Moonshine Phenomena, Supersymmetry, and Quantum Codes

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Field theory in condensed matter:
a symposium in honor of Nick Read

Yale, April 12, 2019

a project with Jeff Harvey

….. still in progress ….

1 Brief Background On Moonshine

2 Quantum Mukai Theorem & GTVW Model

3 Supercurrents & Codes

4 RR States: MOG Construction Of The Golay Code

5 Extension To Other Moonshine Examples

6 K-Theory Comment & Conclusions

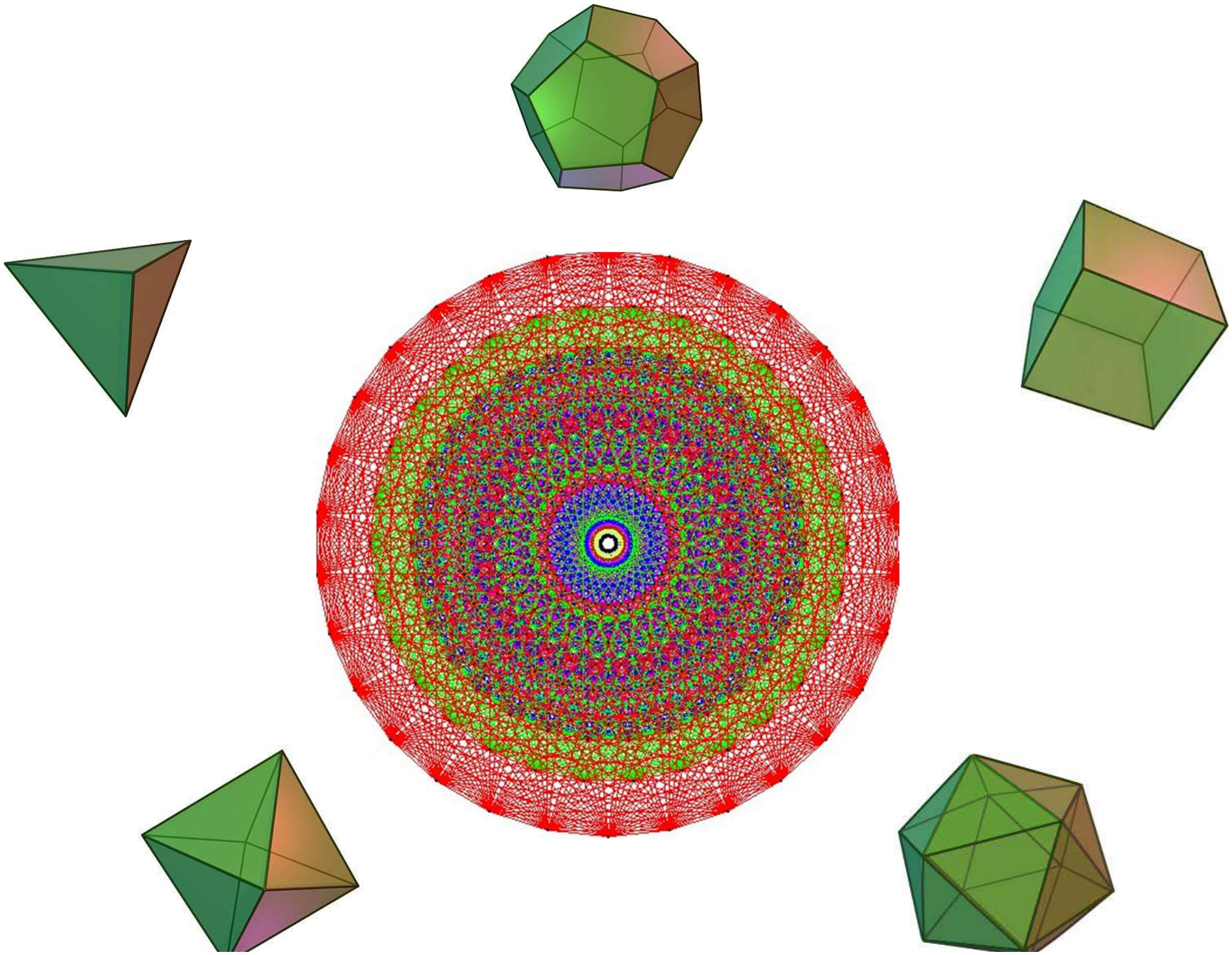
Philosophy – 1/2

We can divide physicists into two classes:

Our world is a random choice drawn from a huge ensemble:

Philosophy – 2/2

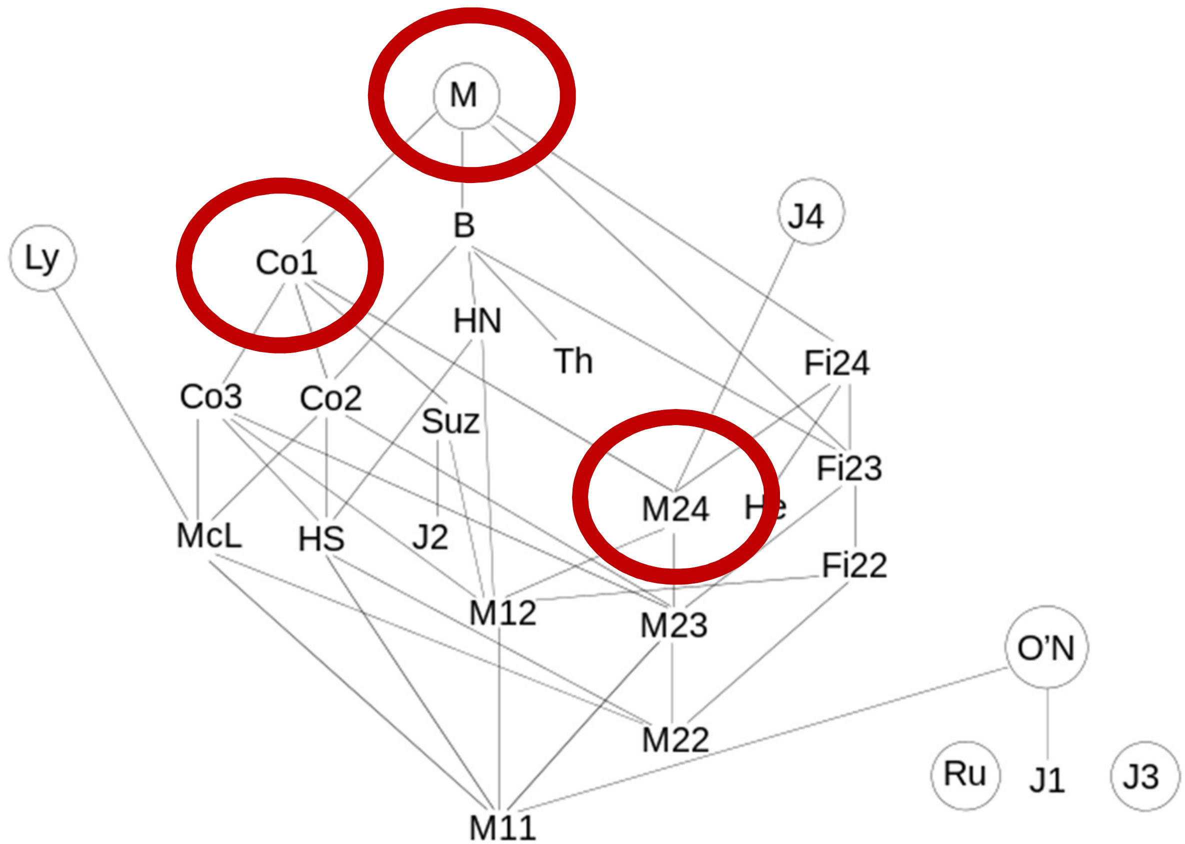
The fundamental laws of nature are based on some beautiful exceptional mathematical structure:



Background: Finite-Simple Groups

Jordan-Holder Theorem: Finite simple groups are the atoms of finite group theory.

$\mathbb{Z}_p$ p = prime $\quad A_n \; n \geq 5 \quad SL_n(\mathbb{F}_p)$ etc.



Background: McKay & Conway-Norton 1978-1979

$$J = \sum_n J_n q^n = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

Now list the dimensions of irreps of $\mathbb{M}$

R_n = 1, 196883, 21296876, 842609326, 18538750076, 19360062527, 293553734298,…. , $\sim 2.6 \times 10^{26}$

$$J_{-1} = R_1 \qquad J_1 = R_1 + R_2$$

$$J_2 = R_1 + R_2 + R_3 \qquad J_3 = 2R_1 + 2R_2 + R_3 + R_4$$

A way of writing J_n as a positive linear combination of the R_j for all n is a ``***solution of the Sum-Dimension Game***.''

There are infinitely many such solutions!!

Background: Characters

Which, if any, of these solutions is interesting?

Every solution defines an infinite-dimensional $\mathbb{Z}$-graded representation of $\mathbb{M}$

$$V = q^{-1} R_1 \oplus q(R_1 \oplus R_2) \oplus q^2(R_1 \oplus R_2 \oplus R_3) \oplus \cdots$$

Now for every $g \in \mathbb{M}$ we can compute the character:

$$\chi(q;g) \coloneqq Tr_V \, g \, q^N$$

A solution of the Sum-Dimension game is ***modular*** if the $\chi(q;g)$ is a modular function in $\Gamma_0(m)$ where $g^m = 1$.

Amazing Fact Of Monstrous Moonshine

There is a unique modular solution of the Sum-Dimension game!

Moreover the $\chi(q;g)$ have very special properties.

Why Cond. Matt. People Should Care

CFT explanation of Monstrous Moonshine by Frenkel, Lepowsky, Meurman, & Borcherds drove many developments in 2d CFT, expecially RCFT

(2d CFT also came from statmech and string theory)

But techniques introduced to explain moonshine – orbifolds, VOA, holomorphic CFT have played a key role as well and have led to many important advances…

e.g. modular tensor categories are a direct descendent of this research --

Chiral Conformal Field Theory

Massless scalar in 1+1 dimensions:

$$\mathrm{x}(\sigma, t) = x_L(\sigma + t) + x_R(\sigma - t)$$

Self-dual (chiral) scalar: Only x_L or x_R

e.g. edge modes in the FQHE

$$\partial_z x^j = -i \sum_n \alpha_n^j e^{inz}$$

$$\mathrm{j} = 1, \ldots, 24$$

$$z = \sigma + \tau$$

For Moonshine:
24 free chiral bosons
with periodicity $\mathbb{R}^{24}/\Lambda$

$$\left[\alpha_n^i, \alpha_m^j\right] = n\delta^{ij}\delta_{n+m,0}$$

$$\alpha_0^j = p^j \in \Lambda$$

Leech & Golay

FLM use torus associated to Leech lattice Λ:

Definition:[Cohn,Kumar,Miller,Radchenko,Viazovska]
$\Lambda \subset \mathbb{R}^{24}$ is the best sphere packing in d=24

$$Aut(\Lambda) = Co_0 \subset SO(24)$$

Λ can be constructed using the Golay code $\mathcal{G} \subset \mathbb{F}_2^{24}$

$\mathcal{G}$ is a special 12-dimensional subspace with nice error-correcting properties. Discovered @ Bell Labs in 1949 and used by Voyager 1&2 to send color photos

Definition: $M24 \subset S_{24}$ is the subgroup of permutations preserving the set $\mathcal{G}$

Special B-field

Moreover, target space torus has a very special ``B-field''

$\frac{1}{2!} B_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \Rightarrow$ ``topological term'' in the action

$$S = \int d^2\sigma \, (\, G_{\mu\nu} \, \partial_i x^{\mu} \, \partial^i x^{\nu} + B_{\mu\nu} \epsilon^{ij} \partial_i x^{\mu} \partial_j x^{\nu})$$

Translation symmetry on zero-modes converted to a magnetic translation group:

$$T\left(\frac{1}{2} v_1\right) T\left(\frac{1}{2} \, v_2\right) = (-1)^{v_1 \cdot v_2} \, T\left(\frac{1}{2} v_2\right) T\left(\frac{1}{2} v_1\right)$$

This is a discrete Heisenberg group: There is a unique irreducible representation: It is $2^{\frac{24}{2}} = 2^{12}$ dimensional.

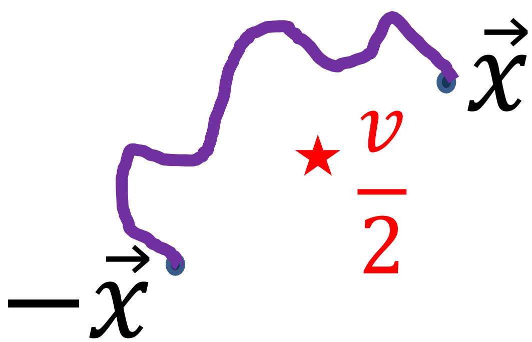
FLM Construction – 2/3

Now ``orbifold'' by $\vec{x} \to -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

''Orbifold by a symmetry G of a CFT'':
Gauge the symmetry

Symmetric twist fields: 2^{24}-dimensional space:

Basis: σ_v where v are the ``TRIM'' $\left[\frac{1}{2}v\right] \in T$



Chiral twist fields span a ``square-root'' of this representation: Very subtle quantum fields.

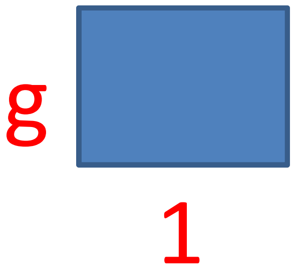
$\mathbb{M}$ As An Automorphism Group

OPE of conformal fields form a VOA: $\wp(\partial^* x) e^{i\, p \cdot x}$

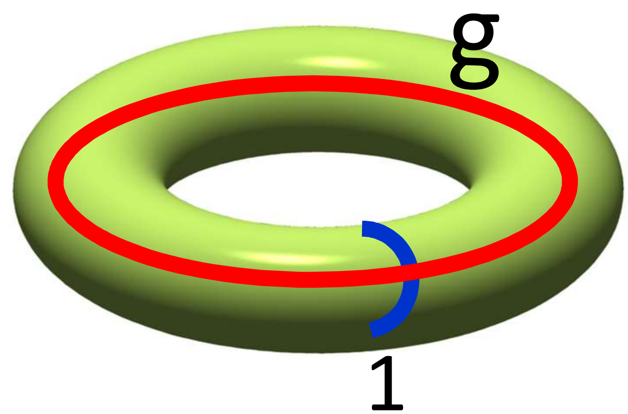
FLM & Borcherds: Automorphisms of the OPE algebra of the quotient theory = $\mathbb{M}$

Magnetic translation group of translations by TRIM + Co_1 + a ``quantum symmetry'' exchanging twisted & untwisted sectors generate the Monster.

Payoff: Conceptual Explanation of Modularity



$$:= Tr_{\mathcal{H}}\, g\, q^{L_0 - \frac{c}{24}} =$$



$\Longrightarrow$ Modularity

This is the gold standard for the conceptual explanation of Moonshine-modularity

(But a truly satisfying conceptual explanation of genus zero properties remains elusive.)

New Moonshine

Eguchi, Ooguri, Tachikawa 2010 + much interesting subsequent work.

Now generalize in two ways:

Generalize the target space torus T to sigma model with target $\mathcal{X}$

Make the theory worldsheet supersymmetric: (x^{μ}, ψ^{μ})

Get a CFT if $\mathcal{X}$ is a complex manifold that solves Einstein's equations: $R_{\mu\nu} = 0$

A K3 surface IS a solution of the Euclidean signature Einstein equations that is also compact and simply connected.

Now CFT has (4,4) superconformal symmetry.

(Super-) Conformal Symmetry:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

Superconformal symmetry $\Rightarrow$ supercurrent:

$$T_F(z) = \sum_r G_r z^{-r-\frac{3}{2}} \qquad T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

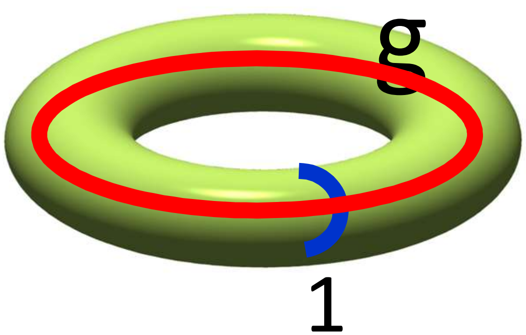
(p, q) superconformal symmetry $\Rightarrow$

p holomorphic $T_F^i(z)$ and q anti-holomorphic $T_F^a(\bar{z})$

Elliptic Genus (Witten index) for K3

for any symmetry group G of the CFT, if $g \in G$

$$\mathcal{E}^{g}_{\mathcal{C}}(z,\tau) = Tr_{\mathcal{H}_{RR}}(-1)^F \cdot g \cdot e^{2\pi i \tau\left(L_0 - \frac{c}{24}\right) + 2\pi i z J_0^3 - 2\pi i \bar{\tau}\left(\tilde{L}_0 - \frac{c}{24}\right)}$$

$=$ 

 Modularity

Example:

$$\mathcal{E}^{g=1}_{\mathcal{C}}(z,\tau) = 8\left[\left(\frac{\vartheta_2(z)}{\vartheta_2(0)}\right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_3(0)}\right)^2 + \left(\frac{\vartheta_4(z)}{\vartheta_4(0)}\right)^2\right]$$

The New Moonshine Phenomena Remain Unexplained – 1/2

Remarkably one can also define $\mathcal{E}^{g}(\tau, z)$ for all $g \in M24$ with the ``right'' modular properties,

<u>AS IF</u> there were an M24 symmetry of the K3 sigma model…..

But there is no obvious M24 action on the K3 sigma model !!

The New Moonshine Phenomena Remain Unexplained – 2/2

There is no known analog of the FLM construction revealing M24 symmetry. .

Despite 9 years of intense effort by a small, but devoted, community of physicists and mathematicians….

We don't understand something about symmetries of 2d conformal field theories.

It might be something important. Or maybe not.

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Quantum Mukai Theorem

Most obvious approach is to find a K3 surface $\mathcal{X}$ with a lot of symmetry, so that the σ $-$model also has a lot of symmetry.

Important no-go theorem: QMT

There is a 1-1 correspondence between

(a.) Symmetry groups of K3 sigma-models commuting with (4,4) supersymmetry.

(b.) Subgroups of Co_0 fixing sublattices of Λ of rank ≥ 4.

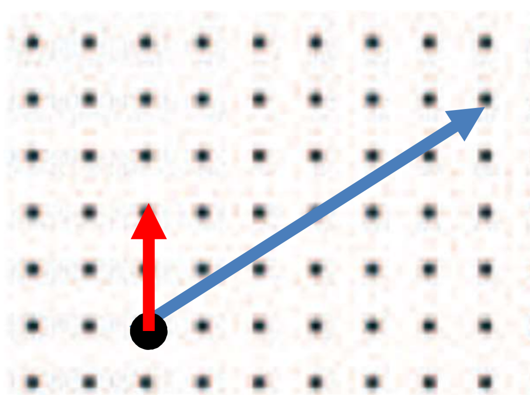
M. Gaberdiel, S. Hohenegger, R. Volpato 2011

Symmetries Preserving Sublattices

Given a symmetric lattice what sublattices fixed by some nontrivial subgroup of the point group ?

In general, a sublattice preserves none of the crystal symmetries of the ambient lattice.

Consider, e.g., the lattice generated by (p,q) in the square lattice in the plane.



Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of the Leech lattice:

99	4	245760	$2^8{:}M_{20}$	$2_{II}^{-2}4_{II}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{II}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4{:}A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{II}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{II}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
104	4	9216	$[2^{10}3^2]$	$2_{II}^{+4}3^{+2}$	0	1	2	1	1	-	Mon_a*
105	4	6144	$[2^{11}3]$	$2_{II}^{-2}4_6^{+2}3^{-1}$	0	1	4	1	1	-	Mon_a
106	4	5760	$2^4{:}A_6$	$4_5^{-1}8_1^{+1}3^{+1}$	2	1	1	1	2	1	M_{23}*
107	4	4096	$2^{1+8}{:}2^3$	4_4^{+4}	0	1	8	1	1	-	Mon_a
108	4	2520	A_7	$3^{+1}5^{+1}7^{+1}$	3	1	1	1	2	1	M_{23}*
109	4	1944	$3^{1+4}{:}2.2^2$	$2_2^{+2}3^{+3}$	1	1	1	1	1	-	S
110	4	1920	$2^4{:}S_5$	$4_3^{-1}8_1^{+1}5^{-1}$	2	1	2	1	3	1	M_{23}
111	4	1344	$2^3{:}L_2(7)$	$4_2^{+2}7^{+1}$	2	1	1	1	3	1	M_{23}*
112	4	1152	$Q(3^2{:}2)$	$8_6^{-2}3^{-1}$	2	1	2	1	2	1	M_{23}*

From the viewpoint of explaining Mathieu Moonshine, the QMT is a huge disappointment:

(a.) Some GHV groups are NOT subgroups of M24

(b.) M24 is not a subgroup of any quotient of any GHV group.

We need a new idea

Moonshine is about the elliptic genus.

Only (4,1) susy is needed to define the elliptic genus

Stab(4,1) is much bigger than Stab(4,4).

Whether it is ``big enough'' is unknown

GTVW Model

Largest group $Stab(4,4) \cong 2^8 : M20$ associated with a distinguished K3 sigma model investigated by Gaberdiel, Taormina, Volpato, Wendland.

2d susy sigma model with target:

$\mathcal{X} = T/\mathbb{Z}_2$ $\quad$ $T = \mathbb{R}^4/L$ $\quad$ L: 4d bcc lattice

Special B-field

$$B(v,w) = g(v,w)\ mod\ 2 \quad v,w \in \pi_1(T)$$

Equivalence To A WZW Model

Amazing result of GTVW:
This model is isomorphic to the product of 6 copies of the ***bosonic*** k=1 SU(2) WZW model !

WZW with $G = SU(2)^6$ and
each factor has WZW term with $k = 1$

$SU(2)$ current algebra with level $k = 1$
has 2 unitary hw irreps: V_0 and V_1

Nonabelian Bosonization

("Witten's nonabelian bosonization" or "FKS construction")

Gaussian model: $S = \frac{R^2}{4\pi}\int \partial x \tilde{\partial} x \qquad x \sim x + 2\pi$

$$e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}+w\,R\right)x}(z) \otimes e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}-w\,R\right)\tilde{x}}(\tilde{z})$$

At R=1 we have a theory equivalent to the $SU(2)_1$ WZW model

$$J^3(z) = \frac{1}{\sqrt{2}}\partial x(z), J^{\pm}(z) = e^{\pm i\sqrt{2}x}(z)$$

$$\tilde{J}^3(\tilde{z}) = \frac{1}{\sqrt{2}}\partial\tilde{x}(\tilde{z})\,, \tilde{J}^{\pm}(\tilde{z}) = e^{\pm i\sqrt{2}\tilde{x}}(\tilde{z})$$

Gives an $su(2)_L \oplus su(2)_R$ current algebra.

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Supersymmetry In A ***<u>Bosonic</u>*** WZW Model?

We need to find a holomorphic current $T_F(z)$ of dimension $\frac{3}{2}$

With OPE:

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

Chiral Fields Of Dimension 3/2

Introduce product of six holomorphic fields in the spin ½

$$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6} := \exp\left(\frac{i\sqrt{2}}{2} (\epsilon_1 X_1 + \epsilon_2 X_2 + \cdots + \epsilon_6 X_6) \right) \quad \epsilon_i \in \{ \pm 1 \}$$

$\Rightarrow$ 2^6 vertex operators of conformal dimension = $\left(\frac{1}{4}\right) \times 6 = \frac{3}{2}$

$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6}$ span a 2^6 dimensional vector space of holomorphic (3/2,0) operators.

Identify this space with the space of states in a system of 6 Qbits. For any $s \in (\mathbb{C}^2)^{\otimes 6}$ write V_s

Which Ones Are Supercurrents?

The V_s have OPE's:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}} T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2} J^A(z_2) + \frac{\bar{s}\Sigma^{AB} s}{z_{12}} J^A J^B(z_2) + \cdots$$

J^A : generators of $SU(2)^6$ affine Lie algebra, $A = 1, \ldots, 3 \cdot 6 = 18$

Σ^A, Σ^{AB} generate 1- and 2- Qbit errors

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

N=1 Generator

Up to global symmetry there is a unique N=1 generator.

Using results of GTVW it is V_Ψ for

$$\Psi = [\emptyset] + i\,[123456] + ([1234] + [3456] + 1256]) + i([12] + [34] + [56]) \\ + ([135] + [245] + [236] + [146]) - i([246] + 235] + [136] + [145])$$

$$[135] \coloneqq | -, +, -, +, -, + \rangle$$

Obtained by tedious translation from the susy for the K3 sigma model….

Is there a <u>*code*</u> governing this quantum state?

Yes!! It is the ``hexacode''

$\mathbb{F}_4$ And The Hexacode

Finite field of $4 = 2^2$ elements: $\mathbb{F}_4 = \{\, 0, 1, \omega, \bar{\omega} \,\}$

Addition: $1 + \omega = \bar{\omega}$ $\quad 1 + \bar{\omega} = \omega$ $\quad \omega + \bar{\omega} = 1$

Multiplication: $\omega \cdot \omega = \bar{\omega}$ $\quad \omega \cdot \bar{\omega} = 1$

Hexacode: $\mathcal{H}_6 \subset \mathbb{F}_4^6$

$$w = \left(a, b, c, \Phi_{abc}(1)\,, \Phi_{abc}(\omega), \Phi_{abc}(\bar{\omega})\right)$$

$$\Phi_{abc}(x) \coloneqq a\,x^2 + b\,x + c$$

Relation To Quaternion Group

$$Q \cong \{\pm 1, \pm i\,\sigma^1, \pm i\,\sigma^2, \pm i\,\sigma^3\} \subset SU(2)$$

Group of special unitary bit-flip and phase-flip errors in theory of QEC.

Associate Pauli operators $h(x)$ to $x \in \mathbb{F}_4$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(\omega) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad h(\bar{\omega}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$h(x)h(y) = c_{x,y}\, h(x+y)$$

$c_{x,y}$: A sign that cannot be removed by redefinitions.

N=1 Generator And The Hexacode

For $w = (x_1, x_2, \dots, x_6) \in \mathbb{F}_4^6$ define

$$h(w) := h(x_1) \otimes h(x_2) \otimes \cdots \otimes h(x_6)$$

$$h(w_1)h(w_2) = \chi(w_1, w_2)h(w_1 + w_2)$$

For general $w_1, w_2 \in \mathbb{F}_4^6$ cannot remove signs χ .

$$\Psi = 2^{-6} \sum_{w \in \mathcal{H}_6} h(w)|+^6\rangle$$

$$h(w_1)h(w_2) = h(w_1 + w_2) \qquad w_1, w_2 \in \mathcal{H}_6 \subset \mathbb{F}_4^6$$

$$\mathrm{Stab}_{Q^6}\Psi \supset \mathcal{H}_6$$

Relation To Other Quantum Codes

This [[6,0,4]] quantum code is related to a well known QEC constructed from a unique [[5,1,3]] code.

It is related to Ψ_{GTVW} by a local unitary transformation $u \in SU(2)^6$

We realized this with TOM MAINIERO.

Mainiero has shown how to formulate a cohomology theory associated to ANY quantum state in a multipartite system $\mathcal{H} = \otimes_i \mathcal{H}_i$

The Poincare polynomial is a surrogate for von Neumann entropy. Tom computed: $P(y) = 432\, y^2$ for both states.

Consequences: 1/2

V_Ψ generates an N=1 superconformal symmetry:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB}s}{z_{12}}J^A J^B(z_2) + \cdots$$

J^A : generators of $SU(2)^6$ affine Lie algebra, $A = 1, \ldots, \ 3 \cdot 6 = 18$

Σ^A, Σ^{AB} generate 1- and 2- qubit errors

$$\bar{\Psi}\Sigma^A\Psi = 0 \quad \& \quad \bar{\Psi}\Sigma^{AB}\Psi = 0$$

Because Ψ is in a QEC. $\Rightarrow$ $T_F = V_\Psi$

Consequences: 2/2

$Stab_{SU(2)^6}(\Psi)$ is a finite group

Again follows from the error-correcting properties of the hexacode because the generators of $SU(2)^6$ are the Σ^A

We do not know exactly what this group is. But:

$Stab_{Q^6}(\Psi)$ is just the hexacode!

(extended by $\mathbb{Z}_2^5$ ....)

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RR Sector And The Golay Code

The RR groundstates of K3 in the WZW description form a rep of $(SU(2)_L \times SU(2)_R)^6$

$$e^{\pm \frac{i}{\sqrt{2}} X_L^{(\alpha)}} \otimes e^{\pm \frac{i}{\sqrt{2}} X_R^{(\alpha)}} \in \left(\frac{1}{2}\right)_L^{(\alpha)} \otimes \left(\frac{1}{2}\right)_R^{(\alpha)}$$

$$\alpha = 1,2,3,4,5,6$$

There is a distinguished basis of RR groundstates:

$$\mathbb{H} \cong \left[\left(\frac{1}{2}\right)_L \otimes \left(\frac{1}{2}\right)_R \right]_{\mathbb{R}}$$

as $SU(2)_L \times SU(2)_R$ representations

The usual basis $1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ of quaternions corresponds to 4 distinguished spin states:

$$1 \leftrightarrow \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle \right) := |1\rangle$$

$$\mathfrak{i} \leftrightarrow \frac{1}{\sqrt{2}}\left(|+,+\rangle + |-,-\rangle \right) := |2\rangle$$

$$\mathfrak{j} \leftrightarrow \frac{i}{\sqrt{2}}\left(|+,+\rangle - |-,-\rangle \right) := |3\rangle$$

$$\mathfrak{k} \leftrightarrow \frac{i}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle \right) := |4\rangle$$

In this basis the action of $h(x)_L \otimes h(x)_R$ is diagonal, e.g $h(1)_L \otimes h(1)_R$ takes:

$$|1\rangle \to |1\rangle, \qquad |2\rangle \to |2\rangle, \qquad |3\rangle \to -|3\rangle, \qquad |4\rangle \to -|4\rangle$$

Column Interpretations Of Hexacode Digits

$|1\rangle \to |1\rangle$, $|2\rangle \to |2\rangle$, $|3\rangle \to -|3\rangle$, $|4\rangle \to -|4\rangle$

$$\begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$h(x)_L \otimes h(x)_R$$

$$\begin{array}{ccccc} x = & 0 & 1 & \omega & \bar{\omega} \\ \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{array} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

So when we consider a general element of $Stab_{Q^6}(\Psi)$ acting on the entire RR sector we get a 4×6 array of 0's and 1's

	$h(x_1)$	$h(x_2)$	$h(x_3)$	$h(x_4)$	$h(x_5)$	$h(x_6)$
$\|1\rangle$						
$\|2\rangle$						
$\|3\rangle$						
$\|4\rangle$						

Example:

	$h(1)$	$h(1)$	$h(\omega)$	$h(\omega)$	$h(\bar{\omega})$	$h(\bar{\omega})$
$\|1\rangle$	0	0	0	0	0	0
$\|2\rangle$	0	0	1	1	1	1
$\|3\rangle$	1	1	0	0	1	1
$\|4\rangle$	1	1	1	1	0	0

Golay Code & The MOG

Nontrivial statement: The length 24 codewords generated from $Stab_{Q^6}(\Psi)$ are Golay code words.

This gives half the Golay code $\mathcal{G}^+$

To get the full Golay code include worldsheet parity (exchanging left- and right-moving dof). This acts as the parity operator in $O(4)$

$$\begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ``odd interpretations of hexacode digits "

Golay Code & The MOG

The action of the stabilizer of $\Psi_L - \Psi_R$ within $\langle P, Q^6 \rangle \subset \left(Pin(4)\right)^6$ in the canonical basis of RR states defines the full Golay code.

This presentation of the Golay code is the Miracle Octad Generator of Curtis and Conway.

Result: A clean physical interpretation of the MOG.

So What?

The Golay code can be found in this action of symmetries commuting with $N = 1$ supersymmetry.

By definition, the automorphism group of the Golay code is $M24$

So M24 is a symmetry group

OF

the group of symmetries…

Is this the long-sought explanation of Mathieu Moonshine?

Not yet: We do not understand why the ``symmetry group OF the group of symmetries'' should imply symmetry properties of the Witten index.

However, along the way we have found some intriguing relations between quantum codes, supersymmetry and Moonshine.

We can ask if that relation persists in other examples exhibiting Moonshine.

1. Brief Background On Moonshine
2. Quantum Mukai Theorem & GTVW Model
3. Supercurrents & Codes
4. RR States: MOG Construction Of The Golay Code
5. Extension To Other Moonshine Examples
6. K-Theory Comment & Conclusions

Other Moonshine Examples

Interestingly, a similar pattern emerges for the other two moonshine examples for Co_1 and $\mathbb{M}$

Co_1 based on $Ising^{\otimes 24}$: There is a unique supercurrent based on a quantum (Golay) code: Essentially follows from work of John Duncan (here at Yale).

Beauty and the Beast: Superconformal Symmetry in a Monster Module

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Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c = 24$. In string terminology, their construction corresponds to compactification on a $\mathbf{Z}_2$ asymmetric orbifold constructed from the torus $\mathbf{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c} = 16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

Important gap: What is the actual supercurrent?
The above ideas will probably allow us
to fill this gap.

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Side Comment: B-Fields On Tori

B-fields on tori have played an important role in our story today.

So, I will repeat a question I posed to the Cond-Matt community when I spoke at Duncan Haldane's 60th birthday conference at PCTS.

In string theory, B-fields lead to ``twisting'' of K-theory. [Witten]

In condensed matter, K-theory is used to classify topological phases of matter.
[Kitaev; Schneider, Ryu, Furusaki, and Ludwig;Read]

Applied to crystallographic topological insulators, phases are classified by twisted equivariant K-theory (of the Brillouin torus). [Freed & Moore]

For normal electrons moving in crystals one finds a ***canonical*** twisting (= B-field)

But! Other interesting twistings of equivariant K-theory of the Brillouin torus exist!

Are they realized by other phases of matter ?

Conclusions

1. New approach to Mathieu Moonshine based on Stab(4,1)

   We do no know yet if this will lead to a solution or not.

2. Interesting connections between QEC and 2d N=1 superconformal symmetry – raises many questions.

HAPPY BIRTHDAY NICK!!!!