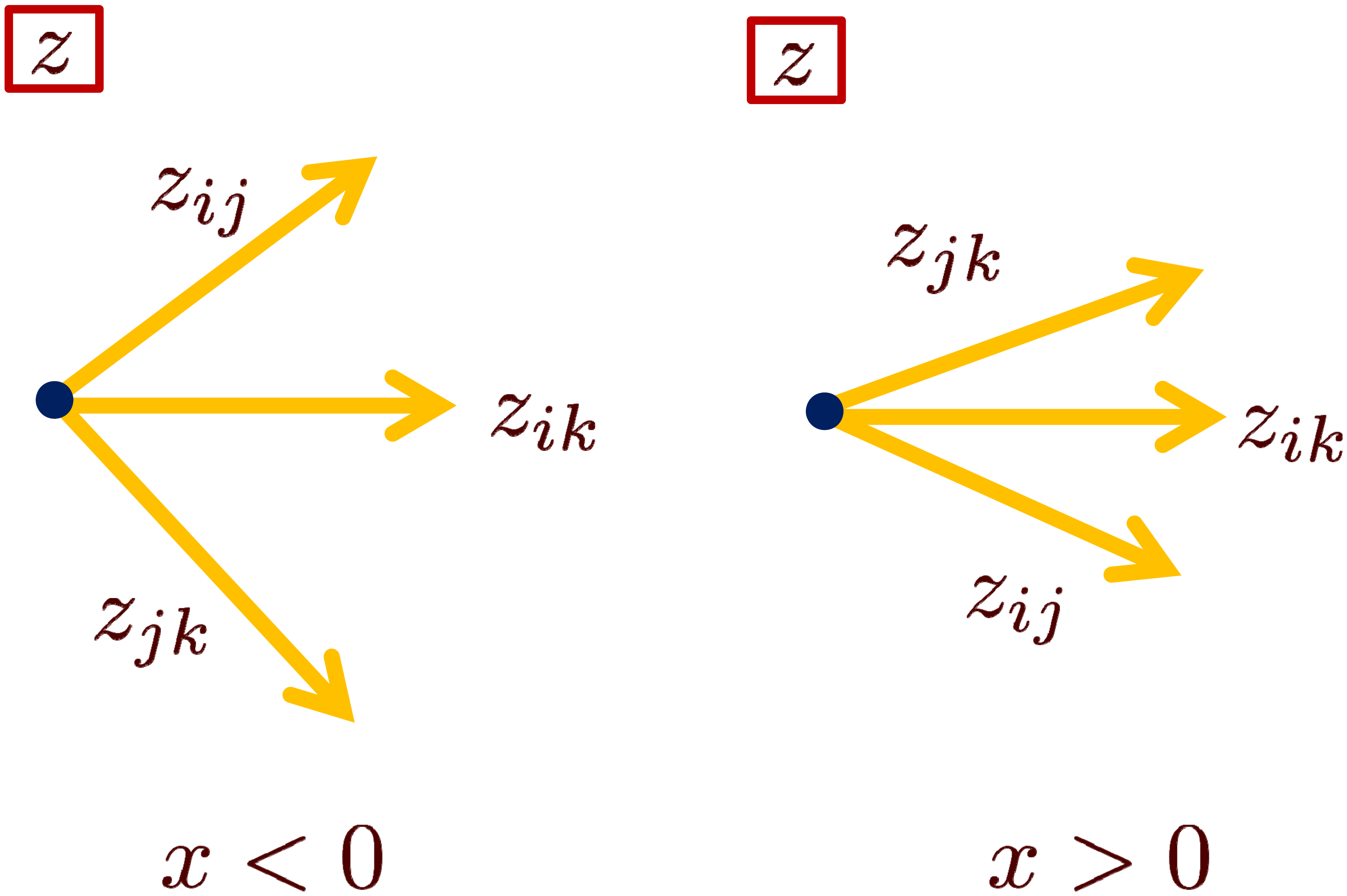
$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

To ***continuous*** $\wp$ we ***want*** to associate an A_∞ functor

$$\mathbb{F}[\wp] : \mathfrak{Br}(\mathcal{T}^{\mathrm{in}}) \to \mathfrak{Br}(\mathcal{T}^{\mathrm{out}})$$

etc.

You can't do that for arbitrary $\wp(x)$!



z
z_{ij}
z_{ik}
z_{jk}
$x < 0$
z
z_{jk}
z_{ik}
z_{ij}
$x > 0$

Categorified Cecotti-Vafa Wall-Crossing

We cannot construct $\mathbb{F}[\wp]$ keeping β and R_{ij} constant!

Existence of suitable Interfaces needed for flat transport of Brane categories implies that the web representation jumps discontinuously:

$$R_{ik}^{\mathrm{out}} - R_{ik}^{\mathrm{in}} =$$

$$(R_{ij}^{+} - R_{ij}^{-}) \otimes \left(R_{jk}^{+} - R_{jk}^{-}\right)$$

Categorified Wall-Crossing

In general: the existence of suitable wall-crossing Interfaces needed to construct a flat parallel transport $F[\wp]$ demands that for certain paths of vacuum weights the web representations (and interior amplitude) must jump discontinuously.

Moreover, the existence of wall-crossing interfaces constrains how these data must jump.

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Summary

1. Motivated by 1+1 QFT we constructed a web-based formalism

2. This naturally leads to L_∞ and A_∞ structures.

3. It gives a natural framework to discuss Brane categories and Interfaces and the 2-category structure

4. There is a notion of flat parallel transport of Brane categories. The existence of such a transport implies categorified wall-crossing formulae

Outlook

1. There are many interesting applications to Physical Mathematics: See Davide Gaiotto’s talk.

2. There are several interesting generalizations of the web-based formalism, not alluded to here.

3. The generalization of the categorified 2d-4d wall-crossing formula remains to be understood.