SAM LECTURES ON BPS WALL-CROSSING

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OUTLINE OF THE MINICOURSE

LECTURE 1: WALL-CROSSING FORMULAE

LECTURE 2: HYPERKÄHLER METRICS & THE KONTSEVICH-SOIBELMAN WCF

LECTURE 3: M5-BRANES, HITCHIN SYSTEMS, AND BPS STATES

LECTURE 4: THE SPECTRUM-GENERATING STOKES MATRIX: FLIPS, TWISTS, & POPS

1. INTRODUCTION & OVERVIEW

THIS MINICOURSE IS ABOUT THE BPS SPECTRUM OF STRING & FIELD THEORIES WITH $N=2, D=4$ SUSY.

THE BPS REPS OF $N=2$ ARE "SMALL" OR "RIGID" AND THEREFORE MORE AMENABLE TO STUDY. IN A SENSE THEY ARE "PIECEWISE CONSTANT" AS A FUNCTION OF PHYSICAL PARAMETERS IN THE THEORY.

THUS, INFORMATION OBTAINED AT WEAK COUPLING CAN BE USED TO OBTAIN INFO AT STRONG COUPLING.

SOME REASONS FOR STUDYING BPS STATES

1. EXACT RESULTS IN FIELD THEORY AND STRING THEORY.

2. MICROSCOPIC ORIGIN OF BLACK HOLE ENTROPY

3. OSV CONJECTURE

4. MATHEMATICS OF "DONALDSON-THOMAS INVARIANTS" AND STABILITY IN DERIVED CATEGORIES.

5. RELATIONS TO AUTOMORPHIC FORMS & ANALYTIC NUMBER THEORY

6. NEW INFINITE DIMENSIONAL ALGEBRAIC STRUCTURES ASSOCIATED TO C.Y. MANIFOLDS ("ALGEBRAS OF BPS STATES")

THE PHYSICAL PARAMETERS:

- COUPLINGS IN THE UV LAGRANGIAN
- CHOICE OF QUANTUM VACUUM

MASSLESS SCALAR FIELDS HAVE VEV'S CHARACTERISING THE MODULI OF VACUA IN THE PATH INTEGRAL ON $\mathbb{R}^4$

THESE ARE BC'S FOR THE FIELDS AT $\vec{x} \to \infty$.

IT TURNS OUT THAT BPS STATES ARE NOT COMPLETELY INDPT. OF PARAMETERS.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF).

BRIEF SUMMARY OF THE MINICOURSE

LECTURE 1:

- BASIC DEFINITIONS OF BPS INDEX
- REVIEW 3 KINDS OF BPS INDICES:

PRIMITIVE, SEMI-PRIMITIVE, MULTIPLICATIVE (KSWCF)

- LIGHTNING REVIEW OF SEIBERG-WITTEN THEORY

- COMPACTIFICATION ON A CIRCLE:

SEMIFLAT METRIC APPROXIMATION.

LECTURE 2:

- HYPERKÄHLER METRICS
- LEADING QUANTUM CORRECTION FROM A 1-LOOP COMPUTATION.
- QUANTUM EFFECTS OF MUTUALLY NONLOCAL PARTICLES
- KSWCF & CONTINUITY OF METRIC

LECTURE 3:

- REVIEW OF M5-BRANE COMPACTIFIED ON A RIEMANN SURFACE: RESULTING $d=4$, $\mathcal{N}=2$ GAUGE THEORIES.
- MAPPING TO HITCHIN SYSTEMS
- DESCRIPTION OF BPS STATES IN THE HITCHIN FRAMEWORK.

LECTURE 4:

- HITCHIN SYSTEMS & FLAT CONNECTIONS
- FOCK-GONCHAROV COORDINATES
- WKB-TRIANGULATIONS
- A NEW CONSTRUCTION OF THE TWISTOR COORDINATES
- HOW TO COMPUTE THE BPS SPECTRUM.

KEY REFERENCES

1. DENEF + MOORE, "SPLIT STATES, ENTROPY ENIGMAS, HOLES + HALOS," hep-th/0702146

2. KONTSEVICH + SOIBELMAN, "STABILITY STRUCTURES, ...", 0811.2435

3. GAIOTTO, MOORE, NEITZKE, "FOUR-DIMENSIONAL WALL-CROSSING, ..." arXiv: 0807.4723

4. GAIOTTO, MOORE, NEITZKE, "WALL-CROSSING, HITCHIN SYSTEMS, & WKB", TO APPEAR

2. GENERAL ASPECTS OF $\mathcal{N}=2, D=4$ WALL-CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$ WITH $\mathcal{N}=2$ SUPERPOINCARÉ SYMMETRY

LET $\mathcal{H}$ BE THE ONE-PARTICLE HILBERT SPACE.

AS A REPRESENTATION OF THE $\mathcal{N}=2$ SUPERPOINCARÉ ALGEBRA $\mathcal{A}$, $\mathcal{H}$ DEPENDS ON THE BOUNDARY VALUES OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED IN THE MODULI SPACE OF VACUA: $\mathcal{B}$

FOR $u \in \mathcal{B}$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{B}$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE A LATTICE Γ OF FLAVOR, ELECTRIC, & MAGNETIC CHARGES.

THE LATTICE OF ELEC. & MAG. CHARGES HAS RANK $2r$:

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \mathcal{A}$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$ OR $Z(\gamma;u)$.

RECALL THE $\mathcal{N}=2$, $D=4$ SUSY ALGEBRA

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

$$\mathcal{A}_0 = \big(\underbrace{\mathrm{Spin}(1,3)}_{\hat{M}_{\mu\nu}} \ltimes \underbrace{\mathbb{R}^4}_{\hat{P}_\mu}\big) \oplus u(2) \oplus \underbrace{\mathbb{R}}_{\hat{Z}}$$

$$\mathcal{A}_1 = \underbrace{\left[\mathrm{Spinor} \otimes \mathbb{C}^2\right]_{\mathbb{R}}}_{Q_{\alpha I},\ \bar{Q}^{\dot{\alpha} I}}$$

$$\{Q_{\alpha A}, \bar{Q}_{\dot{\beta}}{}^{B}\} = 2\hat{P}_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta_A{}^B$$

$$\{Q_{\alpha A}, Q_{\beta B}\} = 2\hat{Z}\, \epsilon_{\alpha\beta} \epsilon_{AB}$$

CONVENTIONS: BAGGER & WESS

$$\bar{Q}_{\dot{\alpha} A} = \left(Q_\alpha{}^A\right)^\dagger$$

LEMMA: (THE BOGOMOLNYI BOUND)

ON A SUBSPACE $\mathcal{H}_z$ WHERE $\hat{Z} = z$, THE ENERGY IS BOUNDED BY

$$E \geq |z|$$

PROOF: $D=4, \mathcal{N}=2$ IS A REDUCTION OF $D=6, \mathcal{N}=1$ ALGEBRA.

$$Spin(1,3) \times Spin(2) \hookrightarrow Spin(1,5)$$

$$2_{1/2} \oplus 2^*_{-1/2} \longleftarrow 4$$

D=6 ALGEBRA HAS SU(2) R-SYMMETRY

$$Q^{(6)} \in (4,2)^+$$

+: Symplectic Majorana

$$\left(Q^{(6)}_{rA}\right)^\dagger = B_r{}^s \, \varepsilon^{AB} \, Q^{(6)}_{sB}$$

chiral

$$\left\{Q^{(6)}_{rA}, Q^{(6)}_{sB}\right\} = -2i \left(C\Gamma^M\right)^+_{rs} \hat{P}_M \, \varepsilon_{AB}$$

$$B\Gamma^M B^{-1} = (\Gamma^M)^*$$

$$C\Gamma^M C^{-1} = -(\Gamma^M)^{tr}$$

USING A PARTICULAR REPRESENTATION FOR Γ^M WE HAVE:

$$\begin{pmatrix} Q^{(6)}_{1A} \\ Q^{(6)}_{2A} \\ Q^{(6)}_{3A} \\ Q^{(6)}_{4A} \end{pmatrix} = \begin{pmatrix} Q^{(4)}_{2A} \\ -Q^{(4)}_{1A} \\ -\bar{Q}^{(4)}_{\dot{1}A} \\ \bar{Q}^{(4)}_{\dot{2}A} \end{pmatrix}$$

WITH THIS IDENTIFICATION WE RECOVER THE $d=4$, $\mathcal{N}=2$ ALGEBRA WITH:

$$\hat{Z} = -i\left(\hat{P}_4 - i\hat{P}_5\right)$$

But for unitary physical irreps

$$(M^{(6)})^2 = E^2 - \vec{P}^2 - |Z|^2 \geq 0$$

$$\Rightarrow \quad E^2 \geq \vec{P}^2 + |Z|^2 \geq |Z|^2 \quad \blacksquare$$

DEF'N: $\mathcal{H}_{BPS}$ IS THE SUBSPACE OF $\mathcal{H}$ WHERE $E = |Z|$.

<u>Remark</u>: Equality requires $\vec{P} = 0$ and $M^{(6)} = 0$

$M^{(6)} = 0 \Rightarrow 1/2$ THE SUSY OPERATORS ARE REPRESENTED AS ZERO ($\mathcal{N}=1$ IS PRESERVED) $\arg Z$ DETERMINES <u>WHICH</u> $\mathcal{N}=1$ SUBALGEBRA.

BOUNDSTATES

- SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS
  - Cecotti, Fendley, Intriligator, Vafa - "A NEW SUSY INDEX" hepth/9204102
  - Seiberg + Witten, "Electric-magnetic duality, monopole condensation, ..." hep-th/9407087
- IN ALL EXAMPLES I KNOW

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$

(DOES IT FOLLOW FROM GENERAL PRINCIPLES?)

- SO, BY THE TRIANGLE INEQUALITY

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)| \right) \leq 0$$

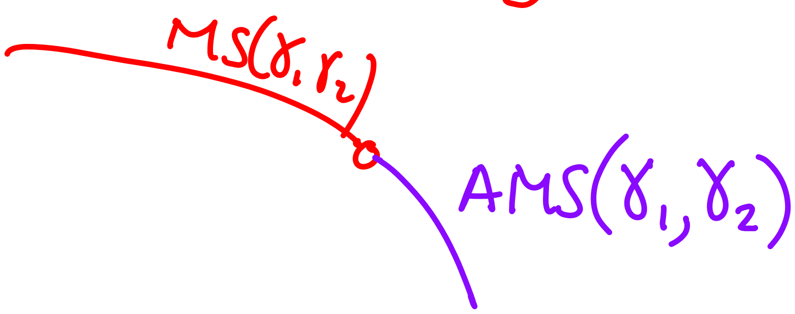
WE ARE DISCUSSING DECAY WHERE WE DON'T TRY TO DISSOCIATE THE BOUNDSTATE BY PUTTING ENERGY IN.

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY WHERE THE BINDING ENERGY IS ZERO:

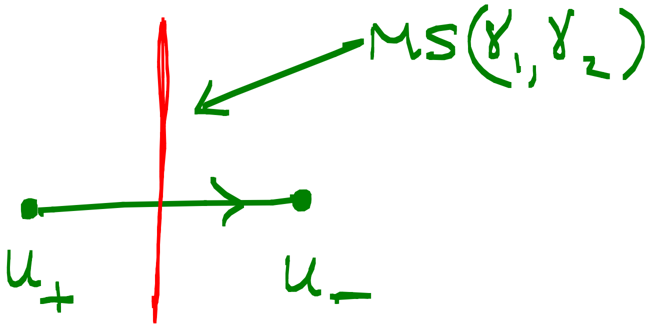
$$\arg Z_{\gamma_1}(u) = \arg Z_{\gamma_2}(u)$$

$$MS(\gamma_1,\gamma_2) := \{ u \mid Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

- Real Codimension One
- Ends on boundary or on a zero



SO A BOUNDSTATE OF PARTICLES WITH CHARGES γ_1, γ_2 <u>MIGHT</u> DECAY



WE WANT TO SAY <u>HOW MANY STATES DECAY</u>.

$$\dim \mathcal{H}^{BPS}_{\gamma, u_+} - \dim \mathcal{H}^{BPS}_{\gamma, u_-} = ?$$

$\Longrightarrow$ <u>WALL-CROSSING FORMULA</u>

MORE ABOUT SHORT MULTIPLETS

When $E = |Z|$ A LINEAR COMBINATION OF SUPERCHARGES

$$\{Q, Q^\dagger\} = 0$$

MUST BE REPRESENTED BY ZERO.

REMAINING GENERATORS FORM A CLIFFORD ALGEBRA..

THE CLIFFORD VACUUM MIGHT HAVE SPIN.

FOR MORE DETAILS: BAGGER & WESS, Ch. 1.

AS A SPIN(3) REP, TAKES THE FORM:

$$\mathcal{H}^{BPS}_{\gamma,u} = \mathcal{H}_{\frac{1}{2}HM} \otimes \tilde{\mathcal{H}}^{BPS}_{\gamma,u}$$

$$\mathcal{H}_{\frac{1}{2}HM} = 2(\underline{0}) + (\underline{\tfrac{1}{2}})$$

IRREP OF SUSY: $\tilde{\mathcal{H}}^{BPS}_{\gamma,u} = (j) =$ IRREP OF SPIN(3)

$j=0$: HM's 4 real scalars. $\Rightarrow$ HM moduli

$j=1/2$: VM's 2 real scalars $\Rightarrow$ VM moduli

- HM + VM PARTICLES CAN BECOME MASSLESS IN PAIRS

- THIS CAN EVEN HAPPEN AS A FUNCTION OF HM MODULI

(HARVEY & MOORE, hep-th/9510182, sec. 3.1)

DEFINE THE BPS INDEX

$$\Omega(\gamma; u) := -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}} (2J_3)^2 (-1)^{2J_3}$$

$$= \mathrm{Tr}_{\widetilde{\mathcal{H}}^{BPS}_{\gamma,u}} (-1)^{2\widetilde{J}_3}$$

EXERCISE: SHOW THAT

$\frac{1}{2}$HM: $\Omega = +1$ VM $= \Omega = -2$

SO THE INDEX COUNTS

Hypermultiplets − Vectormultiplets

- THE BPS INDEX IS PIECEWISE CONSTANT, BUT CAN CHANGE ACROSS WALLS OF MARGINAL STABILITY

- IT IS SOMETIMES USEFUL TO CONSIDER REFINED INVARIANTS

$$\Omega(\gamma; u; x, y) = \mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}} (-y)^{2J_3} (+x)^{R}$$

R: $U(1)$ R-CHARGE OF PRESERVED $N=1$

× DIACONESCU-MOORE, 0706.3193

× DIMOFTE-GUKOV, 0904.1420

<u>Exercise</u>: Coupling just to the spin

$$\Omega(\gamma; u; y) = \mathrm{Tr}(-y)^{2J_3}$$

Compute the order of the zero at $y=1$.

SYMPLECTIC & POISSON STRUCTURE ON Γ

DIRAC QUANTIZATION $\Longrightarrow$ LATTICE OF ELECTRIC + MAGNETIC CHARGES IS SYMPLECTIC

$$\langle \cdot, \cdot \rangle : \quad \Gamma \times \Gamma \longrightarrow \mathbb{Z}$$

BILINEAR (AS A $\mathbb{Z}$-MODULE), ANTISYMMETRIC, NONDEGENERATE.

"DIRAC–SCHWINGER–ZWANZIGER"

<u>EXAMPLE</u>: $r=1$, $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$

$$\langle (e_1, m_1), (e_2, m_2) \rangle = e_1 m_2 - e_2 m_1$$

A PARTICLE WITH CHARGES $(e,m) \in \Gamma$ IS A "DYON."

THE ELECTROMAGNETIC FIELD IN THE PRESENCE OF TWO DYONS HAS SPIN

$$\vec{J} = \frac{1}{2}(e_1 m_2 - e_2 m_1)\, \hat{r}_{12}$$

REMARK:

IT SOMETIMES HAPPENS THAT THERE ARE "FLAVOR SYMMETRIES" IN ADDITION TO GAUGE SYMMETRIES. THEN $\langle \cdot , \cdot \rangle$ IS DEGENERATE ON A SUBLATTICE

$$\Gamma_{\text{flavor}} = \text{RADICAL OF } \langle \cdot , \cdot \rangle$$
$$= \{ \gamma_f \mid \langle \gamma_f , \gamma \rangle = 0 , \forall \gamma \in \Gamma \}$$

THEN WE HAVE:

$$0 \longrightarrow \Gamma_{\text{flavor}} \longrightarrow \Gamma \longrightarrow \Gamma_{\text{gauge}} \longrightarrow 0$$

$\uparrow$

SYMPLECTIC

IT IS OFTEN USEFUL TO THINK OF FLAVOR SYMM'S AS WEAKLY GAUGED. IF $\tilde{\Gamma}$ IS SYMPLECTIC AND $\Gamma_e \subset \tilde{\Gamma}$ AN ISOTROPIC SUBLATTICE $\Gamma_e^{\perp}$ WILL HAVE A DEGENERATE FORM, THIS IS OUR Γ ABOVE.

CONVERSELY, GIVEN Γ AS ABOVE AND A SPLITTING $\Gamma \cong \Gamma_{flav} \oplus \Gamma_{gauge}$

$$\tilde{\Gamma} = \Gamma_{flav} \oplus \Gamma_{flav}^{*} \oplus \Gamma_{gauge}$$

WILL BE SYMPLECTIC.

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3. PRIMITIVE & SEMI-PRIMITIVE WCF

DENEF & MOORE GAVE FORMULAE FOR $\Delta\Omega$ FOR DECAYS $\gamma \to \gamma_1 + \gamma_2$ WHERE AT LEAST ONE OF γ_1, γ_2 ARE PRIMITIVE.

THE DERIVATION IS BASED ON DENEF'S MULTICENTERED SOLUTIONS OF $\mathcal{N}=2$ SUGRA. (THERE IS ALSO A MICROSCOPIC ARGUMENT BASED ON THE QUIVER QUANTUM MECHANICS ASSOCIATED WITH LIGHT MODES WHICH APPEAR NEAR WMS.)

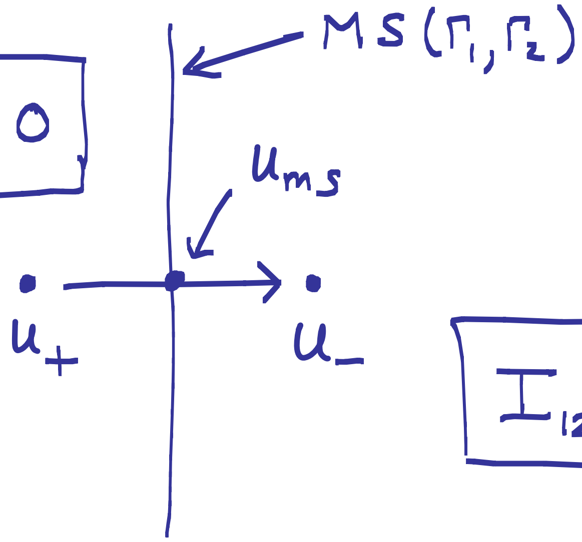
DENEF'S MULTI-CENTERED SOLUTIONS HAVE PARAMETERS $\vec{x}_i \in \mathbb{R}^3$, THE CENTERS. NEAR $\vec{x}_i$ THE SOLUTION ASYMPTOTES TO A SINGLE-CENTERED BPS SPHERICALLY SYMMETRIC B.H. OF CHARGE γ_i.

PRIMITIVE WALL-CROSSING FORMULA:

Γ HAS SYMPLECTIC FORM $\langle \cdot , \cdot \rangle$

LET $I_{12} := \langle \gamma_1 , \gamma_2 \rangle$

$$\boxed{I_{12}\,\mathrm{Im}(Z_1 Z_2^*) > 0}$$



$$\boxed{I_{12}\,\mathrm{Im}(Z_1 Z_2^*) < 0}$$

γ_1, γ_2 PRIMITIVE, u_{ms} GENERIC $\Longrightarrow$

$$\boxed{\mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\gamma_1 ; u_{ms}) \otimes \mathcal{H}(\gamma_2 ; u_{ms})}$$

$$J_{12} = \tfrac{1}{2}\left(|I_{12}| - 1 \right)$$

$$\Delta\Omega = (-1)^{I_{12}-1} |I_{12}| \, \Omega(\Gamma_1, t_{ms}) \Omega(\Gamma_2, t_{ms})$$

DERIVATION OF PRIMITIVE WCF:

CONSIDER BOUNDSTATE OF TWO PRIMITIVE CHARGES:

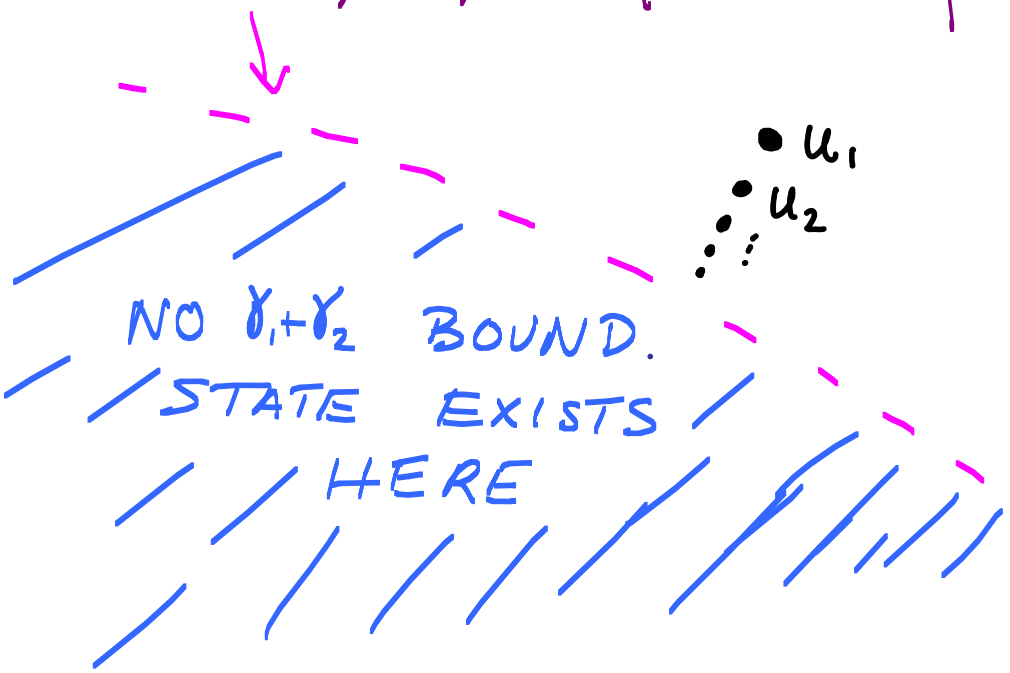
$$\gamma_1 \bullet \xrightarrow{\vec{R}} \bullet\, \gamma_2$$

USING EXPLICIT SOLUTIONS TO SUGRA DENEF DERIVED A KEY FORMULA FOR THE BOUNDSTATE RADIUS OF TWO "CONSTITUENT" BLACK HOLES:

$$\boxed{R_{12} = |\vec{x}_1 - \vec{x}_2| = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{\operatorname{Im}\left(Z_{\gamma_1}(u)\, \overline{Z_{\gamma_2}(u)}\right)}}$$

- NOTE: $\langle \gamma_1, \gamma_2 \rangle \operatorname{Im}(Z_1 \bar{Z}_2) > 0$
- NOTE THAT BY CHANGING u WE CAN MAKE $\operatorname{Im}(Z_1 \bar{Z}_2)\big|_u \to 0$ WHILE $|Z_1 + Z_2|_u \neq 0$

$$MS(\gamma_1,\gamma_2) := \left\{ u \in \mathcal{B} \;\middle|\; \frac{Z_1}{Z_2} \in \mathbb{R}_+ \right\}$$

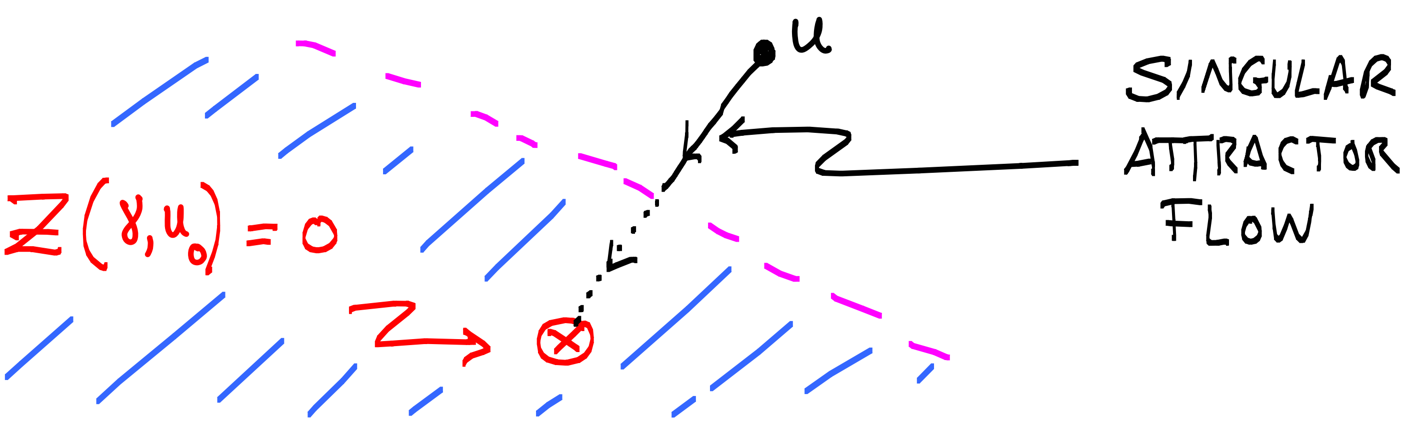


CHANGE BC's @ $r=\infty \Rightarrow$

$R_{12} \longrightarrow \infty$

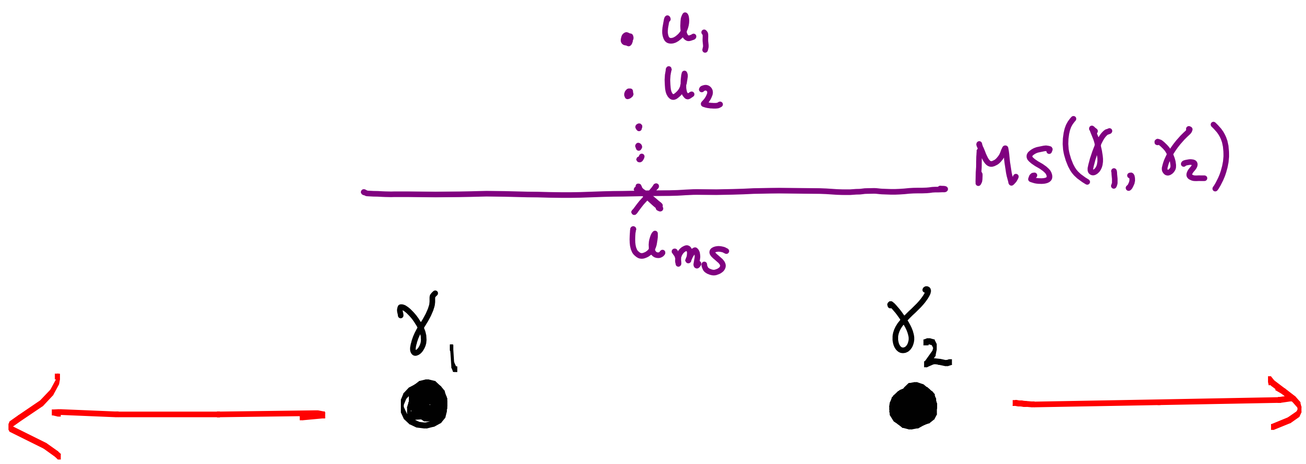
FOR ATTRACTOR EXPERTS:

IF $Z(\gamma; u_0) = 0$ THEN u_0 IS IN THE BLUE REGION.



MACROSCOPIC ARGUMENT FOR WCF:

$$R_{12} = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\operatorname{Im}(Z_1 \bar{Z}_2)_\infty}$$



ELECTROMAGNETIC FIELD OF TWO DYONS HAS SPIN:

$$J_{12} = \frac{1}{2}\left(|\langle \gamma_1, \gamma_2 \rangle| - 1 \right)$$

quantum correction

LOCALITY $\Longrightarrow$ FOR γ_1, γ_2 PRIMITIVE:

STATES LOST FROM $\mathcal{H}(\gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\gamma_1; t_{MS}) \otimes \mathcal{H}(\gamma_2; t_{MS})$$

THE SEMI-PRIMITIVE WCF

- NOTE THAT $MS(\gamma_1, \gamma_2) = MS(N_1\gamma_1, N_2\gamma_2)$ $N_1, N_2 \in \mathbb{Z}_{>0}$. THUS, ACROSS THE SAME WALL MANY KINDS OF BOUNDSTATES CAN DECAY.

- LET US FOCUS ON BOUNDSTATES WITH ONE CONSTITUENT OF CHARGE γ_1 AND N CONSTITUENTS OF CHARGES PROPORTIONAL TO γ_2

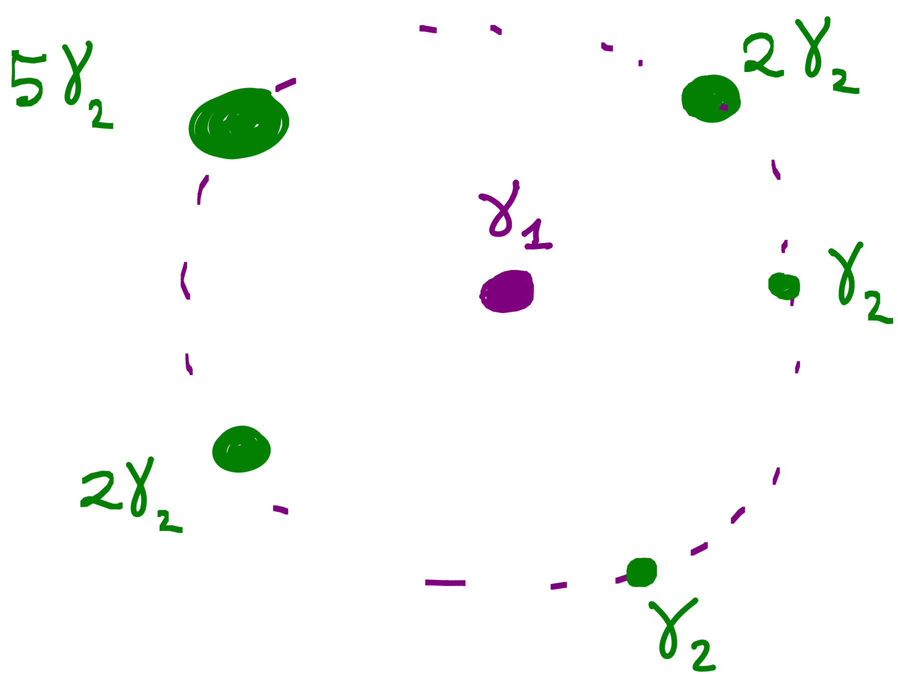
$$\gamma_j = \lambda_j \gamma_2 \qquad \lambda_j > 0 :$$

IT TURNS OUT THAT THE $N=2$ SUGRA SOLUTIONS SATISFY A SIMPLE CONSTRAINT:

$$|\vec{x}_j - \vec{x}_1| = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \bar{Z}_2)}$$

CONVERSELY, ANY SUCH COLLECTION OF CENTERS GIVES A SOLUTION.

THESE ARE CALLED <u>HALO SOLUTIONS</u>



- THE PARTICLES IN THE HALO ARE MUTUALLY BPS – HENCE NON-INTERACTING. THERE IS AGAIN A SPIN-DEGREE OF FREEDOM FROM THE MUTUALLY NONLOCAL SOURCE IN CORE + HALO.

- QUANTIZATION OF SUCH PARTICLES GIVES FINITE-DIMENSIONAL VECTOR SPACES:

$$\left(J_{\gamma_1, k\gamma_2}\right) \otimes \mathcal{H}^{BPS}_{k\gamma_2, u}$$

- SINCE WE CAN ADD SUCH PARTICLES FREELY AND THEY ARE NONINTERACTING THESE "HALO-STATES" COMPRISE A FOCK SPACE OF THE FORM

$$\mathcal{H}_{\gamma_1, u} \otimes \bigotimes_{k=1}^{\infty} \widetilde{\mathcal{F}}\Big(\underbrace{\left(J_{\gamma_1, k\gamma_2}\right) \otimes \mathcal{H}^{BPS}_{k\gamma_2, u}}_{\text{CREATION OPERATORS}} \Big)$$

NOW, WHEN u CROSSES THE WALL $MS(\gamma_1, \gamma_2)$ <u>ALL</u> THE HALO STATES DISAPPEAR — THE ENTIRE FOCK SPACE DISAPPEARS FROM THE SPECTRUM.

AT A GENERIC POINT ON THE WALL THERE WILL BE NO OTHER DECAYS, SO:

$$\Omega(\gamma_1) + \sum_{N\geq 1} \Delta\Omega(\gamma \to \gamma_1 + N\gamma_2) q^N$$

$$= \Omega(\gamma_1) \prod_{k>0} \left(1 - (-1)^{\langle \gamma_1, k\gamma_2 \rangle} q^k\right)^{|\langle \gamma_1, k\gamma_2 \rangle| \Omega(k\gamma_2)}$$

UNFORTUNATELY...

THESE METHODS ARE DIFFICULT TO USE WHEN <u>BOTH</u> γ_1, γ_2 ARE NON-PRIMITIVE

4. THE KONTSEVICH–SOIBELMAN FORMULA

KONTSEVICH & SOIBELMAN PROPOSED

A REMARKABLE W.C.F. FOR

AN INDEX $\Omega^{DT}(\gamma;u)$,

"GENERALIZED DONALDSON–THOMAS INVARIANT OF A CALABI-YAU 3-FOLD"

WE EXPECT THAT

$$\Omega(\gamma;u) = \Omega^{Ph}(\gamma;u) = \Omega^{DT}(\gamma;u)$$

SO THE KS WCF APPLIES TO PHYSICAL BPS DEGENERACIES.

THEIR FORMULA APPLIES TO ALL DECAYS $\gamma \to \gamma_1 + \gamma_2$.

IN THIS COURSE WE PROVE THE KS WCF FOR $\Omega^{Ph}(\gamma;u)$ IN $\mathcal{N}=2$ FIELD THEORIES

3 PIECES OF DATA & 3 INGREDIENTS

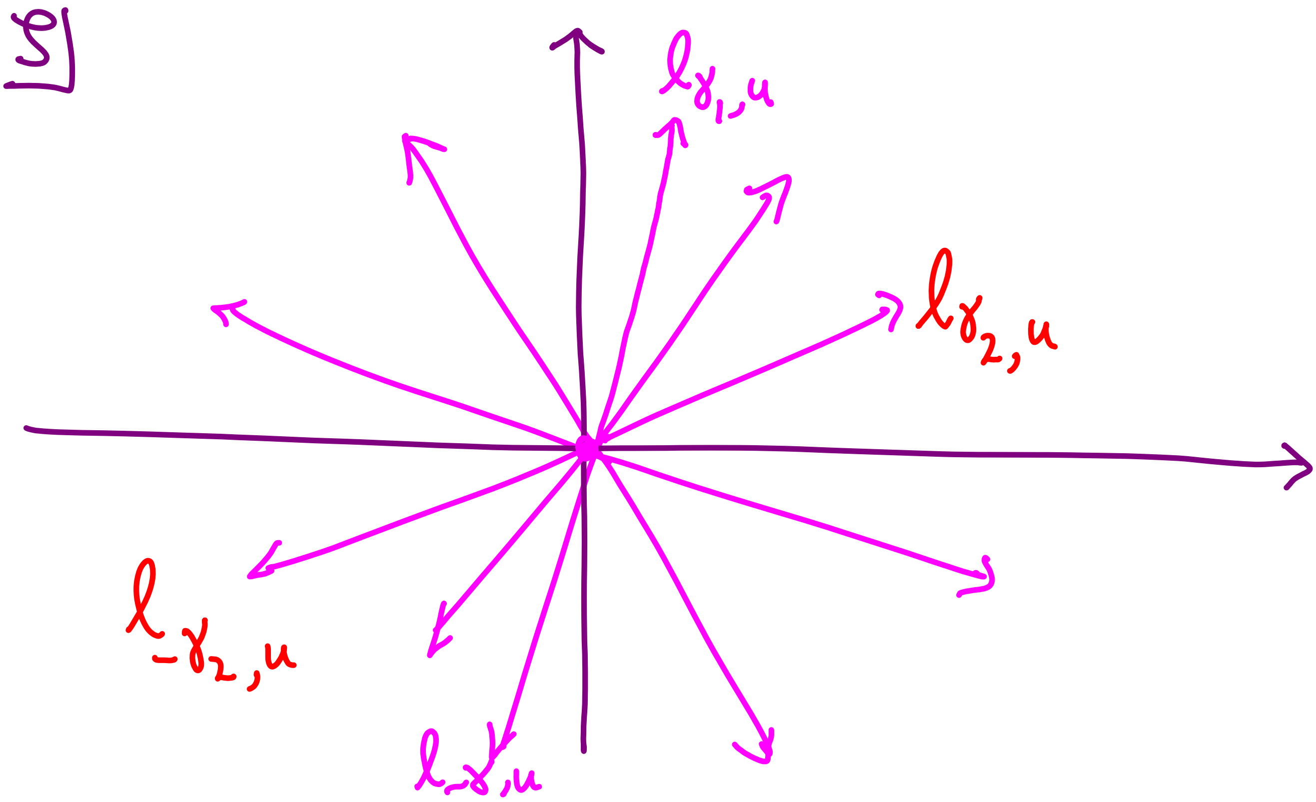
DATA:

1. SYMPLECTIC LATTICE Γ
2. CENTRAL CHARGES $Z_\gamma(u)$, $u \in \mathcal{B}$
3. PIECEWISE CONSTANT $\Omega(\gamma; u) \in \mathbb{Z}$

FIRST INGREDIENT: BPS RAYS:

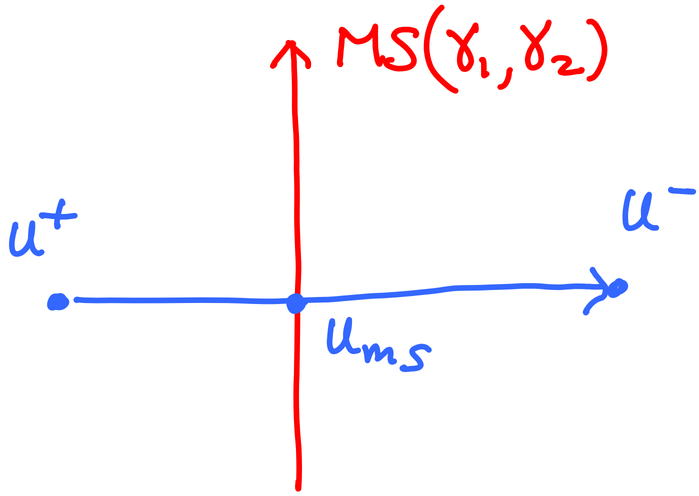
FOR $u \in \mathcal{B}$, $\gamma \in \Gamma$

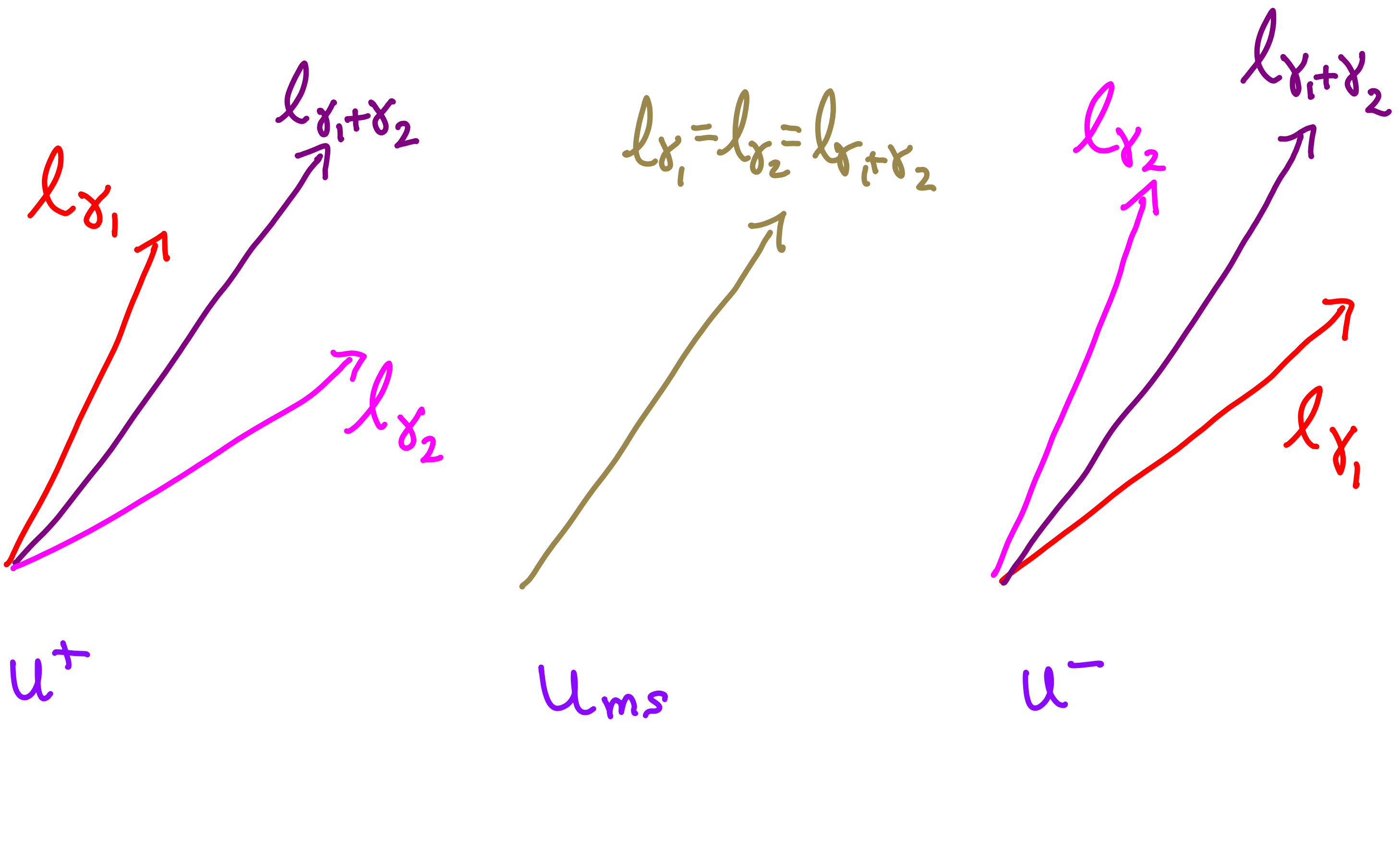
$$\ell_{\gamma,u} := Z_\gamma(u)\mathbb{R}_- = \left\{ \zeta \,\middle|\, \zeta / Z_\gamma(u) \in \mathbb{R}_- \right\}$$



AS u VARIES THE SLOPES OF THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1, \gamma_2)$ BPS RAYS WILL COALESCE





SECOND INGREDIENT: A SYMPLECTIC TORUS:

- INTRODUCE THE COMPLEX TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{2r}$$

CHOOSING A BASIS γ_i FOR $\Gamma \Longrightarrow$

$$\left(e^{\theta_1}, \cdots, e^{\theta_{2r}}\right) \in T, \quad \theta_i \in \mathbb{C}$$

$\gamma \in \Gamma \Longrightarrow$ FUNCTION $X_\gamma : T \to \mathbb{C}^*$

"HOLOMORPHIC FOURIER MODES"

EXPLICITLY: $\gamma = n^i \gamma_i \Longrightarrow$

$$X_\gamma = \exp\left(\sum_{i=1}^{2r} n^i \theta_i\right)$$

N.B.: $X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$

- HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^T := \frac{1}{2}\epsilon^{ij}\frac{dX_{\gamma_i}}{X_{\gamma_i}}\wedge\frac{dX_{\gamma_j}}{X_{\gamma_j}} \qquad \epsilon_{ij} = \langle\gamma_i,\gamma_j\rangle$$

Exercise: Compute the Poisson bracket

$$\{X_\gamma, X_{\gamma'}\} = \langle\gamma,\gamma'\rangle X_\gamma X_{\gamma'}$$
$$= \langle\gamma,\gamma'\rangle X_{\gamma+\gamma'}$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A SYMPLECTOMORPHISM K_γ:

$$K_\gamma : X_{\gamma'} \longrightarrow X_{\gamma'}\left(1-\sigma(\gamma)X_\gamma\right)^{\langle\gamma',\gamma\rangle}$$

$$X_{\gamma'} \to X_{\gamma'}\exp\left[\langle\gamma,\gamma'\rangle\log\left(1-\sigma(\gamma)X_\gamma\right)\right]$$

$$\sigma(\gamma) = \pm 1$$

LIE ALGEBRA VERSION

K & S INTRODUCE A LIE ALGEBRA

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$$

DEFINE A GROUP ELEMENT

$$U_\gamma := \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

AND WORK WITH U_γ INSTEAD OF K_γ

CHOOSE A QUADRATIC REFINEMENT

$$\frac{\sigma(\gamma_1+\gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle\gamma_1,\gamma_2\rangle}$$

$\sigma(\gamma) e_\gamma$ GENERATE THE LIE ALGEBRA OF SYMPLECTIC VECTOR FIELDS.

Exercise: Show $K_\gamma = \exp\left(\sum \frac{\sigma(n\gamma_0)}{n^2} \{X_{n\gamma_0}, \cdot\} \right)$

EXAMPLE

$$r = 1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b), (a',b') \rangle = ab' - a'b$$

$$T \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \qquad y = X_{0,1}$$

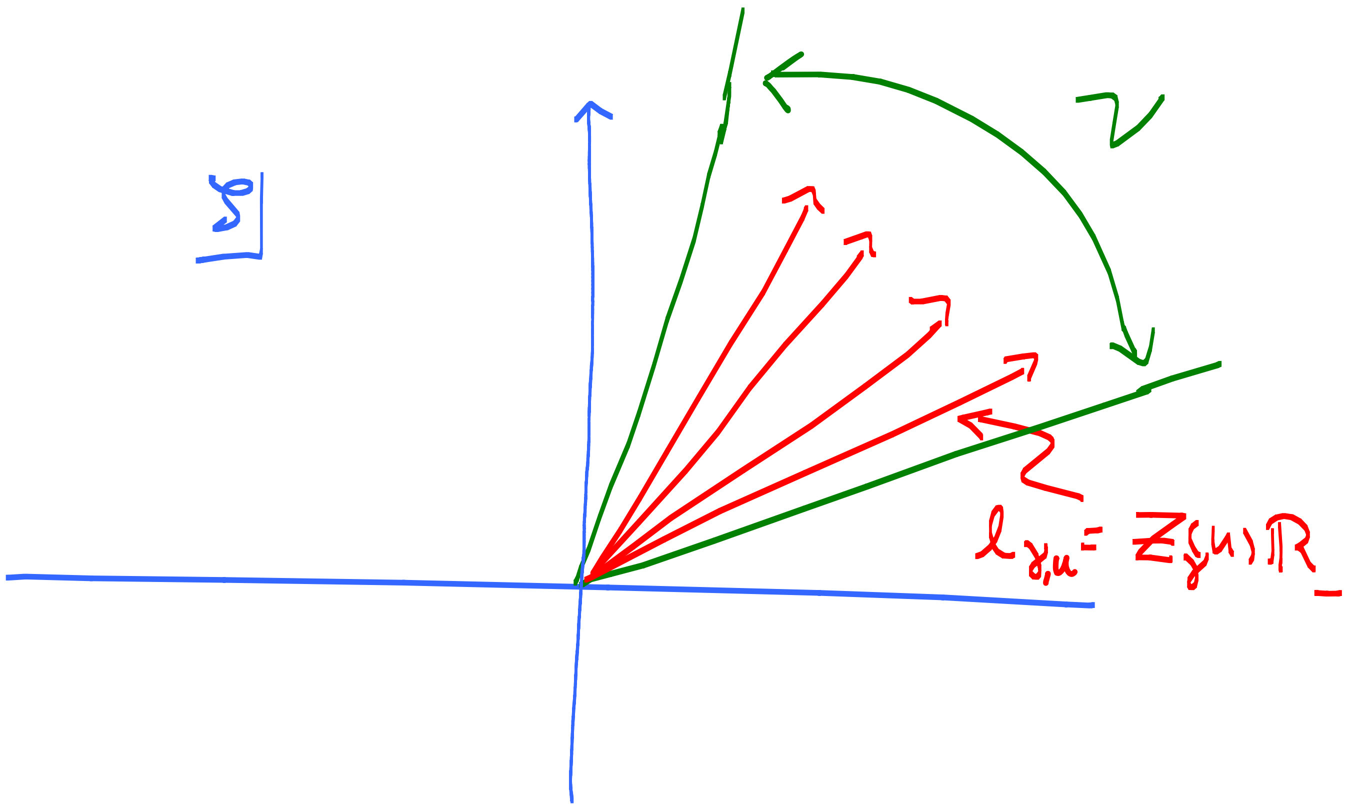
$$\omega^T = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$K_{a,b} : \begin{cases} x \longrightarrow x\left(1 - (-1)^{ab} x^a y^b\right)^b \\ y \longrightarrow y\left(1 - (-1)^{ab} x^a y^b\right)^{-a} \end{cases}$$

Exercise: PROVE THE "PENTAGON IDENTITY"

$$K_{(1,0)} K_{(0,1)} = K_{(0,1)} K_{(1,1)} K_{(1,0)}$$

THIRD INGREDIENT: CONVEX CONE $\mathcal{V}$



FOR EACH BPS RAY DEFINE

$$S_\gamma := \prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma';u)}$$

AND THEN DEFINE:

$$A_{\mathcal{V}} := \overrightarrow{\prod_{\ell_{\gamma,u} \subset \mathcal{V}}} S_\gamma$$

$$\mathcal{A}_\nu := \prod_{\ell_\gamma \subset \nu}^{\rightarrow} S_\gamma = \prod_{-Z_\gamma \in \nu}^{\rightarrow} K_\gamma^{\Omega(\gamma;u)}$$

THE PRODUCT IS TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

$\mathcal{A}_\nu$ DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma;u)$ DEPEND ON u ...

DEFINITION: WE SAY THAT $\Omega(\gamma;u)$ SATISFY THE KS WCF IF,

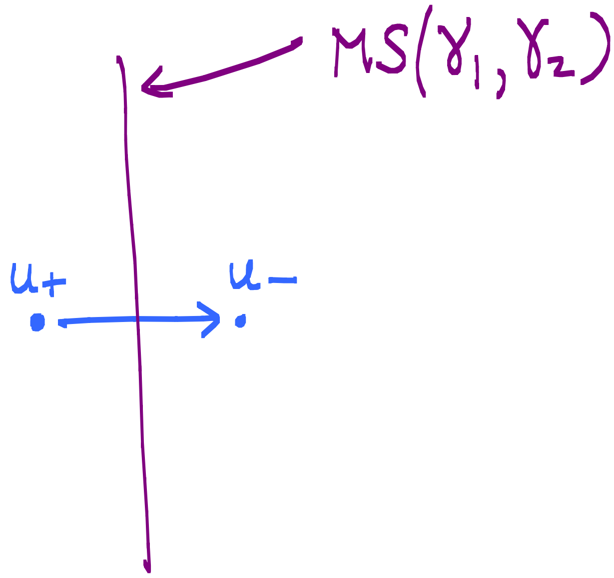
NEVERTHELESS,

$$A_{\nu} = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} K_{\gamma}^{\Omega(\gamma;u)}$$

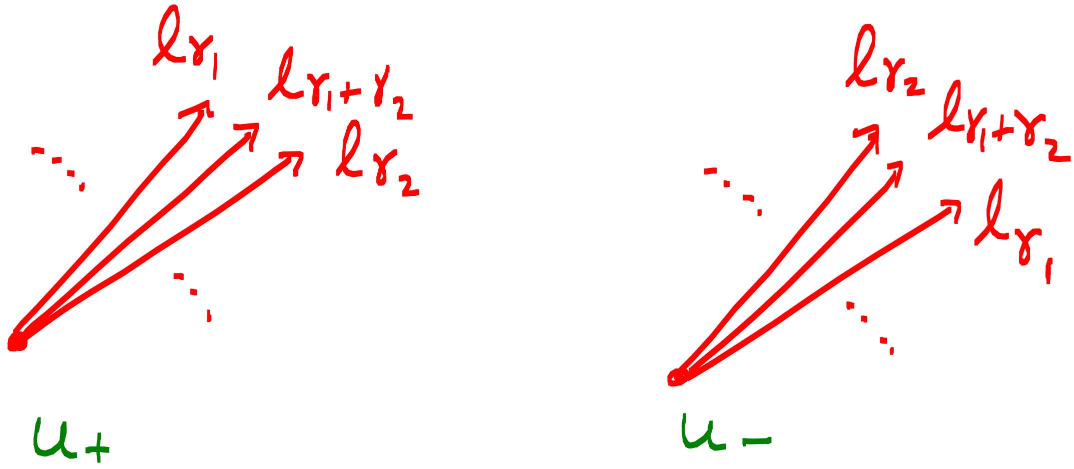
IS CONSTANT IN u AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR ν.

THIS IS A WALL-CROSSING FORMULA ...

AS u CROSSES A WALL



$$\ell_\gamma = Z_\gamma(u) \cdot \mathbb{R}_- \qquad \text{ROTATES}$$



$\Longrightarrow$ EXCHANGE ORDER IN

$$A_\nu = \prod_{\ell_\gamma \subset \nu}^{\curvearrowright} K_\gamma^{\Omega(\gamma;u)}$$

$\Omega(\gamma;u)$ MAKES A COMPENSATING CHANGE

UNIQUENESS LEMMA:

SUPPOSE THERE IS A POSITIVE BASIS γ_i FOR Γ, i.e. $\Omega \neq 0 \Rightarrow$

$\gamma = \sum n_i \gamma_i$ with all $n_i \geq 0$ or all $n_i \leq 0$

THEN FOR $[\vartheta_-, \vartheta_+]$ = A SECTOR OF OPENING $\leq \pi$, ANY ELEMENT IN THE GROUP GENERATED BY K_γ HAS A UNIQUE DECOMPOSITION:

$$g = \prod_{\vartheta_- < -\arg Z_\gamma(u) < \vartheta_+} K_\gamma^{\Omega(\gamma;u)}$$

PROOF: DEFINE A DEGREE $|\gamma| = \sum n_i$. THEN K_γ ARE "UPPER TRIANGULAR" SO EXP. MAP IS 1-1 & ONTO ∎

THUS, GIVEN Ω_- ON ONE SIDE OF THE WALL, Ω_+ ON THE OTHER SIDE ARE UNIQUELY DETERMINED.

THIS IS THUS A COMPLETE SOLUTION OF THE WALL-CROSSING PROBLEM.

[REMARK: IT DOES LEAVE OPEN THE QUESTION OF THE FULL DESCRIPTION OF THE BPS HILBERT SPACES ON BOTH SIDES OF THE WALL....

"CATEGORIFY THE KSWCF"]

RECOVERING THE PRIMITIVE WCF

AT A GENERIC POINT $u \in MS(\gamma_1, \gamma_2)$

$Z(\gamma; u) \parallel Z_1, Z_2 \Longrightarrow$

$$\gamma = \gamma_{a,b} := a\gamma_1 + b\gamma_2$$

(γ_1, γ_2 primitive)

FOR SMALL CONE ANGLE ONLY THE LIE SUBALGEBRA $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ CONTRIBUTES:

$$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)I_{12}} (ad-bc) I_{12} e_{a+c,b+d}$$

DEFINE:

$$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma,mb}}{m^2} \right)$$

$$\prod_{\substack{a/b \uparrow \\ a \geq 0 \\ b \geq 0}} U_{a,b}^{\Omega^-(\Gamma_{a,b})} = \prod_{\substack{a/b \downarrow \\ a \geq 0 \\ b \geq 0}} U_{a,b}^{\Omega^+(\Gamma_{a,b})}$$

LIE ALGEBRA IS FILTERED $\Longrightarrow$

CAN RESTRICT TO

HEISENBERG ALGEBRA
$$\left\{ \begin{array}{l} [e_{0,1}, e_{1,0}] = (-1)^{I_{12}-1} I_{12}\, e_{1,1} \\ e_{1,1} \ \underline{\text{CENTRAL}} \end{array} \right.$$

$$U_{0,1}^{\Omega^-(\Gamma_1)}\, U_{1,1}^{\Omega^-(\Gamma_1+\Gamma_2)}\, U_{1,0}^{\Omega^-(\Gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\Gamma_2)}\, U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)}\, U_{0,1}^{\Omega^+(\Gamma_1)}$$

$$\boxed{U_{01}\, U_{10} = U_{1,1}^{\pm I_{12}} \cdot U_{10}\, U_{01}} \Longrightarrow$$

$$U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)-\Omega^-(\Gamma_1+\Gamma_2)} = U_{0,1}^{\Omega(\Gamma_1)}\, U_{1,0}^{\Omega(\Gamma_2)}\, U_{0,1}^{-\Omega(\Gamma_1)}\, U_{1,0}^{-\Omega(\Gamma_2)}$$

$$= U_{1,1}^{I_{12}\,\Omega(\Gamma_1)\,\Omega(\Gamma_2)}$$

PRIMITIVE W.C. FORMULA!

REMARK: WE DID NOT NEED A SYMPLECTIC STRUCTURE ON Γ; A POISSON STRUCTURE WOULD SUFFICE:

$$\{X_{\gamma_1}, X_{\gamma_2}\} = \langle \gamma_1, \gamma_2 \rangle X_{\gamma_1+\gamma_2}$$

AND K_γ IS A POISSON-MORPHISM.

5. LIGHTNING SUMMARY OF SEIBERG-WITTEN THEORY

G - COMPACT S.S. GAUGE GROUP, RANK = r

$\Longrightarrow$ D = 4, N = 2 FIELD THEORY

(INCLUDE HM's LATER)

$$\mathcal{B} = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r : \quad \begin{aligned} u_2 &= \langle \mathrm{Tr}\, \Phi^2 \rangle \\ u_3 &= \langle \mathrm{Tr}\, \Phi^3 \rangle \\ &\vdots \end{aligned}$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

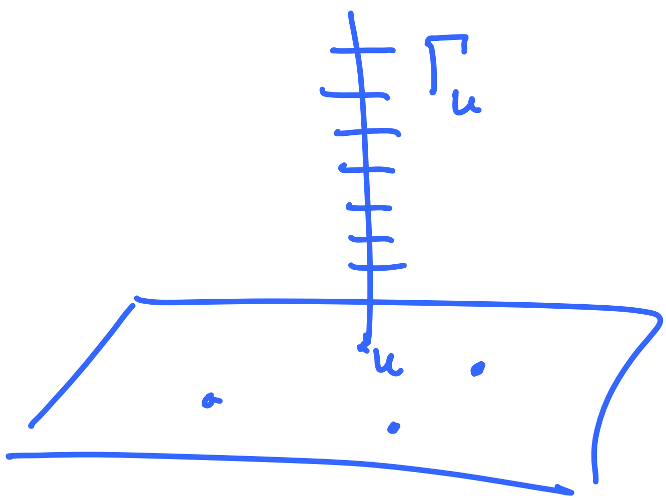
IN TERMS OF

<u>SPECIAL KÄHLER GEOMETRY</u>

REVIEW SPECIAL KÄHLER GEOM:

A USEFUL WAY TO PRESENT THIS FOR US IS TO BEGIN WITH THE LATTICE OF CHARGES Γ. NOW, Γ "DEPENDS ON u" GLOBALLY, BECAUSE THERE CAN BE MONODROMY AROUND COMPLEX COD. 1 DIVISORS IN $\mathcal{B}$.

THIS IS KNOWN AS A "LOCAL SYSTEM"



("LOCAL SYSTEM" IN GENERAL MEANS A BUNDLE WITH A FLAT CONNECTION.)

A.) WE ALSO NEED THE CENTRAL CHARGE FUNCTION Z WHICH IS LINEAR IN γ,

$$Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$$

B.) IN REGIONS OF $\mathcal{B}$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag} \, , \quad \Gamma_{mag} = \Gamma_{el}^{*}$$

$$= \mathrm{Span}\{\alpha^I\} \oplus \mathrm{Span}\{\beta_I\}$$

$$\langle \alpha^I, \alpha^J \rangle = \langle \beta_I, \beta_J \rangle = 0 \qquad \langle \alpha^I, \beta_J \rangle = \delta^I{}_J$$

CHOOSING A DUALITY FRAME:

CENTRAL CHARGE FUNCTION IS:

$$Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg}$$

$$a^I = Z_{\alpha_I}(u) \qquad a_{D,I} = Z_{\beta^I}(u)$$

C.) NOTE THAT $\mathrm{Hom}(\Gamma, \mathbb{C})$ IS A SYMPLECTIC VECTOR SPACE WITH SYMPLECTIC FORM

$$\omega = da^I \wedge da_{D,I}$$

(a^I, $a_{D,I}$ INDEPENDENT)

IN OUR PROBLEM Z IS NOT AN ARBITRARY ELEMENT OF $\mathrm{Hom}(\Gamma, \mathbb{C})$, RATHER, IT IS PARAMETRIZED BY $\mathcal{B}$, AND INDEED $\mathcal{B}$ IS COMPLETELY DETERMINED BY Z.

SO WE MAY REGARD $\mathcal{B}$ AS A SUBSPACE OF $\mathrm{Hom}(\Gamma, \mathbb{C})$, AND THE KEY POINT IS

THAT IT IS A

LAGRANGIAN SUBSPACE:

$$\langle dZ, dZ \rangle = 0$$

BECAUSE $\mathcal{B}$ IS LAGRANGIAN THE

$$a^I = Z_{\alpha_I}(u) \qquad I = 1, \ldots, r$$

ARE LOCAL COORD'S ON $\mathcal{B}$

AND MOREOVER THE $a_{D,I}$

ARE FUNCTIONS OF THE a^J.

BECAUSE $\mathcal{B}$ IS LAGRANGIAN,

LOCALLY THERE IS A

GENERATING "PREPOTENTIAL" $\mathcal{F}$

FOR THIS SUBSPACE:

$$a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}$$

D. THUS

$$\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J} = \tau_{JI} .$$

IN TERMS OF THIS DATA THE EFFECTIVE LAGRANGIAN IS

$$\boxed{\begin{aligned}\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im} \tau_{IJ} \left(da^I * d\bar{a}^J + F^I * F^J \right) \\ + \frac{1}{4\pi} \operatorname{Re} \tau_{IJ} \, F^I \wedge F^J + \ldots .\end{aligned}}$$

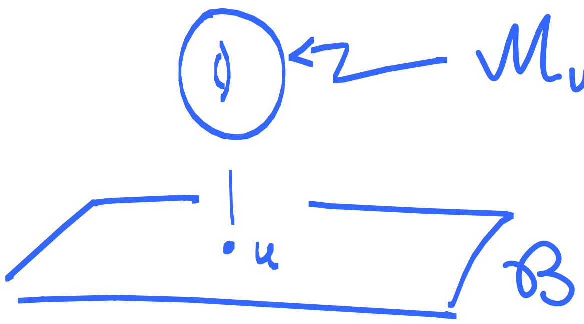
(A SUPERSPACE INTEGRAL OF $\mathcal{F}$, HENCE $\mathcal{N}=2$ SUSY)

E.) CLEARLY, PHYSICALLY SENSIBLE THEORIES HAVE $\mathrm{Im}\,\tau_{IJ} > 0$. THUS τ_{IJ} IS A <u>PERIOD MATRIX</u> FOR AN ABELIAN VARIETY

$$\mathcal{M}_u = \Gamma_u^* \underset{\mathbb{Z}}{\otimes} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

THESE FIT TOGETHER TO MAKE A FIBRATION OF TORI:

$$\begin{tikzcd} \mathcal{M}_u \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ u \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$



$$\boxed{\text{THE SPACE } \mathcal{M} \text{ WILL BE OF CENTRAL IMPORTANCE TO US}}$$

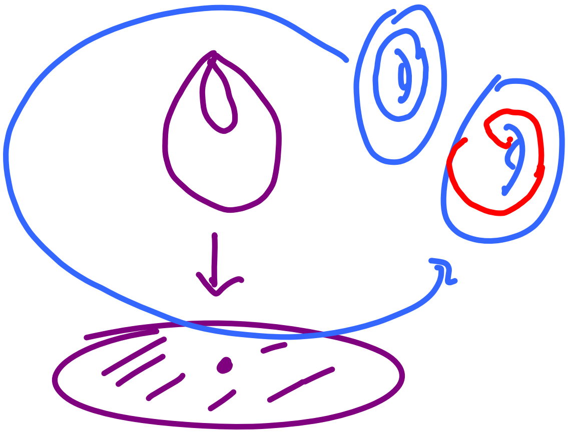
THAT CONCLUDES SPECIAL KÄHLER

IN ORDER TO REALIZE THIS ABSTRACT STRUCTURE:

S & W CONSTRUCT AN EXPLICIT FAMILY OF RIEMANN SURFACES Σ_u, TOGETHER WITH A MEROMORPHIC (1,0)-FORM λ_u ON Σ_u

- $\Gamma_u := H_1(\Sigma_u, \mathbb{Z})$
- $Z_\gamma(u) := \oint_\gamma \lambda_u \qquad \gamma \in \Gamma_u$
- $\mathcal{M}_u := \mathrm{JAC}(\Sigma_u)$

THE MONODROMY COMES FROM LEFSHETZ



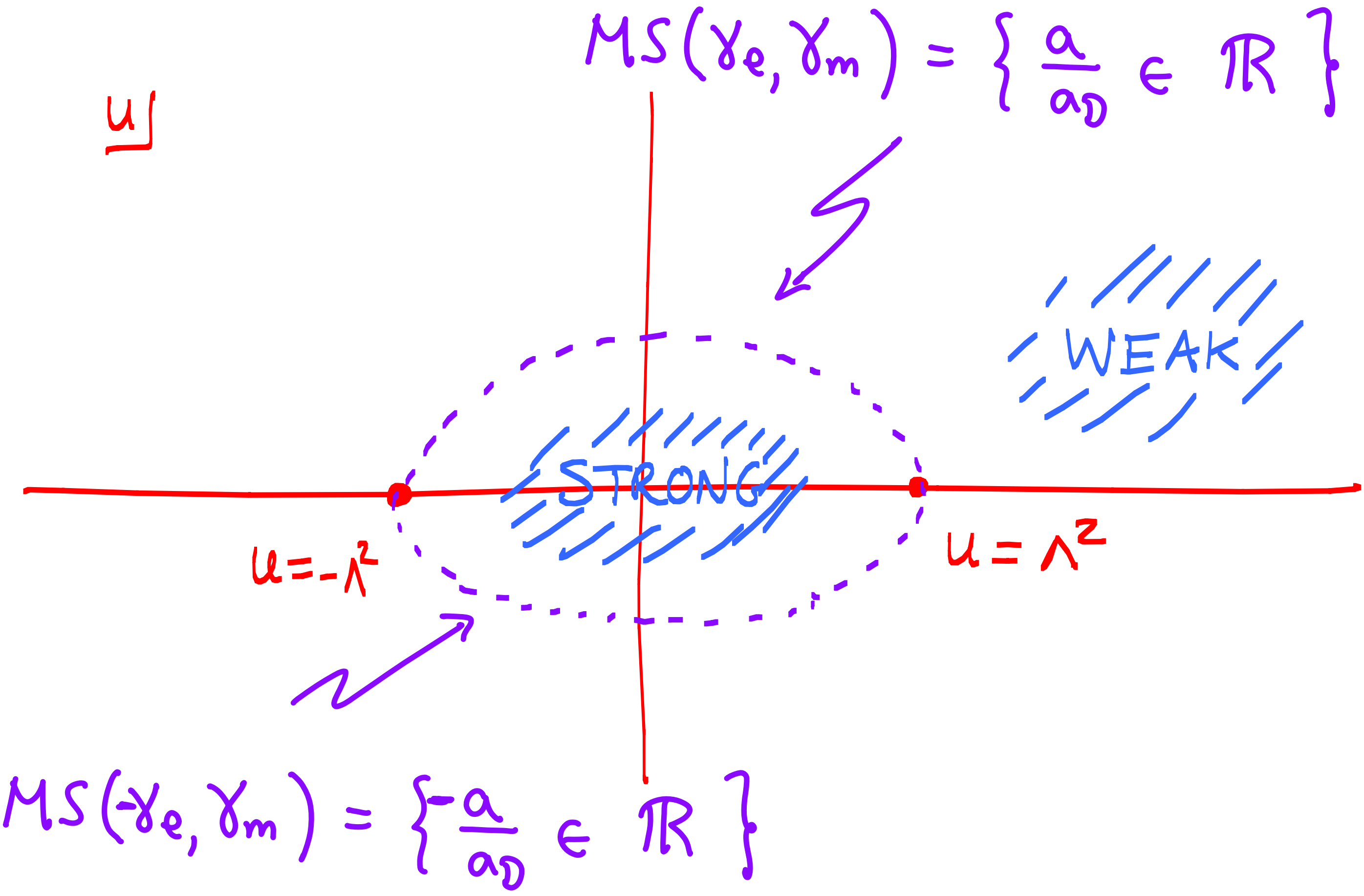
$\gamma \longrightarrow \gamma + \langle \gamma, \nu \rangle \nu$

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u : \quad t + \frac{\Lambda^4}{t} = v^2 - 2u$$

$$\lambda_u = v \frac{dt}{t}$$

$$a = \oint_\alpha \lambda_u \qquad a_D = \oint_\beta \lambda_u$$



AT WEAK COUPLING WE FIND SOLITONIC STATES OF CHARGE $(0,1)$, $Z = a_D$ (MONOPOLE) AND $(2,-1)$ $Z = 2a - a_D$ (DYON)

THESE VANISH ON THE DISCRIMINANT LOCUS $u = +\Lambda^2$ & $u = -\Lambda^2$ WHERE THE ELLIPTIC CURVE DEGENERATES.

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0)$$

$$\oplus \text{ CONJUGATE}$$

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1)$$

$$\oplus \text{ CONJUGATE}$$

[Bilal & Ferrari]

KS IDENTITY:

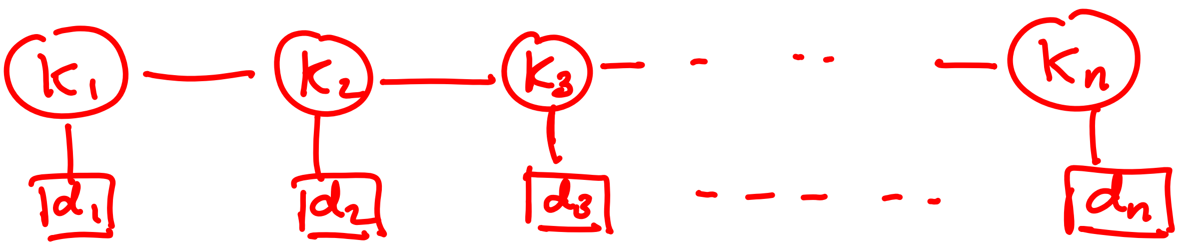
$$K_{2,-1} K_{0,1} = K_{0,1} K_{2,1} K_{4,1} \cdots K_{2,0}^{-2} \cdots K_{6,-1} K_{4,-1} K_{2,-1}$$

<u>IT IS TRUE!!!</u>

ELEMENTARY PROOF: GMN-1, APPENDIX A.

FOR GENERAL $d=4$, $\mathcal{N}=2$ THEORIES SUCH A COMPLETE KNOWLEDGE OF THE BPS SPECTRUM IS NOT KNOWN.

HOWEVER IN LECTURE 3 WE SHOW HOW TO WRITE THE SW CURVE FOR LINEAR QUIVER GAUGE THEORIES



AND IN LECTURE 4 WE GIVE AN ALGORITHM FOR COMPUTING THE SPECTRUM WHEN $k_i = 2$.

6. Compactifying $N=2, D=4$ on a circle

- Now consider the theory on $\mathbb{R}^3 \times S^1_R$.

- Low energy theory is a 3D σ-model: $\mathbb{R}^3 \longrightarrow \mathcal{M}$

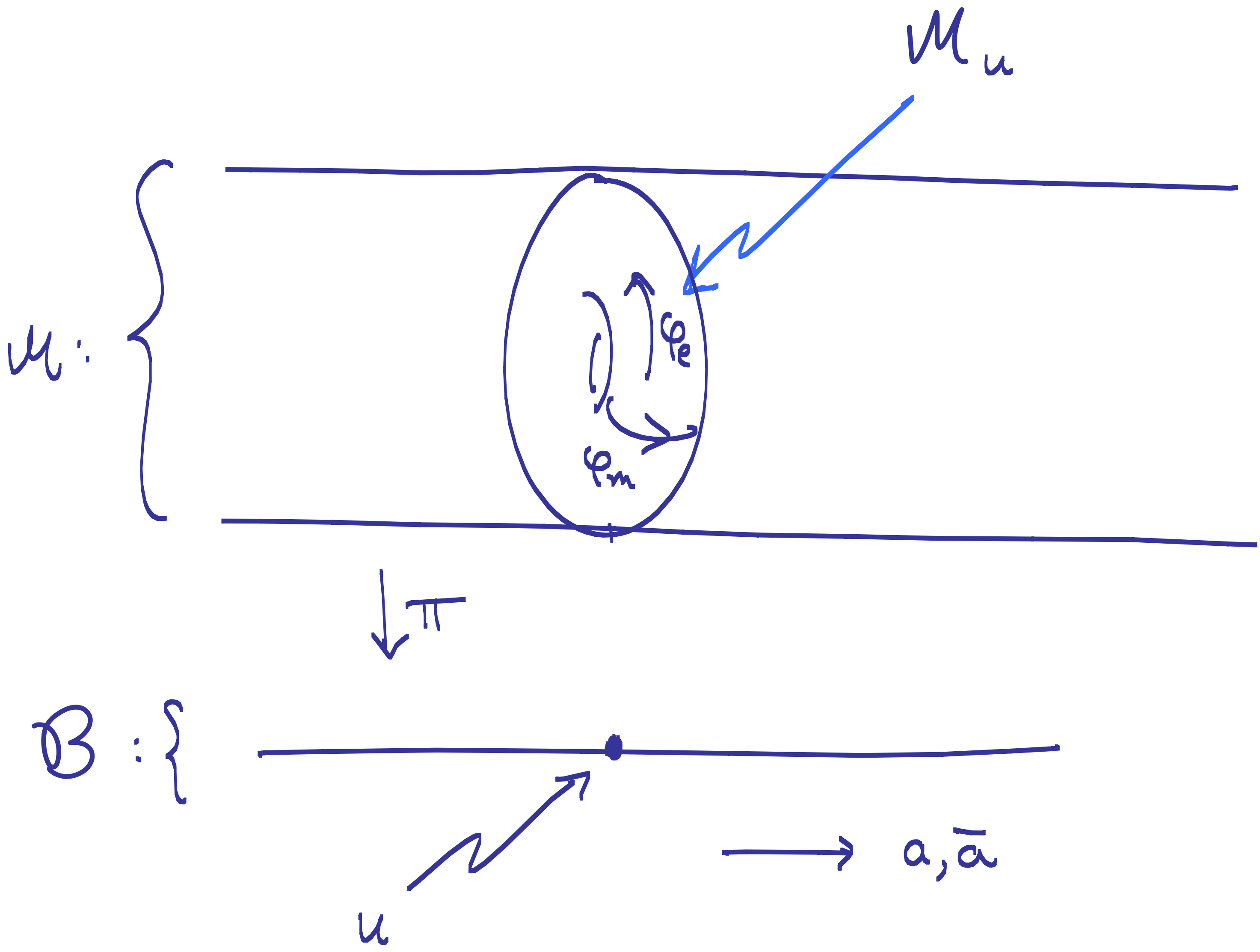
$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_3^I \, dx^3 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,3})_I \, dx^3 \end{aligned}\right\} \text{PERIODIC!}$$

<u>Topologically</u> $\mathcal{M}$ is a torus fibration over $\mathcal{B}$:

IT IS EXACTLY $\mathcal{M} = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$

THAT APPEARED ABOVE:



- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J$$

$$- \frac{1}{2R} (\operatorname{Im}\tau)^{-1, IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R (\operatorname{Im}\tau) |da|^2 + R^{-1} (\operatorname{Im}\tau)^{-1} |dz|^2$$

Details of the computation:

$\mathbb{R}^{1,2} \times S^1$ metric $ds^2 = dx^\mu dx_\mu + R^2 (dx^3)^2$

0 1 2   3 $x^3 \sim x^3 + 2\pi$

KK:

$$a^I \longrightarrow a^I(x^\mu)$$

$$A^I \longrightarrow \varphi_e(x^\mu)\, dx^3 + \bar{A}^I$$

$$F^I = d\varphi_e\, dx^3 + \bar{F}^I \qquad \varphi_e \sim \varphi_e + 1$$

$$da^I \underset{4}{*} d\bar{a}^J = R\,dx^3\; da^I \underset{3}{*} d\bar{a}^J$$

$$F^I \underset{4}{*} F^J = \frac{dx^3}{R}\, d\varphi_e^I \underset{3}{*} d\varphi_e^J + R\,dx^3\, F^I \underset{3}{*} F^J$$

Integrate over x^3 in action:

$$i \int_{\mathbb{R}^{1,2}} -\frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\, da^I \underset{3}{*} d\bar{a}^J - \frac{1}{2R}\,\mathrm{Im}\,\tau_{IJ}\, d\varphi_e^I \underset{3}{*} d\varphi_e^J$$

$$- \frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\, \bar{F}^I \underset{3}{*} \bar{F}^J - \mathrm{Re}\,\tau_{IJ}\, \bar{F}^I\, d\varphi_e^J$$

- Now dualize the gauge field by adding

$$\exp\ i\int \bar{F}^I \wedge d\varphi_{m,I}$$

$$\varphi_{m,I} \sim \varphi_{m,I} + 1.$$

and doing the Gaussian integral on $\bar{F}^I$ in

$$\exp\ i\int -\frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\,\bar{F}^I \underset{3}{*} \bar{F}^J + \bar{F}^I\left(d\varphi_{mI} - \mathrm{Re}\,\tau_{IJ}\,d\varphi_e^J\right)$$

Stationary point

$$\bar{F}^I = -\frac{1}{R}(\mathrm{Im}\,\tau)^{-1,IJ}\left(d\varphi_{mJ} - \mathrm{Re}\,\tau_{JK}\,d\varphi_e^K\right)$$

Stationary action is ½ the linear term ☒

7. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{B}$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.